Generating Kummer Type Formulas for Hypergeometric Functions

Nobuki Takayama

Kobe University, Rokko, Kobe 657-8501, Japan, takayama@math.kobe-u.ac.jp

Abstract. Kummer type formulas are identities of hypergeometric series. A symmetry by the permutations of n-letters yields these formulas. We will present an algorithmic method to derive known and new Kummer type formulas. The algorithm utilizes several algorithms in algebra and geometry for generating Kummer type formulas.

1 Introduction

The Gauss hypergeometric function is defined by the series

$$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(1)_n (c)_n} x^n, \quad (a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$$

where a, b, c are complex parameters and we assume $c \notin \mathbf{Z}_{<0}$. The series in the right hand side converges in the unit disk and $F(a,b,c;x)$ can be analytically continued to $\mathbf{C} \setminus \{0,1\}$ as a multi-valued analytic function. The Gauss hypergeometric function satisfies several attractive formulas. Among them, we will consider Kummer's formula

$$F(a,b,c;x) = (1-x)^{-a} F\left(a, c-b, c; \frac{x}{x-1}\right)$$

and its generalizations in this article.

There are several methods to prove it. Among them, we here refer to a method by the Chu-Vandermonde formula of binomial coefficients [19]. Although this proof presents an interesting interplay between the famous identity in combinatorics and Kummer's formula, it seems hard to generalize this approach.

I.M.Gel'fand suggested an idea to derive and prove Kummer's formula in his series of lectures at Kyoto in 1989. The method is a natural generalization of the method to derive $24 = 4!$ solutions of the Gauss hypergeometric equation by Kummer. The point of his idea is that the method can be also applied to hypergeometric functions associated to the product of simplices, which have the symmetry of permutations of n-letters.

Since his presentation of the idea, there have been only a few tries to study systematically Kummer type formulas (see, e.g., [15]). The author thinks that the reason for it is the lack of integrations of algebra and geometry software systems. As readers will see in the body of this paper, the systematic study of Kummer type formulas requires several kinds of explicit constructions in algebra and geometry including Gröbner basis computation, triangulations, and constructions of fans, for which algorithms and systems have been intensively studied in the last 10 years.

In this article, we will present an algorithm to derive Kummer type formulas for hypergeometric functions associated to the product of simplices developing the original idea by I.M.Gel'fand. We can experience an interplay between algebra, geometry and software systems through deriving Kummer type formulas of hypergeometric functions.

2 Hypergeometric Function Associated to $\boldsymbol{\Delta_{k-1} \times \Delta_{n-k-1}}$

Let us recall the definition of hypergeometric functions associated to a product of linear forms [1], [8]. We fix two numbers k and n satisfying $n \geq 2k \geq 4$. Let α_j be parameters satisfying $\sum_{j=1}^{n} \alpha_j = -k$. *Hypergeometric function* of type $E_{k,n}$ is defined by the integral

$$\Phi(\alpha; z) = \int_C \prod_{j=1}^{n} (\sum_{i=1}^{k} z_{ij} s_i)^{\alpha_j} ds_2 \cdots ds_k$$

where we put $s_1 = 1$ and $z \in M_{k,n}$ = the space of $k \times n$ matrices and C is a bounded $(k-1)$-cell in the hyperplane arrangement defined by $\prod_{j=1}^{n} \sum_{i=1}^{k} z_{ij} s_i = 0$ in the $(s_2, \ldots, s_k)$-space.

When $k = 2$, $n = 4$, it is written as

$$\Phi(\alpha; z) = \int_C (z_{11} + z_{21} s_2)^{\alpha_1} \cdots (z_{14} + z_{24} s_2)^{\alpha_4} ds_2.$$

The integral $\Phi(z)$ agrees with the Gauss hypergeometric integral when $z = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix}$.

The hypergeometric function of type $E_{k,n}$ is quasi-invariant under the action of complex torus $(\mathbf{C}^*)^n$ and the general linear group $GL(k) = GL(k, \mathbf{C})$. In fact, we have for $h = \begin{pmatrix} h_1 & 0 & & 0 \\ 0 & h_2 & & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & h_n \end{pmatrix} \in (\mathbf{C}^*)^n$ and $g \in GL(k)$,

$$\Phi(\alpha; zh) = \left(\prod_j h_j^{\alpha_j} \right) \Phi(\alpha; z), \tag{1}$$

$$\Phi(\alpha; gz) = |g|^{-1} \Phi(\alpha; z). \tag{2}$$

It follows from the quasi-invariant property and the integral representation that the function $\Phi(\alpha; z)$ satisfies a system of first order equations and a system of second order equations respectively.

Theorem 1. (Gel'fand [8]) *The function $\Phi(\alpha; z)$ satisfies*

$$\left(\sum_{i=1}^{k} z_{ip}\frac{\partial}{\partial z_{ip}} - \alpha_p\right) f = 0,\ \ p = 1, \dots, n$$
$$\left(\sum_{p=1}^{n} z_{ip}\frac{\partial}{\partial z_{jp}} + \delta_{ij}\right) f = 0,\ \ i, j = 1, \dots, k$$
$$\left(\frac{\partial^2}{\partial z_{ip}\partial z_{jq}} - \frac{\partial^2}{\partial z_{iq}\partial z_{jp}}\right) f = 0,\ \ i, j = 1, \dots, k, p, q = 1, \dots, n$$

We call this system of equations $E_{k,n}$.

Let $A = (a_{ij})$ be an integer $d \times N$-matrix of rank d and β a vector over complex numbers $(\beta_1, \dots, \beta_d)^T$. The $\mathcal{A}$-*hypergeometric* (GKZ hypergeometric) system $H_A(\beta)$ is the following system of linear partial differential equations for the indeterminate function $f(x_1, \dots, x_N)$:

$$\left(\sum_{j=1}^{N} a_{ij}x_j\partial_j - \beta_i\right) f = 0, \quad \text{for } i = 1, \dots, d$$

$$(\partial^u - \partial^v)f = 0 \quad \text{for all } u, v \in \mathbf{N}_0^N \text{ with } Au = Av.$$

Here, we use the multi-index notation $\partial^u = \prod_{i=1}^N \partial_i^{u_i}$.

The $\mathcal{A}$-hypergeometric system was first introduced and studied by Gel'fand, Kapranov and Zelevinsky [9].

If we restrict the system of differential equations in Theorem 1 to the affine chart

$$\begin{pmatrix} 1 & \cdots & 0 & x_{11} & \cdots & x_{1\ell} \\ 0 & \cdots & 0 & x_{21} & \cdots & x_{2\ell} \\ 0 & \cdots & 0 & & \cdots & \\ 0 & \cdots & 1 & x_{k1} & \cdots & x_{k\ell} \end{pmatrix}, \quad \ell = m - k$$

of $GL(k)\backslash M_{k,n}$, we obtain the A-hypergeometric system associated to the product of simplices $\Delta_{k-1} \times \Delta_{n-k-1}$. More precisely, put

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \ \dots,\ a_{n-k} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \in \mathbf{Z}^{n-k},$$

$$A_{k,n} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \\ . & \cdots & . & . & \cdots & . & & . & \cdots & . \\ . & \cdots & . & . & \cdots & . & \cdot\ \cdot\ \cdot & . & \cdots & . \\ . & \cdots & . & . & \cdots & . & & . & \cdots & . \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 1 & \cdots & 1 \\ a_1 & \cdots & a_{n-k} & a_1 & \cdots & a_{n-k} & & a_1 & \cdots & a_{n-k} \end{pmatrix},$$

and $\beta = (-\alpha_1 - 1, \ldots, -\alpha_k - 1, \alpha_{k+1}, \ldots, \alpha_n)^T$. Then, the restricted system of $E_{k,n}$ is the $\mathcal{A}$-hypergeometric system $H_{A_{k,n}}(\beta)$ with $x_1 = x_{11}, x_2 = x_{12}, \ldots, x_\ell = x_{1\ell}, x_{\ell+1} = x_{21}, \ldots, x_N = x_{k\ell}$. Note that, by using the $GL(k)$-quasi-invariant property (2), solutions of $E_{k,n}$ can be expressed in terms of solutions of $H_{A_{k,n}}(\beta)$. We will present later some explicit constructions of solutions of $E_{2,4}$ and $E_{2,5}$.

3 Configuration Space

We denote by $X_{k,n}$ the quotient space

$$GL(k)\backslash M^*_{k,n}/(\mathbf{C}^*)^n.$$

Here, $M^*_{k,n}$ denotes the set of the $k \times n$ matrices of which any $k \times k$ minor does not vanish. The space $X_{k,n}$ is called the configuration space of n-points in the $(k-1)$-dimensional project space. It is a $(k-1) \times (n-k-1)$ dimensional affine variety. The group of the permutations of n-columns of $M_{k,n}$ acts on $X_{k,n}$. We denote the group by S_n.

Regular functions on $M^*_{k,n}$ which are invariant under the action of $GL(k)$ and $(\mathbf{C}^*)^n$ are regular functions on $X_{k,n}$. Let us denote by

$$y_{i_1,\cdots,i_k}$$

the determinant of $(i_1, \ldots, i_k)$-th columns of the $k \times n$ matrix $z = (z_{ij})$, which is called the *Plücker coordinate*. Then, the affine chart x_{ij} of $GL(k)\backslash M^*_{k,n}$ is expressed as

$$x_{ij} = \frac{y_{1,2,\cdots,i-1,k+j,i+1,\cdots,k}}{y_{1,2,\cdots,k}}, \tag{3}$$

which is invariant under the action of $GL(k)$.

Let us take a vector

$$p = (p_{ij}) \in \mathrm{Ker}\, \mathbf{Z}^{k\times(n-k)} \xrightarrow{A_{k,n}} \mathbf{Z}^n.$$

Then, it is easy to see that $x^p = \prod x_{ij}^{p_{ij}}$ is invariant as a function of $y = (y_{i_1,\ldots,i_k})$ by the actions of $GL(k)$ and $(\mathbf{C}^*)^n$. Hence, x^p is a regular function on the configuration space $X_{k,n}$.

Let σ be an element of S_n. The action of σ on x^p is defined as

$$\sigma \bullet x^p = \prod_{i,j} \left(\frac{y_{\sigma(1),\sigma(2),\cdots,\sigma(i-1),\sigma(k+j),\sigma(i+1),\cdots,\sigma(k)}}{y_{\sigma(1),\sigma(2),\cdots,\sigma(k)}} \right)^{p_{ij}} \tag{4}$$

The action induces a biholomorphic transformation on $X_{k,n}$.

Example 1. $k = 2, n = 4$
The configuration space $X_{2,4}$ is isomorphic to $\mathbf{P}^1 \setminus \{0, 1, \infty\} = \mathbf{C} \setminus \{0, 1\}$. An isomorphism is given by

$$X_{2,4} \ni (z_{ij}) \mapsto \frac{y_{4,2}y_{1,3}}{y_{3,2}y_{1,4}} = x \in \mathbf{C} \setminus \{0, 1\}.$$

We denote the Plücker coordinate y_{ij} by $[ij]$. Then, the action of S_4 on $\mathbf{C}^1 \setminus \{0, 1\}$ is summarized as follows.

$$\begin{aligned}
x &:= \frac{[42][13]}{[32][14]} \\
(12) \bullet x &= \frac{[41][23]}{[31][24]} = \frac{1}{x} \\
(23) \bullet x &= \frac{[43][12]}{[23][14]} = \frac{[43][13]}{[32][14]} - 1 = x - 1 \\
&\quad ([42][13] - [41][23] + [21][43] = 0, \quad \text{Plücker's relation}) \\
(34) \bullet x &= \frac{1}{x}
\end{aligned}$$

Here, for example, (23) means $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \in S_4$.

Example 2. $k = 2, n = 4$
The configuration space $X_{2,5}$ is isomorphic to

$$X = \mathbf{C}^2 \setminus \{(x, y) \,|\, xy(x-1)(y-1)(x-y) = 0\}$$

by the isomorphism

$$X_{2,5} \ni (z_{ij}) \mapsto \left(\frac{[42][13]}{[32][14]}, \frac{[14][52]}{[42][15]} \right) = (x, y) \in X$$

where $[ij] = y_{i,j}$. The action of S_5 is as follows.

$$\begin{aligned}
&(12) \bullet x = \frac{1}{x}, \quad (12) \bullet y = \frac{1}{y} \\
&(23) \bullet x = 1 - x, \quad (23) \bullet y = \frac{1 - xy}{1 - x} \\
&(34) \bullet x = \frac{1}{x}, \quad (34) \bullet y = xy \\
&(45) \bullet x = xy, \quad (45) \bullet y = \frac{1}{y}
\end{aligned}$$

where Plücker relations

$$\sum_{k=0}^{2} (-1)^k y_{i_1,j_k} y_{j_0,\cdots,\hat{j}_k,\cdots,j_2} = 0$$

are used to reduce expressions in y_{ij} into expressions in x and y. The index sets $\{i_1\}$ and $\{j_0, j_1, j_2\}$ run over the subsets of $\{1, 2, 3, 4, 5\}$. For instance, when $i_1 = 1$ and $j_0 = 3, j_1 = 4, j_2 = 5$, the relation is $y_{1,3}y_{4,5} - y_{1,4}y_{3,5} + y_{1,5}y_{3,4} = 0$.

We are interested in deriving explicit expressions of the action of S_n on the configuration space $X_{k,n}$ as in the two examples. It can be done by an elimination. Put $m = (k-1)(n-k-1)$, which is the dimension of the configuration space $X_{k,n}$.

Data : $x_1 = \frac{y^{p(1)}}{y^{q(1)}}, \ldots, x_m = \frac{y^{p(m)}}{y^{q(m)}}$: a system of local coordinates of a compactification of $X_{k,n}$, and $\sigma \in S_n$.

Result : Polynomials f_j and g_j such that $\sigma \bullet x_j = \frac{f_j(x_1,\ldots,x_m)}{g_j(x_1,\ldots,x_m)}$

1. Put

$$P = \{\text{the Plücker relations}, y_{i_1,\cdots,i_k} y^*_{i_1,\cdots,i_k} - 1 \mid i_1, \ldots, i_k \subset \{1, \ldots, n\}\}.$$

   Define

$$I = \langle P, y^{q(i)}x_i - y^{p(i)} \ (i = 1, \ldots, m), (\sigma \bullet y^{q(j)})w_j - (\sigma \bullet y^{p(j)}) \rangle$$

   as an ideal in

$$\mathbf{Q}[y_{1,2,\ldots,k}, \ldots, y_{n-k+1,\ldots,n}, y^*_{1,2,\ldots,k}, \ldots, y^*_{n-k+1,\ldots,n}, x_1, \ldots, x_m, w_j].$$

2. Find the ideal intersection $J = \mathbf{Q}(x_1, \ldots, x_m)I \cap \mathbf{Q}(x_1, \ldots, x_m)[w_j]$.
   It is generated by an element of the form $w_j - \frac{f_j(x)}{g_j(x)}$.
3. Output $\frac{f_j}{g_j}$.

Algorithm 1: Action of S_n on the configuration space

The correctness follows from that the ideal generated by the Plücker relations is prime and hence radical. Here is a sketch of a proof of the primeness suggested by B.Sturmfels. “Establish the SAGBI basis property of the $k \times k$-minors and then to infer that the Plücker ideal is precisely the ideal of all algebraic relations on the $k \times k$-minors. This implies primeness, since the kernel of a map from a domain to a domain is always prime.” Gröbner basis of Plücker relations and the SAGBI argument are described in the chapter three of the book by B.Sturmfels on invariant theory [17].

Elimination can be done by the characteristic set method or the Gröbner basis method. Readers are invited to articles by M.Noro[10], H.Schönemann[14], and D.M.Wang[20] in this volume for algorithms, implementations and related topics in this area.

We close this section with a short remark on efficiency. Although the method above is simple, it seems that it is not the most efficient method to explicitly derive the action of S_n on a coordinate system of the configuration space. It seems more efficient to use a method based on transforming a given $k \times n$ matrix to the normal form $\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 & 1 & * & \cdots & * \\ & & \cdots & & 1 & * & \cdots & * \\ & & \cdots & & 1 & * & \cdots & * \\ 0 & 0 & \cdots & 1 & 1 & * & \cdots & * \end{pmatrix}$ by the left $GL(k)$ and the right $(\mathbf{C}^*)^n$ action.

4 Series solutions

Several methods to construct series solutions of $\mathcal{A}$-hypergeometric system have been known. Among them, we would like to mention here the following two constructions.

Theorem 2. [9]

1. *There is a correspondence between a regular triangulation T of A and a basis of series solutions of $H_A(\beta)$ for generic β.*
2. *A translation of the secondary cone of a regular triangulation T of A is the domain of convergence of the basis of series solutions associated to T.*

Theorem 3. [12] *There is a construction algorithm of series solution basis of $H_A(\beta)$ for any β. The series solutions are constructed by extending the solutions of the initial ideal* $\mathrm{in}_{(-w,w)}(H_A(\beta))$ *with respect to a given weight $(-w, w)$.*

Details on these methods are explained in [9] and section 3.4 and chapter two of the book [12]. Triangulations play an important role to construct series solutions. Readers are invited to consult the article by J.Pfeifle and J.Rambau [11] on algorithms and implementations for triangulations.

We will add one more note here. Series solutions can be analytically continued to the complement of the *singular locus* of $H_{A_{k,n}}(\beta)$, of which defining equation is expressed in terms of sparse resultants. In case of $A_{2,4}$, the defining equation is

$$x_{11}x_{12}x_{21}x_{22}(x_{11}x_{22} - x_{12}x_{21}).$$

Readers are invited to the article by I.Emiris[6] on sparse resultants in this book.

Let $\phi(\alpha; x)$ be a series solution of $A_{k,n}$-hypergeometric system. Then, by the quasi-invariant relation (2), the function

$$\Phi(\alpha; y) = \frac{1}{y_{1,2,\ldots,k}} \phi(\alpha; x(y)) \tag{5}$$

is a solution of $E_{k,n}$ where $x(y)$ is a coordinate system defined in (3).

The invariant property of the system of equations $E_{k,n}$ under the action of S_n yields the following theorem.

Theorem 4. *If the function $\Phi(\alpha; y)$ is a solution of $E_{k,n}$, then the function $\sigma \bullet \Phi(\alpha; y) := \Phi(\sigma \bullet \alpha; \sigma \bullet y)$ is also a solution of $E_{k,n}$ for any $\sigma \in S_n$.*

Example 3. $k = 2, n = 4$.

The matrix $A_{2,4}$ is $\begin{array}{cccc} x_{11} & x_{12} & x_{21} & x_{22} \end{array}$ $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. The hypergeometric system $H_{A_{2,4}}(\beta)$ is generated by

$$x_{11}\partial_{11} + x_{12}\partial_{12} - (-\alpha_1 - 1),\ x_{21}\partial_{21} + x_{22}\partial_{22} - (-\alpha_2 - 1),$$

$$x_{11}\partial_{11} + x_{21}\partial_{21} - \alpha_3,\ x_{12}\partial_{12} + x_{22}\partial_{22} - \alpha_3,\ \partial_{11}\partial_{22} - \partial_{12}\partial_{21}$$

where $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2$. Take the triangulation $T = \{124, 134\}$ or the Gröbner deformation with respect to the weight $w = (0, 1, 1, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the starting terms of series solutions are x^p and x^q where $p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} \alpha_3 & -\alpha_1 - \alpha_3 - 1 \\ 0 & -\alpha_2 - 1 \end{pmatrix}$ and $q = \begin{pmatrix} -\alpha_1 - 1 & 0 \\ -\alpha_2 - \alpha_4 - 1 & \alpha_4 \end{pmatrix}$. Note that

$$\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}^{-1} (z_{ij}) = \begin{pmatrix} 1 & 0 & \frac{[32]}{[12]} & \frac{[42]}{[12]} \\ 0 & 1 & \frac{[13]}{[12]} & \frac{[14]}{[12]} \end{pmatrix}.$$

Then, we put $x_{11} = \frac{[32]}{[12]}, x_{12} = \frac{[42]}{[12]}, x_{21} = \frac{[13]}{[12]}, x_{22} = \frac{[14]}{[12]}$. By extending the starting terms to series solutions, we obtain solutions of $H_{A_{2,4}}(\beta)$ and consequently those of E_{25}:

$$\eta_1 F(-\alpha_3, \alpha_2 + 1, -\alpha_1 - \alpha_3; s)$$
$$\eta_2 F(-\alpha_4, \alpha_1 + 1, -\alpha_2 - \alpha_4; s)$$

where $s = \frac{x_{12}x_{21}}{x_{11}x_{22}} = \frac{[42][13]}{[32][14]}$ and

$$\eta_1 = \frac{1}{[12]} x_{12}^{-\alpha_1} x_{22}^{-\alpha_2} \left(\frac{x_{11}}{x_{12}}\right)^{\alpha_3} \frac{1}{x_{12}x_{22}}$$
$$\eta_2 = \frac{1}{[12]} \left(\frac{x_{21}}{x_{11}}\right)^{\alpha_1} x_{21}^{\alpha_3} x_{22}^{\alpha_4} \frac{x_{21}}{x_{11}},$$

$\sum_{i=1}^{4} \alpha_i = -2$. F is the Gauss hypergeometric function. We obtain $|S_4| = 24$ fundamental sets of solutions by applying the Theorem 4.

5 Deriving Kummer Type Formulas

Let $\eta_1(y)\phi_1(\alpha; x(y)), \eta_2(y)\phi_2(\alpha; x(y)), \ldots, \eta_r(y)\phi_r(\alpha; x(y))$ be a basis of solutions of $E_{k,n}$. It follows from the quasi-invariant property (1) and (2) that the quotient function

$$(\sigma \bullet \eta_i(y))\phi_i(\sigma \bullet \alpha; \sigma \bullet x(y))/\eta_1(y)\phi_1(\alpha; x(y))$$

is a multi-valued analytic function on $X_{k,n}$ for $\sigma \in S_n$. Since, the function $\phi_1(\alpha; x(y))$ is a multi-valued analytic function in $X_{k,n}$, the quotient function

$$(\sigma \bullet \eta_i(y))\phi_i(\sigma \bullet \alpha; \sigma \bullet x(y))/\eta_1(y)$$

is also a multi-valued analytic function on the configuration space $X_{k,n}$.

We take a coordinate system

$$s_1 = x(y)^{p(1)}, s_2 = x(y)^{p(2)}, \ldots, s_m = x(y)^{p(m)} \tag{6}$$

of a compactification of the configuration space $X_{k,n}$ which satisfies the condition: all $\phi_i(\alpha; x(y))$ can be expressed in a convergent power series of $s_1, \ldots, s_m$.

If we take a *compatible basis* $\{p(1), \ldots, p(m)\}$ [9], it satisfies the condition above. And, we need a “small” basis in a suitable sense. However, the author does not know an efficient method to find a compatible basis and does not know what compatible basis is “small” for our purpose. They are questions of geometry.

Now, we are ready to state our method to derive Kummer type formulas of hypergeometric series associated to $E_{k,n}$.

Data : $\eta_i(y)$, $\phi_i(\alpha; x(y))$, $s_j = x(y)^{p(j)}$
Result : Kummer Type Formulas
Suppose that $\eta_i(y)/\eta_1(y), i \geq 2$ are not holomorphic at $s_1 = \cdots = s_m$.

1. Let G be the isotropy group in S_n that fixes the point $s_1 = \cdots = s_m$.
2. **for** $\sigma \in G$ **do**
   consider functions $\{\frac{\sigma\bullet\eta_i}{\eta_1} \mid i = 1, \ldots, r\}$. If the function $\frac{\sigma\bullet\eta_i}{\eta_1}$ is holomorphic at $s_1 = \cdots = s_m = 0$ and other functions $\frac{\sigma\bullet\eta_j}{\eta_1}$, $j \neq i$ are not holomorphic, then output
   $$c\frac{\sigma \bullet \eta_i}{\eta_1}\phi_i(\sigma \bullet \alpha; \sigma \bullet x(y)) = \phi_1(\alpha; x(y)). \tag{7}$$
   Here, c is a suitable constant. Note that the both sides are functions in $s_1, \ldots, s_m$.
   end

Algorithm 2: Deriving Kummer type formulas

Since the formula (7) contains complex power functions, we have to be careful about choices of branches. As to this topics, readers are invited to the article of J.Davenport[4] in this book. The problem of choices of branches in formulas on hypergeometric functions has a long history. For example, É.Goursat gave connection formulas of the Gauss hypergeometric function on the upper half plane and the lower half plane in his long paper in 1881 [7]. This approach of giving connection formulas on simply connected domains is nice, however, most editors of formula books do not take his original nice rigorous approach and list connection formulas without the side conditions on branches. In the several variable case, the problem of finding a nice decomposition into simply connected domains is more interesting and difficult. See, for example, [22, Chapter V] and [13].

Example 4. $k = 2, n = 4$.
By a calculation, we have $\eta_2/\eta_1 = s^{\alpha_1+\alpha_3+1}$, which is not holomorphic at $s = 0$. Put $\sigma = (13) = [3, 2, 1, 4, 5]$. Then, we have

$$\frac{\sigma \bullet \eta_1}{\eta_1} = \left(\frac{[32][14]}{[34][12]}\right)^{\alpha_2+1}$$

and

$$\sigma \bullet F(-\alpha_3, \alpha_2 + 1, -\alpha_1 - \alpha_3; s) = F\left(-\alpha_1, \alpha_2 + 1, -\alpha_1 - \alpha_3; \frac{[42][31]}{[12][34]}\right).$$

Since $[12][34]-[13][24]+[23][14]=0$ we have $\frac{[42][31]}{[12][34]}=\frac{s}{s-1}$. Similarly, we have $\frac{[32][14]}{[12][34]}=1-\frac{s}{s-1}$. Therefore, we obtain

$$F(-\alpha_3,\alpha_2+1,-\alpha_1-\alpha_3;s)=(1-s)^{-\alpha_2-1}F\left(-\alpha_1,\alpha_2+1,-\alpha_1-\alpha_3;\frac{s}{s-1}\right),$$

which is well-known Kummer's formula for the Gauss hypergeometric function.

Example 5. $k=2, n=5$. Put $x_{11}=\frac{[32]}{[12]}, x_{12}=\frac{[42]}{[12]}, x_{13}=\frac{[52]}{[12]}, x_{21}=\frac{[13]}{[12]}, x_{22}=\frac{[14]}{[12]}, x_{23}=\frac{[15]}{[12]}$,

$$\phi_1(x(y))=F_1\left(\alpha_1+1,-\alpha_4,-\alpha_5,\alpha_3+\alpha_1+2;\left(\frac{x_{21}x_{12}}{x_{11}x_{22}}\right),\left(\frac{x_{21}x_{13}}{x_{11}x_{23}}\right)\right),$$
$$\phi_2(x(y))=G_2\left(-\alpha_3,-\alpha_5,\alpha_3+\alpha_1+1,\alpha_5+\alpha_2+1;\left(-\frac{x_{21}x_{12}}{x_{11}x_{22}}\right),\left(-\frac{x_{22}x_{13}}{x_{12}x_{23}}\right)\right),$$
$$\phi_3(x(y))=F_1\left(\alpha_2+1,-\alpha_3,-\alpha_4,\alpha_5+\alpha_2+2;\left(\frac{x_{21}x_{13}}{x_{11}x_{23}}\right),\left(\frac{x_{22}x_{13}}{x_{12}x_{23}}\right)\right),$$

and

$$\begin{aligned}\eta_1&=[12]^{-1}x_{11}^{-\alpha_1-1}x_{21}^{\alpha_3+\alpha_1+1}x_{22}^{\alpha_4}x_{23}^{\alpha_5}\\ \eta_2&=[12]^{-1}x_{11}^{\alpha_3}x_{12}^{-\alpha_1-1-\alpha_3}x_{22}^{-\alpha_2-1-\alpha_5}x_{23}^{\alpha_5}\\ \eta_3&=[12]^{-1}x_{11}^{\alpha_3}x_{12}^{\alpha_4}x_{13}^{\alpha_5+\alpha_2+1}x_{23}^{-\alpha_2-1}.\end{aligned}$$

Here
$$F_1(a,b,b',c;x_1,x_2):=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}(1)_m(1)_n}x_1^m x_2^n$$

is the Appell hypergeometric function and G_2 is the hypergeometric function G_2 in Horn's list. Then, the functions $\eta_i\phi_i$ are solutions of $E_{k,n}$. Put

$$s_1=\frac{x_{21}x_{12}}{x_{11}x_{22}}=\frac{[42][13]}{[32][14]},\ s_2=\frac{x_{22}x_{13}}{x_{12}x_{23}}=\frac{[52][14]}{[42][15]}.$$

The functions $\frac{\eta_i}{\eta_1}\phi_i(x(y))$ are expressed in terms of s_1 and s_2 as follows.

$$\begin{aligned}F_1(s_1,s_1s_2)\\ s_1^{-1}s_1^{-\alpha_1}s_1^{-\alpha_3}G_2(-s_1,-s_2)\\ (s_1s_2)^{-1}(s_1s_2)^{-\alpha_1}(s_1s_2)^{-\alpha_3}s_2^{-\alpha_4}F_1(s_1s_2,s_2)\end{aligned}$$

These functions converge in $|s_1|,|s_2|<1$ and the first function $\phi_1(s_1,s_1s_2)$ is holomorphic at $s_1=s_2=0$.

The isotropy group G at $s_1=s_2=0$ consists of eight elements. The group G is the dihedral group.

$(\sigma \bullet s_1, \sigma \bullet s_2)$	σ
$(\frac{(-s_1+1)s_2}{s_2-1}, \frac{-s_1s_2+s_1}{s_1-1})$	$[5,3,2,4,1]$
$(\frac{s_2}{s_2-1}, \frac{s_1s_2-s_1}{s_1s_2-1})$	$[5,1,2,4,3]$
$(\frac{-s_1s_2+s_1}{s_1-1}, \frac{(-s_1+1)s_2}{s_2-1})$	$[3,5,1,4,2]$
$(\frac{s_1}{s_1-1}, \frac{(s_1-1)s_2}{s_1s_2-1})$	$[3,2,1,4,5]$
$(\frac{(s_1-1)s_2}{s_1s_2-1}, \frac{s_1}{s_1-1})$	$[2,3,5,4,1]$
(s_2, s_1)	$[2,1,5,4,3]$
$(\frac{s_1s_2-s_1}{s_1s_2-1}, \frac{s_2}{s_2-1})$	$[1,5,3,4,2]$
(s_1, s_2)	$[1,2,3,4,5]$

Here, $[i_1, i_2, i_3, i_4, i_5]$ denotes the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i_1 & i_2 & i_3 & i_4 & i_5 \end{pmatrix} \in S_5$.

Here is a list of

$$\left[\frac{\sigma \bullet \eta_1}{\eta_1}, \frac{\sigma \bullet \eta_2}{\eta_1}, \frac{\sigma \bullet \eta_2}{\eta_1}, \sigma \right]$$

generated by Algorithm 1 in this article. For example, the first line means that

$$(\sigma \bullet \eta_1)/\eta_1 = \frac{-s_1s_2+1}{s_1s_2} \left(\frac{s_2-1}{s_2} \right)^{\alpha_4} \left(\frac{-1}{s_1s_2} \right)^{\alpha_1} (-s_1s_2+1)^{\alpha_5} \left(\frac{s_1s_2-1}{s_1s_2} \right)^{\alpha_3}$$

for $\sigma = [5,3,2,4,1]$.

```
[[0,(-s1*s2+1)/(s1*s2)],[4,(s2-1)/(s2)],[1,(-1)/(s1*s2)],[5,-s1*s2+1],
 [3,(s1*s2-1)/(s1*s2)]]
[[0,((s1^2-s1)*s2-s1+1)/(s1*s2-s1)],[4,-s1+1],[1,(s1-1)/(s1*s2-s1)],
 [5,-s1*s2+1],[3,((-s1^2+s1)*s2+s1-1)/(s1*s2-s1)]]
[[0,-s1*s2+1],[4,-s1+1],[1,-1],[5,-s1*s2+1],[3,s1*s2-1]]
[5,3,2,4,1]

[[0,(1)/(s1*s2)],[4,(s2-1)/(s2)],[3,(s1*s2-1)/(s1*s2)],[1,(-1)/(s1*s2)]]
[[0,(1)/(s1*s2-s1)],[3,(s1*s2-1)/(s1*s2-s1)],[1,(-1)/(s1*s2-s1)]]
[[0,(1)/(s1*s2-1)],[1,(-1)/(s1*s2-1)]]
[5,1,2,4,3]

[[0,s1*s2-1],[1,-1],[4,-s1+1],[3,s1*s2-1],[5,-s1*s2+1]]
[[0,((-s1^2+s1)*s2+s1-1)/(s1*s2-s1)],[1,(s1-1)/(s1*s2-s1)],[4,-s1+1],
 [3,((-s1^2+s1)*s2+s1-1)/(s1*s2-s1)],[5,-s1*s2+1]]
[[0,(s1*s2-1)/(s1*s2)],[1,(-1)/(s1*s2)],[4,(s2-1)/(s2)],
 [3,(s1*s2-1)/(s1*s2)],[5,-s1*s2+1]]
[3,5,1,4,2]

[[0,-1],[1,-1],[4,-s1+1],[5,-s1*s2+1],[3,-1]]
[[0,(-s1+1)/(s1)],[1,(-s1+1)/(s1)],[4,-s1+1],[5,-s1*s2+1],
 [3,(-s1+1)/(s1)]]
[[0,(-s1*s2+1)/(s1*s2)],[1,(-s1*s2+1)/(s1*s2)],[4,(-s1*s2+1)/(s2)],
 [5,-s1*s2+1],[3,(-s1*s2+1)/(s1*s2)]]
```

```
[3,2,1,4,5]

[[0,(s1*s2-1)/(s1*s2)],[5,-s1*s2+1],[4,(-s1*s2+1)/(s2)],
 [1,(-s1*s2+1)/(s1*s2)],[3,(-s1*s2+1)/(s1*s2)]]
[[0,(s1-1)/(s1)],[5,-s1*s2+1],[4,-s1+1],[1,(-s1+1)/(s1)],
 [3,(-s1+1)/(s1)]]
[[5,-s1*s2+1],[4,-s1+1],[1,-1],[3,-1]]
[2,3,5,4,1]

[[0,(-1)/(s1*s2)],[4,(1)/(s2)],[3,(1)/(s1*s2)],[1,(1)/(s1*s2)]]
[[0,(-1)/(s1)],[3,(1)/(s1)],[1,(1)/(s1)]]
[[0,-1]]
[2,1,5,4,3]

[[0,(-1)/(s1*s2-1)],[1,(-1)/(s1*s2-1)]]
[[0,(-1)/(s1*s2-s1)],[3,(s1*s2-1)/(s1*s2-s1)],[1,(-1)/(s1*s2-s1)]]
[[0,(-1)/(s1*s2)],[3,(s1*s2-1)/(s1*s2)],[4,(s2-1)/(s2)],[1,(-1)/(s1*s2)]]
[1,5,3,4,2]

[]
[[0,(1)/(s1)],[3,(1)/(s1)],[1,(1)/(s1)]]
[[0,(1)/(s1*s2)],[3,(1)/(s1*s2)],[4,(1)/(s2)],[1,(1)/(s1*s2)]]
[1,2,3,4,5]
```

From this table, we obtain one of Kummer type formulas for the Appell function F_1 firstly studied by Vavasseure [2]

$$\begin{aligned}
&(1-s_1s_2)(1-s_1s_2)^{\alpha_3}(1-s_1)^{\alpha_4}(1-s_1s_2)^{\alpha_5} \\
&\quad \times F_1\left(\alpha_3+1,-\alpha_2,-\alpha_4,\alpha_1+\alpha_3+2;s_1s_2,\frac{s_1(1-s_2)}{s_1-1}\right) \\
&= F_1\left(\alpha_1+1,-\alpha_4,-\alpha_5,\alpha_3+\alpha_1+2;s_1,s_1s_2\right)
\end{aligned}$$

standing for $\sigma=[5,3,2,4,1]$ and $(\sigma\bullet\eta_3)/\eta_1$.

We have sketched a method to generate Kummer type formulas. However, it is not the end of the story and the following problems are arising.

1. Improve efficiency. How far can we derive Kummer type formulas of $E_{k,n}$ for large k and n? Can we find an interesting structure?
2. We need an efficient method to find a "small" compatible basis.
3. We will obtain a lot of Kummer type identities. However, we have no method to classify them by a suitable mathematical meaning.
4. Building a system to study and generate formulas of hypergeometric functions by integrating algebra and geometry systems (`http://www.openxm.org`).
5. How to store generated formulas and proofs in an electronic or digital formula book on hypergeometric functions[5], [21], [18]?

As to the last problem, OpenMath approaches seem to be promissing. Readers are invited to articles by A.Cohen, H.Cuypers, E.R.Barreiro, H.Sterk[3], and A.Solomon[16] in this book on OpenMath activities.

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