Gröbner Bases and Hypergeometric Functions

Bernd Sturmfels* and Nobuki Takayama

October 8, 1997

1 Introduction

The purpose of this tutorial is to illustrate the use of Gröbner bases and Buchberger's algorithm in the algebraic study of linear partial differential equations. Our reference example is the following system of six differential equations for a function $\Phi(x_1, x_2, x_3, x_4)$ in four complex variables:

$$\begin{aligned} \frac{\partial^2\Phi}{\partial x_2 \partial x_3} &= \frac{\partial^2\Phi}{\partial x_1 \partial x_4}, & \frac{\partial^3\Phi}{\partial x_1^2 \partial x_3} &= \frac{\partial^3\Phi}{\partial x_2^3}, \\ \frac{\partial^3\Phi}{\partial x_2 \partial x_4^2} &= \frac{\partial^3\Phi}{\partial x_3^3}, & \frac{\partial^3\Phi}{\partial x_1 \partial x_3^2} &= \frac{\partial^3\Phi}{\partial x_2^2 \partial x_4}, \end{aligned} \qquad (1)$$

$$\begin{aligned} x_1\frac{\partial\Phi}{\partial x_1} + x_2\frac{\partial\Phi}{\partial x_2} + x_3\frac{\partial\Phi}{\partial x_3} + x_4\frac{\partial\Phi}{\partial x_4} &= a \cdot \Phi\,, \\ x_2\frac{\partial\Phi}{\partial x_2} + 3x_3\frac{\partial\Phi}{\partial x_3} + 4x_4\frac{\partial\Phi}{\partial x_4} &= b \cdot \Phi\,, \end{aligned}$$

where a and b are complex parameters. Experts on hypergeometric functions will recognize these equations: this is the $\mathcal{A}$-hypergeometric system of Gel'fand, Kapranov and Zelevinsky (1989) for the particular configuration

$$\mathcal{A} \quad = \quad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}.$$

We shall use Gröbner bases to prove that, for every open ball U in $\mathbf{C}^4$, the dimension of the $\mathbf{C}$-vector space of holomorphic functions Φ on U which

*Partially supported by the David and Lucile Packard Foundation

satisfy (1) is at most five. Moreover, for $(a, b) = (1, 2)$ and suitable U it is exactly five. See Propositions 2.1 and 4.1 for precise statements.

One remark for experts: our example shows that Theorem 2 in (Gel'fand et al. 1989) is incorrect without an additional Cohen-Macaulayness assumption. The statement of this important theorem was corrected in (Gel'fand et al. 1993) but no explicit counterexample was given. This tutorial shows that it is easy to construct and study counterexamples by using Gröbner bases.

In the remainder of the introduction we motivate the equations (1) by describing some special solutions: polynomial solutions, algebraic solutions, rational solutions, and solutions admitting integral representations. In the sequel, any function Φ which satisfies the differential equations (1) is called *$\mathcal{A}$-hypergeometric*. The relation to classical hypergeometric functions, mentioned in the tutorial by Chyzak (1998), can be seen in Theorem 4.1.

We first ask whether the system (1) has any polynomial solutions. A necessary condition for this is that (a, b) lies in the monoid $\mathbf{N}\mathcal{A}$ spanned by the set $\mathcal{A}$ in $\mathbf{N}^2$, because the last two equations in (1) are equivalent to

$$\Phi(sx_1, stx_2, st^3x_3, st^4x_4) \quad = \quad s^a t^b \cdot \Phi(x_1, x_2, x_3, x_4). \tag{2}$$

To see this equivalence, apply the operators $\partial/\partial s|_{s=1}$ and $\partial/\partial t|_{t=1}$ to (2). Conversely, if (a, b) lies in $\mathbf{N}\mathcal{A}$ then the following set is finite and non-empty

$$\mathcal{F}_{(a,b)} = \{(u_1, u_2, u_3, u_4) \in \mathbf{N}^4 \mid u_1 + u_2 + u_3 + u_4 = a,\ u_2 + 3u_3 + 4u_4 = b\},$$

and there is exactly one $\mathcal{A}$-hypergeometric polynomial up to scalar multiples:

$$\Phi_{(a,b)}(x) \quad = \quad \sum_{u \in \mathcal{F}_{(a,b)}} \frac{x_1^{u_1} x_2^{u_2} x_3^{u_3} x_4^{u_4}}{u_1!\, u_2!\, u_3!\, u_4!}, \tag{3}$$

which is a solution of (1).

Next, consider the zeros of the following polynomial in one variable ϕ,

$$f(\phi) \quad = \quad x_1 + x_2\phi + x_3\phi^3 + x_4\phi^4.$$

They can be written in terms of radicals using Cardano's formula. These algebraic functions are $\mathcal{A}$-hypergeometric for $(a, b) = (0, -1)$. In fact, if U is an open ball in $\mathbf{C}^4$ which does not intersect the zero set of the discriminant

$$\Delta_{\mathcal{A}} \;=\; x_1x_4 \cdot (27x_1^2x_3^4 + 4x_2^3x_3^3 + 6x_4x_1x_2^2x_3^2 + 192x_4^2x_1^2x_2x_3 + 27x_4^2x_2^4 - 256x_4^3x_1^3),$$

then the four roots of $f(\phi) = 0$ are well-defined holomorphic functions on U, and they form a $\mathbf{C}$-basis for the space of $\mathcal{A}$-hypergeometric functions on U. More generally, if m is a non-zero integer and $(a, b) = (0, -m)$ then a solution basis is given by the m-th powers of the roots $\phi_1, \phi_2, \phi_3, \phi_4$ of f.

This observation leads to a nice rational $\mathcal{A}$-hypergeometric function:

$$\phi_1^m + \phi_2^m + \phi_3^m + \phi_4^m.$$

Indeed, the m-th power sum of the roots is a Laurent polynomial in $x_1, \ldots, x_4$ which satisfies (1) with parameters $(a, b) = (0, -m)$. For instance, for $m = 1$ we get the $\mathcal{A}$-hypergeometric function x_3/x_4, and for $m = 6$ we get

$$\phi_1^6 + \phi_2^6 + \phi_3^6 + \phi_4^6 \quad = \quad (x_3^6 - 6x_1x_3^2x_4^3 + 6x_2x_3^3x_4^2 + 3x_2^2x_4^4) \, / \, x_4^6.$$

Further algebraic solutions to (1) for other integral parameter values can be found by considering residue integrals such as

$$\frac{\phi_i^m}{f'(\phi_i)} \quad = \quad \frac{1}{2\pi\sqrt{-1}} \int_{C_i} \frac{z^m}{f(z)} dz,$$

where C_i is a small, positively oriented loop around the point ϕ_i in the complex plane. This is an algebraic function of x_1, x_2, x_3, x_4, and it satisfies (1) for $(a, b) = (-1, -1-m)$. If we take a big loop C which positively encircles all four roots, then we get the rational $\mathcal{A}$-hypergeometric function

$$\frac{\phi_1^m}{f'(\phi_1)} + \frac{\phi_2^m}{f'(\phi_2)} + \frac{\phi_3^m}{f'(\phi_3)} + \frac{\phi_4^m}{f'(\phi_4)}.$$

Finally, we mention a general integral representation for $\mathcal{A}$-hypergeometric functions. If the complex parameters a and b are sufficiently generic, then every solution to (1) on an open ball $U \subset \mathbf{C}^4$ can be written as

$$\Phi(x_1, x_2, x_3, x_4) \quad = \quad \int_\Gamma f(z)^a z^{-b-1} dz \tag{4}$$

where Γ is a segment with endpoints belonging to $D = \{0, \phi_1, \phi_2, \phi_3, \phi_4\}$. If the real parts of a and $-b-1$ are less than or equal to -1, then the integral does not converge. In this case, we take the finite part of the integral in the sense of Hadamard. How many segments are linearly independent? We assume that all the roots ϕ_i are real numbers for simplicity. Among $\binom{5}{2} = 10$ segments, only four segments are independent; when all roots are ordered as

$$\phi_0 = 0 < \phi_1 < \phi_2 < \phi_3 < \phi_4,$$

the integral (4) can be expressed as a $\mathbf{C}$-linear combination of the four integrals $\int_{[\phi_i,\phi_{i+1}]} f(z)^a z^{-b-1} dz$. The segments $[\phi_i, \phi_{i+1}]$ are the compact chambers of a 1-dimensional hyperplane arrangement. They form a basis of the twisted homology group $H_1^{lf}(\mathbf{C}\backslash D, \nabla)$; see page 53 in (Aomoto and Kita 1994). Hypergeometric differential equations play an important role in algebraic geometry because for certain rational parameters (a, b) the integral (4) can be regarded as periods of a family of curves; see (Yoshida 1997).

2 Characteristic variety and holonomicity

In this section we show how to prove that the space of $\mathcal{A}$-hypergeometric functions is finite-dimensional, how to evaluate its dimension, and how to compute the characteristic variety. We use Gröbner bases in the Weyl algebra

$$W \quad = \quad k\langle x_1, x_2, x_3, x_4, \partial_1, \partial_2, \partial_3, \partial_4\rangle$$

over a field k, which in our case is either $\mathbf{Q}$ or $\mathbf{Q}(a, b)$. The Weyl algebra W is the free associative algebra over k modulo the commutation rules

$$x_i x_j = x_j x_i, \;\; \partial_i\partial_j = \partial_j\partial_i, \;\; x_i\partial_j = \partial_j x_i \text{ for } i \neq j \,, \;\text{ and } \partial_i x_i = x_i\partial_i + 1.$$

A linear system of partial differential equations with polynomial coefficients corresponds to a left ideal in W. Our system (1) corresponds to the left ideal $H_{\mathcal{A}}$ which is generated by

$$\partial_2\partial_3 - \partial_1\partial_4, \;\; \partial_1^2\partial_3 - \partial_2^3, \;\; \partial_2\partial_4^2 - \partial_3^3, \;\; \partial_1\partial_3^2 - \partial_2^2\partial_4, \tag{5}$$

$$x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - a, \;\; x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4 - b. \tag{6}$$

Here a, b are either specific numbers or indeterminate parameters. A function Φ on $U \subset \mathbf{C}^4$ is $\mathcal{A}$-hypergeometric if it is annihilated by the left W-ideal $H_{\mathcal{A}}$.

The theory (and practise) of Gröbner bases works perfectly well for left ideals in the Weyl algebra W. We quickly review the relevant basics. Every element f in W can be written uniquely as a k-linear combination of *normally ordered* monomials $x^a\partial^b = x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}\partial_1^{b_1}\partial_2^{b_2}\partial_3^{b_3}\partial_4^{b_4}$. This representation of f is called *normally ordered representation.* For example, the monomial $\partial_1 x_1\partial_1$ is not normally ordered. Its normally ordered representation is $x_1\partial_1^2 + \partial_1$.

Consider the commutative polynomial ring in eight variables

$$\mathrm{gr}\,(W) \quad = \quad k[x_1, x_2, x_3, x_4, \xi_1, \xi_2, \xi_3, \xi_4]$$

and the k-linear map $\mathrm{gr}\,: W \to \mathrm{gr}\,(W)$, $x^a\partial^b \mapsto x^a\xi^b$. Let $<$ be any term order on $\mathrm{gr}\,(W)$. This gives a total order among normally ordered monomials in W via $x^A\partial^B > x^a\partial^b \;\Leftrightarrow x^A\xi^B > x^a\xi^b$. For any element $f \in W$ let $in_<(f)$ denote the highest monomial $x^A\partial^B$ in the normally ordered representation of f. If I is a left ideal in W then its *initial ideal* is the ideal $\mathrm{gr}\,(in_<(I))$ in $\mathrm{gr}\,(W)$ generated by all monomials $\mathrm{gr}\,(in_<(f))$ for $f \in I$. Clearly, $\mathrm{gr}\,(in_<(I))$ is generated by finitely many monomials $x^a\xi^b$. A finite subset G of I is called *a Gröbner basis* of I with respect to the term order $<$ if $\{\mathrm{gr}\,(in_<(g))\,|\,g \in G\}$ generates $\mathrm{gr}\,(in_<(I))$. Noting that $in_<(f) \le in_<(g)$ implies $in_<(hf) \le in_<(hg)$ for all $h \in W$, one proves that the reduced Gröbner basis of I is unique and finite, and can be computed using Buchberger's algorithm. An efficient implementation is available in the system `kan` by Takayama (1991).

The definition of the initial ideal extends to partial term orders given by weight vectors. Of importance in algebraic analysis is the partial term order v defined on W by the weight vector $(0,0,0,0,1,1,1,1)$, that is, by the rule:

$$\begin{array}{lcl} x_1^{A_1}x_2^{A_2}x_3^{A_3}x_4^{A_4}\partial_1^{B_1}\partial_2^{B_2}\partial_3^{B_3}\partial_4^{B_4} & \quad\text{if and only if}\quad & B_1+B_2+B_3+B_4 \\ \ge x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}\partial_1^{b_1}\partial_2^{b_2}\partial_3^{b_3}\partial_4^{b_4} & & \ge b_1+b_2+b_3+b_4. \end{array}$$

The v-leading form $in_v(f)$ of an element $f \in W$ is also called the *v-principal part* of f. For an m-th order operator $f = \sum_{B_1+\cdots+B_4\le m} C_{AB}x^A\partial^B$ it equals

$$in_v(f) \quad = \sum_{B_1+\cdots+B_4=m} C_{AB}x^A\partial^B.$$

Let $\mathrm{gr}\,(in_v(I))$ be the ideal in $\mathrm{gr}\,(W)$ generated by the polynomials $\mathrm{gr}\,(in_v(f))$ for all $f \in I$. We call $\mathrm{gr}\,(in_v(I))$ the *characteristic ideal* of $I \subset W$.

The algebraic subset of affine 8-space defined by $\mathrm{gr}\,(in_v(I))$ is the *characteristic variety* of I. It can be computed as follows: let $<$ be any term order which refines the partial term order v and compute the reduced $<$-Gröbner basis G for I. Then $\mathrm{gr}\,(in_v(I))$ is generated by the polynomials $\mathrm{gr}\,(in_v(g))$ for all $g \in G$. This algorithm was introduced by Oaku (1994), (1996). For the general setting of D-modules and computation of functors for D-modules see (Borel et al. 1987), (Kashiwara 1983), (Oaku 1997) and their references.

The *Fundamental Theorem of Algebraic Analysis* (Sato et al. 1973) states that each associated prime of the characteristic ideal is at least middle-dimensional, that is, has dimension ≥ 4. If each associated prime of $\mathrm{gr}\,(in_v(I))$ has dimension exactly 4 then the left ideal I is called *holonomic*. For any connected open subset U of $\mathbf{C}^4$ let $Sol(U,I)$ denote the $\mathbf{C}$-vector space of holomorphic functions on U which are annihilated by all operators in I. If I is holonomic then $dim_{\mathbf{C}}(Sol(U,I))$ is finite, and, moreover, these dimensions have a finite maximum $r(I)$ as U ranges over all *connected* open subsets of $\mathbf{C}^4$. The integer $r(I)$ is called the *rank* of the holonomic ideal I. It is known that $dim_{\mathbf{C}}(Sol(U,I)) = r(I)$ if U is sufficiently small and does not meet the *singular locus* of I. In our example $I = H_{\mathcal{A}}$ the singular locus is the discriminant hypersurface $\Delta_{\mathcal{A}} = 0$.

We shall express the rank $r(I)$ purely algebraically. Consider the ring $R = k(x_1,\ldots,x_4)\langle\partial_1,\ldots,\partial_4\rangle$ of differential operators with rational function coefficients. Let RI be the left ideal generated by I in R. Then

$$r(I) \quad = \quad \dim_{k(x_1,\ldots,x_4)} R/RI. \tag{7}$$

The equivalence of this formula with the analytic definition of $r(I)$ and an algorithm to evaluate the dimension were announced in (Takayama 1987).

To evaluate the rank $r(I)$ we proceed as follows. Fix any block term order

$$x^A\partial^B > x^a\partial^b \quad \Longleftrightarrow \quad \partial^B >_1 \partial^b \;\text{ or }\; (\partial^B = \partial^b \text{ and } x^A >_2 x^a)$$

where $>_1$ is a term order in $k[\partial]$ that refines the partial term order by the weight vector $(1,1,1,1)$ and $>_2$ is a term order in $k[x]$. Compute the reduced Gröbner basis G of I with respect to this block order. Let $\mathrm{gr}\,(in_<(G))$ be the set of leading monomials $x^a\xi^b$ of all elements in G. This set generates the ideal $\mathrm{gr}\,(in_<(I))$ in $\mathrm{gr}\,(W)$. Now substitute $x_1 = x_2 = x_3 = x_4 = 1$ in $\mathrm{gr}\,(in_<(G))$. The remaining monomials generate a monomial ideal M in $k[\xi_1,\xi_2,\xi_3,\xi_4]$. Since G is also a Gröbner basis for RI in R, we have

$$r(I) \quad = \quad dim_k(k[\xi_1,\xi_2,\xi_3,\xi_4]/M).$$

Let us see how this algorithm works for our hypergeometric ideal $H_{\mathcal{A}}$. We select $<$ to be the reverse lexicographic refinement of v such that $\partial_1 < \ldots < \partial_4$ and $x_1 < \ldots < x_4$. We regard a and b as parameters, that is, we work over the rational function field $k = \mathbf{Q}(a,b)$. The reduced Gröbner basis G of $H_{\mathcal{A}}$ with respect to $<$ consists of 34 elements. We list only their leading terms, the 34 minimal generators of the initial ideal $\mathrm{gr}\,(in_<(H_{\mathcal{A}}))$ in $\mathrm{gr}\,(W)$:

$$\begin{gathered}
\xi_2\xi_3,\, \xi_2^3,\, \xi_3^3,\, x_3\xi_1\xi_4^2,\, x_3\xi_2\xi_4^2,\, \xi_2^2\xi_4,\, x_4\xi_4,\, x_3\xi_3,\, \underline{x_4\xi_3^2},\, x_3\xi_1\xi_2\xi_4,\, x_3^3\xi_1^2\xi_4,\, \underline{x_3^2\xi_2\xi_4},\\
x_1x_4\xi_1\xi_2^2, x_1x_3\xi_1\xi_2^2, x_1x_3^2\xi_1^2\xi_4, x_2x_3^2\xi_1^2\xi_4, x_2\xi_2^2, \underline{x_1x_3x_4\xi_2^2}, x_1x_2^2x_4\xi_1^2\xi_3, x_1^2x_4^2\xi_1^2\xi_3,\\
x_1x_2x_4^2\xi_1^2\xi_3, x_2^2x_4^2\xi_1^2\xi_3, x_1x_4^3\xi_1^2\xi_3, \underline{x_2x_3^3\xi_1\xi_4}, x_1x_2^3x_4\xi_1^2\xi_2, x_2x_3^4\xi_1^2\xi_2, x_1x_2x_3^3\xi_1^2\xi_2,\\
x_1x_2^2x_3^2\xi_1^2\xi_2, x_2^2x_3^3\xi_1^2\xi_2, x_1x_2^2x_3x_4\xi_1^2\xi_2, \underline{x_1x_2x_4^3\xi_1\xi_3}, x_1^2x_3^4\xi_1^2\xi_2, \underline{x_2^2x_3^4\xi_1\xi_2}, x_1x_2^3x_3^3\xi_1^3.
\end{gathered}$$

From the Gröbner basis G we can extract a subset of seven elements whose v-leading forms minimally generate the characteristic ideal $\mathrm{gr}\,(in_v(H_{\mathcal{A}}))$:

$$\xi_2\xi_3 - \xi_1\xi_4, \quad \xi_2^3 - \xi_1^2\xi_3, \quad \xi_2^2\xi_4 - \xi_1\xi_3^2, \quad \xi_3^3 - \xi_2\xi_4^2, \tag{8}$$

$$x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_4, \quad x_2\xi_2 + 3x_3\xi_3 + 4x_4\xi_4, \tag{9}$$

$$(b-2)x_1\xi_2^2 + (b-a-1)x_2\xi_1\xi_3 + (b-3a+1)x_3\xi_2\xi_4 + (b-4a+2)x_4\xi_3^2 \tag{10}$$

The polynomials in (8) and (9) obviously lie in $\mathrm{gr}\,(in_v(H_{\mathcal{A}}))$ since they are the v-leading forms of (5) and (6). The polynomial (10) is non-trivial and depends on the parameters (a,b). It is easy to verify that (8), (9) and (10) generate $\mathrm{gr}\,(in_v(H_{\mathcal{A}}))$: just compute their (commutative) Gröbner basis with respect to $<$ and check that you get the same 34 initial monomials as above.

The 34 initial monomials define an ideal of dimension 4. We conclude that the characteristic ideal has dimension 4, and therefore the system $H_{\mathcal{A}}$ is holonomic. To evaluate the rank of $H_{\mathcal{A}}$, we set $x_1 = x_2 = x_3 = x_4 = 1$ in these 34 monomials. The resulting ideal M has codimension 4 in $k[\xi_1, \xi_2, \xi_3, \xi_4]$:

$$M \quad = \quad \langle \xi_1^3,\ \xi_1\xi_2,\ \xi_2^2,\ \xi_3,\ \xi_4 \rangle.$$

Thus the hypergeometric ideal $H_{\mathcal{A}}$ is holonomic of rank 4 for (a,b) generic. By looking at the Gröbner basis G we can see that $\mathrm{gr}\,(in_<(H_{\mathcal{A}}))$ remains the same if a,b are specialized to numerical values, provided $(a,b) \neq (1,2)$.

For $(a,b) = (1,2)$ things are dramatically different: the polynomial (10) vanishes identically and the reduced Gröbner basis G has only 29 elements. To get the 29 monomial generators of $\mathrm{gr}\,(in_<(H_{\mathcal{A}}))$ erase the six underlined monomials above and replace them by $x_4\xi_1\xi_3^2$. Setting $x_i = 1$ we now find

$$M \quad = \quad \langle \xi_1^3,\ \xi_1^2\xi_2,\ \xi_2^2,\ \xi_3,\ \xi_4 \rangle.$$

This ideal has codimension 5 in $k[\xi_1, \xi_2, \xi_3, \xi_4]$. We summarize our results:

Proposition 2.1 *The hypergeometric ideal $H_{\mathcal{A}}$ is holonomic of rank 4 for $(a,b) \neq (1,2)$, and it is holonomic of rank 5 for $(a,b) = (1,2)$.*

We remark that the square of (10) lies in the ideal generated by (8) and (9). This shows that the radical of $\mathrm{gr}\,(in_v(H_{\mathcal{A}}))$ is independent of the parameters (a,b), and hence so is the characteristic variety. The reader will find it instructive to compute (using Gröbner bases) the radical and primary decomposition of the characteristic ideal. Hint: The characteristic variety has four irreducible components, by Theorem 4 in (Gel'fand et al. 1989).

3 Creation operators

An important issue in the study of differential equations is to find operators that create new solutions from known solutions. Many nice families of equations such as hypergeometric systems, Painlevé equations and KP equations admit such operators. In this section we discuss creation operators for the system (1), and we show how to compute them using Gröbner bases.

Suppose we are given any solution $\phi_{a,b}(x)$ to (1), for instance, one of the polynomial or algebraic functions discussed in the introduction, or one of the series solutions in Section 4. By taking partial derivatives we get new solutions for smaller parameter values indicated by subscripts:

$$\begin{aligned} \partial_1\, \phi_{a,b}(x) \;=\; \phi_{a-1,b}(x)\,, \quad \partial_2\, \phi_{a,b}(x) \;=\; \phi_{a-1,b-1}(x), \\ \partial_3\, \phi_{a,b}(x) \;=\; \phi_{a-1,b-3}(x), \quad \partial_4\, \phi_{a,b}(x) \;=\; \phi_{a-1,b-4}(x). \end{aligned} \tag{11}$$

In this context ∂_i is called the i-th *annihilation operator*. Our problem is to construct an inverse operator C_i, called a *creation operator*, which takes solutions of low parameter values to solutions of higher parameter values.

Saito et al. (1997) showed that the creation operators have the form $C_i = r_i/B_i(a,b)$, where r_i is an element of $W = \mathbf{Q}\langle x_1, x_2, x_3, x_4, \partial_1, \partial_2, \partial_3, \partial_4\rangle$ and B_i is a product of linear polynomials in a and b with integer coefficients. For example, the creation operator for $i = 1$ equals $C_1 = r_1/B_1(a,b)$ where

$$B_1(a,b) \quad = \quad (b-2)(4a-b)(4a-b-1)(4a-b-2)(4a-b-3)$$

and r_1 has the following longish expression in the Weyl algebra W:

$$\begin{aligned}
&256x_2x_1^4\partial_2\partial_1^3 + 768x_2^2x_1^3\partial_2^2\partial_1^2 + 864x_2^3x_1^2\partial_3\partial_1^3 + 768x_3x_1^4\partial_3\partial_1^3 + 864x_2^4x_1\partial_4\partial_1^3 \\
&+2560x_3x_2x_1^3\partial_4\partial_1^3 + 1024x_4x_1^4\partial_4\partial_1^3 + 81x_2^5\partial_4\partial_2\partial_1^2 + 3168x_3x_2^2x_1^2\partial_4\partial_2\partial_1^2 \\
&+3072x_4x_2x_1^3\partial_4\partial_2\partial_1^2 + 1728x_3x_2^3x_1\partial_4\partial_2^2\partial_1 + 768x_3^2x_1^3\partial_4\partial_2^2\partial_1 + 3456x_4x_2^2x_1^2\partial_4\partial_2^2\partial_1 \\
&+351x_3x_2^4\partial_4\partial_3\partial_1^2 + 1824x_3^2x_2x_1^2\partial_4\partial_3\partial_1^2 + 1728x_4x_2^3x_1\partial_4\partial_3\partial_1^2 + 1024x_4x_3x_1^3\partial_4\partial_3\partial_1^2 \\
&+1440x_3^2x_2^2x_1\partial_4^2\partial_1^2 + 234x_4x_2^4\partial_4^2\partial_1^2 + 2304x_4x_3x_2x_1^2\partial_4^2\partial_1^2 + 378x_3^2x_2^3\partial_4^2\partial_2\partial_1 \\
&+288x_3^3x_1^2\partial_4^2\partial_2\partial_1 + 1728x_4x_3x_2^2x_1\partial_4^2\partial_2\partial_1 + 448x_3^3x_2x_1\partial_4^2\partial_2^2 + 432x_4x_3x_2^3\partial_4^2\partial_2^2 \\
&+384x_4x_3^2x_1^2\partial_4^2\partial_2^2 + 174x_3^3x_2^2\partial_4^2\partial_3\partial_1 + 576x_4x_3^2x_2x_1\partial_4^2\partial_3\partial_1 + 48x_4x_3^3x_2\partial_4^2\partial_3^2 \\
&+48x_3^4x_1\partial_4^3\partial_1 + 216x_4x_3^2x_2^2\partial_4^3\partial_1 + 37x_3^4x_2\partial_4^3\partial_2 + 64x_4x_3^3x_1\partial_4^3\partial_2 + 3x_3^5\partial_4^3\partial_3 \\
&+4x_4x_3^4\partial_4^4 - 512x_1^4\partial_1^3 + 384x_2x_1^3\partial_2\partial_1^2 + 2304x_2^2x_1^2\partial_2^2\partial_1 + 1944x_2^3x_1\partial_3\partial_1^2 \\
&+3712x_3x_1^3\partial_3\partial_1^2 + 8640x_3x_2x_1^2\partial_4\partial_1^2 + 4608x_4x_1^3\partial_4\partial_1^2 + 1620x_3x_2^3\partial_4\partial_2^2 \\
&+486x_2^4\partial_4\partial_1^2 + 6552x_3x_2^2x_1\partial_4\partial_2\partial_1 + 9216x_4x_2x_1^2\partial_4\partial_2\partial_1 + 2112x_3^2x_1^2\partial_4\partial_2^2 \\
&+6048x_4x_2^2x_1\partial_4\partial_2^2 + 2952x_3^2x_2x_1\partial_4\partial_3\partial_1 + 1296x_4x_2^3\partial_4\partial_3\partial_1 + 2304x_4x_3x_1^2\partial_4\partial_3\partial_1 \\
&+204x_3^3x_2\partial_4\partial_3^2 + 288x_4x_3^2x_1\partial_4\partial_3^2 + 1008x_3^2x_2^2\partial_4^2\partial_1 + 2880x_4x_3x_2x_1\partial_4^2\partial_1
\end{aligned}$$

$+328x_3^3x_1\partial_4^2\partial_2 + 864x_4x_3x_2^2\partial_4^2\partial_2 + 144x_4x_3^2x_2\partial_4^2\partial_3 + 10x_3^4\partial_4^3 - 4608x_1^3\partial_1^2$
$-1488x_2x_1^2\partial_2\partial_1 + 816x_2^2x_1\partial_2^2 + 504x_2^3\partial_3\partial_1 + 3024x_3x_1^2\partial_3\partial_1 + 576x_3^2x_1\partial_3^2$
$+720x_4x_2^2\partial_3^2 + 3984x_3x_2x_1\partial_4\partial_1 + 3264x_4x_1^2\partial_4\partial_1 + 1212x_3x_2^2\partial_4\partial_2$
$+3264x_4x_2x_1\partial_4\partial_2 + 252x_3^2x_2\partial_4\partial_3 + 384x_4x_3x_1\partial_4\partial_3 + 96x_4x_3x_2\partial_4^2$
$-1632x_1^2\partial_1 - 792x_2x_1\partial_2 + 168x_3x_1\partial_3 + 48x_3x_2\partial_4 + 96x_4x_1\partial_4 - 48x_1.$

The creation operator C_1 is a left inverse to ∂_1 modulo $H_{\mathcal{A}}$. This implies

$$r_1\,\phi_{a,b}(x) \quad = \quad B_1(a+1,b)\cdot\phi_{a+1,b}(x).$$

The algorithm to be described below gives us similar formulas for $i = 2,3,4$:

$$\begin{aligned} r_2\,\phi_{a,b}(x) &= B_2(a+1,b+1)\cdot\phi_{a+1,b+1}(x),\\ r_3\,\phi_{a,b}(x) &= B_3(a+1,b+3)\cdot\phi_{a+1,b+3}(x),\\ r_4\,\phi_{a,b}(x) &= B_4(a+1,b+4)\cdot\phi_{a+1,b+4}(x). \end{aligned} \qquad (12)$$

Outside the zero set of the *b-function* B_i, the creation operator $C_i = r_i/B_i$ is well-defined and gives an isomorphism of left W-modules between $\mathcal{A}$-hypergeometric systems with different parameter values. The relations (11) and (12) are also called *contiguity relations* of $\mathcal{A}$-hypergeometric functions. See (Chyzak 1998) for contiguity relations in the setting of Ore algebra.

The relations (12) can be used to generate solutions to (1) for high parameter values from solutions with small parameter values. For instance, all $\mathcal{A}$-hypergeometric polynomials (3) can be generated from the trivial $\mathcal{A}$-hypergeometric function $\Phi_{(0,0)} = 1$ by successively applying the creation operators. More precisely, there exists an identity

$$r_1^{i_1}r_2^{i_2}r_3^{i_3}r_4^{i_4}\,(1) \quad = \quad const\cdot\Phi_{(i_1+i_2+i_3+i_4,\,i_2+3i_3+4i_4)},$$

where the multiplier "*const*" is a positive integer.

We now present an algorithm from (Saito et al. 1997) for computing creation operators. It uses Buchberger's algorithm in the extended Weyl algebra

$$W[a,b] \quad = \quad \mathbf{Q}[a,b]\langle x_1,\ldots,x_4,\partial_1,\ldots,\partial_4\rangle.$$

The two new variables a and b commute with the eight old variables x_j, ∂_j. Let $>$ be any term order on $W[a,b]$ which satisfies the following rule

$$\begin{gathered} a^{C_1}b^{C_2}x_1^{A_1}x_2^{A_2}x_3^{A_3}x_4^{A_4}\partial_1^{B_1}\partial_2^{B_2}\partial_3^{B_3}\partial_4^{B_4} \;\geq\; a^{c_1}b^{c_2}x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}\partial_1^{b_1}\partial_2^{b_2}\partial_3^{b_3}\partial_4^{b_4} \\ \text{whenever}\quad A_1+\cdots+A_4+B_1+\cdots+B_4 \;\geq\; a_1+\cdots+a_4+b_1+\cdots+b_4. \end{gathered}$$

Thus $>$ is an *elimination order* with blocks $\{x_1, \ldots, x_4, \partial_1, \ldots, \partial_4\} > \{a, b\}$.

Let $H_{\mathcal{A}}(a, b)$ be the *parametric hypergeometric ideal*, that is, the left ideal in $W[a, b]$ generated by the six polynomials in (5) and (6). To find the i-th creation operator C_i, we consider the left ideal generated by $H_{\mathcal{A}}(a, b)$ together with the i-th annihilation operator ∂_i. We compute the reduced Gröbner basis G for this ideal with respect to $>$. It is proved in (Saito et al. 1994) that $H_{\mathcal{A}}(a, b) + W[a, b] \cdot \partial_i$ has non-zero intersection with $\mathbf{Q}[a, b]$. The elimination property of $>$ ensures that this intersection is generated by $G \cap \mathbf{Q}[a, b]$.

Choose any element $B_i(a, b)$ of $G \cap \mathbf{Q}[a, b]$. Keeping track of multipliers during the Gröbner basis computation, we find $r_i \in W[a, b]$ such that

$$B_i(a, b) - r_i \cdot \partial_i \quad \in \quad H_{\mathcal{A}}(s).$$

In fact, we may assume $r_i \in W$ since any occurrence of a and b in r_i can be replaced by $x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4$ and $x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4$ respectively.

Here is the punch line: any $\mathcal{A}$-hypergeometric function $\phi_{a,b}(x)$ is annihilated by $B_i(a, b) - r_i \cdot \partial_i$. Thus we can recover $\phi_{a,b}(x)$ by applying the operator $r_i / B_i(a, b)$ to the "lower" $\mathcal{A}$-hypergeometric function $\partial_i \phi_{a,b}(x)$. This is exactly what we wanted: by shifting (a, b) we get the formulas in (11).

4 Series solutions

The study of hypergeometric series is important for many applications, including type II_B string theory (Hosono et al. 1997), singularity theory (Noumi 1984), and Gauss-Schwarz theory (Yoshida 1997). Gröbner bases give a nice description of the asymptotic behavior of $\mathcal{A}$-hypergeometric functions and can be used to construct series expansions.

Let $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ and consider the corresponding *toric ideal*

$$I_{\mathcal{A}} \quad = \quad (\partial^u - \partial^v \ : \ \mathcal{A}u = \mathcal{A}v,\, u, v \in \mathbf{N}^4) \quad \subset \quad k[\partial_1, \ldots, \partial_4].$$

This ideal is generated by the four binomials in (5). Fix $\omega \in \mathbf{R}^4$ which defines a term order for $I_{\mathcal{A}}$. This means that $in_\omega(I_{\mathcal{A}})$ is a monomial ideal in $k[\partial_1, \ldots, \partial_4]$. The set of *all* weight vectors which give the same initial ideal is an open convex polyhedral cone C_ω, called the *Gröbner cone* of $I_{\mathcal{A}}$ and ω:

$$C_\omega \quad = \quad \{\, \omega' \in \mathbf{R}^4 \ \mid \ in_{\omega'}(I_{\mathcal{A}}) = in_\omega(I_{\mathcal{A}}) \,\}.$$

For background on Gröbner bases and Gröbner cones of toric ideals see (Sturmfels 1995).

The weight vector $(-\omega, \omega) \in \mathbf{R}^8$ defines a partial term order on the Weyl algebra W. The initial ideal $in_{(-\omega,\omega)}(H_{\mathcal{A}})$ of the hypergeometric ideal $H_{\mathcal{A}}$ is called the *initial system* of $H_{\mathcal{A}}$ with respect to ω. Note that the initial system contains $in_\omega(I_{\mathcal{A}})$ and the two polynomials in (6). Saito, Sturmfels and Takayama (1998) showed that $in_{(-\omega,\omega)}(H_{\mathcal{A}})$ is generated by $in_\omega(I_{\mathcal{A}})$ and (6) if the parameters (a, b) are generic. For special parameters, the initial system may be larger than the subideal generated by $in_\omega(I_{\mathcal{A}})$ and (6). It can be found easily by computing a Gröbner basis for $H_{\mathcal{A}}$ with respect to a term order refining $(-\omega, \omega)$.

Let $\phi(x_1, x_2, x_3, x_4)$ be any holomorphic solution of (1). Note that it can be analytically continued to any point outside the zeros of the discriminant $\Delta_{\mathcal{A}}$. We introduce a parameter $t \in \mathbf{R}$ and consider the asymptotic behavior of the function

$$\phi(t^\omega x) \quad = \quad \phi(t^{\omega_1}x_1, t^{\omega_2}x_2, t^{\omega_3}x_3, t^{\omega_4}x_4) \qquad \text{for} \quad t \longrightarrow 0.$$

Let $in_\omega(\phi)$ denote the dominant term of $\phi(t^\omega x)$ around $t = 0$. It is easy to see that $in_\omega(\phi)$ is annihilated by the initial system $in_{(-\omega,\omega)}(H_{\mathcal{A}})$.

This observation suggests that we construct series solutions to (1) whose lowest terms are monomials x^u, $u \in \mathbf{C}^4$, which are annihilated by the initial system. This makes sense because the solution space to the initial system $in_{(-\omega,\omega)}(H_{\mathcal{A}})$ is spanned by monomials if (a, b) is generic; cf. (Saito et al. 1998).

Let x^u be any monomial solution to $in_{(-\omega,\omega)}(H_{\mathcal{A}})$. Let L be the integer kernel of the matrix A. This is a rank 2 sublattice of $\mathbf{Z}^4$. Consider the monoid $M_{\omega,u} := \{\, v \in L \mid u_i + v_i \in \mathbf{N} \text{ for all } i \in \{1,2,3,4\} \text{ such that } u_i \in \mathbf{N}\}$.

Theorem 4.1 *The following series is a formal solution to the equations (1):*

$$\Phi_{\omega,u}(x) \quad := \quad \sum_{v \in M_{\omega,u}} x^{u+v} / \prod_{i=1}^{4} \Gamma(u_i + v_i + 1)$$

Moreover, there exists a vector $\tau_\omega \in \mathbf{R}^4$ such that this series converges whenever $(-\log|x_1|, \ldots, -\log|x_4|)$ lies in the translated Gröbner cone $C_\omega + \tau_\omega$.

This construction of series solutions appears in the work of Hosono, Lian and Yau (1997) for explicit constructions of mirror maps. See also (Sturmfels

1992). It differs from the triangulation construction of Gel'fand, Kapranov and Zelevinsky (1989). The key point in the proof of Theorem 4.1 is that the inner product $\omega' \cdot v$ is positive for all non-zero elements v in the monoid $M_{\omega,u}$ and all vectors ω' in the Gröbner cone C_ω.

Let us see how this construction works for the weight vector $\omega = (0,1,2,0)$. For this term order the four generators of $I_{\mathcal{A}}$ are already a Gröbner basis:

$$\underline{\partial_2\partial_3} - \partial_1\partial_4,\ \underline{\partial_1\partial_3^2} - \partial_2^2\partial_4,\ \underline{\partial_2^3} - \partial_1^2\partial_3,\ \underline{\partial_3^3} - \partial_2\partial_4^2.$$

The initial system $in_{(-\omega,\omega)}(H_{\mathcal{A}})$ contains the underlined leading terms and the two polynomials in (6). Suppose that $x^u = x_1^{u_1}x_2^{u_2}x_3^{u_3}x_4^{u_4}$ is annihilated by $in_{(-\omega,\omega)}(H_{\mathcal{A}})$. Then the exponents satisfy following six algebraic equations

$$u_2u_3 \;=\; u_1u_3(u_3-1) \;=\; u_2(u_2-1)(u_2-2) \;=\; u_3(u_3-1)(u_3-2) \;=\; 0$$

$$\text{and} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \;=\; \begin{pmatrix} a \\ b \end{pmatrix}.$$

Suppose that a and b are generic. Then these six equations have four solutions

$$\begin{array}{rcllllll} u^{(1)} & = & (& (4a-b)/4, & 0, & 0, & b/4 &), \\ u^{(2)} & = & (& (4a-b-3)/4, & 1, & 0, & (b-1)/4 &), \\ u^{(3)} & = & (& (4a-b-1)/4, & 0, & 1, & (b-3)/4 &), \\ u^{(4)} & = & (& (4a-b-6)/4, & 2, & 0, & (b-2)/4 &). \end{array}$$

The corresponding monomials $x^{u^{(1)}}$, $x^{u^{(2)}}$, $x^{u^{(3)}}$ and $x^{u^{(4)}}$ form a basis for the solution space of $in_{(-\omega,\omega)}(H_{\mathcal{A}})$. They give rise to $\mathcal{A}$-hypergeometric series

$$\Phi^{(i)} \quad = \quad \sum_{\substack{m,n\in\mathbf{Z}: \\ u_2^{(i)}+4m\geq 3n,\; u_3^{(i)}+n\geq 0}} c_{mn} \cdot x^{u^{(i)}+m(-3,4,0,-1)+n(2,-3,1,0)}, \qquad \text{where}$$

$$c_{mn} = 1/(\Gamma(u_1^{(i)}-3m+2n+1)\Gamma(u_2^{(i)}+4m-3n+1)\Gamma(u_3^{(i)}+n+1)\Gamma(u_4^{(i)}-m+1)).$$

By Theorem 4.1, the series $\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}$ and $\Phi^{(4)}$ converge on an open subset U of $\mathbf{C}^4$ specified by the Gröbner cone. They are linearly independent since their ω-lowest terms $x^{u^{(i)}}$ are distinct. Hence, by Proposition 2.1, they form a basis for the space of $\mathcal{A}$-hypergeometric functions on U for generic (a,b).

Let us specialize the parameters (a, b) to $(1, 2)$. In this case we have

$$\begin{array}{lcl} u^{(1)} & = & (\; 1/2, \; 0, \; 0, \; 1/2 \;), \\ u^{(2)} & = & (\; 1/4, \; 1, \; 0, \; 1/4 \;), \\ u^{(3)} & = & (\; 1/4, \; 0, \; 1, \; -1/4 \;), \\ u^{(4)} & = & (\; -1, \; 2, \; 0, \; 0 \;). \end{array}$$

$\Phi^{(1)}, \Phi^{(2)}$ and $\Phi^{(3)}$ are convergent series solutions, but $\Phi^{(4)} \equiv 0$. For example, the series $\Phi^{(3)}$ with terms ordered by ω-weight starts like this:

$$c_{00} \frac{x_1^{1/4} x_3}{x_4^{1/4}} + c_{0,-1} \frac{x_2^3}{x_1^{7/4} x_4^{1/4}} + c_{11} \frac{x_2 x_3^2}{x_1^{3/4} x_4^{5/4}} + c_{10} \frac{x_2^4 x_3}{x_1^{11/4} x_4^{5/4}} + c_{1,-1} \frac{x_2^7}{x_1^{19/4} x_4^{5/4}}$$
$$+ c_{22} \frac{x_2^2 x_3^3}{x_1^{7/4} x_4^{9/4}} + c_{21} \frac{x_2^5 x_3^2}{x_1^{15/4} x_4^{9/4}} + c_{20} \frac{x_2^8 x_3}{x_1^{23/4} x_4^{9/4}} + c_{2,-1} \frac{x_2^{11}}{x_1^{31/4} x_4^{9/4}} + c_{33} \frac{x_2^3 x_3^4}{x_1^{11/4} x_4^{13/4}} + \ldots$$

By examining the initial system, we find the two additional $\mathcal{A}$-hypergeometric functions x_2^2/x_1 and x_3^2/x_4. Proposition 2.1 now implies

Proposition 4.1 *Let $(a, b) = (1, 2)$. The set $\{\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}, x_2^2/x_1, \; x_3^2/x_4\}$ is a basis for the five-dimensional space of $\mathcal{A}$-hypergeometric functions on U.*

In closing we mention an important result about series solutions. Hotta (1991) showed that all $\mathcal{A}$-hypergeometric ideals are *regular holonomic*. This implies that any formal series solution of the form

$$\sum_{k \in \mathbf{N}^4} c_k (x_1 - a_1)^{k_1} (x_2 - a_2)^{k_2} (x_3 - a_3)^{k_3} (x_4 - a_4)^{k_4}, \quad a_i \in \mathbf{C}$$

converges in a neighborhood of the point (a_1, a_2, a_3, a_4).

References

[1] Aomoto, K., Kita, N. (1994): *Theory of Hypergeometric Functions*, (in Japanese) Springer-Verlag, Tokyo.

[2] Borel, A., Grivel, P.-P., Kaup, B., Haefliger, A., Malgrange, B., Ehlers, F. (1987): *Algebraic D-modules*, Academic Press, Boston.

[3] Chyzak, F. (1998): Gröbner bases, symbolic summation and symbolic integration, in this volume.

[4] Gel'fand, I.M., Zelevinskii, A.V., Kapranov, M.M. (1989): Hypergeometric functions and toral manifolds, *Functional Analysis and its Applications* 23: 94–106.

[5] Gel'fand, I.M., Zelevinskii, A.V., Kapranov, M.M. (1993): Correction to the paper: "Hypergeometric functions and toric varieties", *Functional Analysis and its Applications* 27: 4; see also Math. Reviews `95a:22010`.

[6] Hosono, S., Lian, B.H., Yau, S.-T. (1997): Maximal degeneracy points of GKZ systems, *Journal of the American Math. Society* 10: 427–443.

[7] Hotta, R. (1991): Equivariant D-modules, Proceedings of ICPAM Spring School in Wuhan, ed. by P. Torasso, Travaux en cours, Hermann, Paris, to appear.

[8] Kashiwara, M. (1983): *Systems of Microdifferential Equations*, Birkhäuser, Boston.

[9] Noumi, M. (1984). Expansion of the solutions of a Gauss-Manin system at a point of infinity, *Tokyo Journal of Mathematics* 7: 1–60.

[10] Oaku, T. (1994): Computation of the characteristic variety and the singular locus of a system of differential equations with polynomial coefficients. *Japan Journal of Industrial and Applied Math.* 11: 485–497.

[11] Oaku, T. (1996): Gröbner bases for D-modules on a non-singular affine algebraic variety, *Tohoku Mathematical Journal* 48: 575-600.

[12] Oaku, T. (1997): Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules, *Advances in Applied Mathematics* 19: 61–105.

[13] Saito, M., Sturmfels, B., Takayama, N. (1997): Hypergeometric polynomials and integer programming, submitted.

[14] Saito, M., Sturmfels, B., Takayama, N. (1998): Initial systems of $\mathcal{A}$-hypergeometric equations, in preparation.

[15] Saito, M., Takayama, N. (1994): Restrictions of $\mathcal{A}$-hypergeometric systems and connection formulas of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric function, *International Journal of Mathematics* 5: 537–560.

[16] Sato, M., Kawai, T., Kashiwara, M. (1973): Microfunctions and pseudodifferential equations, in: Springer Lecture Notes in Mathematics, 287, pp. 265–529.

[17] Sturmfels, B. (1992): Asymptotic analysis of toric ideals, *Memoirs of the Faculty of Sciences, Kyushu University, Ser. A: Math.* 46: 217–228.

[18] Sturmfels, B. (1995): *Gröbner Bases and Convex Polytopes*, American Mathematial Society, University Lecture Series 8, Providence, RI.

[19] Takayama, N. (1987): Gröbner bases in the ring of differential operators and its applications (in Japanese), *Sūrikaisekikenkyūsho Kokyūroku* 612: 67–77.

[20] Takayama, N. (1991): *Kan: A system for computation in algebraic analysis*, source code available at `http://www.math.s.kobe-u.ac.jp/KAN/`

[21] Yoshida, M. (1997): *Hypergeometric Function, My Love: Modular Interpretations of Configuration Spaces*, Vieweg Verlag, Wiesbaden.

Bernd Sturmfels, Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-11, Japan and Department of Mathematics, University of California, Berkeley, CA 94720, USA; `bernd@math.berkeley.edu`

Nobuki Takayama, Department of Mathematics, Kobe University, Rokko, Kobe, 657, Japan; `takayama@math.kobe-u.ac.jp`