HGM の不安定性をどう回避するか?

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HGM の三つの step

1. パラメータ付定積分のみたす線形偏微分方程式系を代数的アルゴリズム等で見つける.
2. 初期条件を計算
3. 微分方程式の数値解析で, パラメータ付定積分の値を決める.

Step 3 がむつかしくなる場合の例, 困難の回避の方法.

"Algorithms to Reduce the Instability of the HGM and Tricks Useful for the HGM", preprint (expository, technical).

1. このスライドの PDF: Nobuki Takayama [search]
2. `http://www.math.kobe-u.ac.jp/OpenXM/Math/hgm/ref-hgm.html`

復習: Runge-Kutta 法

$$\frac{dF}{dt} = P(t)F \tag{1}$$

where $P(t)$ is an $r \times r$ matrix and $F(t)$ is a column vector valued unknown function. Let E be the $r \times r$ identity matrix.

$$F_1 = F_0 + hk_1 = (E + hP(t_0))F_0, \quad k_1 = P(t_0)F_0.$$

$$F(t_0+h)-F_1 = F(t_0)+F'(t_0)h+O(h^2)-F_1 = O(h^2), \; F'(t_0) = P(t_0)F_0$$

The 4th order Runge-Kutta (RK) method.

$$\begin{aligned} k_{i+1} &= P(t_0 + c_{i+1}h)(F_0 + a_{i+1}k_i h), \quad k_0 = 0 && (2) \\ F_1 &= F_0 + h(b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4) && (3) \end{aligned}$$

Determine the constants so that $F_1 - F(t_0 + h) = O(h^5)$ where $F(t)$ is the solution with the initial condition $F(t_0) = F_0$. $a_1 = c_1 = 0, b_1 = 1/6, b_2 = 1/3, b_3 = 1/3, b_4 = 1/6, c_2 = c_3 = c_4 = 1/2, a_2 = a_3 = 1/2, a_4 = 1.$

E.Hairer, S.P.Norsett, G.Wanner, Solving ordinary differential equations I, II, 1993, 1996, Springer

Matrix factorial

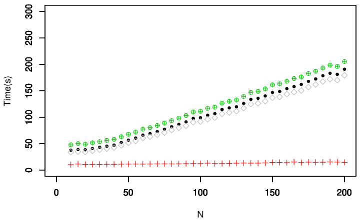
$$Q(t,h) = E + hP(t), \quad \text{for the first order RK}$$

or $Q(t,h)$ is an analogous matrix for the 4th order RK. Then, $F_{k+1} = Q(k)F_k$ ($Q(k) = Q(t_0 + kh, h)$ in short). We call

$$Q(k)Q(k-1)\cdots Q(1)Q(0)$$

the matrix factorial. Applying the matrix factorial to F_0, we obtain F_{k+1} (approximate solution).

Methods for exact evaluation of matrix factorials (the binary splitting and the modular method)*. 以下おまけ話題.



5×5 contingency table, a benchmark test of evaluating the normalizing constant (A-hypergeometric polynomial) with 32 processes from [tgkt]. N is a parameter in the marginal sum.

* [tgkt] Y.Tachibana, Y.Goto, T.Koyama, N.Takayama, Holonomic Gradient Method for Two Way Contingency Tables, arxiv:1803.04170

復習: adaptive Runge-Kutta method

Let F_1 be the vector determined by RK (of the 4th order) of the step size $2h$ (not h). Let F_2 be the vector determined by RK two times with the step size h.

$$|F(t_0 + 2h) - F_1| = \phi(2h)^5 + O(h^6) \tag{4}$$

where ϕ depends only on the solution F and t_0. We also have

$$|F(t_0 + 2h) - F_2| = \phi h^5 + \phi' h^5 + O(h^6) \tag{5}$$

Assume $\phi = \phi'$. Taking the difference of (5) and (4), we have

$$|F_2 - F_1| \sim 30\phi h^5 + O(h^6) \tag{6}$$

The good point of this identity is that we can estimate ϕ without knowing the true solution $F(t)$ and estimate the coeficient of the error. We put $\Delta(h) = 30\phi h^5$.

復習: adaptive Runge-Kutta method 続き

Let us assume

$$\Delta = \varepsilon |F_0| \tag{7}$$

Then, $\phi = |F_0|\varepsilon/(30h^5)$. Then the relative error $|(F(t+h_0) - F_1)/F_0|$ is bounded by

$$\frac{|\phi| h^5}{|F_0|} + O(h^6) = \frac{\varepsilon}{30} + O(h^6) \tag{8}$$

When we want to make the relative error smaller than $\frac{\varepsilon}{30}$, we need to make $\Delta(h)$ (difference of $2h$ step and two times of h step) smaller than $\varepsilon|F_0|$.

In order to choose the next h, use the following relation

$$\frac{h_0}{h_1} = \left(\frac{\Delta(h_0)}{\Delta(h_1)}\right)^{1/5}$$

```
--> load("ak2.rr");
--> QQ=rk_mat2(newmat(2,2,[[0,1],[t,0]])
--> base_replace(QQ[0],QQ[1]);
[ 1/24*h^4*t^2+(1/48*h^5+1/2*h^2)*t+1/6*
[ 1/6*h^3*t^2+(1/6*h^4+h)*t+1/24*h^5+1/2
```

小話: 複素領域で ODE を数値解析 (とばす)

$$\frac{d}{dz}F = P(z)F, \quad z \in \mathbf{C}$$

We want to solve the differential equation along the path

$$z = z_0 + (z_1 - z_0)t, \quad 0 \leq t \leq 1, z_0, z_1 \in \mathbf{C}$$

with the initial value $F(z_0) = F_0$. By $d/dz = (z_1 - z_0)^{-1} d/dt$,

$$\frac{dF}{dt} = (z_1 - z_0)P(z_0 + (z_1 - z_0)t)F \tag{9}$$

Decompose $(z_1 - z_0)P(z_0 + (z_1 - z_0)t)$ into the real part and the imaginary part as $P_1(t) + \sqrt{-1}P_2(t)$ Put $F = u + \sqrt{-1}v$.

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{10}$$

```
c2rsys(P(t));
```

Defusing method (heuristic) 1

$$\frac{dF}{dt} = P(t)F \quad (11)$$

$$F(t_0) = F_0^{\mathrm{true}} \in \mathbf{R}^n \quad (12)$$

F_0^{true} is the initial value of F at $t = t_0$.

Situation

1. The initial value has at most 3 digits of accuracy. We denote this initial value F_0.
2. The property $|F| \to 0$ when $t \to +\infty$ is known, e.g., from a background of the statistics.
3. There exists a solution $\tilde{F}$ of (11) such that $|\tilde{F}| \gg 0$, $t \to +\infty$.

Under this situation, the HGM works only for a very short interval of t because the error of the initial value vector makes the fake solution $\tilde{F}$ dominant and it hides the true solution $F(t)$. We call this bad behavior of the HGM *the instability of the HGM*.

Defusing method 2. 例.

Example

$$\frac{d}{dt}F = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} F$$

The solution space is spanned by $F^1 = (\exp(-t), 0, 0)^T$, $F^2 = (0, \exp(-t), 0)^T$, $F^3 = (1, 1, 1)^T$. The initial value $(1, 0, 0)^T$ at $t = 0$ yields the solution F^1. Add some errors $(1, 10^{-30}, 10^{-30})^T$ to the initial value. Then, we have

t	value F_1 by RK	difference $F_1 - F^1$
50	1.92827e-22	9.99959e-31
60	8.75556e-27	1.00000e-30
70	1.39737e-30	1.00000e-30
80	1.00002e-30	1.00000e-30

We can see the instability.

Defusing method 3. Airy function を以下の例に

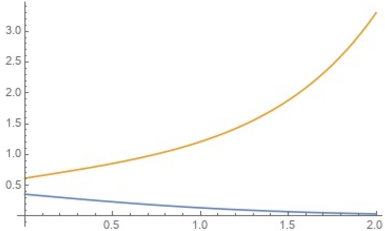
From the Airy differential equation $y''(t) - ty(t) = 0$ by $F = (y(t), y'(t))^T$,

$$P(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

$$\mathrm{Ai}(t) = \frac{1}{\pi} \lim_{b \to +\infty} \int_0^b \cos\left(\frac{s^3}{3} + ts\right) ds$$

(Airy function) is a solution of the Airy differential equation. We want to obtain values of $\mathrm{Ai}(x)$ by RK.†

The figure is a graph of Airy $\mathrm{Ai}(t)$ function and Airy $\mathrm{Bi}(t)$ function drawn by Mathematica. The function $F(t) = (\mathrm{Ai}(t), \mathrm{Ai}'(t))^T$ satisfies the condition 2 of the Situation 1 of the instability problem.



† More advanced method is "S.Chevillard, M.Mezzarobba, Multiple-precision evaluation of the Airy Ai function with reduced cancellation, arxiv:1212.4731".

Defusing method 4. Algorithm

$F_{k+1} = Q(k)F_k$. $Q = Q(N-1)\cdots Q(1)Q(0)$.

Algorithm

1. *Obtain eigenvalues* $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ *(assumption) of* Q *and the corresponding eigenvectors* $v_1, \ldots, v_r$.
2. *Let* λ_m *be the eigenvalue which is almost equal to* 0.
3. *Express the initial value vector* F_0 *containing errors in terms of* v_i*'s as*
$$F_0 = f_1 v_1 + \cdots + f_r v_r, \quad f_i \in \mathbf{R} \tag{13}$$
4. *Choose a constant* c *such that* $F_0' := c(f_m v_m + \cdots + f_r v_r)$ *approximates* F_0.
5. *Determine* F_N *by* $F_N = QF_0'$ *with the new initial value vector* F_0'. 要するに Q の大きい固有値に対応する固有空間の分を初期値から除く.

We call this algorithm the *defusing method*. This is a heuristic algorithm .

Defusing method 5. Airy の例

Example

$t_0 = 0$, $h = 10^{-3}$, $N = 10 \times 10^3$, 4-th order Runge-Kutta scheme. We have $\lambda_1 = 9.708 \times 10^9$, $v_1 = (-5.097, -159.919)^T$ and $\lambda_2 = 3.247 \times 10^{-7}$, $v_2 = (-5.097, 37.16)^T = (a, b)$. Then, $m = 2$. We assume the 3 digits accuracy of the value Ai(0) as 0.355 and set $F_0' = (0.355, 0.355b/a)$. Then, the obtained value F_{5000} at $t = 5$ is $(0.00010808, -0.00024685)$ by the defusing method. We have the following accurate value by Mathematica

`In[1]:= N[AiryAi[5]];    Out[1]= 0.00010834`

On the other hand, we appy the 4th order Runge-Kutta method with $h = 10^{-3}$ for $F_0 = (0.355, -0.259)^T$, which has the accuracy of 3 digits. It gives the value at $t = 5$ as $(-0.147395, -0.322215)$, which is a completely wrong value, and the value at $t = 10$ as $(-102173, -320491)$, which is a blow-up solution.

Example: defusing method for $H_n^k(x,y)$, 1

$$H_n^k(x,y) = \int_0^x t^k e^{-t} {}_0F_1(;n;yt)\mathrm{d}t.$$

Proposition (dots)

The function $u = H_n^k(x,y)$ satisfies

$$\begin{aligned} \{\theta_y(\theta_y + n - 1) + y(\theta_x - \theta_y - k - 1)\} \bullet u &= 0, \\ (\theta_x - \theta_y - k - 1 + x)\,\theta_x \bullet u &= 0. \end{aligned}$$

where $\theta_x = x\frac{\partial}{\partial x}$,$\theta_y = y\frac{\partial}{\partial y}$. The holonomic rank of this system is 4.

The ODE of y direction is unstable for H_n^k.[‡]

[‡] [dots] F.H.Danufane, K.Ohara, N.Takayama, C.Siriteanu, Holonomic Gradient Method-Based CDF Evaluation for the Largest Eigenvalue of a Complex Noncentral Wishart Matrix, `https://arxiv.org/abs/1707.02564`.

余談: $H_n^k(x, y)$ はどう応用される?

Theorem (Kang-Alouini[§])

When the matrix $\Sigma^{-1}MM^{*}$[¶] has the positive eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_s$, then the cummulative distribution function of the largest eigenvalue ϕ_s of S for the threshold x is

$$\mathrm{P}\left(\phi_s \le x\right) = \frac{\exp(-\sum_{i=1}^{s}\lambda_i)}{\Gamma(t-s+1)^s \prod_{1\le i<j\le s}(\lambda_j - \lambda_i)} \det \Psi(x)$$

where $\Psi(x)$ is a matrix valued function of which (i, j) element is

$$H_{t-s+1}^{t-i}(x, \lambda_j) = \int_0^x y^{t-i} \exp(-y)\, {}_0F_1(\,; t-s+1; y\lambda_j)\, dy$$

[§]M. Kang, M. S. Alouini, Largest Eigenvalue of Complex Wishart Matrices and Performance Analysis of MIMO MRC Systems, IEEE Journal on Selected Areas in Communications 21 (2003), 418–426.

[¶]channel matrix H is $N_T \times N_R$ complex valued random matrix. The column vector X satisfies $E[X] = M$ and the convariance is Σ^{-1}. $S = \Sigma^{-1}HH^*$.

Defusing Method for H_n^k, 2.

The ODE of y direction is unstable for H_n^k.
By the `DEtools[formal_sol]` function of Maple, we have

$$\begin{aligned} h_1 &= (xy)^{-1/2(1/2+n)} \exp(-2(xy)^{1/2})(1 + O(1/y^{1/2})), \\ h_2 &= y^{-k-1}(1 + O(1/y)), \\ h_3 &= (xy)^{-1/2(1/2+n)} \exp(2(xy)^{1/2})(1 + O(1/y^{1/2})), \\ h_4 &= y^{1-n+k} \exp(y)(1 + O(1/y)), \end{aligned}$$

when $y \to +\infty$. What is the asymptotic behavior of the function $H_n^k(x, y)$ when x is fixed? We compare the value of h_4 and the value by a numerical integration in Mathematica[‖].

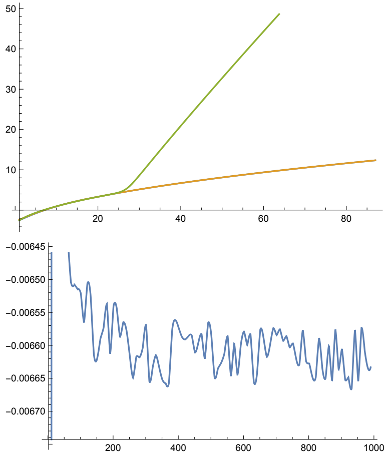
y	Ratio
1000	7.36595030875893e-452
2000	2.64621603289928e-881
3000	2.67723893601667e-1311

where

Ratio $= (H_1^{10}(1/2, y))/(y^{1-n+k} \exp(y))$, which suggests that H_n^k is expressed by h_1, h_2, h_3 without the dominant component h_4.

[‖] The method to evaluate hypergeometric functions in Mathematica is still a black box. It is not easy to give a numerical evaluator of hypergeometric functions which matches to Mathematica in all ranges of parameters and independent variables.

Defusing method for H_n^k, 3



$\log H_1^{10}(1, y)$. Exact value (by numerical integration) and the value by our defusing method agree. The adaptive Runge-Kutta method with the initial relative error 10^{-20} (upper curve) does not agree with the exact value when y is larger than about 25.

The relative error of $H_1^{10}(1, y)$ of our defusing method. The relative error is defined as $(H_d - H)/H$ where H_d is the value by the defusing method and H is the exact value.

小技 1. 例 χ^r 分布, 1.

Koyama** gave an integral formula of a generalization of χ^2 distribution motivated by the work of Marumo, Oaku, Takemura††

Theorem (koyama2019)

The probability density function $f(x) = \frac{d}{dx}P(\sum_{k=1}^{n} X_k^r < x)$ *(* X_k *'s are i.i.d random normal variables with* $m = 0, \sigma = 1$*,* $r \geq 3$*) is expressed by the following integrals.*

$$f(x) = \frac{1}{\pi}\frac{1}{2\pi^{n/2}}\int_0^\infty \exp(-xs)\,\mathrm{Im}\left[\varphi_3(s)\exp(\sqrt{-1}\pi/r) + \varphi_0(s)\right]^n ds, r \ odd$$

$$f(x) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{n/2}\int_0^\infty \exp(-xs)\,\mathrm{Im}\left[\varphi_3(s)\exp(\sqrt{-1}\pi/r)\right]^n ds, r \ even \tag{14}$$

** [koyama2019] T. Koyama, An integral formula for the powered sum of the independent, identically and normally distributed random variables, preprint. Old version is at arxiv https://arxiv.org/abs/1706.03989

†† [mot2014] N.Marumo, T.Oaku, A.Takemura, Properties of powers of functions satisfying second-order linear differential equations with applications to statistics, arxiv:1405.4451

小技 1. 例 χ^r 分布, 2.

Here,

$$\varphi_3(s) = \int_0^\infty \exp(-st^r) \exp\left(-\frac{e^{2\pi\sqrt{-1}/r}}{2} t^2\right) dt \qquad (15)$$

for $s > 0$ and

$$\varphi_0(s) = \int_0^\infty \exp(-st^r - t^2/2) dt \qquad (16)$$

We will evaluate the following integral when $r = 4$ as an example.

$$f(x) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{n/2} \int_0^\infty \exp(-xs) \,\mathrm{Im}\, \left[\varphi_3(s) \exp(\sqrt{-1}\pi/r)\right]^n ds,$$

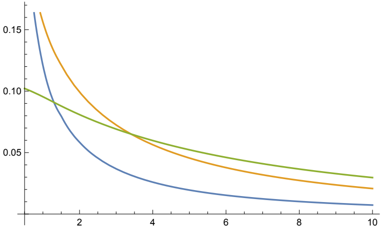
小技 1. 例 χ^r 分布, 3.

It seems that it is not a good method to evaluate $f(x)$ itself by the HGM, because the rank of the holonomic system for the integrand becomes very high when n increases [mot2014].

It will be a good method to generate a table of φ_3 by the HGM and use a one dimensional numerical integration method to obtain the value of the PDF $f(x)$.

Note that the HGM is a good method to generate a table of values.

Trick: use HGM as a subprocedure of a numerical integration.



The PDF $f(x)$ for $r = 4, n = 1, 3, 5$

```
--> load("test-ak2.rr");
--> Ans=hgm_phi3(R=6,X=100)$   // evalua
 ...
Time=[ 41.2335 0 2313312788 41.2705 ]
--> Ans[0];
[100,[ (0.4229-0.012354*@i) ...]]
```

小技 1. 例 χ^r 分布, 4.

Proposition

The cummulative distribution function (CDF) $P(\sum_{i=1}^{n} X_i^r < y)$ is approximately expressed as

$$\int_0^b \frac{1-\exp(-ys)}{s}\xi(s)ds + c_\alpha \frac{b^{-\alpha}}{\alpha} - c_\alpha y^\alpha \int_{by}^{\infty} e^{-t} t^{-\alpha-1} dt \quad (17)$$

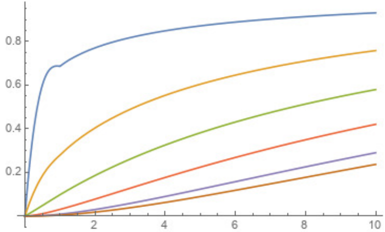
where b is a sufficiently large number, $\alpha = n/r$, and $\xi(s)$ is given in (18) and (19).

$$\xi(s) = \frac{1}{\pi}\frac{1}{(2\pi)^{n/2}} \mathrm{Im}\left[\varphi_3(s)\exp(\sqrt{-1}\pi/r) + \varphi_0(s)\right]^n \quad r \text{ is odd} \quad (18)$$

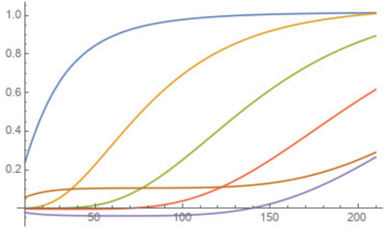
$$\xi(s) = \frac{1}{\pi}\left(\frac{2}{\pi}\right)^{n/2} \mathrm{Im}\left[\varphi_3(s)\exp(\sqrt{-1}\pi/r)\right]^n \quad r \text{ is even} \quad (19)$$

c_α is a constant (see the preprint as to the explicit value).

小技 1. 例 χ^r 分布, 5.



The CDF $F_n(y)$ for $y \in [0, 10]$, $r = 4$, $n = 1, 3, 5, 7, 9, 10$ (from the top to the bottom).



The CDF $F_n(y)$ for $y \in [10, 210]$, $n = 10, 30, 50, 70, 90, 100$. Note that $n = 90, 100$ cases (two lower curves) give wrong values because of numerical error of high powers n.

ところで φ_3 の微分方程式, 1

Put

$$f(x_1, x_2) = \int_{-\infty}^{\infty} \exp(x_1 z^2 + x_2 z^r) dz \tag{20}$$

Lemma

The function f satisfies the following A-hypergeometric system

$$(2\theta_1 + r\theta_2 + 1) \bullet f = 0 \tag{21}$$

$$(\partial_1^{r_1} - \partial_2) \bullet f = 0, \quad (r = 2r_1 \text{ is even}) \tag{22}$$

$$(\partial_1^{r} - \partial_2^2) \bullet f = 0, \quad (r \text{ is odd}) \tag{23}$$

where $\theta_i = x_i \partial_i = x_i \frac{\partial}{\partial x_i}$.

ところで φ_3 の微分方程式, 2

Lemma

Fix x_1 to a number. The function $f(x_1, x_2)$ annihilated by the following ordinary differential operator

$$\left(\frac{-r}{2}\right)^{r_1} \prod_{k=0}^{r_1-1} \left(\theta_2 + \frac{2k+1}{r}\right) - x_1^{r_1} \partial_2 \quad (r \text{ is even}) \tag{24}$$

$$\left(\frac{-r}{2}\right)^{r} \prod_{k=0}^{r-1} \left(\theta_2 + \frac{2k+1}{r}\right) - x_1^{r} \partial_2^2 \quad (r \text{ is odd}) \tag{25}$$

$\varphi_3(s) = f\left(-\frac{e^{2\pi\sqrt{-1}/r}}{2}, -s\right)$.

小技, 例 $E[\chi(M_x)]$, 1.

The expected Euler characteristic for the largest eigenvalue of a real Wishart matrix is numerically evaluated for a small sized Wishart matrix by HGM *. Let $A = (a_{ij})$ be a real $m \times n$ matrix valued random variable with the density

$$p(A)dA, \quad dA = \prod da_{ij}.$$

We assume that $p(A)$ is smooth and $n \geq m \geq 2$. Define a manifold

$$M = \{hg^T \,|\, g \in S^{m-1}, h \in S \in S^{n-1}\} \simeq S^{m-1} \times S^{n-1} / \sim$$

where $(h, g) \sim (-h, -g)$ and h and g are regarded as column vectors and hg^T is a rank 1 $m \times n$ matrix. Put

$$f(U) = \mathrm{tr}(UA) = g^T Ah, \quad U \in M$$

and

$$M_x = \{hg^T \in M \,|\, f(U) = g^T Ah \geq x\}$$

We are interested in $E[\chi(M_x)]$.

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* [euler2019] N.Takayama, L.Jiu, S.Kuriki, Y.Zhang, Computations of the Expected Euler Characteristic for the Largest Eigenvalue of a Real Wishart Matrix, arxiv:1903.10099

小技, 例 $E[\chi(M_x)]$, 2.

Assume $m = n = 2$ and $p(A)$ is a Gaussian distribution

$$p(A)dA = \frac{1}{(2\pi)^{mn/2}\det(\Sigma)^{n/2}} \exp\left\{-\frac{1}{2}\mathrm{Tr}\,(A-M)^T\Sigma^{-1}(A-M)\right\}dA.$$

The mean is expressed by the variable $M = (m_{ij})$. We gave an integral representation of $E(\chi(M_x))$ in [euler2019]. Moreover, we derived an ODE of rank 11 for (26) by the computer algebra package `HolonomicFunctions.m` (C.Koutchan).

$$E[\chi(M_x)] = \frac{1}{2\pi^2}\int_x^\infty d\sigma \int_{-\infty}^\infty db \int_{-\infty}^\infty ds \int_{-\infty}^\infty dt \frac{s_1 s_2(\sigma^2 - b^2)}{(1+s^2)(1+t^2)} \exp\left\{-\frac{1}{2}\tilde{R}\right\} \quad (26)$$

where $\tilde{R}$ is a rational function in $\sigma, b, s, t, s_1, s_2, m_{11}, m_{21}, m_{22}$. More precisely, put

$$\begin{aligned} R &= s_1\left(b\sin\theta\sin\phi + \sigma\cos\theta\cos\phi - m_{11}\right)^2 + s_2\left(\sigma\sin\theta\cos\phi - b\cos\theta\sin\phi - m_{21}\right)^2 \\ &\quad + s_1\left(\sigma\cos\theta\sin\phi - b\sin\theta\cos\phi\right)^2 + s_2\left(b\cos\theta\cos\phi + \sigma\sin\theta\sin\phi - m_{22}\right)^2, \end{aligned}$$

replace $\sin, \cos$ in R by

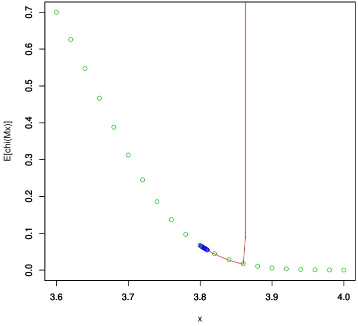
$$\sin\theta = \frac{2s}{1+s^2},\quad \cos\theta = \frac{1-s^2}{1+s^2},\quad \sin\phi = \frac{2t}{1+t^2},\quad \cos\phi = \frac{1-t^2}{1+t^2}.$$

and we set this $\tilde{R}$. We want to evaluate it when $m_{11} = 1, m_{21} = 2, m_{22} = 3$ (means) and $s_1 = 10^3, s_2 = 10^2$,

小技, 例 $E[\chi(M_x)]$, 3.

bigfloat, 冪級数を使うのを躊躇しない

Trick: Do not hesitate to use the bigfloat and powerseries. We use series solutions as a basis of interpolation or extrapolation.



The extrapolation function with powerseries of 20000 terms. Solid line is the extrapolation function, which diverges when $x > 3.8633$. Dots are values by simulations.

We use bigfloat of size 380 to determine series solutions.

計算のチャレンジと問題など 1.

Computational Try

R.Vidunas and A.Takemura[†] derived a system of linear partial differential equations for the outage probability $P(\phi_s \leq x)$. Try to make a numerical analysis of this system with Gröbner basis, the defusing method, or the method to obtain a stabile system.

Problem

Derive a good system of non-linear equations satisfied by $\det \Psi(x)$. The theory of holonomic quantum field and Hirota bilinear equations might help to solve this problem. If we can find such system, try a numerical analysis of it.

Computational Try

Try the defusing method for $H_n^k(x,y)$ upto $y \sim 10^8$, which lies in a range to apply to practical problems.

[†] R.Vidunas, A.Takemura, Differential relations for the largest root distribution of complex non-central Wishart matrices, arxiv:1609.01799

計算のチャレンジと問題など 2.

Computational Try

The defusing method for non-linear equation needs to compute a composition of non-linear functions instead of the matrix factorial. What is the size of a problem feasible by current computer algebra systems?

Computational Try

Marumo, Oaku, Takemura gave a method to derive a linear ODE for φ^n. The function φ_3 for $r = 4$ satisfies a 2nd order linear ODE. Try to make a numerical analysis of the system for φ_3^n with the defusing method, or the method to obtain a stabile system.

Problem

Give a method for a high precision evalution of the hypergeometric function ${}_rF_1$ and ${}_rF_0$. Refer, e.g., to the paper by S.Chevillard and M.Mezzarobba.

Computational Try

Try to make a numerical analysis of the ODE of rank 11 for $E[\chi(M_x)]$ with the defusing method, or the method to obtain a stabile system.