Design and Implementation of a Digital Formula Book for Generalized Hypergeometric Functions

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1 Introduction

Problems in the management of mathematical knowledge are becomming more important, particularly as technologies become more dependent on mathematics and mathematical sciences. One of the most important problems is editing digital formula books. Several approaches have been proposed, a notable example of which is the Mathematical Functions web site by Wolfram Research, which stores more than 37,000 mathematical formulae [8]. It is an impressive work, and allows for the retrieval of most mathematical formulae, but is based on Mathematica and presentation MathML, and does not provide other information such as formula expressions in other computer algebra systems, references, proofs, or verifications.

In this paper, an open design for a machine-readable document format for storing mathematical formulae is proposed based on XML, OpenMath[13] and OpenXM[14]. This structure has already been implemented in the form of a digital formula book for generalized hypergeometric functions and the OpenXM/fb authoring tools.

In the proposed format, mathematical formulae are stored with references and other information as *annotated digital formula* in a collection called a *formula book*.

The structure of the proposed formula book is as follows.

1. XML is used to store annotated digital formulae.

2. The formula book allows for several mathematical notions. In the present implementation, formulae are represented in OpenMath[13], which allows for new definitions of mathematical notions more easily than in MathML.

3. Related information such as references, proofs and verifications are stored in a machine-readable format or in a human-understandable format.

4. The formula book provides authoring tools to find typing mistakes in formulae.

Since the OpenMath XML expression is not easy to read and write, the authors developed a new language called `tfb` to express OpenMath objects. An example of a tfb expression can be found in the example below enclosed in the tag `<tfb>`. In order to write annotated digital formulae, elements such as title, author, and references are required as in most documents. DocBook is gaining popularity for this purpose, as exemplified by its recent use in conjunction with OpenMath in a project on interactive mathematical docuemnts [2]. The DocBook tags are adequate for describing general elements such as authors and titles, and are extended for use in the present formula book, as shown between `<formula>` tags, in the example of a fb-formula document below.

```xml
<?xml version="1.0" encoding="ISO-2022-JP"?>
<formula>
  <title>
    Quadratic transformation of an independent variable
  </title>
  <author> E. Goursat </author>
  <editor> Yasushi Tamura </editor>

  <tfb macroset="http://www.openxm.org/fb/hfb.txt">
    2 * arith1.root(nums1.pi,2)
      * hypergeo0.gamma(a + b + (1 / 2))
      / hypergeo0.gamma(a + (1 / 2))
      / hypergeo0.gamma(b + (1 / 2))
      * hypergeo1.hypergeometric2F1(a,b,1 / 2,x)
    ~relation1.eq~
    (hypergeo1.hypergeometric2F1(2 * a, 2 * b, a + b + (1 / 2),
        1 + arith1.root(x,2) / 2)
      + hypergeo1.hypergeometric2F1(2 * a, 2 * b, a + b + (1 / 2),
          1 ~arith1.minus~ arith1.root(x,2) / 2));
  </tfb>

  <description>
    Quadratic transformation of independent variable
  </description>
  <reference>
    <xref uri="http://www.openxm.org/fb/bib-goursat1.xml"
      linkend="goursat1" page="118"/>
  </reference>
  <evidence checker="Mathematica">
    @@ /. {a->1/2,b->3/5,c->-2/11,x->0.2}
  </evidence>
</formula>
```

This code describes the formula

$$\frac{2\sqrt{\pi}\,\Gamma(a+b+\frac{1}{2})\,{}_2F_1(a,b,\frac{1}{2},x)}{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})}$$

$$= \quad {}_2F_1(2\,a,2\,b,a+b+\frac{1}{2},\frac{1-\sqrt{x}}{2}) + {}_2F_1(2\,a,2\,b,a+b+\frac{1}{2},\frac{1+\sqrt{x}}{2})$$

which was found by E.Goursat more than a century ago [5].

The tag `<evidence>` in the document above is used by our numerical checker. One of the design goals of the formula book is to automatically check formulae in the formula book. However, full automatic proofs are obviously not feasible. One practical solution is to use numerical evaluation to find trivial typing mistakes and input errors. The numerical checker incorporated in the present design evaluates both sides of a given formula numerically for parameter values given in the `evidence` tag, as chosen by the editors.

The present implementation of the numerical checker system is as follows: This checker looks for `<evidence>` tags in fb-formula documents, generates programs for a computer algebra system, and sends the programs by HTTP. At the server side, an httpd daemon translates the requests by http into OpenXM requests and sends the request to an OpenXM server, in this case OpenXM/mathematica or OpenXM/asir. Any computer algebra system, which can numerically evaluate generalized hypergeometric functions, compliant with the OpenXM protocol, can replace the OpenXM/mathematica or OpenXM/asir systems. A checker can be chosen by the checker attribute of the evidence tag. In the present case, a special symbol `@@` is used for the numerical checker system to represent the tfb expression in the `<formula>` tag, which is replaced by the local expression for the numerical checker when the request is sent. In the author's experience in editing formula books of generalized hypergeometric functions, this numerical checker works very well for finding typing mistakes and errors. Moreover to this, our input job became very pleasant work by the checker.

The OpenMath content dictionaries used in the scheme for hypergeometric functions and some of the formulae already added to the formula book can be found at [15].

2   Tfb Expression

Tfb (Tiny Formula Book expression) is a syntax sugared language to express OpenMath Objects[13].

The tfb expression can express the tree structure of the OpenMath objects. XML tree structure is hard to read and write for humans, however the tfb expression is human friendly and rigorous enough.

The following is an example of the tfb expression for $c*(a+b) = c*a+c*b$.

```
<tfb>
  arith1.times(c,arith1.plus(a,b)) ~relation1.eq~
  (arith1.times(c,a) ~arith1.plus~ arith1.times(c,b));
```

```
        </tfb>
```

It is translated to the following OpenMath XML expression by our tfb parser.

```
<OMOBJ>
  <OMA><OMS cd="relation1" name="eq"/>
    <OMA><OMS cd="arith1" name="times"/>
      <OMV name="c"/>
      <OMA><OMS cd="arith1" name="plus"/>
        <OMV name="a"/>
        <OMV name="b"/>
      </OMA>
    </OMA>
    <OMA><OMS cd="arith1" name="plus"/>
      <OMA><OMS cd="arith1" name="times"/>
        <OMV name="c"/>
        <OMV name="a"/>
      </OMA>
      <OMA><OMS cd="arith1" name="times"/>
        <OMV name="c"/>
        <OMV name="b"/>
      </OMA>
    </OMA>
  </OMA>
</OMOBJ>
```

As we see from the output, the tfb expression `cdname.f` is translated to

<div align="center">

`<OMS cd="cdname" name="f"/>` .

</div>

The following is the definition of the grammar of the tfb expression.

S	→	[expression] ';'
expression	→	[term] ('~' string '~' [term])*
term	→	number | string | '(' [expression] ')' | [function]
function	→	string '(' ([expression] (',' [expression])*)? ')'

The translation rule is as follows.

1. The tfb expression `cdname.f(a1,...,an)` is translated to

   `<OMA> <OMS cd="cdname" name="f"/> a1  ...  an </OMA>`

2. The tfb expression `a1 ~cdname.f~ a2` is equivalent to `cdname.f(a1,a2)`.

3. The associativity of the binary expression `a1 ~cdname.f~ a2` is left to right.

4. `OMBIND` and `OMBVAR` are used to express lambda expressions (see the example in Section 4).

For editing formula books for hypergeometric functions, the symbols standing for elementary arithmetics are often used. The tfb allows to use a macro set. The following is a set of macro set for our formula book for hypergeometric functions.

```
+    ~arith1.plus~
-    ~arith1.minus~
*    ~arith1.times~
/    ~arith1.divide~
=    ~relation1.eq~
```

We call the tfb expression with this macro set *the tfb expression with the macro set for fb*. When we use this macroset, the value of the macroset attribute must be specified as

```
<tfb macroset="http://www.openxm.org/fb/hfb.txt">
```

3 Comparison of the tfb expression and other expressions

Let us compare the tfb expression and other expressions for mathematical object by using an example. We use the example of $\sin(1 + x)$. It is encoded as follows.

- OpenMath XML expression

```
<OMOBJ>
  <OMA><OMS cd="transc1" name="sin"/>
    <OMA><OMS cd="arith1" name="plus"/>
      <OMI> 1 </OMI>  <OMV name="x"/>
    </OMA>
  </OMA>
</OMOBJ>
```

- Tfb expression

```
 transc1.sin(1 ~arith1.plus~ x);
```

- Tfb expression with the macro set for fb.

```
 transc1.sin(1+x);
```

- Mathematica

```
 Sin[1+x]
```

- Presentation MathML

```
<math xmlns='http://www.w3.org/1998/Math/MathML'>
    <mrow>
    <mi>sin</mi>
    <mo>(</mo>
      <mrow>
       <mn>1</mn> <mo>+</mo> <mi>x</mi>
      </mrow>
    <mo>)</mo>
    </mrow>
    </math>
```

The presentation MathML is not relevant for describing the semantics. The tfb expression with the macro set for formula books is shorter than the Open-Math XML expression and easier to read and write. Of course, the expression `Sin[1+x]` in Mathematica is the shortest, but we think that the object hierarchy (CD) of OpenMath is systematic and we want to explicitly use and show this object hierarchy to describe mathematical objects in our formula book. These are reasons why we design and use tfb expressions.

4 A Content Dictionary for Hypergeometric Functions

OpenMath is a standard for communicating mathematics across the Web. As is explained in [2], "the representation of mathematics in OpenMath relies on four 'expression tree' constructors, on five basic objects, and on a special, sixth, basic object: symbols, defined in Content Dictionaries(CDs for short). The core CD's are publicly available collections of mathematical definitions". However, OpenMath does not provide a CD for hypergeometric functions. So we have developed it. The following is a part of our CD definition for Gauss's hypergeometric function

$$F(a, b, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} x^k, \quad (a)_k = a(a+1) \cdots (a+k-1).$$

```
<CDDefinition>
<Name> hypergeometric2F1 </Name>
<Description>
Gauss HyperGeometric Function.
    This function has a branch cut on [1,+infinity).
</Description>
<CMP>
    hypergeometric2F1(a,b,c;z)
    =\sum_{n=0}^{+\infty}
      \frac{pochhammer(a,n)pochhammer(b,n)}
        {pochhammer(c,n)pochhammer(1,n)} z^n
</CMP>
```

```
<FMP>
<tfb>
  hypergeometric2F1(a,b,c,z) ~relation1.eq~
    arith1.sum(interval1.integer_interval(0,infty),
      OMBIND(fns1.lambda,OMBVAR(n),
        pochhammer(a,n) ~arith1.times~ pochhammer(b,n) ~arith1.divide~
        (pochhammer(c,n) ~arith1.times~ pochhammer(1,n)) ~arith1.times~
        arith1.power(z,n)));
</tfb>
</FMP>
</CDDefinition>
```

We have developed the following CD's which are obtainable at
`http://www.math.kobe-u.ac.jp/OCD/`

hypergeo0	Supplementary functions to define hypergeometric functions.
hypergeo1	Gauss's hypergeometric function and generalized hypergeometric function $_pF_q$
hypergeo2	Famous hypergeometric functions such as Kummer, Bessel, Hankel, and Airy.
orthpoly1	The orthogonal polynomials of Legendre and Jacobi.
intpath1	Path of integration.
poly1p	Supplementary functions to define A-hypergeometric functions.
set1p	
linalg1p	
logic1p	
hypergeon1	A-hypergeometric functions (n variables) [18]
hypergeon2	Appell's hypergeometric function and Lauricella's hypergeometric funcion.
weylalgebra1	Weyl algebra and the ring of differential operators.

The following symbols are defined in our CD's.

```
hypergeo0: gamma, beta, pochhammer
hypergeo1: hypergeometric0F1, hypergeometric1F1, hypergeometric2F1
           hypergeometric_pFq
hypergeo2: kummer, besselJ, besselY, hankel1, hankel2
           airyAi, airyBi
orthpoly1: legendreP, legendreQ, jacobiG
intpath1: infty, segment, circle, circle_with_starting_point,
          path_in_sector, path_in_sector2, closed_path
linalg1p:  plus_part, minus_part, ones, kernel
poly1p: multi_power, index, indexed_variable, vector_of_indexed_variables
        sorted_set_of_indexed_variables
set1p: cartesian_product_n
weylalgebra1: diffop, gr, partialdiff, act
hypergeon1: raising_multi_factorial, falling_multi_factorial
            a_hypergeometric
hypergeon2: multi_pochhammer, appel_F1, appel_F2, appel_F3, appel_F4
            lauricella_FA, lauricella_FC, lauricella_FD
```

Our CD covers from the classical Gauss hypergeometric functions to the modern $\mathcal{A}$-hypergeometric series substantially introduced by Gel'fand, Kapranov and Zelevinsky [18]. The CD `intpath1` is an experimental CD to describe pathes of complex integrations based on the theory of twisted cycles.

5 Automatic Verifications of Formulae of Hypergeometric Functions

In our project, we store formulae of hypergeometric functions in tfb expressions. We will not only store formulae, but also will try to verify them as far as we can. There are several methods to prove or verify formulae.

1. Proving formulae in human understandable way (traditional method).

2. Proving formulae with an assistance of computers. For example, the famous method by Zeilberger may be classified in this category [16].

3. Numerical verifications.

4. Automatic derivation of formulae [11], [21], [22], [23].

5. Full automatic verifications by a proof checker like Coq.

The first method is traditional. The second method is discussed in [16]. The last method has not been studied and the performance will not be practical for our purpose. We will discuss about the third method in the next section as a practical approach.

6 Examples of Verification by Numerical Evaluation

Among the five methods, the "numerical verification method" is not rigorous, but is very useful and practical. A lot of people have implicitly used the method. Suppose that we are given a formula $A(x) = B(x)$. The numerical verification method numerically evaluates $A(x)$ and $B(x)$ at a point $x = p$. If the two values are not close, the checker warns that there may be a typo or an error in the formula. In this section we will discuss about this method. We will present an example of finding typos and errors in a book by the numerical verification method to convince readers that it is useful.

We have developed two checkers for numerical verification method. As is mentioned in Numerical Recipes [17], it is difficult and impossible to write a perfect code for numerical evaluations of hypergeometric functions which works for all values of parameters. So, it will be useful to have more than one numerical checkers.

1. The first checker is written by using Asir and Java from a scratch. It uses series expansion, connection formulae and integral representations of generalized hypergeometric functions $_pF_{p-1}$ for numerical evaluation. Algorithms used in this checker are discussed in [23]. This checker is faster than Mathematica for some parameter values.

2. The second checker utilizes Mathematica functions `N`, `Hypergeometric2F1` and `HypergeometricPFQ`. Unfortunately, algorithms to evaluate generalized hypergeometric functions used in Mathematica do not seem to be open and the source code is not available.

In this section, we present two examples by the second checker (checker which uses mathematica). The first checker works in a similar way.

Let us consider the following formula, which is one of the identity related to Kummer's 24 solutions [6, p.38].

$$_2F_1(a, b, c; x) = (1 - x)^{(c-a-b)} {}_2F_1(c - a, c - b, c; x)$$

The input of our checker is a tfb expression of this formula. Our checker converts it into an internal expression of a corresponding OpenMath object and outputs a Mathematica program to verify the formula numerically.

When a given tfb expression is translated to Mathematica expression, letters '.' and '_' are removed and the letter just after these letters are capitalized. For example, the tfb expression

```
(c ~arith1.times~ (a ~arith1.plus~ b)) ~relation1.eq~
((c ~arith1.times~ a) ~arith1.plus~ (c ~arith1.times~ b))
```

is translated as follows.

```
relation1Eq[arith1Times[c,arith1Plus[a,b]],
    arith1Plus[arith1Times[c,a],arith1Times[c,b]]]
```

The file `tfb_math.m` is used to translate function names defined in OpenMath CD's into function names for Mathematica. We below present a part of the file.

```
arith1Plus[a_,b_] := Module[{},
    Return[a+b]; ]
arith1UnaryMinus[a_] := Module[{},
    Return[-a];  ]
arith1Times[a_,b_] := Module[{},
    Return[a*b]; ]
```

Let us return to our example.

$$_2F_1(a, b, c; x) = (1 - x)^{(c-a-b)} {}_2F_1(c - a, c - b, c; x)$$

The following is a tfb expression for above formula.

```
<tfb>
  hypergeo1.hypergeometric2F1(a,b,c,x) ~relation1.eq~
  ( (1 ~arith1.minus~ x) ~arith1.power~ (c ~arith1.minus~ a ~arith1.minus~ b)
    ~arith1.times~
    hypergeo1.hypergeometric2F1(c ~arith1.minus~ a, c ~arith1.minus~  b, c,
      x));
</tfb>
```

The following is a Mathematica expression generated by our checker.

```
relation1Eq[hypergeo1Hypergeometric2F1[a,b,c,x],
   arith1Times[arith1Power[arith1Minus[1,x],arith1Minus[arith1Minus[c,a],b]],
      hypergeo1Hypergeometric2F1[arith1Minus[c,a],arith1Minus[c,b],c,x]]]
```

The following is a Mathematical input program converted by `tfb_math.m`.

```
Hypergeometric2F1[a, b, c, x] -
  1.*(1. - 1.*x)^(-1.*a - 1.*b + c)*
   Hypergeometric2F1[-1.*a + c, -1.*b + c, c, x]
```

If we replace a, b, c by numbers which are not integers, and replace x by a complex number of which absolute value is less than 1 then, we get the value 0., which is a numerical evidence that the formula is correct.

We have seen a successful case, but the following identity related to Kummer's 24 solutions in the book [6] could not be verified numerically.

$$x^{-\beta} F(\beta, \beta - \gamma + 1, \beta + 1 - \alpha; \frac{1}{x}) = (-x)^{\alpha - \gamma} (1 - x)^{\gamma - \beta - 1} {}_2 F_1(\beta + 1 - \gamma, 1 - \alpha, \beta + 1 - \alpha; \frac{1}{1 - x})$$

In this case, the numerical value of the difference of the left hand side and the right hand side at $x = 2 + 2\sqrt{-1}$ is $0.0933913 - 0.776621I$ by Mathematica checker. We double-checked the proof and we found a typo in the book [6]. The correct formula seems to be as follows.

$$x^{-\beta} F(\beta, \beta - \gamma + 1, \beta + 1 - \alpha; \frac{1}{x}) = (-x)^{1 - \gamma} (1 - x)^{\gamma - \beta - 1} {}_2 F_1(\beta + 1 - \gamma, 1 - \alpha, \beta + 1 - \alpha; \frac{1}{1 - x})$$

Even after this correction, our numerical checker still do not succeed to verify it numerically. Why does it not succeed? It is because the book[6] does not specify the branch of the power function x^a.

In our CD and Mathematica, the power function x^a is defined as a single valued function on the simply connected domain $\mathbf{C} \setminus [0, -\infty)$ as follows.

$$x^a = \exp(a\pi \arg(x)) \exp(a \log(|x|)), -\pi < \arg(x) < \pi$$

Let us show an example that presents how we should be cautious about branches of power functions. We often make the following simplification

$$(-x)^a = (-1)^a (x)^a.$$

However, this simplification is not correct under our definition. In fact, under our definition, $(-x)^a$ is the composite function of z^a and $z = -x$. The branch cut of the function $(-x)^a$ is $[0, +\infty)$ and we have the following formula for the simplification.

$$(-x)^a = \begin{cases} e^{a\pi i} x^a & \text{Im } x < 0 \\ e^{-a\pi i} x^a & \text{Im } x > 0 \end{cases}$$

Figure 1 explains our simplification formula visually. It is the graph of the real part of
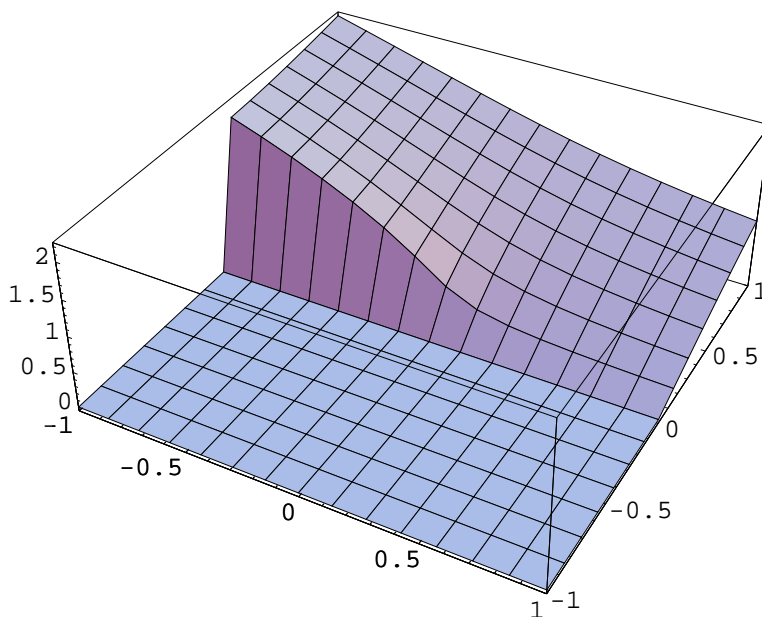
$$(-z)^a - \exp(a\pi i) z^a.$$

Figure 1: The graph of the real part of $(-x - iy)^{\frac{1}{2}} - \exp(\frac{1}{2}\pi i)(x + iy)^{\frac{1}{2}}$

The difference is zero only on $\mathrm{Im}\,(z) < 0$.

Under our definition of the power function, the correct formula is as follows.

$$(-x)^{-\beta} F(\beta, \beta-\gamma+1, \beta+1-\alpha; \frac{1}{x}) = (-x)^{1-\gamma}(1-x)^{\gamma-\beta-1}{}_2F_1(\beta+1-\gamma, 1-\alpha, \beta+1-\alpha; \frac{1}{1-x})$$

After this correction, our checker outputs the difference of 10^{-16}. As we have seen, our numerical checker finds a typo and an error in the book [6]. We want to emphasize, of course, this kind of typos does not spoil this ook. This book is a wonderful book on hypergeometric functions and pioneering one on Painlevé functions.

We have seen in example above, mathematical books often do not care about branches of power functions. However, it is not acceptable in our formula book project. There is no automatic way to make corrections on branches. As to a comprehensive reference on this branch problem, we refer to [4].

7   Design Outline of Annotated Digital Formula Book Format

We will design a machine readable document format to store mathematical formulae. In this paper, we will call mathematical objects stored in a digital format *digital formulae*. We can describe digital formulae with the tfb expression. As we discussed in the previous chapter, it is more useful to store additional

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informations such as proofs, references, verifications than storing only digital formulae.

For example, more than 37,000 mathematical formulae are stored at Mathematical Functions web cite [8], but it does not provide us no other informations such as references, proofs, and verifications. It is one of the unsatisfactory features of Mathematical Functions web cite.

Our document format will give not only a way to describe digital formulae but also a way to describe related informations. As we have seen in the previous chapter, we can store digital formulae in a machine readable format with the OpenMath. We can also use the semantic MathML for the same purpose, but the OpenMath is easier to define new mathematical notions than the semantic MathML. Hence, we chose to use the OpenMath to describe digital formulae.

Let us call digital formulae with related informations *annotated digital formulae*. A collection of annotated digital formulae will be called a *formula book* in this paper.

We will use the XML to describe our digital formula book. The reasons for this choice are as follows. (1) OpenMath mainly uses the XML expression. If we use the XML, the whole digital formula book will be a well-formed XML document and we can use several tools for XML documents. (2) We want to translate our digital formula book into LaTeX and HTML and other several formats. (3) The XML is the best for this purpose and the XML will be the standard for digital documents, especially for digital documents for the world wide web.

Following these basic decisions, we design a digital formula book with the following architecture.

1. Related informations to digital formulae such as references, and numerical verifications should be stored in a machine readable format.

2. Our formula book should be flexible so that it can describe several mathematical notions. We will use the OpenMath [13], which can define new mathematical notions relatively easily. We use the tfb expression to describe the OpenMath objects.

3. We should utilize the XML technology which is now dominant in the computer networks. We will use the DOM(document object model) and the XSL to build our authoring tools for our digital formula book.

4. Autoring tools to find typos and errors of formulae are necessary. As we have seen in the previous chapter, we could find a typo in the list of Kummer's 24 solutions in a book by a numerical evaluation. We will use this numerical checker as an authoring tool to find typos and errors.

In the next section, we will discuss our document format of our digital formula book based on the architecture above.

8 Document Structures of Digital Formula Books

In order to write digital formula books in the XML, we need elements such as the title, the author, references and so on which appears in most documents. The DocBook [3] is a standard to describe documents such as books and papers in the XML. The DocBook is getting popular in the mathematical community. For example, in [2], they report a project on interactive mathematical documents, which is based on the DocBook and the OpenMath. The DocBook tags are good enough for describing general elements such as authors, titles and so on, we will utilize the DocBook tags for our formula book. The DocBook provides several document formats such as the book style and the article style. For our purpose, the article style will be enough. We make our own definitions to describe formulae. Our extensions appear in `<formula>` tags, as will hereinafter be described in detail.

The top node for annotated digital formulae is the `<formula>` tag. Our digital formula books are described as DocBook article documents extended with the `<formula>` tag.

Let us define our document styles. Documents of which top node is `<formula>` and which describe annotated digital formulae are called *fb-formula documents*. In the fb-formula documents, the OpenMath objects are expressed by the tfb expression. Fb formula documents of which tfb expressions are replaced by the OpenMath XML expressions are called *fbOM-formula documents*. Documents written with the DocBook article tags and `<formula>` tags are called *fb-DocBook-article documents*, of which top node must be `<article>`. Fb-DocBook-article documents of which tfb expressions are replaced by OpenMath XML expressions are called *fbOM-DocBook-article documents*.

Here is an example of a fb-formula document:

```
<?xml version="1.0" encoding="ISO-2022-JP"?>
<?xml-stylesheet type="text/xsl" href="fb.xsl"?>
<formula>
  <title>
    Quadratic transformation of independent variable
  </title>

  <author> E. Goursat </author>

  <editor> Yasushi Tamura </editor>

  <tfb macroset="http://www.openxm.org/fb/hfb.txt">
    2 * arith1.root(nums1.pi,2)
      * hypergeo0.gamma(a + b + (1 / 2))
      / hypergeo0.gamma(a + (1 / 2))
      / hypergeo0.gamma(b + (1 / 2))
      * hypergeo1.hypergeometric2F1(a,b,1 / 2,x)
    ~relation1.eq~
    (hypergeo1.hypergeometric2F1(2 * a, 2 * b, a + b + (1 / 2),
        1 + arith1.root(x,2) / 2)
```

```
      + hypergeo1.hypergeometric2F1(2 * a, 2 * b, a + b + (1 / 2),
          1 ~arith1.minus~ arith1.root(x,2) / 2));
  </tfb>

  <description>
    Quadratic transformation of independent variable
  </description>

  <reference>
    <xref uri="http://www.openxm.org/fb/bib-goursat1.xml"
      linkend="goursat1" page="118"/>
  </reference>

  <evidence checker="Mathematica">
    @@ /. {a->1/2,b->3/5,c->-2/11,x->0.2}
  </evidence>

</formula>
```

The following is the document type definition of the fb-formula document.

```
<!DOCTYPE formula [
<!ELEMENT formula (title?,author?,editor?,tfb,description*,
                   proof*,reference*,evidence*)>
<!ELEMENT title (#PCDATA)>
<!ELEMENT author (#PCDATA)>
<!ELEMENT editor (#PCDATA)>
<!ELEMENT tfb (#PCDATA)>
<!ELEMENT description (#PCDATA | xref)*>
<!ELEMENT proof (#PCDATA | xref)*>
<!ELEMENT reference (#PCDATA | xref)*>
<!ELEMENT evidence (#PCDATA)>
<!ATTLIST evidence checker CDATA #REQUIRED>
<!ELEMENT xref EMPTY>
<!ATTLIST xref uri CDATA  linkend CDATA #REQUIRED
               page CDATA #REQUIRED>
]>
```

The following table is an explanation of tags that come under the node `<formula>`.

`<title>`	The title of the formula.
`<author>`	The author name of the formula.
`<editor>`	We call the person who inputs the formula in the fb document format *editor*. The name of the editor comes here.
`<tfb>`	The mathematical content of the formula is written in the tfb expression under this tag. The fb document must contain one `<tfb>` tag. In the fbOM-formula document, this tag and the content are replaced by the `<OMOBJ>` tag and the content expressed by the OpenMath XML expression.
`<description>`	An explanation about the formula comes here. The hyperlink tag `<xref>` may be used under this tag.
`<proof>`	Proofs (for humans) are written under this tag. The hyperlink tag `<xref>` may be used under this tag.
`<reference>`	Original references of the formula. The hyperlink tag `<xref>` may be used under this tag.
`<evidence>`	A program to check the formula is put under this tag.

Remark: The hyperlink tags `<xref>` in fb-formula documents usually refer to places in other documents. This is a different use of the `<xref>` tag comparing to the DocBook specification. The labels for `<xref>` must consists of a universal resource identifier and a label in that document. Here is an example:

```
<reference>
  <xref uri="http://www.openxm.org/fb/formula/bib-iksy.xml"
    linkend="iksy"  page="38-39"/>
</reference>
```

The tags `<title>`, `<author>` , `<editor>`, `<tfb>` must appear only one times at most in `<formula>`. The content of the formula is put under the `<tfb>` tag with the tfb expression.

Our system now accepts the tag `<tex>` to embed documents written in LATEX, which is used to describe definitions, facts, and proofs for humans. It will be a practical approach now. However, from a semantic XML-minded point of view, it will not be allowed. It is a future problem to remove the tag @verb@¡tex¿@ from our formula book (cf. [12]).

9   Implementation of Numerical Checker

One of the design goals of our formula books is to automatically check formulae in our formula books. When we edit a formula book, we often generate typos. However, formula books are required to be trustful and we have to edit the formula book as trustful as possible. Moreover, mathematicians sometimes do not care about branches of analytic functions and there are formulae in literatures of which choices of branches are wrong in view of our rigorous and machine

treatable definition of hypergeometric functions. We also need to find this kind of troubles by our automatic checker.

Our checker evaluates the both sides of a given formula numerically for given value of parameters. The values of parameters are chosen by editors. We call it the *fb numerical checker system*. We use this system, because the numerical check is a practical solution to find typos of formulae.

Our implementation of the fb numerical checker system is as follows: The tfb expressions of digital formulae are translated to tree structures of OpenMath objects. These OpenMath objects are used as intermediate expressions and expressions for Mathematica or Risa/Asir are generated from these OpenMath objects. Our fb numerical checker system consists of a client and a server. This client looks for <evidence> tags in fb-formula documents, and generates programs for a server and sends programs by the HTTP protocol. At the server side, an httpd daemon is working and this daemon translates the requests by http into OpenXM requests and send them to an OpenXM server: OpenXM/mathematica or OpenXM/asir. If we have a computer algebra system compliant to the OpenXM protocol, we can replace OpenXM/mathematica or OpenXM/asir by the system.

Currently Mathematica and Risa/Asir are working as fb numerical checkers. We can choose a checker by the checker attribute of the evidence tag. (In the example of the previous section, we choose Mathematica.) We use a special symbol @@ for our fb numerical checker systems. This symbol stands for the tfb expression in the <formula> tag and this is replaced by the local expression for the numerical checker when the checker client sends the request.

The decision if the formula passed the check or not is made by the editor. Our checker displays three numerical values {value of the left-hand side − the value of the right-hand side, the value of the left-hand side, the value of the right-hand side }. Because when both sides are extremely small, the difference is also small and we cannot judge the validity of the formula only by the difference of values.

Let us see an example. When we send the fb-formula document in the previous section to our fb numerical checker, we get the following reply.

```
HTTP/0.9 200 OK
Connection: close
Content-length: 62
Content-Type: text/plain

{8.881784197001252*^-16, 3.806381231714452, 3.806381231714451}
```

The last list is the result. In this example, since the difference of the value of the left hand side and the value of the right hand side is relatively smaller than the values of the both sides, we can conclude that the correctness of the formula will be highly probable. This decision is done by the editor and selections of values of parameters are also done by the editor.

10 Authoring Tools for Annotated Digital Formulae

We provide the following tools for dealing with annotated digital formulae. These tools read data from the standard input and write the results to the standard output.
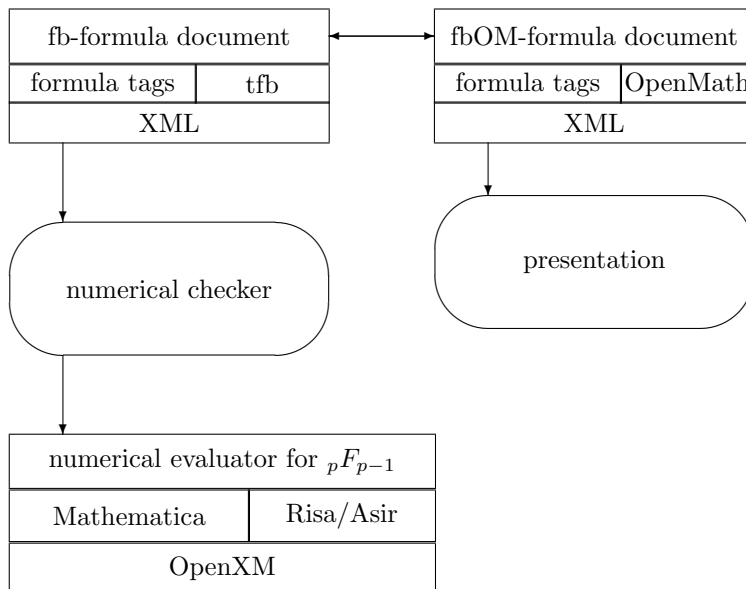
tfb2fb.sh	This tool converts tfb expressions to OpenMath XML expressions. When the input is a fb-formula document, the output is the fbOM-formula document. When the input is a fb-DocBook-article document, the output is the fbOM-DocBook-article document. If the input is a content dictionary(CD) expressed in terms of the tfb expression, the output is the OpenMath CD.
tfb2math-check.sh	This tool is a client for checking numerically the formula in a fb-formula document by Mathematica.
tfb2asir-check.sh	This tool is a client for checking numerically the formula in a fb-formula document by Risa/asir.
tfb2asir-check2.sh	
fb2tex.sh	This tool converts a fbOM-DocBook-article document to a LaTeX document.
formula2tex.sh	This tool converts a fbOM-formula document to a LaTeX document.

FbOM-DocBook-article documents and fbOM-formula documents are well-formed XML documents. We use the XSL, which is a standard system to translate XML documents into several formats, to translate digital formulae expressed in the OpenMath XML expression into LaTeX expressions.

The following is the output by the translation by `tfb2fb.sh | formula2tex.sh` of the fb-formula document in the previous section.

$$
\begin{aligned}
& (\frac{\frac{((2 \cdot {}^2\sqrt{\pi}) \cdot \Gamma(((a+b)+\frac{1}{2})))}{\Gamma((a+\frac{1}{2}))}}{\Gamma((b+\frac{1}{2}))} \cdot {}_2F_1(a, b, \frac{1}{2}; x)) \\
= \quad & ({}_2F_1((2 \cdot a), (2 \cdot b), ((a+b)+\frac{1}{2}); \frac{(1+{}^2\sqrt{x})}{2}) \\
& + {}_2F_1((2 \cdot a), (2 \cdot b), ((a+b)+\frac{1}{2}); \frac{(1-{}^2\sqrt{x})}{2}))
\end{aligned}
$$

Edited by Yasushi Tamura
Quadratic transformation of independent variable

17

fb-formula document		fbOM-formula document	
formula tags	tfb	formula tags	OpenMath
XML		XML	

↓ (left) ↔ (between documents) ↓ (right)

| numerical checker | presentation |

↓

numerical evaluator for $_pF_{p-1}$	
Mathematica	Risa/Asir
OpenXM	

11 Formula Book for Hypergeometric Functions

We are editing a formula book for hypergeometric functions. We use the fb-formula document format.

We have referred to the following literatures as input sources up to now.

1. From Gauss to Painleve [6]

2. Iwanami mathematical formula book III [9]

3. Generalized hypergeometric functions [20]

4. Sur l'Equation Differentielle Lineaire qui Admet pour Integrale la Serie Hypergeometrique [5]

5. Some transformations of generalized hypergeometric series. [24]

6. Algebraic values of G-functions [1]

We have input 131 formulae in our CVS repository up to now. After inputting formulae, we use `tfb2math-check.sh` to find typos. The formulae are classifed into the following categories.

1. Contiguity relations (automatically generated [21]) such as $(a-c)\ _2F_1(-1+a,b,c,z)+(-2\,a+c+(a-b)\,z)\ _2F_1(a,b,c,z)+a\,(1-z)\ _2F_1(1+a,b,c,z) = 0$

2. Kummer type identities (semi-authomatically generated [22]) such as
$$_2F_1(a, b, c, x) = (1 - x)^{-\mathtt{a} - b + \mathtt{c}} \, _2F_1(-a + c, -b + c, c, x)$$

3. Connection formulae (automatically generated for degenerate cases, [23]) such as

$$F(a_1, a_1, b_1; z)$$
$$= \frac{\Gamma(b_1)(-z)^{-a_1}}{\Gamma(a_1)\Gamma(-a_1 + b_1)} \lim_{a_2 \to a_1} \frac{\partial}{\partial a_2} \, _2F_1(a_1, 1 + a_1 - b_1, 1 + a_1 - a_2, 1/z)$$
$$- \frac{\Gamma(b_1)(-z)^{-a_1}}{\Gamma(a_1)\Gamma(-a_1 + b_1)} \lim_{a_2 \to a_1} \frac{\partial}{\partial a_2} \, _2F_1(a_2, 1 + a_2 - b_1, 1 - a_1 + a_2, 1/z)$$
$$- \frac{\gamma\Gamma(b_1) + \Gamma(b_1)\psi(a_1)}{\Gamma(a_1)\Gamma(-a_1 + b_1)} (-z)^{-a_1} \, _2F_1(a_1, 1 + a_1 - b_1, 1, 1/z)$$
$$- \frac{\gamma\Gamma(b_1) - \Gamma(b_1)\log(-z) + \Gamma(b_1)\psi(-a_1 + b_1)}{\Gamma(a_1)\Gamma(-a_1 + b_1)} (-z)^{-a_1} \, _2F_1(a_1, 1 + a_1 - b_1, 1, 1/z),$$

here, $\psi(z)$ is the derivative of $\log(\Gamma(z))$ and γ is $-\psi(1)$.

4. Quadratic transformation of independent variables such as
$$_2F_1(4\,a, \tfrac{1}{4} + 2\,a, \tfrac{1}{2} + 4\,a, x) = \frac{_2F_1(a, \frac{1}{4} + a, \frac{3}{4} + 2\,a, \frac{16\,(1-x)\,x^2}{(2-x)^4})}{\left(1 - \frac{x}{2}\right)^{4\,a}}.$$

5. Special values such as
$$_2F_1\left(\tfrac{1}{12}, \tfrac{5}{12}, \tfrac{1}{2}, \tfrac{1323}{1331}\right) = \frac{3\,11^{\frac{1}{4}}}{4}.$$

In our experience of editing this formula book of generalized hypergeometric functions, the fb numerical checker works very fine both for finding typos and for making the input job exciting for editors. Our formula book is obtainable from `http://www.math.kobe-u.ac.jp/OCD/`

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A Content Dictionaries for Hypergeometric Functions

A.1 `hypergeo0.tfb`

```xml
<?xml version="1.0"?>
<CD>
<!-- $Id: iamc-2003.tex,v 1.5 2003/07/14 03:07:28 taka Exp $ -->
<CDName> hypergeo0 </CDName>
<CDURL> http://www.math.kobe-u.ac.jp/OCD/ </CDURL>
<CDReviewDate> 2003-04-01 </CDReviewDate>
<CDDate> 2002-11-29 </CDDate>
<CDVersion> 0 </CDVersion>
<CDRevision> 0 </CDRevision>
<CDStatus> experimental </CDStatus>
<CDUses>
  <CDName>alg1</CDName>
  <CDName>arith1</CDName>
  <CDName>calculus1</CDName>
  <CDName>complex1</CDName>
  <CDName>fns1</CDName>
  <CDName>integer1</CDName>
  <CDName>interval1</CDName>
  <CDName>logic1</CDName>
  <CDName>nums1</CDName>
  <CDName>relation1</CDName>
</CDUses>
<Description>
    This CD defines some basic hypergeometric integrals and
    symbols necessary to define hypergeometric functions.
    These functions are described in the following books.
    (1) Handbook of Mathematical Functions, Abramowitz, Stegun
    (2) Higher transcendental functions. Vol. III.
        Krieger Publishing Co., Inc., Melbourne, Fla., 1981,
        Erdlyi, Arthur; Magnus, Wilhelm; Oberhettinger,
        Fritz; Tricomi, Francesco G.
    (3) From Gauss to Painleve, Vieweg, Katsunori Iwasaki, Hironobu Kimura,
        Shun Shimomura, Masaaki Yoshida
</Description>
```

```
<CDDefinition>
<Name> gamma </Name>
<Description>
    Euler's gamma function
</Description>
<CMP>gamma(z)=\int_0^{+\infty} t^{z-1} e^{-z} dt (Re(z)>0)</CMP>

<FMP>
<tfb>
  complex1.real(z) ~relation1.gt~ 0 ~logic1.implies~
  (hypergeo0.gamma(z) ~relation1.eq~
    calculus1.defint(
      interval1.interval(0,nums1.infinity),
      OMBIND(fns1.lambda,OMBVAR(t),
        arith1.power(t,z ~arith1.minus~ 1) ~arith1.times~
        arith1.power(nums1.e, arith1.unary_minus(z)))));
</tfb>
</FMP>

<Example> gamma(n) = (n-1)! (n \in N)
<tfb>
  n ~set1.in~ setname1.N ~logic1.implies~
  (hypergeo0.gamma(n) ~relation1.eq~
    integer1.factorial(n ~arith1.minus~ 1));
</tfb>
</Example>
</CDDefinition>
```

```
<CDDefinition>
<Name> beta </Name>
<Description>
    Euler's beta function
</Description>
<CMP>
  beta(p,q)=\frac{gamma(p)gamma(q)}{gamma(p+q)}(p,q \not\in Z_{&lt;=0})
</CMP>

<FMP>
<tfb>
  arith1.unary_minus(p) ~set1.notin~ setname1.N ~logic1.and~
  (arith1.unary_minus(q) ~set1.notin~ setname1.N) ~logic1.implies~
  (hypergeo0.gamma(p) ~arith1.times~ hypergeo0.gamma(q) ~arith1.divide~
  hypergeo0.gamma(p ~arith1.plus~ q));
</tfb>
</FMP>

<Example> beta(p,q)=\int_0^1 t^{p-1} (1-t)^{q-1} dt (Re(p),Re(q)&gt;0)
<tfb>
  complex1.real(p) ~relation1.gt~ 0 ~logic1.and~
  (complex1.real(q) ~relation1.gt~ 0) ~logic1.implies~
  (hypergeo0.beta(p,q) ~relation1.eq~
    calculus1.defint(
      interval1.interval(0,1),
      OMBIND(fns1.lambda,OMBVAR(t),
        arith1.power(t,p ~arith1.minus~ 1) ~arith1.times~
        arith1.power(1 ~arith1.minus~ t,q ~arith1.minus~ 1))));
</tfb>
</Example>
</CDDefinition>

<CDDefinition>
<Name> pochhammer </Name>
<Description>
    Pochhammer symbol
</Description>
<CMP>pochhammer(a,n) = gamma(a+n)/gamma(a)</CMP>

<FMP>
<tfb>
  hypergeo0.pochhammer(alpha,n) ~relation1.eq~
    (hypergeo0.gamma(alpha ~arith1.plus~ n)
      ~arith1.divide~ hypergeo0.gamma(alpha));
</tfb>
</FMP>

<Example>
 pochhammer(a,n) = \prod_0^{n-1} (a+i)
<tfb>
  hypergeo0.pochhammer(a,n) ~relation1.eq~
    arith1.product(
      interval1.integer_interval(0,n ~arith1.minus~ 1),
      OMBIND(fns1.lambda,OMBVAR(i),
        a ~arith1.plus~ i));
</tfb>
</Example>
</CDDefinition>
</CD>                            23
```

A.2 `hypergeo1.tfb`

```xml
<?xml version="1.0"?>
<CD>
<!-- The Gauss hypergeometric function, confluent hypergeometric -->
<!-- functions, and generalized hypergeometric functions in one -->
<!-- variables. -->
<CDName> hypergeo1 </CDName>
<CDURL> http://www.math.kobe-u.ac.jp/OCD/ </CDURL>
<CDReviewDate> 2003-07-26 </CDReviewDate>
<CDDate> 2002-11-29 </CDDate>
<CDVersion> 0 </CDVersion>
<CDRevision> 0 </CDRevision>
<CDStatus> experimental </CDStatus>
<CDUses>
  <CDName>alg1</CDName>
  <CDName>arith1</CDName>
  <CDName>calculus1</CDName>
  <CDName>fns1</CDName>
  <CDName>hypergeo0</CDName>
  <CDName>interval1</CDName>
  <CDName>linalg1</CDName>
  <CDName>linalg4</CDName>
  <CDName>nums1</CDName>
  <CDName>relation1</CDName>
</CDUses>
<Description>
    This CD defines the Gauss hypergeometric function, confluent
    hypergeometric functions, and generalized hypergeometric functions
    in one variable.
    These functions are described in the following books.
    (1) Handbook of Mathematical Functions, Abramowitz, Stegun
    (2) Higher transcendental functions. Vol. III.
        Krieger Publishing Co., Inc., Melbourne, Fla., 1981,
        Erdlyi, Arthur; Magnus, Wilhelm;
        Oberhettinger, Fritz; Tricomi, Francesco G.
    (3) From Gauss to Painleve, Vieweg, Katsunori Iwasaki,
        Hironobu Kimura, Shun Shimomura, Masaaki Yoshida.

</Description>
```

```
<CDDefinition>
<Name> hypergeometric0F1 </Name>
<Description>
    Hypergeometric function {}_0 F_1.
</Description>

<CMP>
    hypergeometric0F1(;a;z)
    =\sum_{n=0}^{+\infty}
      \frac{1}{pochhammer(a,n)pochhammer(1,n)} z^n
</CMP>

<FMP>
<tfb>
  hypergeo1.hypergeometric0F1(a,z) ~relation1.eq~
    arith1.sum(interval1.integer_interval(0,nums1.infinity),
      OMBIND(fns1.lambda,OMBVAR(n),
        1 ~arith1.divide~ hypergeo0.pochhammer(a,n) ~arith1.divide~
        hypergeo0.pochhammer(1,n) ~arith1.times~
        arith1.power(z,n)));
</tfb>
</FMP>
</CDDefinition>
```

```
<CDDefinition>
<Name> hypergeometric1F1 </Name>
<Description>
    Kummer's confluent hypergeometric function.
</Description>

<CMP>
    hypergeometric1F1(a,b;z)
    =\sum_{n=0}^{+\infty}
      \frac{pochhammer(a,n)}{pochhammer(1,n)pochhammer(b,n)} z^n
</CMP>

<FMP>
<tfb>
  hypergeo1.hypergeometric1F1(a,b,z) ~relation1.eq~
    arith1.sum(interval1.integer_interval(0,nums1.infinity),
      OMBIND(fns1.lambda,OMBVAR(n),
        hypergeo0.pochhammer(a,n) ~arith1.divide~
        (hypergeo0.pochhammer(b,n)
          ~arith1.times~ hypergeo0.pochhammer(1,n))
        ~arith1.times~  arith1.power(z,n)));
</tfb>
</FMP>
</CDDefinition>
```

```
<CDDefinition>
<Name> hypergeometric2F1 </Name>
<Description>
    The Gauss hypergeometric function.
    This function has a branch cut on [1,+infinity).
</Description>

<CMP>
    hypergeometric2F1(a,b,c;z)
    =\sum_{n=0}^{+\infty}
      \frac{pochhammer(a,n)pochhammer(b,n)}{pochhammer(c,n)pochhammer(1,n)}
        z^n
</CMP>

<FMP>
<tfb>
  hypergeo1.hypergeometric2F1(a,b,c,z) ~relation1.eq~
    arith1.sum(interval1.integer_interval(0,nums1.infinity),
      OMBIND(fns1.lambda,OMBVAR(n),
        hypergeo0.pochhammer(a,n) ~arith1.times~ hypergeo0.pochhammer(b,n)
         ~arith1.divide~
         (hypergeo0.pochhammer(c,n) ~arith1.times~ hypergeo0.pochhammer(1,n))
         ~arith1.times~ arith1.power(z,n)));
</tfb>
</FMP>

<Example>
 z (1-z) d^2 F/dz^2 + (c - (a+b+1) z) d F/dz - a b F = 0
<tfb>
  z ~arith1.times~ (1 ~arith1.minus~ z) ~arith1.times~
  calculus1.diff(OMBIND(fns1.lambda,OMBVAR(z),
    calculus1.diff(OMBIND(fns1.lambda,OMBVAR(z),
      hypergeo1.hypergeometric2F1(a,b,c,z))))) ~arith1.plus~
  ((c ~arith1.minus~ (a ~arith1.plus~ b ~arith1.plus~ 1) ~arith1.times~ z)
    ~arith1.times~ calculus1.diff(OMBIND(fns1.lambda,OMBVAR(z),
      hypergeo1.hypergeometric2F1(a,b,c,z)))) ~arith1.minus~
  (a ~arith1.times~ b ~arith1.times~ hypergeo1.hypergeometric2F1(a,b,c,z))
  ~relation1.eq~ 0;
</tfb>
</Example>
</CDDefinition>
```

```
<CDDefinition>
<Name> hypergeometric_pFq </Name>
<Description>
    Generalized hypergeometric function.
    This function has a branch cut on [1,+infinity).
</Description>

<CMP>
    hypergeometric_pFq(a,b;z)
    =\sum_{n=0}^{+\infty}
      \frac{\Pi_i pochhammer(a_i,n)}{\Pi_i pochhammer(b_i,n)pochhammer(1,n)}
        z^n
</CMP>

<FMP>
<tfb>
  hypergeo1.hypergeometric_pFq(a,b,z) ~relation1.eq~
    arith1.sum(interval1.integer_interval(0,nums1.infinity),
      OMBIND(fns1.lambda,OMBVAR(n),
        arith1.product(interval1.integer_interval(1,linalg4.size(a)),
          OMBIND(fns1.lambda,OMBVAR(i),
            hypergeo0.pochhammer(lialg1.vector_selector(i,a),n)))
        ~arith1.divide~
        arith1.product(interval1.integer_interval(1,linalg4.size(b)),
          OMBIND(fns1.lambda,OMBVAR(i),
            hypergeo0.pochhammer(linalg1.vector_selector(i,b),n)))
        ~arith1.divide~  hypergeo0.pochhammer(1,n)
        ~arith1.times~
        arith1.power(z,n)));
</tfb>
</FMP>
</CDDefinition>
</CD>
```

B Examples of Using Authoring Tools

Input file: `h21-0160.tfb` (a fb-formula document)

```
<?xml version="1.0" encoding="ISO-2022-JP"?>
<?xml-stylesheet type="text/xsl" href="fb.xsl"?>
<formula>

  <editor> Hiromasa Nakayama </editor>

  <tfb macroset="http://www.openxm.org/fb/hfb.txt">
   hypergeo1.hypergeometric2F1(1 / 12, 5 / 12, 1 / 2, 1323 / 1331)
   =
   ((3 / 4) * arith1.root(11, 4));
  </tfb>

  <description>
    Value of the Gauss hypergeometric function at z = 1323 / 1331
  </description>

  <reference>
    <xref uri="http://www.openxm.org/fb/bib-beukers-1993.xml"
      linkend="beukers-1993" page="45"/>
  </reference>

  <evidence checker="Mathematica">
    @@ /. {}
  </evidence>

</formula>
```

Command: `tfb2math-check.sh < h21-0160.tfb` (check numerical by Mathematica)
Output:

```
HTTP/0.9 200 OK
Connection: close
Content-length: 63
Content-Type: text/plain

{-1.110223024625156*^-15, 1.365870215128402, 1.365870215128403}
```

Command: `tfb2fb.sh < h21-0160.tfb` (output a fbOM-formula document)

```xml
<?xml version="1.0" encoding="ISO-2022-JP"?>
<?xml-stylesheet type="text/xsl" href="fb.xsl"?>
<formula>
  <editor> Hiromasa Nakayama </editor>
  <OMOBJ>
  <OMA><OMS cd="relation1" name="eq"/>
    <OMA><OMS cd="hypergeo1" name="hypergeometric2F1"/>
      <OMA><OMS cd="arith1" name="divide"/>
        <OMI> 1 </OMI> <OMI> 12 </OMI>
      </OMA>
      <OMA><OMS cd="arith1" name="divide"/>
        <OMI> 5 </OMI>  <OMI> 12 </OMI>
      </OMA>
      <OMA><OMS cd="arith1" name="divide"/>
        <OMI> 1 </OMI>   <OMI> 2 </OMI>
      </OMA>
      <OMA><OMS cd="arith1" name="divide"/>
        <OMI> 1323 </OMI>  <OMI> 1331 </OMI>
      </OMA>
    </OMA>
    <OMA><OMS cd="arith1" name="times"/>
      <OMA><OMS cd="arith1" name="divide"/>
        <OMI> 3 </OMI>  <OMI> 4 </OMI>
      </OMA>
      <OMA><OMS cd="arith1" name="root"/>
        <OMI> 11 </OMI>  <OMI> 4 </OMI>
      </OMA>
    </OMA>
  </OMA>
</OMOBJ>

  <description>
    Value of the Gauss hypergeometric function at z = 1323 / 1331
  </description>
  <reference>
    <xref uri="http://www.openxm.org/fb/bib-beukers-1993.xml"
      linkend="beukers-1993" page="45"/>
  </reference>
  <evidence checker="Mathematica">
    @@ /. {}
  </evidence>
</formula>
```

Command: `tfb2fb.sh < h21-0160.tfb | fb2tex.sh` (output TeX document)
Output:

```
\documentclass{jarticle}
\usepackage{html}
\usepackage{makeidx}

\def\OMS#1{
        {\tt #1}
}

\title{}
\author{}
   \index{}
 {\small Edited by
  Hiromasa Nakayama
 }

\begin{center}
  $ \OMS{hypergeo1.hypergeometric2F1}(\frac{ 1 }{ 12 },\frac{ 5 }{ 12 },
      \frac{ 1 }{ 2 },\frac{ 1323 }{ 1331 })=
    (\frac{ 3 }{ 4 }\cdot {}^ 4 \sqrt{ 11 })  $
\end{center}

    Value of the Gauss hypergeometric function at z = 1323 / 1331

  Reference:
  \cite[45]{beukers-1993}
```

The LaTeX system typesets as follows.

Edited by Hiromasa Nakayama

$$_2F_1\bigl(\tfrac{1}{12}, \tfrac{5}{12}, \tfrac{1}{2}, \tfrac{1323}{1331}\bigr) = \bigl(\tfrac{3}{4} \cdot {}^4\sqrt{11}\bigr)$$

Value of the Gauss hypergeometric function at z = 1323 / 1331
Reference: [1, 45]