# Twisted period relations associated with confluent hypergeometric functions

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## 1 Introduction

Formulas of intersection numbers were given in [1] and [4] for twisted de-Rham cohomology groups and twisted homology groups defined by the connection $\nabla_\omega = d + \omega$ associated with a 1-form $\omega$ with simple poles on the projective line $\mathbf{P}$. In this paper, we present a definition of intersection pairing and a formula of intersection numbers between twisted homology groups associated with a general rational 1-form $\omega$ on $\mathbf{P}$. This pairing is naturally induced by the intersection pairing between twisted de-Rham cohomology groups with respect to $\pm\omega$ studied in [6]. Our formula together with results in [6] gives explicit forms of twisted period relations, which yield quadratic relations for generalized confluent hypergeometric functions. Different approaches to this theory are studied in [2]. The authors are grateful to H.Majima for useful discussions on asymptotic analysis.

## 2 Twisted de-Rham cohomology groups

Let $n_1, \ldots, n_m$ be natural numbers such that

$$n_1 \geq n_2 \geq \ldots \geq n_m, \quad n = \sum_{i=1}^{m} n_i \geq 3$$

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and let $x_1, \ldots, x_m$ be distinct $m$ points on the complex projective line $\mathbf{P}$. Put

$$x = \{x_1, \ldots, x_m\}, \quad X = \mathbf{P} \setminus x,$$

$$\omega = \sum_{i=1}^{m} \left( \frac{\alpha_{i;1}}{t - x_i} + \frac{\alpha_{i;2}}{(t - x_i)^2} + \cdots + \frac{\alpha_{i;n_i-1}}{(t - x_i)^{n_i-1}} + \frac{\alpha_{i;n_i}}{(t - x_i)^{n_i}} \right) dt,$$

where $\alpha_{i;k} \in \mathbf{C}$, $\alpha_{i;n_i} \neq 0$ and $\alpha_{i;1}$'s satisfy that $\alpha_{i;1} \notin \mathbf{Z}$ if $n_i = 1$ and that

$$\sum_{i=1}^{m} \alpha_{i;1} = 0.$$

*This condition on the parameters $\alpha_{i;k}$ is assumed throughout this paper.* For the meromorphic 1-form $\omega$ on $\mathbf{P}$, we denote the polar divisor of $\omega$ and the connection with respect to $\omega$ on $X$ by

$$D = \sum_{i=1}^{m} n_i x_i, \quad \nabla_\omega = d + \omega\wedge,$$

respectively; note that

$$\nabla_\omega \circ \nabla_\omega = 0.$$

A smooth function $f$ defined on a neighborhood $U_i$ of $x_i$ is called rapidly decreasing at $x_i$ if $f$ satisfies

$$\lim_{t \to x_i} \frac{1}{|t - x_i|^r} \frac{\partial^{p+q}}{\partial t^p \partial \bar{t}^q} f(t, \bar{t}) = 0$$

for any $p, q, r \in \{0, 1, 2, \ldots\}$, where $t$ is a *complex* coordinate system around $x_i$. Let $\mathcal{S}^0(x)$ be the vector space of smooth functions on $\mathbf{P}$ which are rapidly decreasing at $x_i$ for any $i$, and let $\mathcal{S}^k(x)$ be the vector space of smooth $k$-froms $\zeta$ on $\mathbf{P}$ such that the coefficient functions of the expression of $\zeta$ in terms of a complex coordinate system $t$ around $x_i$ are rapidly decreasing at $x_i$ for any $i$.

A smooth function $f$ defined on $U_i \setminus \{x_i\}$ is called polynomial growth at $x_i$ if there exists $r \in \mathbf{N}$ such that

$$\lim_{t \to x_i} |t - x_i|^r \frac{\partial^{p+q}}{\partial t^p \partial \bar{t}^q} f(t, \bar{t}) = 0$$

for any $p, q \in \{0, 1, 2, \ldots\}$. Let $\mathcal{P}^0(x)$ be the vector space of smooth functions $f$ on $X$ which are polynomial growth at $x_i$ for any $i$, and let $\mathcal{P}^k(x)$ be the

vector space of smooth $k$-froms $\zeta$ such that the coefficient functions of the expression of $\zeta$ in terms of a complex coordinate system $t$ around $x_i$ are polynomial growth at $x_i$ for any $i$.

Note that

$$\mathcal{S}^k(x) \subset \mathcal{P}^k(x)$$

and that $\mathcal{P}^{2-k}(x)$ is regarded as a subspace in the dual of $\mathcal{S}^k(x)$ by the integral

$$\int_X \xi^k \wedge \eta^{2-k}, \quad \xi_i^k \in \mathcal{S}^k(x),\ \eta^{2-k} \in \mathcal{P}^{2-k}(x). \tag{1}$$

We define three complexes with differential $\nabla_\omega$ :

$$\begin{aligned}
(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_\omega) &: \Omega^0_{\mathbf{P}}(*D) \xrightarrow{\nabla_\omega} \Omega^1_{\mathbf{P}}(*D) \xrightarrow{\nabla_\omega} 0 \longrightarrow 0, \\
(\mathcal{S}^\bullet(x), \nabla_\omega) &: \mathcal{S}^0(x) \xrightarrow{\nabla_\omega} \mathcal{S}^1(x) \xrightarrow{\nabla_\omega} \mathcal{S}^2(x) \xrightarrow{\nabla_\omega} 0, \\
(\mathcal{P}^\bullet(x), \nabla_\omega) &: \mathcal{P}^0(x) \xrightarrow{\nabla_\omega} \mathcal{P}^1(x) \xrightarrow{\nabla_\omega} \mathcal{P}^2(x) \xrightarrow{\nabla_\omega} 0,
\end{aligned}$$

where $\Omega^k_{\mathbf{P}}(*D)$ is the vector space of meromorphic $k$-forms on $\mathbf{P}$ with poles only on $x$. The $k$-th cohomology groups $H^k(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_\omega)$, $H^k(\mathcal{S}^\bullet(x), \nabla_\omega)$ and $H^k(\mathcal{P}^\bullet(x), \nabla_\omega)$ of above complexes are called rational, rapidly decreasing and polynomial growth twisted de-Rham cohomology groups with respect to $\nabla_\omega$, respectively. The inclusions

$$(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_\omega) \subset (\mathcal{P}^\bullet(x), \nabla_\omega), \quad (\mathcal{S}^\bullet(x), \nabla_\omega) \subset (\mathcal{P}^\bullet(x), \nabla_\omega)$$

of complexes induce the following isomorphisms among twisted de-Rham cohomology groups.

**Theorem 2.1** *We have*

$$H^k(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_\omega) \simeq H^k(\mathcal{P}^\bullet(x), \nabla_\omega) \simeq H^k(\mathcal{S}^\bullet(x), \nabla_\omega).$$

*Especially,* $H^k(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_\omega)$, $H^k(\mathcal{P}^\bullet(x), \nabla_\omega)$ *and* $H^k(\mathcal{S}^\bullet(x), \nabla_\omega)$ *vanish for* $k \neq 1$.

It is shown in [3] that the rank of $H^1(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_\omega)$ is $n-2$. We note that niether of the first cohomology groups for the complexes consisting of the spaces of smooth $k$-forms on $X$ with compact support and of those of smooth $k$-forms on $X$ with differential $\nabla_\omega$ are isomorphic to $H^1(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_\omega)$, if there exists $n_i$ such that $n_i > 1$. This is the reason why we introduced rapidly decreasing and polynomial growth twisted de-Rham cohomology groups.

# 3 Twisted homology groups

Let $\mathcal{L}_\omega$ be the locally constant sheaf on $X$ of analytic functions which belong to the kernel of the connection $\nabla_{-\omega}$ with respect to $-\omega$. Since solutions of the differential equation $\nabla_{-\omega} f(t) = 0$ can be locally expressed as

$$c \exp\left(\int_s^t \omega\right), \quad c \in \mathbf{C}$$

around any point $s$ in $X$, $\mathcal{L}_\omega$ is regarded as determined by the multi-valued function

$$u(t) = \prod_{i=1}^{m} (t - x_i)^{\alpha_{i;1}} \exp\left(\frac{\alpha_{i;2}}{(t-x_i)} + \frac{\alpha_{i;3}}{2(t-x_i)^2} + \cdots + \frac{\alpha_{i;n_i}}{(n_i-1)(t-x_i)^{n_i}}\right)$$

on $X$. Let $C_k^{lf}(\mathcal{L}_\omega)$ be the vector space of locally finite sums of pairs $\rho_i \otimes u_{\rho_i}(t)$ of $k$-chain $\rho$ and a branch of $u_{\rho_i}(t)$ on $\rho_i$ such that

$$\sup_i(\sup_{t\in\rho_i} |u_{\rho_i}(t)|) < \infty, \ \sum_i |\rho_i| \sup_{t\in\rho_i} |u_{\rho_i}(t)| < \infty, \ \sum_i |\partial\rho_i| \sup_{t\in\partial\rho_i} |u_{\rho_i}(t)| < \infty,$$

where $|\rho_i|$ denotes the volume of $\rho_i$. We define a boundary operator $\partial_\omega$ on $C_\bullet^{lf}(\mathcal{L}_\omega)$ by

$$\partial_\omega : \rho \otimes u_\rho(t) \mapsto \partial\rho \otimes u_\rho(t)|_{\partial\rho},$$

where $\partial$ is the ordinary boundary operator and $u_\rho(t)|_{\partial\rho}$ is the restriction of $u_\rho(t)$ on $\partial\rho$. Since

$$\partial_\omega \circ \partial_\omega = 0,$$

we have a complex with boundary operator $\partial_\omega$

$$(C_\bullet^{lf}(\mathcal{L}_\omega), \partial_\omega) : C_2^{lf}(\mathcal{L}_\omega) \xrightarrow{\partial_\omega} C_1^{lf}(\mathcal{L}_\omega) \xrightarrow{\partial_\omega} C_0^{lf}(\mathcal{L}_\omega) \longrightarrow 0.$$

The locally finite $k$-th twisted homology group $H_k^{lf}(X, \mathcal{L}_\omega)$ with respect to $\mathcal{L}_\omega$ is defined as the $k$-th homology group of the complex $(C_\bullet^{lf}(\mathcal{L}_\omega), \partial_\omega)$.

The space $C_k^{lf}(\mathcal{L}_\omega)$ is regarded as a subspace in the dual of $\mathcal{S}^k(x)$ by the integral

$$\langle \varphi, \gamma \rangle = \sum_i b_i \int_{\rho_i} u_{\rho_i}(t)\varphi,$$

where

$$\varphi \in \mathcal{S}^k(x), \quad \gamma = \sum_i b_i \rho_i \otimes u_{\rho_i}(t) \in C_k^{lf}(\mathcal{L}_\omega).$$

Since we have

$$\langle \varphi + \nabla_\omega f,\ \gamma + \partial_\omega g \rangle = \langle \varphi, \gamma \rangle,$$

for

$$\varphi \in \ker(\nabla_\omega : \mathcal{S}^k(x) \to \mathcal{S}^{k+1}(x)), \quad f \in \mathcal{S}^{k-1}(x),$$
$$\gamma \in \ker(\partial_\omega : C_k^{lf}(\mathcal{L}_\omega) \to C_{k-1}^{lf}(\mathcal{L}_\omega)), \quad g \in C_{k+1}^{lf}(\mathcal{L}_\omega),$$

by the Stokes theorem, the pairing decents to that between $H^k(\mathcal{S}^\bullet(x), \nabla_\omega)$ and $H_k^{lf}(X, \mathcal{L}_\omega)$.

**Theorem 3.1** *The pairing between $H^1(\mathcal{S}^\bullet(x), \nabla_\omega)$ and $H_1^{lf}(X, \mathcal{L}_\omega)$ is perfect.*

# 4 Intersection pairings

The integral (1) for $\mathcal{S}^1$ and $\mathcal{P}^1$ induces a pairing $\langle\ ,\ \rangle$ between $H^1(\mathcal{S}^\bullet(x), \nabla_\omega)$ and $H^1(\mathcal{P}^\bullet(x), \nabla_{-\omega})$, which is called the intersection pairing. By the isomorphisms in Theorem 2.1, this intersection pairing naturally induces the paring between $H^1(\Omega_{\mathbf{P}}^\bullet(*D), \nabla_{\pm\omega})$. The intersection numbers for suitable bases of $H^1(\Omega_{\mathbf{P}}^\bullet(*D), \nabla_{\pm\omega})$ computed in [6] will yield explicit formulas of twisted period relations if we properly define the intersection pairing between the twisted homology groups.

In order to define the intersection pairing between $H_1^{lf}(X, \mathcal{L}_{\pm\omega})$ by the duality in Theorem 3.1, we introduce spaces of temperate currents. By a similar semi-norm with Schwartz's space of rapidly decreasing functions, the space $\mathcal{S}^k$ becomes the Fréche space. The space $\check{\mathcal{S}}^{2-k}$ of continuous $\mathbf{C}$-linear functionals on $\mathcal{S}^k$ is called the temperate currents of degree $2-k$. For the complex $(\mathcal{S}^\bullet(x), \nabla_\omega)$, we have a complex with differential $\nabla_{-\omega}$

$$(\check{\mathcal{S}}^\bullet(x), \nabla_{-\omega}) : \check{\mathcal{S}}^0(x) \xrightarrow{\nabla_{-\omega}} \check{\mathcal{S}}^1(x) \xrightarrow{\nabla_{-\omega}} \check{\mathcal{S}}^2(x) \xrightarrow{\nabla_{-\omega}} 0.$$

Since

$$\int_X \nabla_\omega(\xi) \wedge \eta = (-1)^{k+1} \int_X \xi \wedge \nabla_{-\omega}(\eta), \quad \xi \in \mathcal{S}^k,\ \eta \in \mathcal{P}^{1-k},$$

$$\langle \nabla_\omega(\xi), \gamma \rangle = \langle \xi, \partial_\omega(\gamma) \rangle, \quad \xi \in \mathcal{S}^k,\ \gamma \in C_{k+1}^{lf}(\mathcal{L}_\omega),$$

we have the natural inclusions of complexes

$$(\mathcal{P}^\bullet(x), \nabla_{-\omega}) \subset (\check{\mathcal{S}}^\bullet(x), \nabla_{-\omega}), \quad (C_\bullet^{lf}(\mathcal{L}_\omega), \partial_\omega) \subset (\check{\mathcal{S}}^\bullet(x), \nabla_{-\omega}),$$

which induce maps

$$\begin{aligned} \iota_1 &: H^1(\mathcal{P}^\bullet(x), \nabla_{-\omega}) \to H^1(\check{\mathcal{S}}^\bullet(x), \nabla_{-\omega}), \\ \iota_2 &: H_1^{lf}(X, \mathcal{L}_\omega) \to H^1(\check{\mathcal{S}}^\bullet(x), \nabla_{-\omega}), \end{aligned}$$

respectively, where $H^1(\check{\mathcal{S}}^\bullet(x), \nabla_{-\omega})$ is the first cohomology group of the complex $(\check{\mathcal{S}}^\bullet(x), \nabla_{-\omega})$. We have the following duality of homology and cohomology groups.

**Theorem 4.1** *The maps $\iota_1$ and $\iota_2$ are injective. Moreover,*

$$\iota_1(H^1(\mathcal{P}^\bullet(x), \nabla_{-\omega})) = \iota_2(H_1^{lf}(X, \mathcal{L}_\omega)).$$

We conjecture

$$H^1(\mathcal{P}^\bullet(x), \nabla_{-\omega}) \simeq H^1(\check{\mathcal{S}}^\bullet(x), \nabla_{-\omega}) \simeq H_1^{lf}(X, \mathcal{L}_\omega).$$

By Theorem 2.1 and 4.1, we have maps

$$\begin{aligned} \jmath^+ &: H_1^{lf}(X, \mathcal{L}_{-\omega}) \to H^1(\mathcal{S}^\bullet(x), \nabla_\omega), \\ \jmath^- &: H_1^{lf}(X, \mathcal{L}_\omega) \to H^1(\mathcal{P}^\bullet(x), \nabla_{-\omega}). \end{aligned}$$

The intersection number of $\gamma^+ \in H_1^{lf}(X, \mathcal{L}_\omega)$ and $\gamma^- \in H_1^{lf}(X, \mathcal{L}_{-\omega})$ is defined by

$$\langle \gamma^+, \gamma^- \rangle = \langle \jmath^-(\gamma^+), \jmath^+(\gamma^-) \rangle = \int_X \jmath^-(\gamma^+) \wedge \jmath^+(\gamma^-).$$

**Theorem 4.2** *If the set $\cup_{i,j}(\rho_i^+ \cap \rho_j^-)$ is finite and $\rho_i^+$ and $\rho_j^-$ intersect transversally at each point of $\rho_i^+ \cap \rho_j^-$ for*

$$\gamma^+ = \sum_i b_i \rho_i^+ \otimes u_{\rho_i^+}(t) \in H_1^{lf}(X, \mathcal{L}_\omega), \quad \gamma^- = \sum_j b_j \rho_j^- \otimes u_{\rho_j^-}^{-1}(t) \in H_1^{lf}(X, \mathcal{L}_{-\omega}),$$

*then the intersection number $\langle \gamma^+, \gamma^- \rangle$ is equal to*

$$\langle \gamma^+, \gamma^- \rangle = \sum_{i,j} \sum_{v \in \rho_i^+ \cap \rho_j^-} b_i b_j [u_{\rho_i^+}(t)]_{t=v} [u_{\rho_j^-}^{-1}(t)]_{t=v} I_v(\rho_i^+, \rho_j^-),$$

*where $I_v(\rho_i^+, \rho_j^-)$ is the topological intersection number of $\rho_i^+$ and $\rho_j^-$ at $v \in X$.*

# 5 Twisted period relations

Let $\{\varphi_\mu^\pm\}$ and $\{\gamma_\mu^\pm\}$ be bases of $H^1(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_{\pm\omega})$ and $H_1^{lf}(X, \mathcal{L}_{\pm\omega})$, respectively. We define four $(n-2, n-2)$-matrices as

$$\Pi^+ = \langle \varphi_\mu^+, \gamma_\nu^+ \rangle_{\mu,\nu}, \quad \Pi^- = \langle \varphi_\mu^-, \gamma_\nu^- \rangle_{\mu,\nu},$$

$$I_{ch} = \langle \varphi_\mu^+, \varphi_\nu^- \rangle_{\mu,\nu}, \quad I_h = \langle \gamma_\mu^+, \gamma_\nu^- \rangle_{\mu,\nu}.$$

The naturality of pairings between twisted cohomology and homology groups implies the following.

**Theorem 5.1** *We have twisted period relations with respect to* $\pm\omega$ :

$$\Pi^+ \, {}^t I_h^{-1} \, {}^t\Pi^- = I_{ch}, \quad i.e. \quad {}^t\Pi^- \, I_{ch}^{-1} \, \Pi^+ = {}^t I_h.$$

**Example 5.1** We put

$$\omega = -t dt, \quad D = 3\infty.$$

The spaces $H^1(\Omega^\bullet_{\mathbf{P}}(*D), \nabla_{\pm\omega})$ and $H_1^{lf}(X, \mathcal{L}_{\pm\omega})$ are of rank 1, we put $\varphi_1^\pm = dt$ and

$$\gamma_1^+ = \sum_{n\in\mathbf{Z}} [n, n+1] \otimes e^{\frac{-t^2}{2}}, \quad \gamma_1^- = \sum_{n\in\mathbf{Z}} \left([n, n+1]\sqrt{-1}\right) \otimes e^{\frac{t^2}{2}}.$$

The intersection number $\langle dt, dt \rangle$ is $2\pi\sqrt{-1}$ as given in [6] and Theorem 4.2 implies $\langle \gamma^+, \gamma^- \rangle = 1$. Since

$$\langle dt, \gamma^+ \rangle = \int_{-\infty}^{+\infty} e^{\frac{-t^2}{2}} dt, \quad \langle dt, \gamma^- \rangle = \int_{-\sqrt{-1}\infty}^{+\sqrt{-1}\infty} e^{\frac{t^2}{2}} dt = \sqrt{-1} \int_{-\infty}^{+\infty} e^{\frac{-t^2}{2}} dt,$$

we have the twisted period relation

$$\left(\int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} dt\right) \cdot 1 \cdot \left(\sqrt{-1} \int_{-\infty}^{\infty} e^{\frac{-t^2}{2}} dt\right) = 2\pi\sqrt{-1},$$

which yields the identity

$$\int_{-\infty}^{+\infty} e^{\frac{-t^2}{2}} dt = \sqrt{2\pi}.$$

We present two more examples without further explanations (see [7] for details). The twisted period relations imply that the inversion formula for the gamma function

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$$

and that Lommel's formula

$$(J_a(z)J_{-a+1}(z) - J_{a-1}(z)J_{-a}(z)) \left(\frac{z}{2}\right) = -\frac{\sin(\pi a)}{\pi},$$

for the Bessel function with parameter $a \in \mathbf{C}$

$$J_a(z) = \left(\frac{z}{2}\right)^a \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(a+k+1)} \left(\frac{z}{2}\right)^k,$$

where $z \in \{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$ and the argument of $z/2$ is in $(-\pi/2, \pi/2)$.

# References


[1] K. Cho and K. Matsumoto, Intersection theory for twisted cohomologies and twisted Riemann's period relations I, Nagoya Math. J. **139** (1995), 67–86.

[2] Y. Haraoka, Quadratic relations for confluent hypergeometric functions on $Z_{2,n+1}$, preprint.

[3] H. Kimura, On rational de Rham cohomology associated with the generalized confluent hypergeometric functions I, $\mathbf{P}^1$ case, Proc. Roy. Soc. Edinburgh, **127A** (1997), 145–155.

[4] M. Kita and M. Yoshida, Intersection theory for twisted cycles I, II, Math. Nachr., **166** (1994), 287–304, **168** (1994), 171–190.

[5] K. Matsumoto, Intersection numbers for logarithmic $k$-forms, preprint.

[6] K. Matsumoto, Intersection numbers for 1-forms associated with confluent hypergeometric functions, preprint.

[7] K. Matsumoto and N. Takayama, Intersection theory for generalized confluent hypergeometric functions, in preparation.