# Tangent cone algorithm for homogenized differential operators

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**Abstract**

We extend Mora's tangent cone or the écart division algorithm to a homogenized ring of differential operators. This allows us to compute standard bases of modules over the ring of analytic differential operators with respect to sufficiently general orderings which are needed in the $D$-module theory.

*Key words:* tangent cone algorithm, division, standard base, Gröbner base, differential operator, $D$-module

## 1 Introduction

In the theory of $D$-modules, one often needs to compute standard or Gröbner bases of ideals of, or modules over, the ring $\mathcal{D}$ of analytic differential operators with respect to some ordering. In terms of the coordinate system $x = (x_1, \ldots, x_n)$ of $\mathbb{C}^n$, an element $P$ of $\mathcal{D}$ is written in a finite sum $P = \sum_{\beta\in\mathbb{N}^n} a_\beta(x)\partial^\beta$ with $a_\beta(x)$ belonging to $\mathbb{C}\{x\}$, the ring of convergent power series. Here we use the notation $\partial^\beta = \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$ with $\partial_i = \partial/\partial x_i$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$).

Let $D$ be the Weyl algebra, i.e., the ring of differential operators with polynomial coefficients, which is a subring of $\mathcal{D}$. A $D$-module is a global object in the sense that it is considered to be defined on the affine space $\mathbb{C}^n$. On the other hand, a $\mathcal{D}$-module is a local object; in fact, it is regarded as a stalk of a sheaf of modules in the context of analytic $D$-module theory. Hence in order to compute local invariants of $D$-modules, we need standard bases over $\mathcal{D}$ rather than Gröbner bases over $D$ although in some cases, e.g., as in the computation of $b$-functions and restrictions of $D$-modules (cf. [9]), we can extract local information as well from the latter.

¿From the computational viewpoint, we are mostly interested in a submodule $N$ of $\mathcal{D}^r$ which is 'algebraic' in the sense that it is generated by elements of $D^r$. Our main purpose is to compute a standard basis of $N$ with respect to a sufficiently general ordering which is compatible with the left $\mathcal{D}$-module structure of $\mathcal{D}^r$. For example, a standard bases with respect to an ordering compatible with what is called the V-filtration is needed in order to compute local invariants such as the indicial polynomial (or the $b$-function), the restriction, and the local cohomology group of a $\mathcal{D}$-module.

If an ordering is defined first by a well-ordering on the derivations, and then by a reverse well-ordering on the coefficients (polynomials or power series) as a tie-breaker, then one can apply the tangent cone or the écart division algorithm of Mora [8] directly to the coefficients. However, this is not the case with, e.g., the ordering compatible with the V-filtration. For this reason, we adopt homogenization of differential operators following [1] by using a new variable which we denote $h$. Working in this homogenized ring $\mathcal{D}^{(h)}$ of $\mathcal{D}$, we can extend Mora's tangent cone algorithm for power series in its extended form given by Gräbe [3] and the Singular team [6] (see also [2]) to algebraic submodules of $(\mathcal{D}^{(h)})^r$ with respect to sufficiently general monomial orderings.

Mora's tangent cone algorithm can be regarded as an algebraic counterpart of the Weierstrass-Hironaka division theorem for power series. Our tangent cone algorithm is an algebraic counterpart of the division theorem of [1] for $\mathcal{D}^{(h)}$, or of its vector version given in [4] (see also [5]).

By using this tangent cone algorithm, we obtain an algorithm to compute standard bases and syzygies of algebraic modules over $\mathcal{D}^{(h)}$. In fact, we prove analogues of Buchberger's criterion for generators to be a standard base, and of Schreyer's theorem on syzygies. We remark that our division theorem is essentially used in proving the correctness of these analogues. As is presented in [1], standard bases over $\mathcal{D}^{(h)}$ give standard bases over $\mathcal{D}$ via dehomogenization $h = 1$. As an application, we obtain an algorithm to compute minimal filtered free resolutions of $\mathcal{D}^{(h)}$-modules defined in [4], for which the local standard base computation is essential instead of the global Gröbner base computation.

Standard bases can also be computed by bihomogenization, which is a generalization of Lazard's method [7] to algebraic modules over $\mathcal{D}^{(h)}$. Hence there are at least two methods to compute standard bases over $\mathcal{D}^{(h)}$ (and hence over $\mathcal{D}$). We give some examples comparing these two methods by using software Kan/sm1 [12].

## 2 Tangent cone algorithm for algebraic differential operators

By the homogenization process, we can switch from $\mathcal{D}$-modules to modules over the ring $\mathcal{D}^{(h)}$ of homogenized differential operators, which are easier to handle from the computational as well as the theoretical viewpoint. Especially, we have the Weierstrass-Hironaka type division theorem for free modules over $\mathcal{D}^{(h)}$ with respect to sufficiently general monomial orderings as was shown in [1], [4]. Our purpose is to prove its algebraic and algorithmic analogue.

The ring $\mathcal{D}^{(h)}$ is the $\mathbb{C}$-algebra generated by $\mathbb{C}\{x\}$, $\partial_1, \dots, \partial_n$, and a new variable $h$ with the commuting relations

$$ha = ah, \quad h\partial_i = \partial_i h, \quad \partial_i\partial_j = \partial_j\partial_i, \quad \partial_i a - a\partial_i = \frac{\partial a}{\partial x_i} h$$

for any $a \in \mathbb{C}\{x\}$ and $i, j \in \{1, \dots, n\}$. It is a non-commutative graded $\mathbb{C}$-algebra with the grading

$$\mathcal{D}^{(h)} = \bigoplus_{d \geq 0} (\mathcal{D}^{(h)})_d \quad \text{with} \quad (\mathcal{D}^{(h)})_d := \bigoplus_{|\beta|+k=d} \mathbb{C}\{x\}\partial^\beta h^k.$$

An element $P$ of $\mathcal{D}^{(h)}$ is uniquely written as a finite sum $P = \sum_{\beta \in \mathbb{N}^n, k \in \mathbb{N}} a_{\beta k}(x)\partial^\beta h^k$ with $a_{\beta k} \in \mathbb{C}\{x\}$.

Let us denote by $\mathbb{C}[x]_0$ the subring of $\mathbb{C}\{x\}$ consisting of rational functions whose denominators do not vanish at $0 \in \mathbb{C}^n$. Then we put

$$\mathcal{D}^{(h)}_{\mathrm{alg}} := \Big\{ P = \sum_{\beta,k} a_{\beta k}(x)\partial^\beta h^k \ \in \mathcal{D}^{(h)} \mid a_{\beta k}(x) \in \mathbb{C}[x]_0 \Big\},$$

which is a subring of $\mathcal{D}^{(h)}$. We also denote by $h_{(\mathbf{0},\mathbf{1})}(D)$ the subring of $\mathcal{D}^{(h)}_{\mathrm{alg}}$ consisting of operators with polynomial coefficients:

$$h_{(\mathbf{0},\mathbf{1})}(D) := \Big\{ P = \sum_{\beta,k} a_{\beta k}(x)\partial^\beta h^k \ \in \mathcal{D}^{(h)} \mid a_{\beta k}(x) \in \mathbb{C}[x] \Big\}.$$

These two rings are graded $\mathbb{C}$-subalgebras of $\mathcal{D}^{(h)}$. Note that $\mathcal{D}^{(h)}$ is faithfully flat over $\mathcal{D}^{(h)}_{\mathrm{alg}}$, while $\mathcal{D}^{(h)}_{\mathrm{alg}}$ is flat, but not faithfully flat, over $h_{(\mathbf{0},\mathbf{1})}(D)$.

Graded free modules over $\mathcal{D}^{(h)}$, $\mathcal{D}^{(h)}_{\mathrm{alg}}$, and $h_{(\mathbf{0},\mathbf{1})}(D)$ are specified by the rank $r$ and a shift vector $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}^r$, which we denote by

$$
\begin{aligned}
(\mathcal{D}^{(h)})^r[\mathbf{n}] &:= \bigoplus_{d\in\mathbb{Z}} \left( (\mathcal{D}^{(h)})_{d-n_1} \oplus \cdots \oplus (\mathcal{D}^{(h)})_{d-n_r} \right), \\
(\mathcal{D}^{(h)}_{\mathrm{alg}})^r[\mathbf{n}] &:= \bigoplus_{d\in\mathbb{Z}} \left( (\mathcal{D}^{(h)}_{\mathrm{alg}})_{d-n_1} \oplus \cdots \oplus (\mathcal{D}^{(h)}_{\mathrm{alg}})_{d-n_r} \right), \\
h_{(\mathbf{0},\mathbf{1})}(D)^r[\mathbf{n}] &:= \bigoplus_{d\in\mathbb{Z}} \left( h_{(\mathbf{0},\mathbf{1})}(D)_{d-n_1} \oplus \cdots \oplus h_{(\mathbf{0},\mathbf{1})}(D)_{d-n_r} \right).
\end{aligned}
$$

A homogeneous element, i.e. an element of the $d$th direct summand of one of these graded modules is said to be $(\mathbf{0},\mathbf{1})$-homogeneous of degree $d$ (with respect to $\mathbf{n}$). In the sequel, we mainly work in $h_{(\mathbf{0},\mathbf{1})}(D)^r[\mathbf{n}]$.

We take another (arbitrary) shift vector $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r$ for the $(-\mathbf{1},\mathbf{1})$-homogenization. For a vector of operators $P \in h_{(\mathbf{0},\mathbf{1})}(D)^r$ of the form

$$
P = \sum_{i=1}^{r} \sum_{k\in\mathbb{N},\alpha,\beta\in\mathbb{N}^n} a_{\alpha\beta ki} x^\alpha \partial^\beta h^k e_i
$$

with $e_1 := (1, 0, \ldots, 0), \ldots, e_r := (0, \ldots, 0, 1) \in \mathbb{Z}^r$ and $a_{\alpha\beta ki} \in \mathbb{C}$, put

$$
m := \min\{|\beta| - |\alpha| + v_i \mid a_{\alpha\beta ki} \neq 0\}.
$$

Then the $(-\mathbf{1},\mathbf{1})$-homogenization $P^{(s)}$ of $P$ is an element of $(h_{(\mathbf{0},\mathbf{1})}(D)[s])^r$ defined by

$$
P^{(s)} := \sum_{i=1}^{r} \sum_{k\in\mathbb{N},\alpha,\beta\in\mathbb{N}^n} a_{\alpha\beta ki} s^{|\beta|-|\alpha|+v_i-m} x^\alpha \partial^\beta h^k e_i
$$

with a new variable $s$. In general, an element

$$
Q = \sum_{i=1}^{r} \sum_{k,\nu\in\mathbb{N},\alpha,\beta\in\mathbb{N}^n} a_{\nu\alpha\beta ki} s^\nu x^\alpha \partial^\beta h^k e_i
$$

of $(h_{(\mathbf{0},\mathbf{1})}(D)[s])^r$ is said to be $(-\mathbf{1},\mathbf{1})$-homogeneous of degree $p$ if there exists an integer $p$ such that $a_{\nu\alpha\beta ki} = 0$ unless $|\beta| - |\alpha| - \nu + v_i = p$. If, in addition, $Q$ is a $(\mathbf{0},\mathbf{1})$-homogeneous element of $h_{(\mathbf{0},\mathbf{1})}(D)^r[\mathbf{n}]$ of degree $d$, then we call $Q$ *bihomogeneous of bidegree* $(d, p)$. The product of a bihomogeneous element of $h_{(\mathbf{0},\mathbf{1})}(D)$ and a bihomogeneous element of $(h_{(\mathbf{0},\mathbf{1})}(D))^r$ is also bihomogeneous. (For $h_{(\mathbf{0},\mathbf{1})}(D)$, we take the shift vector $(0)$ both for the $(\mathbf{0},\mathbf{1})$- and the $(-\mathbf{1},\mathbf{1})$-homogeneity.)

Introducing commutative variables $\xi = (\xi_1, \ldots, \xi_n)$ corresponding to $\partial$, the (total) symbol of $P = \sum_{i=1}^r \sum_{\alpha,\beta\in\mathbb{N}^n,k\in\mathbb{N}} a_{\alpha\beta ki} x^\alpha \partial^\beta h^k e_i \ \in (\mathcal{D}^{(h)})^r$ is defined to be $\sum_{i=1}^r \sum_{\alpha,\beta\in\mathbb{N}^n,k\in\mathbb{N}} a_{\alpha\beta ki} x^\alpha \xi^\beta h^k e_i \ \in (\mathbb{C}\{x\}[\xi, h])^r$. We fix an ordering $\prec$ among the monomials $\{x^\alpha \xi^\beta h^k e_i\}$ in $\mathbb{C}[x, \xi, h]^r$ which is compatible with multiplication (i.e. a monomial ordering) and satisfies the conditions

$$
\begin{aligned}
&|\beta| + k + n_i < |\beta'| + k' + n_j \quad \Rightarrow \quad x^\alpha \xi^\beta h^k e_i \prec x^{\alpha'} \xi^{\beta'} h^{k'} e_j, && (2.1)\\
&x^\alpha e_i \preceq e_i \text{ for any } \alpha \in \mathbb{N}^n \text{ and } i = 1, \dots, r, && (2.2)\\
&h e_i \prec x_j \xi_j e_i \text{ for any } i = 1, \dots, r \text{ and } j = 1, \dots, n. && (2.3)
\end{aligned}
$$

Note that the condition (2.1) is not really needed because we shall deal only with $(\mathbf{0}, \mathbf{1})$-homogeneous operators. With respect to this ordering, the leading monomial of a nonzero vector

$$
P = \sum_{i=1}^{r} \sum_{k \in \mathbb{N}, \alpha, \beta \in \mathbb{N}^n} a_{\alpha\beta k i} x^\alpha \partial^\beta h^k e_i \ \in (\mathcal{D}^{(h)})^r
$$

is the maximum in $\prec$:

$$
\mathrm{LM}_\prec(P) := \max{}_\prec \{ x^\alpha \xi^\beta h^k e_i \mid a_{\alpha\beta k i} \neq 0 \}.
$$

We call the $i$ such that $\mathrm{LM}_\prec(P) = x^\alpha \xi^\beta h^k e_i$ the *leading position* of $P$, denoted $\mathrm{LP}_\prec(P)$. Note that $\mathrm{LM}_\prec(QP) = \mathrm{LM}_{\prec_i}(Q)\mathrm{LM}_\prec(P)$ holds for $Q \in \mathcal{D}^{(h)}$ and $P \in (\mathcal{D}^{(h)})^r$ with $\mathrm{LP}_\prec(P) = i$, where $\prec_i$ is the ordering for $\mathcal{D}^{(h)}$ defined by

$$
x^\alpha \xi^\beta h^k \prec_i x^{\alpha'} \xi^{\beta'} h^{k'} \quad \Leftrightarrow \quad x^\alpha \xi^\beta h^k e_i \prec x^{\alpha'} \xi^{\beta'} h^{k'} e_i,
$$

in view of the condition (2.3).

Now we define an ordering $\prec_s$ among monomials $\{ s^\nu x^\alpha \xi^\beta h^k e_i \mid k, \nu \in \mathbb{N},\ \alpha, \beta \in \mathbb{N}^n,\ i = 1, \dots, r \}$ of $(h_{(\mathbf{0},\mathbf{1})}(D)[s])^r$ by

$$
\begin{aligned}
& s^\nu x^\alpha \xi^\beta h^k e_i \prec_s s^{\nu'} x^{\alpha'} \xi^{\beta'} h^{k'} e_j \\
\Leftrightarrow \quad & \begin{cases}
\nu + k + |\alpha| + n_i - v_i < \nu' + k' + |\alpha'| + n_j - v_j \\
\text{or} \quad (\nu + k + |\alpha| + n_i - v_i = \nu' + k' + |\alpha'| + n_j - v_j \\
\text{and } x^\alpha \xi^\beta h^k e_i \prec x^{\alpha'} \xi^{\beta'} h^{k'} e_j).
\end{cases}
\end{aligned}
$$

Note that $\prec_s$ is a well-ordering.

**Lemma 2.1** *Suppose that $P, Q \in (h_{(\mathbf{0},\mathbf{1})}(D)[s])^r$ are bihomogeneous of the same bidegree. Then $\mathrm{LM}_\prec(P|_{s=1}) \prec \mathrm{LM}_\prec(Q|_{s=1})$ holds if and only if $\mathrm{LM}_{\prec_s}(P) \prec_s \mathrm{LM}_{\prec_s}(Q)$.*

Proof: First assume that $P, Q$ are monomials $P = s^\nu x^\alpha \partial^\beta h^k e_i$, $Q = s^{\nu'} x^{\alpha'} \partial^{\beta'} h^{k'} e_j$. Then by the bihomogeneity we have

$$
|\beta| + k + n_i = |\beta'| + k' + n_j, \quad |\beta| - |\alpha| - \nu + v_i = |\beta'| - |\alpha'| - \nu' + v_j.
$$

This implies

$$\nu + k + |\alpha| + n_i - v_i = \nu' + k' + |\alpha'| + n_j - v_j.$$

Hence the assertion follows from the definition of the ordering $\prec_s$. We can prove the assertion in the general case by the same argument. $\square$

Let $P, Q$ be nonzero elements of $(h_{(\mathbf{0},\mathbf{1})}(D)[s])^r$. If $\mathrm{LM}_{\prec_s}(Q)$ divides $\mathrm{LM}_{\prec_s}(P)$, let $U \in h_{(\mathbf{0},\mathbf{1})}(D)[s]$ be the monomial whose total symbol is $\mathrm{LM}_{\prec_s}(P)/\mathrm{LM}_{\prec_s}(Q)$. (Here the canonical generators $e_1, \dots, e_r$ are regarded as commutative indeterminates rather than vectors. ) Then we define $\mathrm{Red}(P,Q)$ to be a list

$$\mathrm{Red}(P,Q) = [R,\, U] \quad \text{with } R := P - UQ.$$

Then $\mathrm{LM}_{\prec_s}(R) \prec_s \mathrm{LM}_{\prec_s}(P)$ holds if $R \neq 0$. Suppose $P, Q \in (h_{(\mathbf{0},\mathbf{1})}(D))^r$ are bihomogeneous. Then $R$ is also bihomogeneous of the same bidegree as $P$ and $\mathrm{LM}_{\prec}(R|_{s=1}) \prec \mathrm{LM}_{\prec}(P|_{s=1})$ holds if $R \neq 0$. The latter assertion follows from Lemma 2.1.

By using the bihomogeneity, we can extend a homogenized version of Mora's écart algorithm ([3],[6], we follow the presentation in [2]) for polynomials to free modules over $\mathcal{D}_{\mathrm{alg}}^{(h)}$ as follows:

**Algorithm 2.2 (écart division algorithm for $h_{(\mathbf{0},\mathbf{1})}(D)$)**
Input: $P, P_1, \dots, P_m$: homogeneous nonzero elements of $h_{(\mathbf{0},\mathbf{1})}(D)^r[\mathbf{n}]$.
Output: $a(x) \in \mathbb{C}[x]$, $Q = (Q_1, \dots, Q_m) \in h_{(\mathbf{0},\mathbf{1})}(D)^m$, and $R \in h_{(\mathbf{0},\mathbf{1})}(D)^r$ such that

- $a(x)P = Q_1P_1 + \cdots + Q_mP_m + R$,
- $a(0) \neq 0$,
- $\mathrm{LM}_{\prec}(Q_iP_i) \preceq \mathrm{LM}_{\prec}(P)$ if $Q_i \neq 0$,
- $\mathrm{LM}_{\prec}(R)$ is not divisible by $\mathrm{LM}_{\prec}(P_i)$ for any $i$ if $R \neq 0$.

$\mathcal{G} := [P_1^{(s)}, \dots, P_m^{(s)}]$ (a list), $\quad R := P^{(s)}, \quad A := 1$
$Q = (Q_1, \dots, Q_m) := (0, \dots, 0) \in h_{(\mathbf{0},\mathbf{1})}(D)^m$
IF $R \neq 0$
  THEN $\mathcal{F} := \{P' \in \mathcal{G} \mid \mathrm{LM}_{\prec_s}(P') \text{ divides } \mathrm{LM}_{\prec_s}(s^\ell R) \text{ for some } \ell \in \mathbb{N}\}$
  ELSE $\mathcal{F} := \emptyset$ (an empty set)
$\mathcal{H} := [\,]$ (an empty list)
WHILE ($R \neq 0$ AND $\mathcal{F} \neq \emptyset$) DO
  Choose $P' \in \mathcal{F}$ with $\ell$ minimal, which is the $i$-th element of $\mathcal{G}$
  IF $\ell > 0$ THEN
    $\mathcal{G} := \mathcal{G} \cup [R]$ (append $R$ to $\mathcal{G}$ as the last element)
    $\mathcal{H} := \mathcal{H} \cup [[A, Q]]$ (append a list $[A, Q]$ to $\mathcal{H}$ as the last element)
  $[R, U] := \mathrm{Red}(s^\ell R, P')$
  IF $i \leq m$ THEN $Q_i := Q_i + U$

IF $i > m$ THEN
  $[A', Q'] := \mathcal{H}[i-m]$ (the $(i-m)$-th element of $\mathcal{H}$)
  $A := A - UA'$
  FOR $j = 1, \dots, m$ DO $Q_j := Q_j - UQ'_j$
IF $R \neq 0$ THEN
  $\nu :=$ the highest power of $s$ dividing $R$
  $R := R/s^\nu$
  $\mathcal{F} := \{P' \in \mathcal{G} \mid \mathrm{LM}_{\prec_s}(P')$ divides $\mathrm{LM}_{\prec_s}(s^\ell R)$ for some $\ell \in \mathbb{N}\}$
FOR $j = 1, \dots, m$ DO $Q_j := Q_j|_{s=1}$
$R := R|_{s=1}, \quad a := A|_{s=1}$

We call $a(x)^{-1}R$ a remainder of $P$ on division by $P_1, \dots, P_m$, which is not necessarily unique. Note that $\mathrm{LM}_\prec(a(x)^{-1}R) = \mathrm{LM}_\prec(R) \preceq \mathrm{LM}_\prec(P)$ holds if $R \neq 0$ in view of the condition (2.2). By factoring out the denominators of the input and applying Algorithm 2.2, we get

**Theorem 2.3** *Let $P, P_1, \dots, P_m$ be homogeneous elements of $(\mathcal{D}^{(h)}_{\mathrm{alg}})^r[\mathbf{n}]$. Then one can obtain algorithmically homogeneous $Q_1, \dots, Q_r \in \mathcal{D}^{(h)}_{\mathrm{alg}}$ and $R \in (\mathcal{D}^{(h)}_{\mathrm{alg}})^r[\mathbf{n}]$ satisfying $P = \sum_{k=1}^m Q_k P_k + R$ such that $\mathrm{LM}_\prec(R)$ is not divisible by any of $\mathrm{LM}_\prec(P_k)$ $(k = 1, \dots, m)$ if $R \neq 0$, and $\mathrm{LM}_\prec(Q_k P_k) \preceq \mathrm{LM}_\prec(P)$ if $Q_k \neq 0$, for $k = 1, \dots, m$.*

**Example 2.4** We work in $h_{(\mathbf{0},\mathbf{1})}(D)$ with $n = 2$, $\mathbf{n} = (0)$, $\mathbf{v} = (0)$, and $x = x_1$, $y = x_2$, $\partial_x = \partial_1$, $\partial_y = \partial_2$. Let $\prec$ be an arbitrary monomial ordering which satisfies $x\partial_x \prec \partial_y$, $y\partial_y \prec \partial_x$ as well as (2.2),(2.3), i.e., $x \prec 1$, $y \prec 1$, $h \prec x\partial_x$, $h \prec y\partial_y$. Put

$$P := xy\partial_x\partial_y, \quad P_1 := x\partial_x + xy\partial_y, \quad P_2 := y\partial_y + xy\partial_x.$$

Then Algorithm 2.2 proceeds as follows (the underlined part is the leading monomial with respect to $\prec_s$):

  $R := \underline{xy\partial_x\partial_y} =: P'_3$,
  $\mathcal{G} := [P'_1, P'_2]$ with $P'_1 := P_1^{(s)} = \underline{sx\partial_x} + xy\partial_y$, $P'_2 := P_2^{(s)} = \underline{sy\partial_y} + xy\partial_x$,
  $A := 1, \quad Q := (0, 0), \quad \mathcal{H} := [\,], \quad \mathcal{F} = \{P'_1, P'_2\}$.
1st pass of the WHILE loop (choose $P'_1$ with $\ell = 1$):
  $R := sR - y\partial_y P'_1 = -\underline{xy^2\partial_y^2} - xy\partial_y h =: P'_4$,
  $\mathcal{G} := [P'_1, P'_2, P'_3], \quad \mathcal{H} := [[1, (0, 0)]]$,
  $Q := Q + (y\partial_y, 0) = (y\partial_y, 0)$,
  $\mathcal{F} = \{P'_2\}$.
2nd pass (choose $P'_2$ with $\ell = 1$):
  $R := sR - (-xy\partial_y)P'_2 = \underline{x^2y^2\partial_x\partial_y} + x^2y\partial_x h$,
  $\mathcal{G} := [P'_1, P'_2, P'_3, P'_4], \quad \mathcal{H} := [[1, (0, 0)], [1, (y\partial_y, 0)]]$,
  $Q := Q + (0, -xy\partial_y) = (y\partial_y, -xy\partial_y)$,
  $\mathcal{F} = \{P'_1, P'_2, P'_3\}$.

3rd pass (choose $P'_3$ with $\ell = 0$):

$R := R - xyP'_3 = \underline{x^2y\partial_x h} =: P'_5$,

$A := 1 - xy$, $Q := Q - xy(0,0) = (y\partial_y, -xy\partial_y)$ (by using the 1st element of $\mathcal{H}$),

$\mathcal{F} = \{P'_1\}$.

4th pass (choose $P'_1$ with $\ell = 1$):

$R := sR - xyhP'_1 = -\underline{x^2y^2\partial_y h} =: P'_6$,

$\mathcal{G} := [P'_1, P'_2, P'_3, P'_4, P'_5]$,

$\mathcal{H} := [[1,\ (0,0)],\ [1, (\ y\partial_y, 0)],\ [1 - xy,\ (y\partial_y, -xy\partial_y)]]$,

$Q := Q + (xyh, 0) = (y\partial_y + xyh, -xy\partial_y)$,

$\mathcal{F} = \{P'_2\}$.

5th pass (choose $P'_2$ with $\ell = 1$):

$R := sR - (-x^2yh)P'_2 = \underline{x^3y^2\partial_x h}$,

$\mathcal{G} := [P'_1, P'_2, P'_3, P'_4, P'_5, P'_6]$,

$\mathcal{H} :=$
$[[1,\ (0,0)],\ [1,\ (y\partial_y, 0)],\ [1 - xy,\ (y\partial_y, -xy\partial_y)],\ [1 - xy,\ (y\partial_y + xyh, -xy\partial_y)]]$,

$Q := Q - (0, x^2yh) = (y\partial_y + xyh, -xy\partial_y - x^2yh)$,

$\mathcal{F} = \{P'_1, P'_5\}$.

6th pass (choose $P'_5$ with $\ell = 0$):

$R := R - xyP'_5 = 0$,

$A := A - xy(1 - xy) = (1 - xy)^2$,

$Q := Q - xy(y\partial_y, -xy\partial_y) = (y\partial_y - xy^2\partial_y + xyh, -xy\partial_y + x^2y^2\partial_y - x^2yh)$,

$\mathcal{F} = \{\}$.

Hence we have $R = 0$ and

$$(1 - xy)^2 P = (y\partial_y - xy^2\partial_y + xyh)P_1 + (-xy\partial_y + x^2y^2\partial_y - x^2yh)P_2.$$

Let us prove the correctness of Algorithm 2.2. We denote by $\langle G \rangle$ the ideal generated by a set of monomials $G$ in the polynomial ring. In Algorithm 2.2, $R$ is added to $\mathcal{G}$ only if $s^\ell \mathrm{LM}_{\prec_s}(R)$ is divisible by $\mathrm{LM}_{\prec_s}(\mathcal{G}) = \{\mathrm{LM}_{\prec_s}(P') \mid P' \in \mathcal{G}\}$ with some $\ell > 0$ but $\mathrm{LM}_{\prec_s}(R)$ is not. This implies

$$\langle \mathrm{LM}_{\prec_s}(\mathcal{G}) \rangle \subsetneq \langle \mathrm{LM}_{\prec_s}(\mathcal{G} \cup \{R\}) \rangle, \quad \langle \mathrm{LM}_{\prec}(\mathcal{G}|_{s=1}) \rangle = \langle \mathrm{LM}_{\prec}(\mathcal{G}|_{s=1} \cup \{R|_{s=1}\}) \rangle.$$

Hence the monomial ideal $\langle \mathrm{LM}_{\prec}(\mathcal{G}|_{s=1}) \rangle$ remains unchanged throughout the algorithm, and $\langle \mathrm{LM}_{\prec_s}(\mathcal{G}) \rangle$ stays unchanged after, say, the $k$-th pass of the WHILE loop in view of Dickson's lemma. This implies that after the $k$-th pass, $\mathcal{G}$ itself stays unchanged, and consequently the procedure afterwards is nothing but the usual division algorithm with respect to the well-ordering $\prec_s$. Thus the algorithm terminates and the leading monomial $\mathrm{LM}_{\prec}(R)$ of the final output $R$ is, if nonzero, not divisible by $\mathrm{LM}_{\prec}(P_i)$ for any $i = 1, \dots, m$.

We denote $R$, $Q = (Q_1, \dots, Q_m)$, $i$, $\ell$, $\mathcal{G}$, etc. at the end of the $k$-th pass of the WHILE loop by $R_k$, $Q_{(k)} = (Q_{1k}, \dots, Q_{mk})$, $i(k)$, $\ell(k)$, $\mathcal{G}_k$, etc. and prove the properties

$$
\begin{aligned}
&A_k \in \mathbb{C}[x,s] \text{ with } A_k(0,1) = 1,\\
&(A_k|_{s=1})P = (Q_{1k}|_{s=1})P_1 + \cdots + (Q_{mk}|_{s=1})P_m + R_k|_{s=1}, \\
&\mathrm{LM}_{\prec}(Q_{ik}|_{s=1}P_i) \preceq \mathrm{LM}_{\prec}(P) \text{ if } Q_{ik} \neq 0
\end{aligned}
\tag{2.4}
$$

by induction on $k$. When $k = 0$, these properties are trivially satisfied. By the reduction at the $k$-th pass, we have

$$
s^{\ell(k)} R_{k-1} = U_k P'_{i(k)} + s^{\nu(k)} R_k \quad (\exists \nu(k) \in \mathbb{N}), \tag{2.5}
$$

where $P'_{i(k)}$ is the $i(k)$-th element of $\mathcal{G}_k$. By the induction hypothesis we also have

$$
(A_{k-1}|_{s=1})P = (Q_{1,k-1}|_{s=1})P_1 + \cdots + (Q_{m,k-1}|_{s=1})P_m + R_{k-1}|_{s=1}. \tag{2.6}
$$

First assume $i(k) \leq m$. Then we get $A_k = A_{k-1}$ and

$$
(A_k|_{s=1})P = (Q_{1,k-1}|_{s=1})P_1 + \cdots + (Q_{m,k-1}|_{s=1})P_m + (U_k|_{s=1})P_{i(k)} + R_k|_{s=1}.
$$

Hence (2.4) is satisfied at the $k$-th pass.

Next assume $i(k) > m$. Then $P'_{i(k)} = R_j$ with some $j < k-1$. Hence we have

$$
s^{\ell(k)} R_{k-1} = U_k R_j + s^{\nu(k)} R_k. \tag{2.7}
$$

Since $R_{k-1}$ and $R_j$ are $(\mathbf{0},\mathbf{1})$-homogeneous of the same degree by induction using (2.5), $U_k$ is $(\mathbf{0},\mathbf{1})$-homogeneous of degree zero, that is, a monomial in $\mathbb{C}[x,s]$. In view of the remark preceding Algorithm 2.2, we have

$$
\mathrm{LM}_{\prec}(R_k|_{s=1}) \prec \mathrm{LM}_{\prec}(R_{k-1}|_{s=1}) \prec \cdots \prec \mathrm{LM}_{\prec}(R_j|_{s=1}).
$$

Hence $U_k$ does not belong to $\mathbb{C}[s]$, and consequently $U_k(0,1) = 0$ holds. Thus $A_k = A_{k-1} - U_k A_j$ also belongs to $\mathbb{C}[x,s]$ and satisfies $A_k(0,1) = A_{k-1}(0,1) = 1$. It follows from the induction hypothesis that

$$
(A_j|_{s=1})P = (Q_{1,j}|_{s=1})P_1 + \cdots + (Q_{m,j}|_{s=1})P_m + R_j|_{s=1}. \tag{2.8}
$$

Combining the equations (2.6), (2.7), (2.8), we get

$$
\begin{aligned}
&(A_{k-1} - U_k A_j)|_{s=1} P \\
&\quad = (Q_{1,k-1} - U_k Q_{1,j})|_{s=1} P_1 + \cdots + (Q_{m,k-1} - U_k Q_{m,j})|_{s=1} P_m + R_k|_{s=1}.
\end{aligned}
$$

Since $A_k = A_{k-1} - U_k A_j$ and $Q_{i,k} = Q_{i,k-1} - U_k Q_{i,j}$ for $i = 1, \ldots, m$, (2.4) is also satisfied at the end of the $k$-th pass. This completes the correctness proof of Algorithm 2.2.

**Remark 2.5** For the second homogenization $P^{(s)}$, we can use an arbitrary weight vector of the form $(-u_1, \dots, -u_n, u_1, \dots, u_n)$ with positive integers $u_1, \dots, u_n$ instead of $(-\mathbf{1}, \mathbf{1})$.

**Remark 2.6** Algorithm 2.2 also works in the Weyl algebra $D$ (i.e. without the $(\mathbf{0}, \mathbf{1})$-homogenization in terms of $h$) if we use an ordering satisfying (2.1) with $k = k' = 0$ and (2.2) since the order (i.e. the total degree in $\partial$ shifted by $\mathbf{n}$) of $R$ does not increase in the WHILE loop of the algorithm.

## 3 Computation of standard bases and syzygies

The tangent cone algorithm (Theorem 2.3) enables us to compute standard or Gröbner bases of $\mathcal{D}_{\mathrm{alg}}^{(h)}$-modules with respect to a sufficiently large class of orderings. For the sake of simplicity, we assume that there exist a monomial ordering $\prec_1$ on $\{x^\alpha \xi^\beta h^k \mid (\alpha, \beta, k) \in \mathbb{N}^{2n+1}\}$ such that

$$|\beta| + k < |\beta'| + k' \quad \Rightarrow \quad x^\alpha \xi^\beta h^k \prec_1 x^{\alpha'} \xi^{\beta'} h^{k'} \qquad (\forall \alpha, \alpha' \in \mathbb{N}^n), \tag{3.1}$$
$$x^\alpha \xi^\beta h^k \preceq_1 \xi^\beta h^k \qquad (\forall \alpha, \beta \in \mathbb{N}^n,\ \forall k \in \mathbb{N}), \tag{3.2}$$
$$h \prec_1 x_j \xi_j \qquad (\forall j = 1, \dots, n), \tag{3.3}$$

an ordering $<'$ on $\{1, \dots, r\}$, and monomials $A_i = x^{\alpha^{(i)}} \xi^{\beta^{(i)}} h^{k^{(i)}}$ $(i = 1, \dots, r)$, so that

$$x^\alpha \xi^\beta h^k e_i \prec x^{\alpha'} \xi^{\beta'} h^{k'} e_j$$
$$\Leftrightarrow \quad \begin{cases} x^\alpha \xi^\beta h^k A_i \prec_1 x^{\alpha'} \xi^{\beta'} h^{k'} A_j \\ \text{or} \quad (x^\alpha \xi^\beta h^k A_i = x^{\alpha'} \xi^{\beta'} h^{k'} A_j \quad \text{and} \quad i <' j) \end{cases}$$

It is easy to see that this ordering $\prec$ satisfies the conditions (2.1),(2.2),(2.3) with $n_i := |\beta^{(i)}| + k^{(i)}$.

For two nonzero vectors $P, Q \in (\mathcal{D}^{(h)})^r$ with a common leading position $i$, their S-vector is defined to be

$$\mathrm{S}(P, Q) := SP - TQ,$$

where $S$ and $T$ are 'monomials' in $\mathcal{D}^{(h)}$ whose symbols are

$$\frac{\mathrm{LCM}(\mathrm{LM}_\prec(P)/e_i,\ \mathrm{LM}_\prec(Q)/e_i)}{\mathrm{LM}_\prec(P)/e_i}, \quad \frac{\mathrm{LCM}(\mathrm{LM}_\prec(P)/e_i,\ \mathrm{LM}_\prec(Q)/e_i)}{\mathrm{LM}_\prec(Q)/e_i}$$

respectively, where LCM denotes the least common multiple of monomials.

**Definition 3.1** Let $N$ be a left submodule of $(\mathcal{D}^{(h)})^r$ (or of $(\mathcal{D}^{(h)}_{\mathrm{alg}})^r$) and let $G$ be a subset of $N \setminus \{0\}$. Then $G$ is called a *standard base* or a *Gröbner base* of $N$ with respect to the ordering $\prec$ if it satisfies the following two conditions:

(1) $G$ generates $N$.
(2) For any $P \in N \setminus \{0\}$, its leading monomial $\mathrm{LM}_\prec(P)$ is divisible by (i.e., is a monomial times) $\mathrm{LM}_\prec(Q)$ for some $Q \in G$.

Then we have the following criterion of Buchberger's type.

**Theorem 3.2** *Let $\prec$ be an ordering defined as above by using an ordering $\prec_1$ satisfying (3.1), (3.2), (3.3). Let $P_1, \dots, P_m$ be nonzero homogeneous elements of $(\mathcal{D}^{(h)}_{\mathrm{alg}})^r[\mathbf{n}]$ and $N$ be the left submodule of $(\mathcal{D}^{(h)})^r$ generated by $G := \{P_1, \dots, P_m\}$. Then the following two conditions are equivalent:*

*(1) $G$ is a standard base of $N$ with respect to $\prec$.*
*(2) For any $(i,j) \in \Lambda := \{(i,j) \mid 1 \le i < j \le m,\ \mathrm{LP}_\prec(P_i) = \mathrm{LP}_\prec(P_j)\}$, there exist $Q_{ijk} \in \mathcal{D}^{(h)}_{\mathrm{alg}}$ $(k = 1, \dots, m)$ such that $Q_{ijk}P_k$ are homogeneous of the same degree as $\mathrm{S}(P_i, P_j)$ and*

$$\mathrm{S}(P_i, P_j) = S_{ji}P_i - S_{ij}P_j = \sum_{k=1}^{m} Q_{ijk}P_k, \tag{3.4}$$

$$\mathrm{LM}_\prec(Q_{ijk}P_k) \prec \mathrm{LCM}(\mathrm{LM}_\prec(P_i)/e_\ell,\ \mathrm{LM}_\prec(P_j)/e_\ell) \tag{3.5}$$

*with $\ell := \mathrm{LP}_\prec(P_i)$ if $Q_{ijk} \neq 0$.*

Proof: Assume (1). Then for any $(i,j) \in \Lambda$, we can find $Q_{ijk} \in \mathcal{D}^{(h)}_{\mathrm{alg}}$ and $R_{ij} \in (\mathcal{D}^{(h)}_{\mathrm{alg}})^r$ such that

$$\mathrm{S}(P_i, P_j) = \sum_{k=1}^{m} Q_{ijk}P_k + R_{ij},$$
$$\mathrm{LM}_\prec(Q_{ijk}P_k) \preceq \mathrm{LM}_\prec(\mathrm{S}(P_i, P_j)) \text{ if } Q_{ijk} \neq 0,$$
$$\mathrm{LM}_\prec(R_{ij}) \text{ is not divisible by any of } \mathrm{LM}_\prec(P_k) \text{ if } R_{ij} \neq 0$$

by Theorem 2.3. Then the assumption (1) and the fact that $R_{ij} \in N$ implies $R_{ij} = 0$. Hence (2) holds.

Now assume (2). By Robbiano's theorem ([10]), there exist vectors $\mathbf{w}_i = (w_{i,1}, \dots, w_{i,n}; w_{i,n+1}, \dots, w_{i,2n}; w_{i,2n+1}) \in \mathbb{R}^{2n+1}$ such that the ordering $\prec_1$ is equivalent to the lexicographic ordering with respect to

$$(\langle \mathbf{w}_0, (\alpha, \beta, k)\rangle,\ \langle \mathbf{w}_1, (\alpha, \beta, k)\rangle,\ \cdots,\ \langle \mathbf{w}_p, (\alpha, \beta, k)\rangle).$$

By the condition (3.1), we may assume that $\mathbf{w}_0 = (0, \dots, 0; 1, \dots, 1; 1)$. For

$\varepsilon \in \mathbb{R}$, put

$$\mathbf{w}(\varepsilon) = (w_1(\varepsilon), \dots, w_{2n+1}(\varepsilon)) := \mathbf{w}_1 + \varepsilon \mathbf{w}_2 + \cdots + \varepsilon^{p-1} \mathbf{w}_p.$$

Then by virtue of conditions (3.2), (3.3) we have

$$w_i(\varepsilon) < 0, \quad w_i(\varepsilon) + w_{n+i}(\varepsilon) > w_{2n+1}(\varepsilon) \quad (i = 1, \dots, n) \tag{3.6}$$

for any $\varepsilon > 0$ sufficiently small. By using this vector, we define a new ordering $\prec_1^\varepsilon$ for $\mathcal{D}^{(h)}$ by

$$x^\alpha \xi^\beta h^k \prec_1^\varepsilon x^{\alpha'} \xi^{\beta'} h^{k'}$$
$$\Leftrightarrow \quad \begin{cases} |\beta| + k < |\beta'| + k' \\ \text{or } (|\beta| + k = |\beta'| + k' \text{ and } \langle \mathbf{w}(\varepsilon), (\alpha, \beta, k) \rangle < \langle \mathbf{w}(\varepsilon), (\alpha', \beta', k') \rangle) \\ \text{or } (|\beta| + k = |\beta'| + k' \text{ and } \langle \mathbf{w}(\varepsilon), (\alpha, \beta, k) \rangle = \langle \mathbf{w}(\varepsilon), (\alpha', \beta', k') \rangle \\ \quad \text{and } x^\alpha \xi^\beta h^k \prec_1 x^{\alpha'} \xi^{\beta'} h^{k'}) \end{cases}$$

and define a new ordering $\prec^\varepsilon$ in terms of $\prec_1^\varepsilon$ in the same way as $\prec$ is defined in terms of $\prec_1$.

Now take a nonzero homogeneous $P \in N$. Then there exist homogeneous $Q_1, \dots, Q_m \in \mathcal{D}^{(h)}$ such that

$$P = Q_1 P_1 + \cdots + Q_m P_m. \tag{3.7}$$

There exists a finite set of monomials of $P$ to which the leading monomial of $P$ with respect to any monomial ordering satisfying (2.1),(2.2),(2.3) belongs. It follows that the leading terms of $P, P_i, Q_{ijk}$ and the inequality (3.5) stay the same if we replace $\prec$ by $\prec^\varepsilon$ with $\varepsilon > 0$ small enough. We fix an $\varepsilon > 0$ which satisfies this condition as well as (3.6). If the leading monomial of some $Q_i P_i$ is greater than that of $P$, then rewriting the right hand side of (3.7) by using (3.4), we can replace $Q_1, \dots, Q_m$ in expression (3.7) by $Q'_1, \dots, Q' \in \mathcal{D}^{(h)}$ so that

$$\max{}_{\prec^\varepsilon} \{\mathrm{LM}_{\prec^\varepsilon}(Q'_i P_i) \mid i = 1, \dots, m,\ Q'_i \neq 0\}$$
$$\prec^\varepsilon \max{}_{\prec^\varepsilon} \{\mathrm{LM}_{\prec^\varepsilon}(Q_i P_i) \mid i = 1, \dots, m,\ Q_i \neq 0\}.$$

Let $P$ be homogeneous of degree $d$. Then by (3.6) the set

$$\{(\alpha, \beta, k, i) \ : \ |\beta| + k + n_i = d,\ \mathrm{LM}_{\prec^\varepsilon}(P) \prec^\varepsilon x^\alpha \xi^\beta h^k e_i\}$$

is finite. Hence we can take $Q_i$ in (3.7) so that $\mathrm{LM}_{\prec^\varepsilon}(Q_i P_i) \preceq^\varepsilon \mathrm{LM}_{\prec^\varepsilon}(P)$. Thus $\mathrm{LM}_{\prec}(P) = \mathrm{LM}_{\prec^\varepsilon}(P)$ is divisible by some $\mathrm{LM}_{\prec}(P_i) = \mathrm{LM}_{\prec^\varepsilon}(P_i)$. This completes the proof. □

Hence the Buchberger algorithm with the écart division (Algorithm 2.2) gives an algorithm for computing a standard base of a module generated by a given finite set of generators. In the écart division, we can use an arbitrary shift vector $\mathbf{v}$ for the bihomogenization and may discard the 'denominator' $a(x)$.

**Corollary 3.3** *Let $N_{\mathrm{alg}}$ be a left $\mathcal{D}^{(h)}_{\mathrm{alg}}$-submodule of $(\mathcal{D}^{(h)}_{\mathrm{alg}})^r$ and let $N$ be the left $\mathcal{D}^{(h)}$-submodule of $(\mathcal{D}^{(h)})^r$ generated by $N_{\mathrm{alg}}$. Then for any nonzero element $P$ of $N$ and an ordering $\prec$ as above, there exists an element $P'$ of $N_{\mathrm{alg}} \cap (h_{(\mathbf{0},\mathbf{1})}(D))^r$ such that $\mathrm{LM}_\prec(P) = \mathrm{LM}_\prec(P')$.*

Proof: By the above theorem, there is a standard base of $N$ consisting of elements of $N_{\mathrm{alg}}$. By dividing out the denominators, we may further assume that each element of the base belongs to $(h_{(\mathbf{0},\mathbf{1})}(D))^r$. Hence the assertion follows from the definition of standard base. □

Schreyer's theorem on syzygies also holds in this situation:

**Theorem 3.4** *Let $\prec$ be an ordering as above. Let $P_1, \dots, P_m$ be nonzero homogeneous elements of $(\mathcal{D}^{(h)}_{\mathrm{alg}})^r[\mathbf{n}]$ and $N$ be the left submodule of $(\mathcal{D}^{(h)})^r$ generated by $G := \{P_1, \dots, P_m\}$. Assume that $G$ is a standard base of $N$ with respect to $\prec$. Take $Q_{ijk} \in \mathcal{D}^{(h)}_{\mathrm{alg}}$ which satisfy (3.4) and (3.5) and put*

$$V_{ij} := (0, \dots, S_{ji}, \dots, -S_{ij}, \dots, , 0) - (Q_{ij1}, \dots, Q_{ijm}) \qquad ((i,j) \in \Lambda).$$

*Define an ordering $\prec'$ for $(\mathcal{D}^{(h)})^m$ by*

$$\begin{aligned} & x^\alpha \xi^\beta h^k e'_i \prec' x^{\alpha'} \xi^{\beta'} h^{k'} e'_j \\ \Leftrightarrow & \begin{cases} x^\alpha \xi^\beta h^k \mathrm{LM}_\prec(P_i) \prec x^{\alpha'} \xi^{\beta'} h^{k'} \mathrm{LM}_\prec(P_j) \\ \text{or } (x^\alpha \xi^\beta h^k \mathrm{LM}_\prec(P_i) = x^{\alpha'} \xi^{\beta'} h^{k'} \mathrm{LM}_\prec(P_j) \text{ and } i > j), \end{cases} \end{aligned}$$

*where $e'_1, \dots, e'_m$ are the canonical generators of $(\mathcal{D}^{(h)})^m$. Then $\{V_{ij} \mid (i,j) \in \Lambda\}$ is a standard base with respect to $\prec'$ of the syzygy module*

$$\mathrm{Syz}(P_1, \dots, P_m; \mathcal{D}^{(h)}) := \Big\{(Q_1, \dots, Q_m) \in (\mathcal{D}^{(h)})^r \mid Q_1 P_1 + \cdots + Q_m P_m = 0\Big\}$$

*over $\mathcal{D}^{(h)}$.*

Proof: First we show that $\{V_{ij} \mid (i,j) \in \Lambda\}$ is a standard base of the syzygy module

$$\mathrm{Syz}(P_1, \dots, P_m; \mathcal{D}^{(h)}_{\mathrm{alg}}) := \Big\{(Q_1, \dots, Q_m) \in (\mathcal{D}^{(h)}_{\mathrm{alg}})^r \mid Q_1 P_1 + \cdots + Q_m P_m = 0\Big\}$$

over $\mathcal{D}^{(h)}_{\mathrm{alg}}$ with respect to $\prec'$. In fact, suppose that a nonzero $Q = (Q_1, \dots, Q_m) \in (\mathcal{D}^{(h)}_{\mathrm{alg}})^m$ satisfies $Q_1 P_1 + \cdots + Q_m P_m = 0$ and let $K$ be the subset of $\{1, \dots, m\}$

consisting of $k$'s such that $\mathrm{LM}_{\prec}(Q_k P_k)$ attains the maximum. Then we have

$$\sum_{k \in K} \mathrm{LM}_{\prec_1}(Q_k)\mathrm{LM}_{\prec}(P_k) = 0.$$

It follows that $\mathrm{LM}_{\prec'}(Q)$ is divisible by some $\mathrm{LM}_{\prec'}(V_{ij}) = \mathrm{LM}_{\prec_1}(S_{ji})e'_i$. Hence the remainder of $Q$ on division by $V_{ij}$'s is zero. This means that the $V_{ij}$'s are a standard base of the syzygy module over $\mathcal{D}^{(h)}_{\mathrm{alg}}$ with respect to $\prec'$.

In particular, there is an exact sequence

$$(\mathcal{D}^{(h)}_{\mathrm{alg}})^{\Lambda} \xrightarrow{\psi} (\mathcal{D}^{(h)}_{\mathrm{alg}})^{m} \xrightarrow{\varphi} N_{\mathrm{alg}} \to 0,$$

where $N_{\mathrm{alg}}$ is the left $\mathcal{D}^{(h)}_{\mathrm{alg}}$-module generated by $\{P_1, \ldots, P_m\}$, $\varphi$ and $\psi$ are $\mathcal{D}^{(h)}_{\mathrm{alg}}$-homomorphisms defined by $\varphi(e_i) = P_i$ and $\psi(e'_{(i,j)}) = V_{ij}$ respectively with $\{e'_{(i,j)} \mid (i,j) \in \Lambda\}$ being the canonical base of $(\mathcal{D}^{(h)}_{\mathrm{alg}})^{\Lambda}$. In view of the flatness of $\mathcal{D}^{(h)}$ over $\mathcal{D}^{(h)}_{\mathrm{alg}}$, the above sequence yields an exact sequence

$$(\mathcal{D}^{(h)})^{\Lambda} \xrightarrow{\psi} (\mathcal{D}^{(h)})^{m} \xrightarrow{\varphi} N \to 0.$$

This implies that the syzygy module over $\mathcal{D}^{(h)}$ is generated by $V_{ij}$'s. The first part of the proof and Corollary 3.3 ensure that the $V_{ij}$'s are a standard base of the syzygy module over $\mathcal{D}^{(h)}$ with respect to $\prec'$. $\square$

In Theorem 3.4, put $\mathrm{LM}_{\prec}(P_k) = B_k e_{l_k}$ with a monomial $B_i$. Then we have

$$\begin{aligned}
& x^{\alpha}\xi^{\beta}h^{k}e'_i \prec' x^{\alpha'}\xi^{\beta'}h^{k'}e'_j \\
\Leftrightarrow & \begin{cases} x^{\alpha}\xi^{\beta}h^{k}B_iA_{l_i} \prec_1 x^{\alpha'}\xi^{\beta'}h^{k'}B_jA_{l_j} \\ \text{or } (x^{\alpha}\xi^{\beta}h^{k}B_iA_{l_i} = x^{\alpha'}\xi^{\beta'}h^{k'}B_jA_{l_j} \text{ and } l_i <' l_j), \\ \text{or } (x^{\alpha}\xi^{\beta}h^{k}B_iA_{l_i} = x^{\alpha'}\xi^{\beta'}h^{k'}B_jA_{l_j} \text{ and } l_i = l_j \text{ and } i > j). \end{cases}
\end{aligned}$$

In particular, $\prec'$ satisfies the same condition as what we imposed on $\prec$. Hence if we take $\prec_1$ for which the second weight vector $\mathbf{w}_1$ is of the form $(u, v, 0) = (u_1, \ldots, u_n; v_1, \ldots, v_n; 0)$, repeated applications of Theorem 3.4 yield a $(u, v)$-filtered free resolution of $(\mathcal{D}^{(h)})^r/N$ in the sense of [4]. Finally, step by step minimalization process over $\mathcal{D}^{(h)}_{\mathrm{alg}}$ as is described in [4] over $\mathcal{D}^{(h)}$ produces a minimal $(u, v)$-filtered resolution (see examples in [4]).

**Remark 3.5** Under the assumptions of Theorems 3.2 and 3.4, the dehomogenizations $P_1|_{h=1}, \ldots, P_m|_{h=1}$ are a standard base (with respect to the restriction of $\prec$) of the submodule of $\mathcal{D}^r$ which they generate, and $V_{ij}|_{h=1}$ generate the syzygy module of $P_1|_{h=1}, \ldots, P_m|_{h=1}$ over $\mathcal{D}$.

## 4 The écart division versus bihomogenization

The second homogenization parameter $s$ was used only in Algorithm 2.2, not in the Buchberger algorithm as a whole. However, we can also compute a standard base with $s$ and the usual (not écart) division algorithm with respect to the well-ordering $\prec_s$. This can be regarded as a differential operator version of Lazard's method ([7]).

**Theorem 4.1** *Let $P_1, \dots, P_m$ be bihomogeneous elements of $(h_{(\mathbf{0},\mathbf{1})}(D)[s])^r[\mathbf{n}]$ (resp. $(-\mathbf{1},\mathbf{1})$-homogeneous elements of $D[s]^r$) with respect to shifts $\mathbf{n}, \mathbf{v} \in \mathbb{Z}^r$. Assume that $P_1, \dots, P_m$ are a Gröbner base of the left submodule of $(h_{(\mathbf{0},\mathbf{1})}(D)[s])^r$ (resp. of $D[s]^r$) which they generate, with respect to the ordering $\prec_s$ defined in the same way as in Section 2 from an ordering $\prec$ of Section 3 (resp. the restriction of $\prec_s$ to $D[s]^r$). Then $P_1|_{s=1}, \dots, P_m|_{s=1}$ are a standard base of the left submodule of $(\mathcal{D}^{(h)})^r$ (resp. of $\mathcal{D}^r$) which they generate.*

Proof: It is easy to see by the standard argument that $P_1|_{s=1}, \dots, P_m|_{s=1}$ are a standard base of the left $\mathcal{D}^{(h)}_{\mathrm{alg}}$-submodule of $(\mathcal{D}^{(h)}_{\mathrm{alg}})^r$ which they generate. Then Corollary 3.3 implies the assertion. The case for $\mathcal{D}^r$ is similar in view of Remark 2.6. □

We give some comparisons between the two methods to compute standard bases by our implementation using software Kan ([12]). For a polynomial $f$ of $x$, we take the annihilator ideal $I_f$ of $\delta(t - f(x))$ in $\mathcal{D}^{(h)}$, which is generated by $(\mathbf{0}, \mathbf{1})$-homogeneous operators

$$t - f, \quad \partial_i + \frac{\partial f}{\partial x_i}\partial_t \quad (i = 1, \dots, n).$$

We use an ordering $\prec$ for $\mathcal{D}^{(h)}$ which is defined lexicographically with respect to the weights given by

$$\begin{array}{ccccccccc} t & x_1 & \cdots & x_n & \partial_t & \partial_1 & \cdots & \partial_n & h \\ \hline -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \end{array} \qquad (4.1)$$

with a reverse-lexicographic order as the tie-breaker. Then a standard base of $I_f$ is adapted to the V-filtration with respect to $t = 0$. Such a base is closely connected with the singularity structure, e.g., the Bernstein-Sato polynomial and the local cohomology, attached to the hypersurface $f(x) = 0$ (cf. [9],[4]). In the table below, we show the number of elements of a minimal (or interreduced in the terminology of [6]) standard base of $I_f$ with computation time in parentheses by using a 1GHz Pentium III processor and 2G memory. (E) means the method described in Theorem 3.2 with the écart division; (L) means the method of Theorem 4.1 with bihomogenization; (G) means the usual (global) Gröbner base computation in the homogenized Weyl algebra (see [9],[11]) with respect to an ordering defined by the first two rows of (4.1) and a reverse lexicographic tie-breaking order, which does not give a local standard base but is shown only for reference.

| $f$ | (E) | (L) | (G) |
|---|---|---|---|
| $x^3 - y^2z^2 - w^2$ | 23 (0.21s) | 128 (13.1s) | 36 (0.24s) |
| $x^3 + z^4 + y^3w + w^8$ | 145 (228.9s) | ? ($>$ 1 day) | 82 (47.6s) |

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