Restrictions of $\mathcal{A}$-hypergeometric systems and connection formulas of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric function

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Introduction

This paper consists of two parts. In the first part, we study *the restriction as a $\mathcal{D}$-module* of the $\mathcal{A}$-hypergeometric system to its singularities and give a combinatorial description of the restriction. In the second part, we derive a complete set of the connection formulas among the series solutions of the $\mathcal{A}$-hypergeometric system of the general prism $\Delta_1 \times \Delta_{n-1}$, which is a generalization of the well-known connection formula of the Gauss hypergeometric series :

$$\begin{aligned}
F(\alpha,\beta,\gamma;x) =& \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-x)^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1,1/x) \\
&+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}(-x)^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1,1/x).
\end{aligned} \tag{0.1}$$

Moreover, we give a set of formulas that is regarded as *a multiplicative* 1-*cocycle* of the permutation group $\mathfrak{S}_n$. We utilize the results of the first part to derive connection formulas by the method of boundary values ([Hek;2]).

The general theory of the hypergeometric functions has been developed by K.Aomoto (the theory of twisted cycles), I.M.Gel'fand (hypergeometric functions on the Grassmann manifold etc.) and their respective joint workers. Especially, I.M.Gel'fand, A.V.Zelevinsky and M.M.Kapranov ([GZK2]) defined the $\mathcal{A}$-hypergeometric system for a given set of points $\mathcal{A}$. We can naturally regard the system as a $\mathcal{D}$-module $M_\mathcal{A}$ on ^{n}. The $\mathcal{D}$-module $M_\mathcal{A}$ is holonomic and the sheaf of the holomorphic solutions $\mathcal{H}om_{\mathcal{D}_n}(M_\mathcal{A},\mathcal{O}_n)$ is a constructible sheaf of finite rank by the theorem of Kashiwara ([K1]). In [GZK2], they gave an explicit expression of the characteristic cycle of $M_\mathcal{A}$ and proved that the solution sheaf $\mathcal{H}om_{\mathcal{D}_n}(M_\mathcal{A},\mathcal{O}_n)$ is a locally constant sheaf of which rank is the volume of $\mathcal{A}$ on the generic stratum $X'_\mathcal{A}$ that is the complement of the zero set of the principal $\mathcal{A}$-determinant $E_\mathcal{A}$ ([GZK1]). Moreover, they defined the notion of a regular triangulation and explicitly gave fundamental sets of series solutions on $X'_\mathcal{A}$ determined by regular triangulations.

The results above raise the following problems :

A) to study the $\mathcal{D}$-module $M_\mathcal{A}$ on non-generic strata,

B) to find the connection formulas among the series solutions determined by the regular triangulations.

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As readers may think, the answer to A) solves B) when the geometry of $X'_{\mathcal{A}}$ has a nice property (e.g. Theorem 1.4). In [GZK3; 270p, line 9], they gave an answer to the problem A) in a quite abstract way; they expressed the solution sheaf by the twisted cohomology. Unfortunately, their answer is hard to use for solving the problem B). In Section 1, we will give a description of restrictions of $M_{\mathcal{A}}$ to non-generic strata by utilizing the secondary polytope $\Sigma(\mathcal{A})$ ([GZK1; 3A.2] or [BFS]) and a generalization of the theory of b-functions ([S1]). The description also yields a description of the solution sheaf on non-generic strata.

Let us turn to the problem B). Among various $\mathcal{A}$-hypergeometric systems, the most fundamental and important one is the system when $\mathcal{A}$ is the general prism $\mathcal{A}_n = \Delta_1 \times \Delta_{n-1}$ where Δ_k is the k-simplex. The $\mathcal{A}_n$-hypergeometric system admits the Lauricella function F_D of $n-1$ variables as a solution. The system is also obtained by restricting the hypergeometric system on the Grassmann manifold $G_{2,n+2}$ to an affine chart.

In Theorem 5.1, we will give a set of the connection formulas among the series solutions of the $\mathcal{A}_n$-hypergeometric system, which is the main result of the second part. In case of $n=2$, the $\mathcal{A}_n$-hypergeometric system is essentially equivalent to the Gauss hypergeometric equation. The square $\Delta_1 \times \Delta_1$ admits 2 regular triangulations and our connection formula among the 2 sets of the fundamental solutions determined by the 2 regular triangulations is essentially equivalent to the formula (0.1).

Let us briefly summarize the contents of each section and show the techniques used in this paper.

In Section 1, we define the notion of formal restriction of the $\mathcal{A}$-hypergeometric system and prove that the restriction as a $\mathcal{D}$-module of the $\mathcal{A}$-hypergeometric system is a quotient of the formal restriction. We use a generalization of the theory of b-functions for $\mathcal{A}$-hypergeometric system (cf. [S1]) to prove it. The theory is summarized in Appendix. We also illustrate a general method of deriving the connection formulas among the series solutions by utilizing the formal restriction. This section is the first part of this paper.

The second part of this paper starts from Section 2. In Section 2, we quickly state the known results about the secondary polytope of $\Delta_1 \times \Delta_{n-1}$ in a suitable form for our study. The secondary polytope of $\Delta_1 \times \Delta_{n-1}$ is the permutahedron ([BFS]). In section 3, we give series solutions of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric system. In Section 4, we decompose ^{2n} into simply connected domains to give our connection formulas on these domains. In Section 5, we give our connection formulas in a recursive form. Let $\mathcal{F}$ be a field and suppose that a group G acts on $\mathcal{F}$. A set of matrices $\{C(g) \in GL(m, \mathcal{F}) \,|\, g \in G\}$ that satisfies the condition

$$C(gh) = C(h)C(g)^h, \quad g, h \in G$$

is called *a multiplicative* 1-*cocycle* of the group G. The set of our connection formulas is realized as a multiplicative 1-cocycle of the permutation group $\mathfrak{S}_n$. Notice that the group $\mathfrak{S}_n$ can be understood as the group generated by the operations of restructuring ([GZK1;3A.7]) of the triangulations of $\Delta_1 \times \Delta_{n-1}$. In Section 6, we prove the connection formulas. We can reduce the proof to a problem in lower dimensions by utilizing the formal restriction.

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1. Formal restriction of $\mathcal{A}$-hypergeometric system

We start with reviewing the definition of the $\mathcal{A}$-hypergeometric system ([GZK2]).

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a set of n-points in ^{d} which satisfies the conditions:

(1.1) there exists a vector $c \in ({}^d)^*$ such that

$$\langle c, a_i \rangle = 1, \quad i = 1, \ldots, n;$$

(1.2)
$$\mathcal{A} = a_1 + \cdots + a_n = {}^d.$$

We regard the a_i as the column vector and denote the (i,j) component of the matrix $(a_1, \ldots, a_n)$ by a_{ij}. Put

$$p_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j - \alpha_i, \qquad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, d$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a fixed vector of complex numbers. Let

$$\mathcal{D}_{\mathcal{A}} = \mathcal{O}_{\mathcal{A}} \langle \partial_1, \ldots, \partial_n \rangle, \quad \mathcal{O}_{\mathcal{A}} = \mathcal{O}_n$$

be the sheaf of the differential operators on the $\mathcal{A}$-space ^{n}. The $\mathcal{A}$-hypergeometric system $M_{\mathcal{A}}$ is defined by

$$M_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}} / H_{\mathcal{A}}, \quad H_{\mathcal{A}} = \sum_{i=1}^{d} \mathcal{D}_{\mathcal{A}} p_i + I_{\mathcal{A}}$$

where $I_{\mathcal{A}}$ is the left ideal of $\mathcal{D}_{\mathcal{A}}$ generated by

$$\triangle_b = \prod_{b_j > 0} \partial_j^{b_j} - \prod_{b_j < 0} \partial_j^{-b_j}, \quad b = (b_1, \ldots, b_n) \in \ker(a_1, \ldots, a_n) \cap {}^n.$$

When we need to emphasize the dependency of $p_i, H_{\mathcal{A}}$ and $M_{\mathcal{A}}$ on the parameter α, we denote them by $p_i(\alpha), H_{\mathcal{A}}(\alpha)$ and $M_{\mathcal{A}}(\alpha)$ respectively.

Next, we quickly review the definition of the regular polyhedral subdivision ([GZK1], [BFS]). Let $(\omega_1, \dots, \omega_n)$ be a vector in ^{n}. Consider the convex hull H of the points

$$\{(a_1, \omega_1), \dots, (a_n, \omega_n)\}$$

where a_i are vectors in ^{d}. Let

$$\pi \ : \ {}^{d+1} \ni (y_1, \dots, y_{d+1}) \longmapsto (y_1, \dots, y_d) \in {}^d$$

be the projection. The projection by π of the convex hull H induces the polyhedral subdivision of the convex hull $\mathrm{conv}(\mathcal{A})$. The polyhedral subdivision obtained by this way is called the *regular* polyhedral subdivision. When the polyhedral subdivision is the triangulation of $\mathcal{A}$, the polyhedral subdivision is called the *regular* triangulation. The set of all regular polyhedral subdivisions is the poset (partially ordered set) by the refinement. It is shown in [GZK1] (or [BFS]) that there exists the secondary polytope $\Sigma(\mathcal{A})$ such that the face lattice of the secondary polytope is anti-isomorphic to the poset of the regular polyhedral subdivisions.

We fix a regular polyhedral subdivision $\{\Gamma^{(1)}, \dots, \Gamma^{(r)}\}$ of $\mathrm{conv}(\mathcal{A})$ expressed by indices of the n-points. We can assume $\Gamma^{(1)} = \{1, \dots, m\} \subset \{1, \dots, n\}$ without loss of generality and put $\Gamma = \Gamma^{(1)}$.

Put

$$q_i(\beta) = \sum_{j=1}^{m} a_{ij} x_j \partial_j - \beta_i, \quad i = 1, \dots, d, \quad \beta = (\beta_1, \dots, \beta_d)$$

and

$$\triangle_b = \prod_{b_i > 0} \partial_i^{b_i} - \prod_{b_j < 0} \partial_j^{-b_j}, \quad b = (b_1, \dots, b_m) \in \ker(a_1, \dots, a_m) \cap {}^m.$$

Let $\lambda_1 = 0, \lambda_2, \dots, \lambda_v \in \sum_{j=1}^n {}_{\geq 0} a_j$ be a set of the representatives of the additive group $\mathcal{A}/\Gamma$. We can take the representatives from the semigroup $\sum_{i=m+1}^n {}_{\geq 0} a_i$ because of the following lemma.

Lemma 1.1

$$\Gamma + \sum_{i=m+1}^{n} {}_{\geq 0} a_i = \mathcal{A}.$$

Proof. Put $v = |\mathcal{A}/\Gamma| > 0$. Then $-v a_j \in \Gamma$. Hence, $-a_j \in \Gamma + \sum_{i=m+1}^n {}_{\geq 0} a_i$. □

We denote by $\mathcal{D}_\Gamma$ the sheaf of the differential operators of m-variables $x_1, \dots, x_m$. Put

$$M_\Gamma(\alpha - \lambda_k) = \mathcal{D}_\Gamma / H_\Gamma(\alpha - \lambda_k)$$
$$H_\Gamma(\alpha - \lambda_k) = \sum_{i=1}^{d} \mathcal{D}_\Gamma q_i(\alpha - \lambda_k) + I_\Gamma$$

where I_Γ is the left ideal of $\mathcal{D}_\Gamma$ generated by

$$\triangle_b, \quad b \in \ker(a_1, \ldots, a_m) \cap {}^m.$$

The *formal restriction* of $M_\mathcal{A}$ to the submanifold $x_{m+1} = \cdots = x_n = 0$ is defined to be

$$\oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k), \quad v = |\mathcal{A}/\Gamma|.$$

The formal restriction is the left $\mathcal{D}_\Gamma$-module. If the parameter α is generic, the formal restriction does not depend on the choice of the representatives $\lambda_1, \ldots, \lambda_v$ by virtue of Theorem 1.1. If we regard Γ as the new ^{d} and forget the old lattice structure $\mathcal{A} = {}^d$, then the left $\mathcal{D}_\Gamma$-module $M_\Gamma(\alpha - \lambda_k)$ is nothing but the hypergeometric system defined by the set of points Γ.

Put

$$X_\Gamma = \{x = (x_1, \ldots, x_n) \in {}^n \,|\, x_{m+1} = \cdots = x_n = 0\} \tag{1.3}$$

and let j be the injection from X_Γ to ^{n}. The *restriction as a $\mathcal{D}$-module* of the left $\mathcal{D}_\mathcal{A}$-module $M_\mathcal{A}$ to X_Γ is defined by

$$\mathcal{O}_\Gamma \otimes_{j^{-1}\mathcal{O}_\mathcal{A}} j^{-1} M_\mathcal{A} = j^{-1}\left(\mathcal{D}_\mathcal{A} / \sum_{i=m+1}^{n} x_i \mathcal{D}_\mathcal{A} \otimes_{\mathcal{D}_\mathcal{A}} M_\mathcal{A}\right) = j^{-1}\left(\mathcal{D}_\mathcal{A} / \left(\sum_{i=m+1}^{n} x_i \mathcal{D}_\mathcal{A} + H_\mathcal{A}\right)\right)$$

and is denoted by $j^* M_\mathcal{A}$ ([K2]). The restriction $j^* M_\mathcal{A}$ is the left holonomic $\mathcal{D}_\Gamma$-module ([K2]). We will study a relation between the restriction $j^* M_\mathcal{A}$ and the formal restriction $\oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k)$.

As a first step, we note that there exists a $\mathcal{D}_\Gamma$-morphism r from the formal restriction $\oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k)$ to $j^* M_\mathcal{A}$ by defining

$$r \quad : \quad \oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k) \ni (\ell_k)_{k=1}^{v} \longmapsto \sum_{k=1}^{v} \ell_k \partial^{\lambda_k} \in j^* M_\mathcal{A}$$

where $\partial^{\lambda_k} = \prod \partial_i^{\lambda_{ki}}$ and $\lambda_k = \sum_{i=m+1}^{n} \lambda_{ki} a_i$. The map r is well-defined, because

$$\begin{aligned}
& q_i(\alpha - \lambda_k)\partial^{\lambda_k} \\
= & \left(p_i(\alpha - \lambda_k) - \sum_{j=m+1}^{n} a_{ij} x_j \partial_j\right) \partial^{\lambda_k} \\
= & \partial^{\lambda_k} p_i(\alpha) - \sum_{j=m+1}^{n} a_{ij} x_j \partial_j \partial^{\lambda_k} \\
\in & \sum_{j=m+1}^{n} x_j \mathcal{D}_\mathcal{A} + H_\mathcal{A}(\alpha)
\end{aligned}$$

and

$$I_\Gamma \subset I_\mathcal{A}.$$

Moreover, the map r induces a map among solutions. Let us denote the sheaf of the holomorphic solutions of $M_{\mathcal{A}}$ by $\mathcal{H}om_{\mathcal{D}_{\mathcal{A}}}(M_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}})$. Then we have the induced map

$$ {}^t r \quad : \quad \mathcal{H}om_{\mathcal{D}_\Gamma}(j^* M_{\mathcal{A}}(\alpha), \mathcal{O}_\Gamma) \longrightarrow \mathcal{H}om_{\mathcal{D}_\Gamma}(\oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k), \mathcal{O}_\Gamma). $$

As a second step, we need the theory of b-functions of $\mathcal{A}$-hypergeometric system without assuming the normality condition. Although the theory of b-function with the normality condition has been already studied in [S1], we need to drop the condition for our theorem.

For given $\chi \in {}_{\geq 0}\Gamma$, we consider the natural morphism :

$$ f_\chi \;:\; M_\Gamma(\alpha - \chi) \ni \ell \longmapsto \ell \partial^\chi \in M_\Gamma(\alpha), $$

where $\partial^\chi = \prod \partial_i^{b_i},\ \chi = \sum_{i=1}^m b_i a_i,\ \Gamma = \{a_1, \ldots, a_m\}$. We consider a sufficient condition for the morphism f_χ to be the isomorphism. The set of the facets of the cone defined by Γ is denoted by $\mathcal{F}_\Gamma$. Let $\tau \in \mathcal{F}_\Gamma$ be a facet and

$$ F_\tau \;:\; {}^d = \Gamma \longrightarrow $$

be the integral supporting function of the facet τ; the function F_τ is linear, vanishes on $\tau \cap \Gamma$, takes positive values on $\Gamma \setminus \tau$ and is surjective. The supporting function F_τ can be extended as the function on $\Gamma \otimes \;\; = {}^d$ and we also denote it by F_τ.

THEOREM 1.1 *Assume that* $F_\tau(\alpha) \notin {}_{\geq 0}$ *for any* $\tau \in \mathcal{F}_\Gamma$. *Then the morphism*

$$ f_\chi \;:\; M_\Gamma(\alpha - \chi) \longrightarrow M_\Gamma(\alpha) $$

is isomorphic for all $\chi \in {}_{\geq 0}\Gamma$.

Notice that we regard the parameter α as the vector in $\Gamma \otimes$ in this theorem. This theorem is a generalization of [S1] in which the normality condition is essentially assumed. We will give a proof of the theorem in Appendix.

As the last step, we state our main theorem of the first part.

THEOREM 1.2 *Assume that* $F_\tau(\alpha) \notin {}_{\geq 0} + \sum_{j=m+1}^{n} {}_{\geq 0} F_\tau(a_j)$ *for any* $\tau \in \mathcal{F}_\Gamma$. *Then the morphism*

$$ r \quad : \quad \oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k) \longrightarrow j^* M_{\mathcal{A}}(\alpha) $$

is surjective.

Corollary 1.1 *Assume that* $F_\tau(\alpha) \notin {}_{\geq 0} + \sum_{j=m+1}^{n} {}_{\geq 0} F_\tau(a_j)$ *for any* $\tau \in \mathcal{F}_\Gamma$. *The induced map*

$$ {}^t r \quad : \quad j^{-1}\mathcal{H}om_{\mathcal{D}_{\mathcal{A}}(\alpha)}(M_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}}) \longrightarrow \mathcal{H}om_{\mathcal{D}_\Gamma}(\oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k), \mathcal{O}_\Gamma) $$

is injective.

In order to prove the theorem, we need the following lemmas.

Lemma 1.2 *Suppose that* $F_\tau(\alpha) \notin {}_{\geq 0}$ *for any* $\tau \in \mathcal{F}_\Gamma$. *For any vector* $(c_1, \ldots, c_m) \in {}^{m}_{\geq 0}$, *there exists* $p \in \mathcal{D}_\Gamma$ *such that*

$$ p \prod_{i=1}^{m} \partial_i^{c_i} - 1 \in H_\Gamma(\alpha). $$

Proof. The map

$$ \prod_{i=1}^{m} \partial_i^{c_i} \quad : \quad M_\Gamma(\alpha - \sum_{j=1}^{m} c_j a_j) \longrightarrow M_\Gamma(\alpha) $$

is isomorphism from Theorem 1.1. Let p be the inverse image of $1 \in M_\Gamma(\alpha)$. We have $p \prod_{i=1}^{m} \partial_i^{c_i} = 1$ in $M_\Gamma(\alpha)$. ∎

Lemma 1.3 *Assume that* $F_\tau(\alpha) \notin {}_{\geq 0} + \sum_{j=m+1}^{n} {}_{\geq 0} F_\tau(a_j)$ *for any* $\tau \in \mathcal{F}_\Gamma$. *Then for any* $(c_{m+1}, \ldots, c_n) \in {}^{n-m}_{\geq 0}$, *there exist an operator* $q \in \mathcal{D}_\Gamma$ *and an index* $1 \leq k \leq v$ *such that*

$$ \prod_{i=m+1}^{n} \partial_i^{c_i} - q\partial^{\lambda_k} \in H_{\mathcal{A}} + \sum_{i=m+1}^{n} x_i \mathcal{D}_{\mathcal{A}}. $$

Proof. By the definition of λ_k's, there exist $1 \leq k \leq v$ and $e_1, \ldots, e_m \in$ such that $\sum_{j=m+1}^{n} c_j a_j = \lambda_k + \sum_{i=1}^{m} e_i a_i$. Hence, we have

$$ \prod_{j=m+1}^{n} \partial_j^{c_j} \prod_{e_i<0} \partial_i^{-e_i} - \left(\prod_{e_i>0} \partial_i^{e_i} \right) \partial^{\lambda_k} \in I_{\mathcal{A}}. $$

By Lemma 1.2, there exists an operator $P \in \mathcal{D}_\Gamma$ such that $P \prod_{e_i<0} \partial_i^{-e_i} - 1 \in H_\Gamma(\alpha - \sum_{j=m+1}^{n} c_j a_j)$. Since we have

$$ \prod_{j=m+1}^{n} \partial_j^{c_j} H_\Gamma \left(\alpha - \sum_{j=m+1}^{n} c_j a_j \right) \subset H_{\mathcal{A}} + \sum_{i=m+1}^{n} x_i \mathcal{D}_{\mathcal{A}}, $$

we conclude

$$ \prod_{j=m+1}^{n} \partial_j^{c_j} - P \prod_{e_i>0} \partial_i^{e_i} \partial^{\lambda_k} \in H_{\mathcal{A}} + \sum_{i=m+1}^{n} x_i \mathcal{D}_{\mathcal{A}}. \quad ∎ $$

Proof of Theorem 1.2. Any element $f \in \mathcal{D}_{\mathcal{A}}$ can be written as

$$\sum_{\gamma=(\gamma_{m+1},\ldots,\gamma_n)\in_{\geq 0}^{n-m}} f_\gamma \partial^\gamma$$

where $f_\gamma \in \mathcal{O}_{\mathcal{A}}\langle \partial_1, \ldots, \partial_m \rangle$ and $\partial^\gamma = \partial_{m+1}^{\gamma_{m+1}} \cdots \partial_n^{\gamma_n}$. It follows from Lemma 1.3 that there exist $q_\gamma \in \mathcal{D}_\Gamma$ and λ_{k_γ} such that

$$\partial^\gamma - q_\gamma \partial^{\lambda_{k_\gamma}} \in H_{\mathcal{A}} + \sum_{i=m+1}^{n} x_i \mathcal{D}_{\mathcal{A}}.$$

Since f_γ commutes with x_i $(m+1 \leq i \leq n)$, we have

$$f - \sum (f_\gamma q_\gamma)_{|X_\Gamma} \partial^{\lambda_{k_\gamma}} \in H_{\mathcal{A}} + \sum_{i=m+1}^{n} x_i \mathcal{D}_{\mathcal{A}}.$$

Put

$$\ell_k = \sum_{k_\gamma = k} (f_\gamma q_\gamma)_{|X_\Gamma}, \quad k = 1, \ldots, v.$$

Then, we have

$$r\left((\ell_1, \ldots, \ell_v)\right) = f$$

and we have completed the proof. $\square$

Proof of Corollary 1.1. It follows from Theorem 1.2 that the induced map ${}^t r$ is injective. Since $M_{\mathcal{A}}$ is regular holonomic ([Hot; 6.2]), we have

$$\mathcal{H}om_{\mathcal{D}_\Gamma}(j^* M_{\mathcal{A}}, \mathcal{O}_\Gamma) \simeq j^{-1} \mathcal{H}om_{\mathcal{D}_{\mathcal{A}}}(M_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}})$$

by the Riemann-Hilbert correspondence ([M; Th 2.2.3] or [K3]). $\square$

Let $f \in \mathcal{O}_{\mathcal{A}}$ be a solution of $M_{\mathcal{A}}$ that is holomorphic at a point in X_Γ. We have

$$({}^t r j^{-1})(f) = (\partial^{\lambda_1} f, \ldots, \partial^{\lambda_v} f)_{|X_\Gamma} \in \mathcal{H}om_{\mathcal{D}_\Gamma}(\oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k), \mathcal{O}_\Gamma).$$

Corollary 1.1 asserts the uniqueness of the solution of $M_{\mathcal{A}}$ around X_Γ. The solution f is determined only by the values of the derivatives $\partial^{\lambda_k} f$ on the submanifold X_Γ. Although the uniqueness is enough to obtain our connection formulas, we shall prove that the map ${}^t r$ is isomorphic for generic values of α.

Theorem 1.3 *Let T be a regular triangulation which is a refinement of the regular polyhedral subdivision $\cup \Gamma^{(k)}$. If the parameter α is T-nonresonant* ([GZK2; Def 3]) *and $F_\tau \notin {}_{\geq 0} + \sum_{j=m+1}^{n} {}_{\geq 0} F_\tau(a_j)$ for any $\tau \in \mathcal{F}_\Gamma$, then we have the isomorphism*

$$ {}^t r \quad : \quad j^{-1} \mathcal{H}om_{\mathcal{D}_{\mathcal{A}}}(M_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}}) \simeq \mathcal{H}om_{\mathcal{D}_\Gamma}(\oplus_{k=1}^{v} M_\Gamma(\alpha - \lambda_k), \mathcal{O}_\Gamma)$$

on the generic stratum of X_Γ.

Proof. Since the parameter α is T-nonresonant, we have $\mathrm{vol}_{\mathcal{A}}(\mathcal{A})$ linearly independent series solutions defined by T ([GZK2; Th3]) where $\mathrm{vol}_{\mathcal{A}}(S)$ generally denotes the volume of the convex hull of the set S with respect to the lattice $\mathcal{A} = {}^d$. Among these solutions, we choose $\mathrm{vol}_{\mathcal{A}}(\Gamma)$ solutions standing for the simplices $T \cap \mathrm{conv}(\Gamma)$. These solutions converge on a point $p \in X_\Gamma \subset {}^n$. Therefore we have

$$\dim \mathcal{H}om_{\mathcal{D}_\mathcal{A}}(M_\mathcal{A}, \mathcal{O}_\mathcal{A})_p \geq \mathrm{vol}_\mathcal{A}(\Gamma).$$

On the other hand, we have

$$\dim \mathcal{H}om_{\mathcal{D}_\Gamma}(M_\Gamma(\alpha - \lambda_k), \mathcal{O}_\Gamma)_p \leq \mathrm{vol}_\Gamma(\Gamma)$$

and $\mathrm{vol}_\Gamma(\Gamma) = \mathrm{vol}_\mathcal{A}(\Gamma)/v$. Since the map ${}^t r$ is injective from Corollary 1.1, we have the conclusion on the generic stratum of X_Γ. $\square$

Notice that the proof above also shows that the series solutions standing for the simplices $T \cap \mathrm{conv}(\Gamma)$ span the solution space $j^{-1}\mathcal{H}om_{\mathcal{D}_\mathcal{A}}(M_\mathcal{A}, \mathcal{O}_\mathcal{A})$ on the generic stratum of X_Γ.

Finally, we will illustrate how to use Theorem 1.2 to derive connection formulas. We use the method of boundary values ([Hek]) and the restrictions of the hypergeometric $\mathcal{D}_\mathcal{A}$-module.

Let T and T' be regular triangulations which are refinements of the regular polyhedral subdivision $\cup\Gamma^{(k)}$ such that $T = T'$ on $\cup_{k\geq 2}\mathrm{conv}(\Gamma^{(k)})$. Put

$$P = T \cap \mathrm{conv}(\Gamma), \quad P' = T' \cap \mathrm{conv}(\Gamma) \quad \text{and} \quad Q = T \cap \cup_{k\geq 2}\mathrm{conv}(\Gamma^{(k)}).$$

We denote the vector of series solutions defined by the collection of simplices T by ϕ_T. Using this notation, we have

$$\phi_T = {}^t(\phi_P, \phi_Q), \quad \phi_{T'} = {}^t(\phi_{P'}, \phi_Q).$$

Let $C_P, C_{P'}$ and C_Q be the domains of convergence of the vectors of the series $\phi_P, \phi_{P'}$ and ϕ_Q respectively. Notice that $C_P \cap X_\Gamma \neq \emptyset$ and $C_{P'} \cap X_\Gamma \neq \emptyset$. Put $X = {}^n \setminus \overline{\mathrm{Sing}\, M_\mathcal{A} \setminus X_\Gamma}$ and take points

$$\begin{aligned} &u \in C_P \cap C_Q, \quad u' \in C_{P'} \cap C_Q, \\ &v \in C_P \cap X_\Gamma \cap X, \quad v' \in C_{P'} \cap X_\Gamma \cap X. \end{aligned} \tag{1.4}$$

We, moreover, take the paths α, α' and γ which satisfy the following conditions;

(1.5)

γ is a path in $C_Q \cap (X \setminus X_\Gamma)$ that starts from u and arrives at u'.

α is a path in $C_P \cap X$ that starts from v and arrives at u.

α' is a path in $C_{P'} \cap X$ that starts from v' and arrives at u'.

THEOREM 1.4 *Suppose that the path $\alpha \circ \gamma \circ \alpha'^{-1}$ can be continuously deformed into a path γ' of $X \cap X_\Gamma$ in X. If we are given a connection formula in X_Γ*

$$\gamma'^* \left(\partial^{\lambda_k} \phi_{P|_{X_\Gamma}} \right) = M_{\gamma'} \left(\partial^{\lambda_k} \phi_{P'|_{X_\Gamma}} \right), \quad M_{\gamma'} \in GL(\mathrm{vol}(\Gamma))$$

that is valid for any k $(1 \le k \le v)$, then we have a connection formula in $X \setminus X_\Gamma$

$$\gamma^* \phi_T = \begin{pmatrix} M_{\gamma'} & 0 \\ 0 & I \end{pmatrix} \phi_{T'}$$

where I is the $(\mathrm{vol}(\mathcal{A}) - \mathrm{vol}(\Gamma)) \times (\mathrm{vol}(\mathcal{A}) - \mathrm{vol}(\Gamma))$ identity matrix and $\gamma^* \phi_T$ denotes the analytic continuation of the function ϕ_T along the path γ.

Proof. Since $\gamma \subset C_Q$, we have $\gamma^* \phi_Q = \phi_{Q'}$. We also have $\alpha^* \phi_P = \phi_P$ and $\beta^* \phi_{P'} = \phi_{P'}$, because of $\alpha \subset C_P \cap X$ and $\alpha' \subset C_{P'} \cap X$. Moreover, we have

$$\begin{aligned} &\alpha'^{-1^*} (\gamma^* \phi_P - M_{\gamma'} \phi_{P'}) \\ =& (\gamma \circ \alpha'^{-1})^* \phi_P - M_{\gamma'} \phi_{P'} \\ =& (\alpha^{-1} \circ \gamma')^* \phi_P - M_{\gamma'} \phi_{P'} \\ =& \gamma'^* \phi_P - M_{\gamma'} \phi_{P'}. \end{aligned}$$

It follows from Corollary 1.1 and the assumption that we have $\gamma'^* \phi_P - M_{\gamma'} \phi_{P'} = 0$. Therefore we have $\gamma^* \phi_P - M_{\gamma'} \phi_{P'} = 0$. $\square$

2. Triangulations of the general prism $\Delta_1 \times \Delta_{n-1}$ ([BFS])

We have used the symbols n ,α and p_i and have fixed the meaning of them in Section 1, but readers should forget the fixed meaning from this section.

The k-simplex Δ_k is the convex hull of

$$e_1, \ldots, e_{k+1}$$

where e_i denotes the i-th standard basis vector of ${}^{k+1}$. We consider the general prism $\Delta_1 \times \Delta_{n-1}$ in ${}^2 \oplus {}^n = {}^{2+n}$ of which vertices are

$$e_i \oplus e_j, \quad (i = 1, 2, 1 \le j \le n, e_i \in {}^2, e_j \in {}^n).$$

By a triangulation of $\mathcal{A}_n = \{e_i \oplus e_j\}$, we mean a triangulation of $\Delta_1 \times \Delta_{n-1}$ with vertices in $\mathcal{A}_n$.

Let

$$\tau^{(i)} = \{(1,1), (1,2), \ldots, (1, n-i+1), (2, n-i+1), (2, n-i+2), \ldots, (2, n)\}$$

be the n-simplex where (p,q) denotes the point $e_p \oplus e_q$. The collection

$$T = \{\tau^{(1)}, \dots, \tau^{(n)}\}$$

is a triangulation of $\mathcal{A}_n$ and will be cited as the stair-case triangulation. The n-simplex $\tau^{(i)}$ is often figured, for example in case of $n = 4$, as follows;

$$\tau^{(1)} = \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad \tau^{(2)} = \begin{pmatrix} * & * & * & 0 \\ 0 & 0 & * & * \end{pmatrix}, \tau^{(3)} = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & * & * \end{pmatrix}, \tau^{(4)} = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & * \end{pmatrix}.$$

Let us note that the general prism $\Delta_1 \times \Delta_{n-1}$ admits the action of the group of all permutations of n-letters $\mathfrak{S}_n$;

$$\sigma : \Delta_1 \times \Delta_{n-1} \ni e_i \oplus e_j \longmapsto e_i \oplus e_{\sigma(j)} \in \Delta_1 \times \Delta_{n-1}, \quad \sigma \in \mathfrak{S}_n.$$

Therefore, we can get a new triangulation T^σ by the action of the group $\mathfrak{S}_n$ from the stair-case triangulation T. In case of the general prism, we can obtain all other triangulations of $\mathcal{A}_n$ by the procedure above. Furthermore, they are *regular* triangulations in the sense of [GZK1;3A] (see also [BFS;2]).

Proposition 2.1 ([BFS;5]) *The set of all triangulations of $\mathcal{A}_n$ is the set $\{T^\sigma \,|\, \sigma \in \mathfrak{S}_n\}$. Furthermore, each triangulation T^σ is a regular triangulation.*

More precisely, we have the following by specializing the result of [GZK1; Theorem 3A.5] and [BFS; Lemma 5.2 and example 5.6]. Refer to [BFS] for secondary polytope.

Proposition 2.2 ([GZK1], [BFS]) *The secondary polytope $\Sigma(\mathcal{A}_n)$ is the $(n-1)$-dimensional zonotope which is the Minkowskii sum of $\binom{n}{2}$ segments. The $n!$ vertices of the secondary polytope $\Sigma(\mathcal{A}_n)$ are in one-to-one correspondence with the regular triangulations $\{T^\sigma \,|\, \sigma \in \mathfrak{S}_n\}$.*

3. The $\mathcal{A}$-hypergeometric system of the general prism

Using a symmetry of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric system, we will explicitly give series solutions and compute the integral supporting functions.

Put $\chi_{ij} = e_i \oplus e_j, \quad (i = 1,2, j = 1,\dots,n)$ and we regard χ_{ij} as the column vector in $^{2+n}$. Put

$$\chi = (\chi_{11}, \chi_{12}, \dots, \chi_{1n}, \chi_{21}, \dots, \chi_{2n})$$

which is the $(2+n) \times 2n$ matrix. For example,

$$\chi = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

in case of $n = 2$. Let α_i $(i = 1, \ldots, n), \beta_1, \beta_2$ be complex numbers which satisfy

$$\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{2} \beta_i = n.$$

The $\mathcal{A}$-hypergeometric system of the general prism is the system of partial differential equations

$$p_i f = 0, \quad (i = 1, \ldots, n+2) \tag{3.1}$$

$$\triangle_a f = 0, \quad a \in \mathrm{Ker}\chi \cap {}^{2n} \tag{3.2}$$

where

$$\begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_{2+n} \end{pmatrix} = \chi \begin{pmatrix} u_{11}\partial_{11} \\ u_{12}\partial_{12} \\ \cdot \\ \cdot \\ u_{1n}\partial_{1n} \\ u_{21}\partial_{21} \\ \cdot \\ \cdot \\ u_{2n}\partial_{2n} \end{pmatrix} - \begin{pmatrix} -\beta_1 \\ -\beta_2 \\ \alpha_1 - 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n - 1 \end{pmatrix}, \tag{3.3}$$

$$\triangle_a = \prod_{a_{ij}>0} \partial_{ij}^{a_{ij}} - \prod_{a_{ij}<0} \partial_{ij}^{-a_{ij}}, \quad a = (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}) \in {}^{2n}, \tag{3.4}$$

u_{ij} $(i = 1, 2, j = 1, \ldots, n)$ are variables and $\partial_{ij} = \frac{\partial}{\partial u_{ij}}$.

Let χ' be the $(1+n) \times 2n$ matrix obtained by dropping the first row of the matrix χ. The $\mathcal{A}$-hypergeometric system (3.1) and (3.2) is nothing but the $\mathcal{A}$-hypergeometric system defined by the matrix χ', but we use the redundant operator p_1 and the redundant parameter $-\beta_1$ to keep a symmetry of the system.

In the previous section, we have seen that the group $\mathfrak{S}_n$ acts on the general prism $\Delta_1 \times \Delta_{n-1}$. The group also acts on the solutions of the $\mathcal{A}$-hypergeometric system of the general prism.

PROPOSITION 3.1 *Let $f(\alpha_1, \ldots, \alpha_n; \beta_1, \beta_2; u)$ be a solution of the $\mathcal{A}$-hypergeometric system* (3.1) *and* (3.2). *Then the function*

$$f^\sigma = f(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}; \beta_1, \beta_2; u^\sigma)$$

is also the solution of the system where

$$u^\sigma = \begin{pmatrix} u_{1\sigma(1)} & \cdots & u_{1\sigma(n)} \\ u_{2\sigma(1)} & \cdots & u_{2\sigma(n)} \end{pmatrix}.$$

In [GZK2;1.1], they constructed the fundamental sets of series solutions of the $\mathcal{A}$-hypergeometric system determined by the regular triangulations of $\mathcal{A}$. We follow their construction in the case of the general prism. For the simplex $\tau^{(k)} \in T$, let

$$\gamma^{(k)} = {}^t(\gamma_{11}^{(k)}, \ldots, \gamma_{1n}^{(k)}, \gamma_{21}^{(k)}, \ldots, \gamma_{2n}^{(k)}) \in {}^{2n}$$

be the unique column vector which satisfies the linear equation

$$\chi\gamma^{(k)} = {}^t(-\beta_1, -\beta_2, \alpha_1 - 1, \ldots, \alpha_n - 1)$$

and the constraint

$$\gamma_{ij}^{(k)} = 0 \quad \text{when } (i,j) \notin \tau^{(k)}.$$

Put

$$\phi_k = u^{\gamma^{(k)}} \sum_{p \in \mathrm{Ker}\chi \cap {}^{2n}} u^p / \Gamma(\gamma^{(k)} + p + 1), \quad k = 1, \ldots, n \tag{3.5}$$

where

$$u^{\gamma^{(k)}} = \prod_{i,j} u_{ij}^{\gamma_{ij}^{(k)}}, \quad u^p = \prod_{i,j} u_{ij}^{p_{ij}}$$

and

$$\Gamma(\gamma^{(k)} + p + 1) = \prod_{i,j} \Gamma(\gamma_{ij}^{(k)} + p_{ij} + 1).$$

It follows from [GZK2;Theorem 3 and 5] that $\{\phi_k\}$ is a base of holomorphic solutions of the hypergeometric system (3.1) and (3.2) at a generic point when the T-non-resonant condition is satisfied ([GZK2; Def 3]). The existence of a common domain of convergence of $\{\phi_k\}$ is proved by using the "regularity" of the triangulation in [GZK2;Proposition 2]. Let us state the T-non-resonant condition of the $\mathcal{A}_n$-hypergeometric system.

Proposition 3.2 *If*

$$\sum_{i \in I} \alpha_i + \beta_j \notin \tag{3.6}$$

for any j and for any subset I of $\{1, \ldots, n\}$ such that $1 \le |I| \le n/2$, then the n lattices

$$\gamma^{(k)} + (\ker \chi \cap {}^{2n}), \quad k = 1, \ldots, n$$

are disjoint each other (T-non-resonant) and $\{\phi_k\}$ is the fundamental set of solutions.

Finally, we give the integral supporting functions of the cone defined by the set of points $\Delta_1 \times \Delta_{n-1} \setminus \{(1,n)\}$.

Proposition 3.3 *The cone* $(\Delta_1 \times \Delta_{n-1} \setminus \{(1,n)\}) \cup \{0\}$ *has* $n+2$ *facets and the integral supporting functions of them are*

$$s_i \ (i=1,\ldots,n), t_1, t_2 - s_n.$$

where $\sum_{i=1}^n s_i - t_1 - t_2 = 0$.

4. Simply connected domains in ^{2n}

In most of the literatures, the connection formulas of the Gauss hypergeometric function are given in the upper half plane and in the lower half plane. In order to give connection formulas of the $\mathcal{A}$-hypergeometric system of the general prism in a similar way, we need to decompose u-space ^{2n} into "nice" simply connected domains. Connection formulas can also be given as a representation of the fundamental groupoid of the domain of the definition of the hypergeometric function. Unfortunately, it is a difficult problem to explicitly obtain the fundamental groupoid. We can avoid to study the fundamental groupoid by finding a decomposition into simply connected domains. Notice that the problem of finding a "nice" decomposition was first considered in [Sek1] in order to give connection formulas for the zonal spherical systems.

Let $\{\theta_{ij} \,|\, i=1,2, j=1,\ldots,n\}$ be a set of coordinates of ^{2n} and consider an arrangement $\mathcal{S}'$ of lines defined by

$$\begin{cases} \theta_{ij} = -\pi, 0, \pi, \\ \theta_{1i} - \theta_{2i} = \pm k\pi, \quad (k=0,1,2) \\ (\theta_{1i} - \theta_{2i}) - (\theta_{1j} - \theta_{2j}) = \pm k\pi, \quad (k=0,1,2,3,4). \end{cases}$$

We denote by $\mathcal{S}$ the set of $2n$-cells of $\mathcal{S}'$ which are contained in the domain $\{(\theta_{ij}) \,|\, -\pi < \theta_{ij} < \pi\}$. For an element S of $\mathcal{S}$, we put

$$D(S) = \{(r_{ij}e^{i\theta_{ij}}) \in {}^{2n} \,|\, r_{ij} > 0, (\theta_{ij}) \in S\} \subset \{(u_{ij})\} = {}^{2n}.$$

The set $\{D(S) \,|\, S \in \mathcal{S}\}$ has the following properties.

Proposition 4.1

(1) *The domain* $D(S)$ *is connected and simply connected.*

(2) *There is no singularity of the* $\mathcal{A}$*-hypergeometric system of the general prism* (3.1) *and* (3.2) *in* $D(S)$.

(3) *The set* $\{D(S) \,|\, s \in \mathcal{S}\}$ *is invariant under the action of* $\mathfrak{S}_n$*;*

$$(u_{ij}) \longmapsto (u_{i\sigma(j)}), \quad \sigma \in \mathfrak{S}_n.$$

(4) *The functions*

$$\frac{[ij]^a}{(-[ij])^a} \tag{4.1}$$

and

$$(4.2)\qquad [ij]^a\frac{u_{1i}^a u_{2j}^a}{u_{1j}^a u_{2i}^a},\quad (i\neq j)$$

are constant on $D(S)$. Here, we define

$$[ij]=\frac{u_{1j}u_{2i}}{u_{1i}u_{2j}}$$

and regard z^a as "the single valued" function on $\setminus(-\infty,0)$ which is the analytic continuation of the function $e^{a\log z}$ on $z>0$. Moreover, the value of (4.1) on $D(S)$ is $e^{\pi ia}$ or $e^{-\pi ia}$ and that of (4.2) is $e^{-4\pi ia}$ or $e^{-2\pi ia}$ or 1 or $e^{2\pi ia}$ or $e^{4\pi ia}$.

Proof. (1) The domain $D(S)$ is homeomorphic to the direct product of the simply connected domains S and $^{n}_{>0}$. Therefore, $D(S)$ is simply connected.

(2) Since the singularity of the $\mathcal{A}$-hypergeometric system of the general prism is

$$\prod_{i=1,2,j=1,\dots,n} u_{ij}\prod_{1\le k<\ell\le n}\begin{vmatrix}u_{1k} & u_{1\ell}\\ u_{2k} & u_{2\ell}\end{vmatrix}=0,$$

we get (2).

(3) is easily shown from the definition of $\mathcal{S}$.

(4) Let us notice that the following formulas hold;

$$(4.3)\qquad x^\mu=c_1(\mu;x)(-x)^\mu,$$

$$(4.4)\qquad (xy)^\mu=c_2(\mu;x,y,xy)x^\mu y^\mu,$$

where

$$c_1(\mu;a)=\begin{cases}e^{\pi i\mu} & \text{when }\ \operatorname{Im}a>0\\ e^{-\pi i\mu} & \text{when }\ \operatorname{Im}a<0,\end{cases}$$

and

$$c_2(\mu;a,b,c)=\begin{cases}e^{-2\pi i\mu} & \text{when }\operatorname{Im}a>0,\operatorname{Im}b>0,\operatorname{Im}c<0\\ e^{2\pi i\mu} & \text{when }\operatorname{Im}a<0,\operatorname{Im}b<0,\operatorname{Im}c>0\\ 1 & \text{in the other cases.}\end{cases}$$

(4) follows easily from the formulas (4.3) and (4.4) and a case by case study. □

Now, we can define an analytic continuation of the solution ϕ_k (3.5). Consider the intersection of the domain $D(S)$ and the domain of convergence of ϕ_k. The intersection is non-empty, connected and simply-connected. Therefore, the function ϕ_k has the unique analytic continuation to the larger domain $D(S)$. The analytic continuation is denoted by φ_k.

5. Connection formulas among the series solutions corresponding to the regular triangulations of the general prism

Let ϕ_k be the solution (3.5) of the $\mathcal{A}$-hypergeometric system of the general prism. The function φ_k is the analytic continuation of the function ϕ_k to the domain $D(S),\ S \in \mathcal{S}$ (Section 4) and is the single valued function on $D(S)$. Put $\Phi = {}^t(\varphi_1, \ldots, \varphi_n)$ and $\Phi^\sigma = {}^t(\varphi_1^\sigma, \ldots, \varphi_n^\sigma), \quad \sigma \in \mathfrak{S}_n$. Notice that φ_k^σ is the unique analytic continuation of the series ϕ_k^σ. Since the functions Φ and Φ^σ are fundamental sets of solutions of the $\mathcal{A}$-hypergeometric system of the general prism, there exists the connection matrix $C(\sigma)$ such that

$$\Phi = C(\sigma)\Phi^\sigma \quad \text{on } D(S).$$

For a given $\tau \in \mathfrak{S}_n$, we have

$$\Phi^\tau = C(\sigma)^\tau \Phi^{\sigma\tau}$$

where

$$C(\sigma)^\tau = C(\sigma)_{|\alpha_i \mapsto \alpha_{\tau(i)}, u_{ij} \mapsto u_{i\tau(j)}}.$$

Then we have

$$\Phi = C(\tau)C(\sigma)^\tau \Phi^{\sigma\tau} = C(\sigma\tau)\Phi^{\sigma\tau},$$

which means

$$C(\sigma\tau) = C(\tau)C(\sigma)^\tau. \tag{5.1}$$

The relation (5.1) is the 1-cocycle property. In order to obtain the general connection matrix $C(\sigma)$, we need to explicitly get $C(s_i)$ only for the generators $s_i = (i, i+1)$ of the group $\mathfrak{S}_n$ by virtue of the 1-cocycle property. Notice that T^{s_i} $(i = 1, \ldots, n-1)$ is the triangulation obtained from T by the restructuring s_i ([GZK1;3A.7]).

The connection formula is given by the following recurrence formula with respect to the dimension of the general prism.

THEOREM 5.1 *Assume the non-resonant condition* (3.6) *and the condition* $\alpha_i - 1, -\beta_j \notin {}_{\geq 0}$. *Define* $p \times p$ *matrix* C_p *by the recurrence relations*

$$C_p(s_i; \alpha_1, \ldots, \alpha_p; \beta_1, \beta_2; 1, \ldots, p) = 1 \oplus C_{p-1}(s_i; \alpha_1, \ldots, \alpha_{p-1}; \beta_1, \beta_2 + \alpha_p - 1; 1, \ldots, p-1)$$

for $1 \leq i < p-1$,

$$C_p(s_{p-1}; \alpha_1, \ldots, \alpha_p; \beta_1, \beta_2; 1, \ldots, p) = C_{p-1}(s_{p-2}; \alpha_2, \ldots, \alpha_p; \beta_1 + \alpha_1 - 1, \beta_2; 2, \ldots, p) \oplus 1$$

and

$$C_2(s_1; \alpha_1, \alpha_2; \beta_1, \beta_2; i, j) = \begin{pmatrix} q_{ij}(\alpha_1, \beta_2) & q_{ij}(-\beta_2, -\alpha_1) \\ q_{ij}(-\beta_1, -\alpha_2) & q_{ij}(\alpha_2, \beta_1) \end{pmatrix}$$

where q_{ij} is the Heaviside function on the arrangement $\mathcal{S}$ defined by

$$q_{ij}(\alpha_1,\beta_2)=\frac{\frac{[ij]^{-2\alpha_1}}{(-[ij])^{-2\alpha_1}}-1}{\frac{[ij]^{-2\alpha_1-2\beta_2}}{(-[ij])^{-2\alpha_1-2\beta_2}}-1}[ij]^{-\beta_2}\frac{u_{1j}^{\beta_2}u_{2i}^{\beta_2}}{u_{1i}^{\beta_2}u_{2j}^{\beta_2}},\quad [ij]=\frac{u_{1j}u_{2i}}{u_{1i}u_{2j}}.$$

Then, the matrix

$$C_n(s_i;\alpha_1,\ldots,\alpha_n;\beta_1,\beta_2;1,\ldots,n)$$

is the connection matrix among the set of solutions Φ and Φ^{s_i}.

The theorem 5.1 tells us that the connection matrices of the $\mathcal{A}$-hypergeometric system of the general prism $\Delta_1\times\Delta_{n-1}$ can be obtained from the connection matrices of the $\mathcal{A}$-hypergeometric system of $\Delta_1\times\Delta_{n-2}$. Repeating the process recursively, we can express the connection matrices in terms of the connection matrix of the $\mathcal{A}$-hypergeometric system of $\Delta_1\times\Delta_1$, which is essentially the Gauss hypergeometric equation.

We notice that the connection formulas of the q-analogue of the Lauricella function F_D has already been obtained as a multiplicative 1-cocycle of the permutation group ([AKM]). The authors think that it is an interesting problem to derive our connection formulas by putting $q=1$ or by using the connection formulas of F_D realized as a multiplicative 1-cocycle of the Braid group ([MSTY;5]).

6. Proof

In this section, we will give a proof to the recurrence formula of the connection matrices of the $\mathcal{A}$-hypergeometric function of the general prism $\Delta_1\times\Delta_{n-1}$. Our proof utilizes the formal restriction of $M_{\mathcal{A}}$ defined by a regular polyhedral subdivision and the existence of the recurrence stands for the fact that the formal restriction is essentially the $\mathcal{A}$-hypergeometric system of the smaller general prism $\Delta_1\times\Delta_{n-2}$.

First, put

$$\Gamma=\mathcal{A}_n\setminus\{(1,n)\}\quad \mathcal{A}_n=\Delta_1\times\Delta_{n-1}.$$

Then

$$\mathrm{conv}(\Gamma)\cup\mathrm{conv}(\{(1,n),(2,1),\ldots,(2,n)\})\tag{6.1}$$

is a regular polyhedral subdivision of $\mathcal{A}_n$. Notice that $\mathrm{conv}(\Gamma)$ is the cone over $\Delta_1\times\Delta_{n-2}$. Consider the formal restriction M_Γ. Since the system of differential equations of M_Γ contains the equation

$$u_{2n}\partial_{2n}f=(\alpha_n-1)f,$$

any solution f of M_Γ can be written as

$$f=u_{2n}^{\alpha_n-1}f'(u_{1,1},\ldots,u_{1,n-1},u_{2,1},\ldots,u_{2,n-1}).$$

Inserting the expression above into the system of differential equations of M_Γ, we have the following proposition.

PROPOSITION 6.1 *The differential equation for f' is the $\mathcal{A}$-hypergeometric system of $\Delta_1 \times \Delta_{n-2}$ with the parameters*

$$ {}^t(-\beta_1, -(\beta_2+\alpha_n-1), \alpha_1-1, \ldots, \alpha_{n-1}-1). $$

Figure `zono4.ps`

About the figure: Figure of the secondary polytope $\Sigma(\Delta_1 \times \Delta_{4-1})$. The hexagon 1234-1324-3124-3214-2314-2134 is the face of the secondary polytope and stands for the regular polyhedral subdivision (6.1). The vertex σ corresponds to the solution Φ^σ. Theorem 5.1 gives the connection formulas among the solutions standing for the vertices.

Secondly, we have the following proposition from Theorem 1.3, Proposition 3.2, Proposotion 3.3 and Theorem 4 of [GZK2] on the characteristic variety of the $\mathcal{A}$-hypergeometric system.

PROPOSITION 6.2 *If the non-resonant condition* (3.6) *and the condition $\alpha_i, \beta_j \notin$ are satisfied, then we have the isomorphism*

$$ r:\ j^{-1}\mathcal{H}om_{\mathcal{D}_\mathcal{A}}(M_\mathcal{A}, \mathcal{O}_\mathcal{A}) \simeq \mathcal{H}om_{\mathcal{D}_\Gamma}(M_\Gamma, \mathcal{O}_\Gamma). $$

on $D_n \cap X_\Gamma$ where

$$ D_n = \left\{ u \in {}^{2n} \,\middle|\, \prod_{i=1,2,j=1,\ldots,n,(i,j)\neq(1,n)} u_{ij} \prod_{1\le k<\ell\le n} \begin{vmatrix} u_{1k} & u_{1\ell} \\ u_{2k} & u_{2\ell} \end{vmatrix} \neq 0 \right\} $$

and $X_\Gamma = \{u \in {}^{2n} \,|\, u_{1n} = 0\}$.

Finally, we give a proof of Theorem 5.1. We can easily apply Theorem 1.4 for our case, because the domain of the definition of the $\mathcal{A}_n$-hypergeometric function $D_n \setminus X_\Gamma$ has relatively simple topological structure.

Proof of Theorem 5.1. We prove the theorem by the induction on n. The case $n = 2$ can be easily shown by the connection formula of the Gauss hypergeometric function (see e.g. 14.51 of [WW]), which can be proved, for example, by using the Barnes integral representation.

Let us consider the case $1 \leq i < n-1$. Since $\varphi_1 = \varphi_1^{s_i}$, we consider the difference of the functions

$$\delta = \begin{pmatrix} \varphi_2 \\ \cdot \\ \cdot \\ \varphi_n \end{pmatrix} - C_{n-1}(s_i; \alpha_1, \ldots, \alpha_{n-1}; \beta_1, \beta_2 + \alpha_n - 1; 1, \ldots, n-1) \begin{pmatrix} \varphi_2^{s_i} \\ \cdot \\ \cdot \\ \varphi_n^{s_i} \end{pmatrix}$$

on $D(S)$. Let C_P be the domain of convergence of the vector of series $(\phi_2, \ldots, \phi_n)$ and C_Q be the domain of convergence of the series ϕ_1. We can easily see that $C_P \cap X_\Gamma \neq \emptyset$ and $C_{P'} \cap X_\Gamma \neq \emptyset$ $(C_{P'} = (C_P)^{s_i})$. We can take paths α, α' and γ of which interior lie in the simply connected domain $D(S)$ and which satisfy the condition (1.5). The path $\alpha \circ \gamma \circ {\alpha'}^{-1}$ can be continuously deformed into a path in

$$D'(S) = \{(r_{ij} e^{i\theta_{ij}}) \in {}^{2n} \,|\, r_{ij} > 0,\ (i,j) \neq (1,n),\ (\theta_{ij}) \in S, r_{1n} = 0\} \subset X_\Gamma.$$

It follows from the induction hypothesis, Proposition 6.1, Proposition 6.2 and Theorem 1.4 that we have $\delta = 0$ on $D(S)$.

The case of s_{n-1} can be shown in a similar way. In this case, we restrict the difference to $u_{21} = 0$. $\Box$

Appendix

We will give a proof of Theorem 1.1 in this appendix. Throughout the appendix, ^{d} means the lattice Γ and the readers are advised to forget the lattice $\mathcal{A} \simeq {}^d \supset \Gamma$.

We denote by Λ the semigroup

$$_{\geq 0}\Gamma = \sum_{i=1}^{m} {}_{\geq 0} a_i$$

and we put

$$\Lambda' = \{\chi \in {}^d \,|\, F_\tau(\chi) \geq 0,\ \forall \tau \in \mathcal{F} := \mathcal{F}_\Gamma\}.$$

The following lemma is the key to prove Theorem 1.1 without the normality condition.

Lemma 1 *There exists $\chi' \in \Lambda$ such that*

$$\chi' + \Lambda' \subset \Lambda \subset \Lambda'.$$

In order to prove the lemma, we need preparatory lemmas.

There exist $\mu_1, \ldots, \mu_k \in \Lambda'$ such that

$$\Lambda' = \sum_{i=1}^{k} {}_{\geq 0}\mu_i + \Lambda$$

by Gordan's lemma. We fix such $\mu_1, \ldots, \mu_k$. Since $\mu_i \in \Gamma$, we have the following lemma.

LEMMA 2 *For any i, there exists $m_i \in {}_{\geq 0}$ such that $m_i \mu_i \in \Lambda$.*

We fix such m_i. We define a function $h(\lambda)$ on ^{d} which characterizes the semigroups Λ and Λ'. The function $h(\lambda)$ is defined by

$$h(\lambda) = \max_{\lambda = \sum_{j=1}^{m} \ell_j a_j,\ \ell_j \in} \left(\min_j (\ell_j) \right).$$

LEMMA 3

(1) *$h(\lambda) \geq 0$ if and only if $\lambda \in \Lambda$.*

(2) *For any $\lambda, \mu \in {}^d$, we have*

$$h(\lambda + \mu) \geq h(\lambda) + h(\mu).$$

Proof. The statement (1) can be easily shown from the definition. Let us prove (2). Take representations

$$\lambda = \sum_{i=1}^{m} \ell_i a_i, \quad \mu = \sum_{i=1}^{m} \ell'_i a_i$$

that attain the value $h(\lambda)$ and $h(\mu)$ and assume $h(\lambda) = \min(\ell_i)$ and $h(\mu) = \min(\ell'_i)$. It follows from the definition that we have

$$h(\lambda + \mu) \geq \min(\ell_i + \ell'_i) \geq \min(\ell_i) + \min(\ell'_i) = h(\lambda) + h(\mu). \quad \square$$

LEMMA 4 *There exists a number $c \in {}_{\geq 0}$ such that $h(\lambda) \geq -c$ for all $\lambda \in \Lambda'$.*

Proof. Put

$$-c = \min_{0 \leq d_i < m_i,\ i} h\left(\sum_{i=1}^{k} d_i \mu_i \right).$$

Take an element λ of $\sum_{i=1}^{k} {}_{\geq 0}\mu_i + \Lambda$. The vector λ can be expressed as

$$\lambda = \sum_{i=1}^{k} \ell_i \mu_i + \sum_{j=1}^{m} n_j a_j$$

where $\ell_i, a_j \in {}_{\geq 0}$ and

$$\ell_i = p_i m_i + q_i, \quad q \leq q_i < m_i.$$

We have

$$\begin{aligned} h(\lambda) &\geq h\left(\sum_{i=1}^{k} \ell_i \mu_i\right) && \text{(Lemma 3)} \\ &\geq h\left(\sum_{i=1}^{k} q_i \mu_i\right) && \text{(Lemma 2 and 3)} \\ &\geq -c. \quad \square \end{aligned}$$

Now, we are ready for proving the key lemma.

Proof of Lemma 1. The inclusion $\Lambda \subset \Lambda'$ is trivial. Put $\chi' = c \sum_{i=1}^{m} a_i$ where the number c is the one in Lemma 4. Take an arbitrary vector $\lambda \in \Lambda'$. Then we have

$$h(\chi' + \lambda) \geq h(\chi') + h(\lambda) \geq c - c \geq 0$$

by Lemma 3 and 4. Lemma 3 (1) yields $\chi' + \lambda \in \Lambda$. $\square$

We start to prove Theorem 1.1 by utilizing Lemma 1. It is enough for proving it to show that the left ideal of the Weyl algebra generated by ∂^{χ}, $q_i(\alpha)$ $(i = 1, \ldots, d)$ and $\triangle_r$ $(r \in \ker(a_1, \ldots, a_m) \cap {}^{d})$ is the trivial ideal generated by 1. We will show this fact step by step and begin with defining a decomposition of an ideal.

Let W be the Weyl algebra $\langle x_1, \ldots, x_m, \partial_1, \ldots, \partial_m \rangle$. We denote by I' the left ideal of W generated by $\triangle_r$ $(r \in \ker(a_1, \ldots, a_m) \cap {}^{m})$ and by $I'(\chi_0)$ the left ideal generated by I' and ∂^{χ_0}.

Similarly, the left ideal of W generated by $\triangle_r$ $(r \in \ker(a_1, \ldots, a_m) \cap {}^{m})$ and ∂^{χ} $(F_\tau(\chi) \geq F_\tau(\chi_0))$ is denoted by $I'(\tau, \chi_0)$.

Proposition 1 (cf. [S1; Proposition 4.3]) *For any* $\chi_0 \in \Lambda$, *we have*

$$\cap_{\tau \in \mathcal{F}} I'(\tau, \chi_0 + \chi') \subset I'(\chi_0) \subset \cap_{\tau \in \mathcal{F}} I'(\tau, \chi_0).$$

Proof. Let π be the natural projection of W onto W/I'. Clearly we have $I'(\chi_0) \subset \cap_{\tau \in \mathcal{F}} I'(\tau, \chi_0)$. As $[x_1, \ldots, x_m]$-modules, W/I', $\pi(I'(\tau, \chi_0))$ and $\pi(I'(\chi_0))$ are free modules with basis $\{\, \partial^{\chi} \,|\, \chi \in \Lambda \,\}$, $\{\, \partial^{\chi} \,|\, F_\tau(\chi) \geq F_\tau(\chi_0) \,\}$ and $\{\, \partial^{\chi_0 + \chi} \,|\, \chi \in \Lambda \,\}$ respectively. Hence we have $\cap_{\tau \in \mathcal{F}} \pi(I'(\tau, \chi_0 + \chi')) \subset \pi(I'(\chi_0))$ by Lemma 1. Since we have $\pi^{-1}\pi(I'(\chi_0)) = I'(\chi_0)$ and $\pi^{-1}\pi(I'(\tau, \chi_0 + \chi')) = I'(\tau, \chi_0 + \chi')$, we conclude $\cap_{\tau \in \mathcal{F}} I'(\tau, \chi_0 + \chi') \subset I'(\chi_0)$. $\square$

We denote by $W[s]$ the ring $[s_1, \ldots, s_d] \otimes W$ where each s_i commutes with s_j, x_j and ∂_j. Let I be the left ideal of $W[s]$ generated by $q_i(s)$ $(i = 1, \ldots, d)$ and $\triangle_r$ $(r \in \ker(a_1, \ldots, a_m) \cap {}^d)$. Similarly, the left ideal generated by $q_i(s)$ $(i = 1, \ldots, m)$ and $I'(\chi_0)$ is denoted by $I(\chi_0)$ and the left ideal generated by $q_i(s)$ $(i = 1, \ldots, d)$ and $I'(\tau, \chi_0)$ by $I(\tau, \chi_0)$.

By using a similar argument to [S1; Proposition 4.4], we have the following decomposition.

PROPOSITION 2 *For any $\chi_0 \in \Lambda$, we have*

$$\cap_{\tau \in \mathcal{F}} I(\tau, \chi_0 + \chi') \subset I(\chi_0) \subset \cap_{\tau \in \mathcal{F}} I(\tau, \chi_0).$$

Put $B(\chi_0) = [s] \cap I(\chi_0)$ and $B(\tau, \chi_0) = [s] \cap I(\tau, \chi_0)$. The polynomials in $B(\chi_0)$ are called b-functions with respect to χ_0 (cf. [S1; 5]). Since the left ideal $I(\chi_0)$ is generated by $q_i(s)$ $(i = 1, \ldots, d)$, $\triangle_r$ $(r \in \ker(a_1, \ldots, a_m))$ and ∂^{χ_0}, we have the following proposition from the definition of $B(\chi_0)$.

PROPOSITION 3 (cf. [S1; Corollary 5.4]) *If there exists a polynomial $b(s) \in B(\chi_0)$ such that $b(\beta) \neq 0$, then the morphism*

$$f_{\chi_0} \ : \ M_\Gamma(\beta - \chi_0) \longrightarrow M_\Gamma(\beta)$$

is isomorphic.

It follows from Proposition 2 and the definitions of $B(\chi_0)$ and $B(\tau, \chi_0)$ that the ideal $B(\chi_0)$ can be characterized as follows.

PROPOSITION 4 *For any $\chi_0 \in \Lambda$, we have*

$$\cap_{\tau \in \mathcal{F}} B(\tau, \chi_0 + \chi') \subset B(\chi_0) \subset \cap_{\tau \in \mathcal{F}} B(\tau, \chi_0).$$

The proposition above admits us to study the ideals $B(\chi_0)$ by using simpler ideals $B(\tau, \chi_0 + \chi')$.

The final thing we have to do to prove Theorem 1.1 is to find a nice element in the ideal $B(\tau, \chi_0 + \chi')$. Fortunately, by virtue of [S1; Proposition 5.6 and Lemma 6.1], we have the nice element as follows.

PROPOSITION 5

$$\prod_{m=0}^{F_\tau(\chi_0 + \chi')} (F_\tau(s) - m) \in B(\tau, \chi_0 + \chi').$$

Now, we can complete the proof of Theorem 1.1. In fact, since we have

$$\prod_{\tau\in\mathcal{F}} \prod_{m=0}^{F_\tau(\chi_0+\chi')} (F_\tau(s)-m) \in \cap_{\tau\in\mathcal{F}} B(\tau,\chi_0+\chi') \subset B(\chi_0)$$

from Propositions 4 and 5 and since $F_\tau(\alpha) \notin {}_{\geq 0}$ for all $\tau \in \mathcal{F}$, Proposition 3 completes the proof of Theorem 1.1.

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