Amoebae of type E

Yasuhiko Yamada
Department of Mathematics, Faculty of Science,
Kobe University, Hyogo 657-8501, Japan
(November 10, 2007)

Abstract. Tropical representations of all the second order q-Painlevé equations are given. Corresponding amoebae arising as the integral curves of autonomous cases are also presented.

1 Introduction

A birational transformation is called "tropical" if it can be represented as a subtraction free expression[14]. Such tropical transformations have combinatorial analog through the ultra-discretization procedure[27]

$$a + b \to \max(a, b), \quad a \times b \to a + b. \tag{1}$$

The resulting Max-Plus algebra has been interested from various points of view such as combinatorics[14][23], representation theory[1][2], tropical geometry[18][19] and integrable systems[15][16][29].

It is known that some of the q-difference Painlevé equations and their higher order generalizations have a tropical form[13] [28]. In this note, we give a tropical representation for the Weyl group for all the second order q-Painlevé equations in Sakai's list[24].

$$\begin{array}{c} \hspace{26em} A_1^{(1)\prime} \\ \hspace{24em} \nearrow \\ E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to (A_2 + A_1)^{(1)} \to (A_1 + A_1)^{(1)} \to A_1^{(1)} \to A_0^{(1)} \end{array}$$

This diagram also represent the degeneration of 5d $N = 2$ $SU(2)$ gauge theories[26][8][30]. Corresponding Weyl group actions are given as the Cremona transformations on $\mathbb{P}^2$[17][5]. In case of $E_7^{(1)}$ and $E_8^{(1)}$, however, these actions are not tropical in the original coordinates on $\mathbb{P}^2$. We find the coordinate transformations which makes these formulas tropical form.

Under some special condition on the root parameters, these Painlevé equations become autonomous and integrable. We give the explicit formulae of the integral curves together with the corresponding amoebae[18][19]. The results can be identified with the $[p, q]$-five brane web diagrams(see for example[3][4][9]).

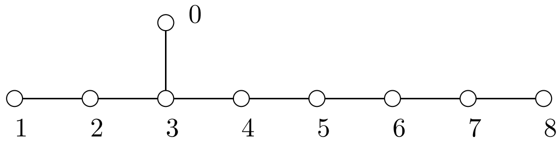
2 Tropical representations of affine Weyl groups

Our construction of discrete Painlvé equations is based on representations of affine Weyl groups, where the difference equations arises as the actions of translations in the affine Weyl group[22].

In what follows, s_i denotes the simple reflections whose labelings are given in the Dynkin diagrams. The action of s_i on multiplicative root parameters b_j are standard one $s_i(b_j) = b_j b_i^{-C_{ij}}$ with corresponding Cartan matrix (C_{ij}), and we will suppress them in the following data. Trivial actions $s_i(x) = x$ will also be omitted.

Case $E_8^{(1)}$. The Weyl group is $W(E_8^{(1)}) = \langle s_0, \ldots, s_8 \rangle$.

$E_8^{(1)}$

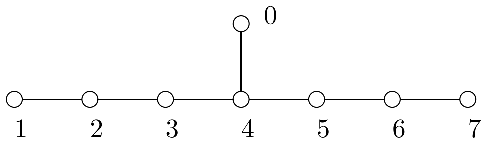


Nontrivial actions on dependent variables (f, g) are given as follows:

$$\begin{aligned} &s_0(f) = \frac{f}{b_0}, && s_2(g) = b_2 g, \\ &s_3(f) = f\frac{b_3 + b_3 f + g}{1 + f + g}, && s_3(g) = g\frac{1 + b_3 f + g}{b_3(1 + f + g)}. \end{aligned} \tag{2}$$

Case $E_7^{(1)}$. The Weyl group is $W(E_7^{(1)}) = \langle s_0, \ldots, s_7 \rangle$ extended by its diagram automorphism $\mathrm{Aut}(E_7^{(1)}) = \langle p_1 \rangle \simeq \mathfrak{S}_2$.

$E_7^{(1)}$

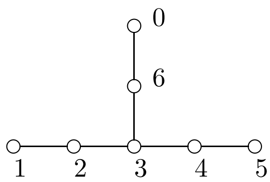


Nontrivial actions on dependent variables (f, g) are given as follows:

$$\begin{aligned} &s_0(f) = \frac{f}{b_0}, && s_3(g) = b_3 g, \\ &s_4(f) = f\frac{b_4 + b_4 f + g}{1 + f + g}, && s_4(g) = g\frac{1 + b_4 f + g}{b_4(1 + f + g)}, \\ &p_1(f) = \frac{b_0}{f}, && p_1(g) = \frac{b_4(1 + f)(b_0 + f)}{fg}. \\ &p_1(b_{\{0,1,2,3,4,5,6,7\}}) = b_{\{0,7,6,5,4,3,2,1\}}. \end{aligned} \tag{3}$$

Case $E_6^{(1)}$. The Weyl group is $W(E_6^{(1)}) = \langle s_0, \ldots, s_6 \rangle$ extended by its diagram-automorphism $\mathrm{Aut}(E_6^{(1)}) = \langle p_1, p_2 \rangle \simeq \mathfrak{S}_3$.

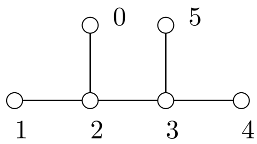
$E_6^{(1)}$



$$
\begin{aligned}
&s_2(f) = \frac{f}{b_2}, \\
&s_3(f) = f\frac{b_3 + b_3 f + g}{1 + f + g}, && s_3(g) = g\frac{1 + b_3 f + g}{b_3(1 + f + g)}, \\
&s_6(g) = b_6 g, \\
&p_1(f) = \frac{1}{f}, && p_1(g) = \frac{g}{b_3 f}, \\
&p_2(f) = g, && p_2(g) = f, \\
&p_1(b_{\{0,1,2,3,4,5,6\}}) = b^{-1}_{\{5,1,2,3,6,0,4\}}, && p_2(b_{\{0,1,2,3,4,5,6\}}) = b^{-1}_{\{1,0,6,3,4,5,2\}}.
\end{aligned}
\tag{4}
$$

Case $D_5^{(1)}$. The Weyl group is $W(D_5^{(1)}) = \langle s_0, \ldots, s_5 \rangle$:

$D_5^{(1)}$



The nontrivial actions are:

$$
\begin{aligned}
&s_0(f) = b_0 f, \quad s_0(g) = g\frac{1 + b_0 f}{1 + f}, \\
&s_2(f) = \frac{f}{b_2}, \\
&s_3(f) = f\frac{b_3 + b_3 f + g}{1 + f + g}, \quad s_3(g) = g\frac{1 + b_3 f + g}{b_3(1 + f + g)}, \\
&s_5(g) = b_5 g.
\end{aligned}
\tag{5}
$$

The diagram automorphism is generated by p_1, p_2:

$$
\begin{aligned}
&p_1(b_{\{0,1,2,3,4,5\}}) = b^{-1}_{\{5,4,3,2,1,0\}}, \quad p_1(f) = \frac{1 + f}{g}, \quad p_1(g) = \frac{1 + f + g}{fg}, \\
&p_2(b_{\{0,1,2,3,4,5\}}) = b^{-1}_{\{0,1,2,3,5,4\}}, \quad p_2(f) = \frac{1}{f}, \quad p_1(g) = \frac{g}{b_3 f}.
\end{aligned}
\tag{6}
$$

Case $A_4^{(1)}$. The Weyl group is $W(A_4^{(1)}) = \langle s_0, \ldots, s_4 \rangle$:

$$
\begin{aligned}
&s_0(f) = f\frac{b_0 + g}{b_0(1 + g)}, \quad s_0(g) = \frac{g}{b_0}, \\
&s_1(g) = b_1 g, \\
&s_2(f) = f\frac{b_2 + b_2 f + g}{1 + f + g}, \quad s_2(g) = g\frac{1 + b_2 f + g}{b_2(1 + f + g)}, \\
&s_3(f) = \frac{f}{b_3}, \\
&s_4(f) = b_4 f, \quad s_4(g) = g\frac{1 + b_4 f}{1 + f}.
\end{aligned}
\tag{7}
$$

The diagram automorphism is generated by p_1, p_2:

$$
\begin{aligned}
&p_1(b_{\{0,1,2,3,4\}}) = b_{\{1,2,3,4,0\}}, \quad p_1(f) = \frac{1 + f + g}{fg}, \quad p_1(g) = \frac{1 + f}{g}, \\
&p_2(b_{\{0,1,2,3,4\}}) = b^{-1}_{\{2,1,0,4,3\}}, \quad p_2(g) = \frac{1 + f}{g}.
\end{aligned}
\tag{8}
$$

Case $(A_1+A_2)^{(1)}$**.** The Weyl group is product of $W(A_2^{(1)})=\langle s_0,s_1,s_2\rangle$ and $W(A_1^{(1)})=\langle r_0,r_1\rangle$. The corresponding multiplicative root parameters are denoted by $\{a_0,a_1,a_2\}$ and $\{b_0,b_1\}$ $(a_0a_1a_2=b_0b_1)$. The tropical action on f,g variables is given by

$$\begin{aligned}
&s_0(f)=f\frac{b_1f+g}{b_1f+a_0g},\quad s_0(g)=a_0g\frac{b_1f+g}{b_1f+a_0g},\\
&s_1(f)=a_1f,\quad s_1(g)=g\frac{1+a_1f}{1+f},\\
&s_2(f)=f\frac{a_2+g}{a_2(1+g)},\quad s_2(g)=\frac{g}{a_2},\\
&r_0(f)=f\frac{b_1(a_1a_2f+g+a_1fg)}{a_1a_2(b_1f+a_0g+a_0a_1fg)},\quad r_0(g)=g\frac{a_0a_2(a_1a_2f+g+a_1fg)}{a_2b_1f+a_0a_2g+b_1fg},\\
&r_1(f)=f\frac{b_1+b_1f+g}{1+f+g},\quad r_1(g)=g\frac{1+b_1f+g}{b_1(1+f+g)}.
\end{aligned}\tag{9}$$

The diagram automorphism is generated by p_1,p_2:

$$\begin{aligned}
&p_1(a_{\{0,1,2\}})=a_{\{1,2,0\}},\quad p_1(b_{\{0,1\}})=b_{\{1,0\}},\quad p_1(f)=\frac{g}{a_2},\quad p_1(g)=\frac{a_0g}{b_1f},\\
&p_2(a_{\{0,1,2\}})=a^{-1}_{\{0,2,1\}},\quad p_2(b_{\{0,1\}})=b^{-1}_{\{1,0\}},\quad p_2(f)=\frac{a_2}{g},\quad p_2(g)=\frac{1}{a_1f}.
\end{aligned}\tag{10}$$

Case $(A_1+A_1)^{(1)}$**.** The symmetry group is semi-direct product of $W(A_1^{(1)})=\langle s_0,s_1\rangle$, $\mathbb{Z}=\langle p_1\rangle$ and $\mathfrak{S}_2=\langle p_2\rangle$. Their relations are $p_2^2=(p_1p_2)^2=s_0p_2s_1p_2=1$, $p_1^2s_i=s_ip_1^2$. The parameters are $\{a_0,a_1,b\}$.
The tropical representation is given as follows.

$$\begin{aligned}
&p_1(a_{\{0,1\}})=a_{\{1,0\}},p_1(b)=a_1b, && p_1(f)=g, && p_1(g)=\frac{1+g}{bfg},\\
&p_2(a_{\{0,1\}})=a^{-1}_{\{1,0\}}, && p_2(f)=\frac{1+g}{bfg}, && p_2(g)=g,\\
&s_1(a_1)=\frac{1}{a_1},\qquad s_1(a_0)=a_0a_1^2, s_1(b)=ba_1, && s_1(f)=f\frac{1+f+a_1g}{1+f+g}, && s_1(g)=g\frac{a_1+f+a_1g}{1+f+g},\\
&s_0(a_1)=a_1a_0^2,\qquad s_0(a_0)=\frac{1}{a_0},\quad s_0(b)=\frac{b}{a_0}, && s_0(f)=\frac{a_0f(1+bfg)}{a_0+bfg}, && s_0(g)=\frac{g(a_0+bfg)}{a_0(1+bfg)}.
\end{aligned}\tag{11}$$

Case $A_1^{(1)}$**.** The symmetry group is semi-direct product of $\mathbb{Z}=\langle p_1\rangle$ and $\mathfrak{S}_2=\langle p_2\rangle$ with relations $p_2^2=(p_1p_2)^2)=1$.

$$\begin{aligned}
&p_1(a_0)=\frac{1}{a_1},p_1(a_1)=a_1^2a_0,p_1(f)=\frac{a_1(1+fg)}{f},p_1(g)=\frac{1}{fg},\\
&p_2(a_0)=\frac{1}{a_1},p_2(a_1)=\frac{1}{a_0}, \qquad p_2(g)=\frac{1}{fg}.
\end{aligned}\tag{12}$$

Case $A_1^{(1)'}$**.** The symmetry group is semi-direct product of $\mathfrak{D}_8=\langle p_1,p_2\rangle$ and $W(A_1^{(1)})=\langle s_1,s_0\rangle$ with relations $p_1^4=p_2^2=(p_1p_2)^2=s_i^2=1$ and $s_1p_2=p_2s_0$. The actions are as follows:

$$\begin{aligned}
&p_1(a_0)=a_1,\quad p_1(a_1)=a_0,\ p_1(f)=g, && p_1(g)=\frac{a_0a_1}{f},\\
&p_2(a_0)=\frac{1}{a_0},\quad p_2(a_1)=\frac{1}{a_1},p_2(f)=\frac{g}{a_0a_1}, && p_2(g)=\frac{f}{a_0a_1},\\
&s_1(a_0)=a_0^2a_1,s_1(a_1)=\frac{1}{a_1},s_1(f)=f\frac{a_1f+g}{f+a_1g}, && s_1(g)=g\frac{a_1f+g}{f+a_1g}.
\end{aligned}\tag{13}$$

Case $A_0^{(1)}$. The symmetry group is dihedral group $\mathfrak{D}_6 = \langle p_1, p_2 | p_1^3 = p_2^2 = (p_2 p_1)^2 = 1 \rangle$.

$$\begin{aligned} p_1(a) &= a,\ p_1(f) = \frac{g}{f}, p_1(g) = \frac{1}{f}, \\ p_2(a) &= \frac{1}{a}, p_2(f) = \frac{1}{g}, p_2(g) = \frac{1}{f}. \end{aligned} \tag{14}$$

3 Realization as Cremona transformations

The tropical representations given in previous section are essentially the same as those given by the Cremona transformations[24](see also[5]). However, the representations obtained in that way are not tropical for $E_7^{(1)}$ and $E_8^{(1)}$ cases. This is because the relevant configurations of 9points on $\mathbb{P}^2$ do not admit the torus action in these cases. The tropical representations for $E_7^{(1)}$ and $E_8^{(1)}$ in previous section were obtained by extending the case $E_6^{(1)}$. Since the representation of $E_6^{(1)}$ on variables (f, g) are nontrivial only near the trivalent vertex, one can easily generalize it for any trivalent Dynkin diagrams T_{pqr}[28].

In this section, we make explicit relation between the tropical representation and geometric ones, in order to understand the systems of type $E_7^{(1)}$ and $E_8^{(1)}$.

Consider the configuration of ten points P_i $(i = 1, \dots, 10)$ on $\mathbb{P}^2$. Let s_{ij} $(1 \le i < j \le 9)$ be the exchange of P_i and P_j and let s_{ijk} $(1 \le i < j < k \le 9)$ be the standard Cremona transformation with base points P_i, P_j, P_k (namely, $(x : y : z) \mapsto (yz : zx : xy)$ in the coordinate $P_i = (1 : 0 : 0), P_j = (0 : 1 : 0), P_k = (0 : 0 : 1)$.) It has been known that the transformations s_{ij} $(1 \le i < j \le 9)$ and s_{ijk} $(1 \le i < j < k \le 9)$ generate the affine Weyl group $W(E_8^{(1)})$[5]. This affine Weyl group action is the geometric origin of the (discrete) Painlevé equations[24]. Note that the tenth point P_{10} plays the role of dependent variables (f, g) in this formulation[21][12].

Case $E_6^{(1)}$ Let us begin with the case $E_6^{(1)}$ for comparison. The configuration is

$$\begin{aligned} &P_1 = (1 : 0 : -1),\ P_2 = (b_2 : 0 : -1), \quad P_3 = (b_1 b_2 : 0 : -1), \\ &P_4 = (0 : -1 : 1),\ P_5 = (0 : -1 : b_6), \quad P_6 = (0 : -1 : b_6 b_0), \\ &P_7 = (-1 : b_3 : 0), P_8 = (-1 : b_3 b_4 : 0), P_9 = (-1 : b_3 b_4 b_5 : 0). \end{aligned} \tag{15}$$

Then our tropical representation is realized as

$$\begin{aligned} &s_0 = s_{56}, s_1 = s_{23}, s_2 = s_{12}, \quad s_3 = s_{147}, \quad s_4 = s_{78}, \\ &s_5 = s_{89}, s_6 = s_{45}, p_1 = s_{05} s_{46}, p_1 = s_{01} s_{26}. \end{aligned} \tag{16}$$

The 9 points are on a degenerate cubic $G(x, y, z) = xyz = 0$. When the parameters satisfy the relation $b_0 b_1 b_2^2 b_3^3 b_4^2 b_5 b_6^2 = 1$, then there exist one-parameter family of cubic curves $F(x, y, z) - HG(x, y, z) = 0$ passing through the 9 points. Then the ratio $H = F(x, y, z)/G(x, y, z)$ gives the integral of the autonomous system. In terms of the inhomogeneous coordinate $f = x/z, g = y/z$, this gives the integral eq.(33) in next section.

Case $E_7^{(1)}$ The configuration is

$$\begin{aligned} &P_1 = (b_3 b_0 : 0 : -1), P_2 = (b_2 b_3 b_0 : 0 : -1), P_3 = (b_1 b_2 b_3 b_0 : 0 : -1), \\ &P_4 = P(b_0^{-1}), \qquad P_5 = P(1), \qquad P_6 = P(b_4), \\ &P_7 = P(b_4 b_5), \qquad P_8 = P(b_4 b_5 b_6), \qquad P_9 = P(b_4 b_5 b_6 b_7). \end{aligned} \tag{17}$$

where $P(u) = (u^{-2} : u^{-1} : 1)$. In this case, the Cremona transformation is not tropical. For instance the action of s_{145} in coordinate $(x : y : 1)$ is given by

$$\begin{aligned} s_{145}(x) &= \frac{x^2 - xy - xyb_0 + y^2 b_0 + xb_0 b_3 - y^2 b_0 b_3}{b_3(x - y^2 - yb_3 + y^2 b_3 + b_0 b_3 - yb_0 b_3)}, \\ s_{145}(y) &= -\frac{y(-x + y - b_0 + yb_0)}{x - y^2 - yb_3 + y^2 b_3 + b_0 b_3 - yb_0 b_3}. \end{aligned} \tag{18}$$

However, it can be repaired into tropical form under the following change of variables:

$$x = \frac{f^2 + f + b_0(1 + f + g)}{g(1 + g)}, \quad y = -\frac{f}{1 + g}. \tag{19}$$

$$f + y(1 + g) = 0, \quad (x - y^2)g = (y - 1)(y - b_0)$$

As a result, our tropical representation is realized as the Cremona transformations written in terms of the variables f, g:

$$\begin{array}{ll} s_0 = s_{45}, s_1 = s_{23}, s_2 = s_{12}, s_3 = s_{145}, & s_4 = s_{56}, \\ s_5 = s_{67}, s_6 = s_{78}, s_7 = s_{89}, p_1 = s_{456}s_{17}s_{28}s_{39}. & \end{array} \tag{20}$$

When $b_0^2 b_1 b_2^2 b_3^3 b_4^4 b_5^3 b_6^2 b_7 = 1$, the equations of cubic pencil is given by

$$\begin{array}{l} x^3 + x^2 m_1 + x m_2 + m_3 - y m_3 w_1 + y^2(m_3 w_2 - m_2) - y^3 m_3 w_3 + \\ x y^2(m_3 w_4 - m_1) - x^2 y m_3 w_5 + H y(x - y^2) = 0, \end{array} \tag{21}$$

where

$$\begin{array}{l} (1 + b_3 b_0 z)(1 + b_2 b_3 b_0 z)(1 + b_1 b_2 b_3 b_0 z) = 1 + m_1 z + \cdots + m_3 z^3, \\ (1 + z)(1 + \dfrac{z}{b_0})(1 + b_4 z)(1 + b_4 b_5 z)(1 + b_4 b_5 b_6 z)(1 + b_4 b_5 b_6 b_7 z) = 1 + w_1 z + \cdots + w_6 z^6. \end{array} \tag{22}$$

Integral $H = H(x, y)$ is also become tropical under the coordinate transformation (19).

Case $E_8^{(1)}$

$$P_i = (u_i : \frac{u_0 - u_i^3}{u_i} : 1), \quad (i = 1, \ldots, 9) \tag{23}$$

In this case, the nontrivial Cremona actions are computed as follows:

$$\begin{aligned} & s_i(u_i) = u_{i+1}, \quad s_i(u_{i+1}) = u_i, \quad (i = 1, \ldots, 8) \\ & s_0(u_0) = \frac{u_0}{u_1 u_2 u_3}, \quad s_0(u_1) = \frac{u_0}{u_2 u_3}, \quad s_0(u_2) = \frac{u_0}{u_1 u_3}, \quad s_0(u_3) = \frac{u_0}{u_1 u_2}, \\ & s_0(x) = \frac{x^2 u_0 k_1 - u_0(u_0 - k_3) + x y u_0 - x u_0 k_2}{y k_3 + x u_0 k_1 - u_0 k_2 - x^2(u_0 - k_3)}, \\ & s_0(y) = \frac{y^2 u_0 k_3 - y u_0^2 k_2 + u_0^2 k_1(u_0 - k_3) - x^2 u_0 k_2(u_0 - k_3) + x y u_0 k_1 k_3 + x u_0(u_0 - k_3)^2}{k_3(y k_3 + x u_0 k_1 - u_0 k_2 - x^2(u_0 - k_3))}. \end{aligned} \tag{24}$$

Again, this action can be recasted into our tropical form through the variable change:

$$\begin{aligned} & u_3(1 + f) + x g = 0, \\ & (x^3 + x y - u_0) u_1 u_2 u_3 f + (x - u_1)(x - u_2)(x - u_3) u_0 = 0 \\ & b_0 = \frac{u_0}{u_1 u_2 u_3}, \quad b_i = \frac{u_i}{u_{i+1}}, \quad (i > 0). \end{aligned} \tag{25}$$

When $u_0^3 = u_1 u_2 \cdots u_9$, the cubic pencil is given by

$$\begin{array}{l} m_8(x^2 y + y^2 + x u_0) u_0^{-2} - m_7(x^2 + y) u_0^{-1} + m_6 - m_5 x + m_4 x^2 - m_3 x^3 - m_2 x(x y - u_0) \\ -m_1(x y^2 + x^2 u_0 - y u_0) - m_9(y^3 - 3x^3 u_0 + 3u_0^2) u_0^{-3} + u_0 H(x^3 + x y - u_0) = 0, \end{array} \tag{26}$$

where

$$\prod_{i=1}^{9}(1 + u_i z) = \sum_{i=0}^{9} m_i z^i. \tag{27}$$

By construction, these pencils(21)(26) coincide with the corresponding Seiberg-Witten curves [8][30][20][7][6], where b_i and H plays the role of multiplicative flavor masses and the u-parameter. In fact, by comparing the Weierstrass normal forms, we find that the curve (26) coincides with the curve (B1) in [6] with $H = 54 - u$. The role of cubic pencils in discrete Painlevé equations was clarified in [12]. The differential cases and their relation to $d = 4$ Seiberg-Witten curves were given in [11].

4 Integral curves of autonomous cases

In this section, we give explicit integrals of autonomous q-Painlevé equatins. In the Fig.(2)-(12), the parameters (b_i) are given in additive notation.

Case $E_8^{(1)}$. (Fig.12) Autonomous condition : $b_0b_1^2b_2^4b_3^6b_4^5b_5^4b_6^3b_7^2b_8=1$. The integral is

$$H=\frac{1}{f^2g^3}\left(\sum_{i=0}^{6}\sum_{j=0}^{6-i}c_{ij}f^ig^j\right). \tag{28}$$

Where

$$\begin{aligned}
&c_{0,0}=\frac{k}{l_3^2},\quad c_{0,1}=\frac{2kl_1}{l_3^2},\quad c_{0,2}=\frac{k(l_1^2+2l_2)}{l_3^2},\quad c_{0,3}=\frac{2k(l_1l_2+l_3)}{l_3^2},\\
&c_{0,4}=\frac{k(l_2^2+2l_1l_3)}{l_3^2},\quad c_{0,5}=\frac{2kl_2}{l_3},\quad c_{0,6}=k,\quad c_{1,0}=\frac{3(k+l_3)}{l_3^2},\quad c_{1,1}=\frac{4kl_1+4l_3l_1+m_1l_3}{l_3^2},\\
&c_{1,2}=\frac{l_1^2k^3+3l_2k^3+l_1^2l_3k^2+m_1l_1l_3k^2+3l_2l_3k^2+m_5l_3^2}{k^2l_3^2},\\
&c_{1,3}=\frac{l_1l_2k^3+3l_3k^3+3l_3^2k^2+m_1l_2l_3k^2+l_1l_2l_3k^2+m_5l_1l_3^2}{k^2l_3^2},\\
&c_{1,4}=\frac{l_1k^3+m_1l_3k^2+l_1l_3k^2+m_5l_2l_3}{k^2l_3},\quad c_{1,5}=\frac{m_5l_3}{k^2},\\
&c_{2,0}=\frac{3(k^2+3l_3k+l_3^2)}{kl_3^2},\quad c_{2,1}=\frac{2(l_1k^2+m_1l_3k+4l_1l_3k+m_1l_3^2+l_1l_3^2)}{kl_3^2},\\
&c_{2,2}=\frac{l_2k^4+l_1^2l_3k^3+m_1l_1l_3k^3+4l_2l_3k^3+m_2l_3^2k^2+m_1l_1l_3^2k^2+l_2l_3^2k^2+m_5l_3^2k+m_5l_3^3}{k^3l_3^2},\\
&c_{2,3}=6,\quad c_{2,4}=\frac{m_4l_3}{k^2},\quad c_{3,0}=\frac{(k+l_3)(k^2+8l_3k+l_3^2)}{k^2l_3^2},\\
&c_{3,1}=\frac{m_1k^2+4l_1k^2+4m_1l_3k+4l_1l_3k+m_1l_3^2}{k^2l_3},\\
&c_{3,2}=\frac{l_2k^3+m_2l_3k^2+m_1l_1l_3k^2+l_2l_3k^2+m_2l_3^2k+m_5l_3^2}{k^3l_3},\\
&c_{3,3}=\frac{m_3l_3}{k^2},\quad c_{4,0}=\frac{3(k^2+3l_3k+l_3^2)}{k^2l_3},\quad c_{4,1}=\frac{2(km_1+l_3m_1+kl_1)}{k^2},\quad c_{4,2}=\frac{m_2l_3}{k^2},\\
&c_{5,0}=\frac{3(k+l_3)}{k^2},\quad c_{5,1}=\frac{m_1l_3}{k^2},\quad c_{6,0}=\frac{l_3}{k^2},
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
&k=b_0b_1b_2^2,\\
&(1+z)(1+b_2z)(1+b_1b_2z)=1+l_1z+l_2z^2+l_3z^3,\\
&(1+\frac{z}{b_3})(1+\frac{z}{b_3b_4})\cdots(1+\frac{z}{b_3b_4\cdots b_8})=1+m_1z+\cdots+m_6z^6.
\end{aligned} \tag{30}$$

Case $E_7^{(1)}$ (Fig11) Autonomous condition: $b_0^2b_1b_2^2b_3^3b_4^4b_5^3b_6^2b_7=1$. The integral is

$$\begin{aligned}
H=&\frac{f^3}{g^2}+\frac{b_0^2}{fg^2}+\frac{2f^2(1+b_0)}{g^2}+\frac{2b_0(1+b_0)}{g^2}+\frac{f(1+4b_0+b_0^2)}{g^2}+\frac{b_0^2m_1}{fg}+\frac{b_0^2m_2}{f}+\frac{gb_0^2m_3}{f}\\
&+\frac{g^2b_0^2m_4}{f}+\frac{f^2m_5}{g}+\frac{b_0(m_1+b_0m_1+m_5)}{g}+\frac{f(b_0m_1+m_5+b_0m_5)}{g}+fm_6+gm_7
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
&(1+z)(1+b_3z)(1+b_2b_3z)(1+b_1b_2b_3z)=1+m_1z+\cdots+m_4z^4,\\
&(1+\frac{z}{b_4})(1+\frac{z}{b_4b_5})(1+\frac{z}{b_4b_5b_6})(1+\frac{z}{b_4b_5b_6b_7})=1+m_5z+\cdots+m_8z^4.
\end{aligned} \tag{32}$$

Case $E_6^{(1)}$ (Fig.10) Autonomous condition: $b_0b_1b_2b_3b_4b_5b_6=1$. The integral is

$$\begin{aligned} H=&\frac{f^2}{gb_1b_2^2}+\frac{(b_1b_2+b_2+1)f}{gb_1b_2^2}+\frac{(b_4b_5+b_5+1)f}{b_1b_2^2b_3b_4b_5}+\frac{b_2b_1+b_1+1}{gb_1b_2}\\ &+\frac{g(b_5b_4+b_4+1)}{b_1b_2^2b_3^2b_4^2b_5}+\frac{g^2b_0b_6^2}{f}+\frac{gb_6(b_6b_0+b_0+1)}{f}+\frac{b_0b_6+b_6+1}{f}+\frac{1}{gf} \end{aligned} \tag{33}$$

Case $D_5^{(1)}$. (Fig.9) Autonomous condition: $b_0b_1b_2^2b_3^2b_4b_5=1$. The integral is

$$H=\frac{1}{b_5fg}+\frac{1+b_5}{b_5f}+\frac{g}{f}+\frac{1+b_1+b_1b_2}{b_1b_2b_5g}+b_0g+\frac{(1+b_2+b_1b_2)f}{b_1b_2^2b_5g}+b_0b_3(1+b_4)f+\frac{f^2}{gb_1b_2^2b_5}. \tag{34}$$

The q-Painlevé equation with $D_5^{(1)}$ symmetry is the q-$P_{\rm VI}$ [10], which is originally given in $\mathbb{P}^1\times\mathbb{P}^1$ rather than $\mathbb{P}^2$. The coordinates $(\tilde f,\tilde g)$ for $\mathbb{P}^1\times\mathbb{P}^1$ are given by $\tilde f=f, \tilde g g=f+b_1b_2$ and the corresponding amoeba is given in Fig.8.

Case $A_4^{(1)}$. (Fig.7) Autonomous condition: $b_0b_1b_2b_3b_4=1$. The integral is

$$H=g+\frac{1+b_1}{b_1b_4f}+\frac{g}{b_4f}+\frac{1+b_3}{b_1b_3b_4g}+\frac{1}{b_1b_4fg}+\frac{f}{b_1b_3b_4g}. \tag{35}$$

Case $(A_2+A_1)^{(1)}$. (Fig.6) Autonomous condition: $a_0a_1a_2=1$. The integral is

$$H=\frac{1}{a_1b_1f}+f+\frac{a_2}{g}+\frac{a_2f}{g}+\frac{g}{b_1}+\frac{g}{a_1b_1f}. \tag{36}$$

Case $(A_1+A_1)^{(1)}$. (Fig.5) Autonomous condition: $a_0a_1=1$. The integral is

$$H=\frac{1}{a_1bf}+\frac{f}{a_1}+\frac{1}{a_1bg}+\frac{1}{a_1bfg}+g. \tag{37}$$

Case $A_1^{(1)}$. (Fig.4) Autonomous condition: $a_0a_1=1$. The integral is

$$H=\frac{a_1}{f}+f+\frac{a_1}{fg}+a_1g. \tag{38}$$

Case $A_1^{(1)'}$. (Fig.3) Autonomous condition: $a_0a_1=1$. The integral is

$$H=\frac{a_1}{f}+a_1f+\frac{1}{g}+g. \tag{39}$$

Case $A_0^{(1)}$. (Fig.2) Autonomous condition: $a=1$. The integral is

$$H=f+\frac{1}{g}+\frac{g}{f}. \tag{40}$$

Normalize the above integrals H as $\tilde H=NH$ with the normalization constants in Table 1. Then $\tilde H$ is invariant under the affine Weyl group actions.

The Newton polygons of the integrals for Mul.4-Mul.10 cases are given in Fig(1). These polygons can also be interpreted as toric diagram which reflect the blowing up structure of the corresponding rational surfaces.

Case	Normalization factor	Case	Normalization factor
$E_8^{(1)}$	1	$E_7^{(1)}$	$b_0^{-3/2} b_1^{-1/2} b_2^{-1} b_3^{-3/2}$
$E_6^{(1)}$	$b_1^{2/3} b_2^{4/3} b_3 b_4^{2/3} b_5^{1/3}$	$D_5^{(1)}$	$b_1^{2/5} b_2^{-1/5} b_3^{1/5} b_4^{3/5}$
$A_4^{(1)}$	$b_0^{-1/2} b_3^{-1/2} b_4^{-1/4} b_5^{1/4}$	$(A_2 + A_1)^{(1)}$	$a_0^{1/3} a_1^{2/3} b_1^{1/2}$
$(A_1 + A_1)^{(1)}$	$a_1^{1/2}$	$A_1^{(1)}$	1
$A_1^{(1)\prime}$	$a_1^{-1/2}$	$A_0^{(1)}$	1

Table 1: Normalization factor

References

[1] A.Berenstein, S.Fomin and A.Zelevinsky, *Parametrization of canonical bases and totally positive matrices*, Adv. in Math. 122(1996), 49-149.

[2] A.Berenstein and D.Kazhdan, *Geometric and unipotent crystals*, GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. 2000, Special Volume, Part I, 188-236.

[3] O.DeWolfe, T.Hauer, A.Iqbal and B.Zweibach, *Five-branes, Seven-branes and Five-dimensional E_n field theories*, JHEP 9903 006 (1999).

[4] O.DeWolfe and B.Zwiebach, *String Junctions for Arbitrary Lie Algebra Representations.* Nuclear Physics **B541** 509 (1999).

[5] I.Dolgachev and D.Ortland, *Point sets in projective spaces and the theta functions*, Astérisque **165**(1998) Soc. Math. de France.

[6] T.Eguchi and K.Sakai, *Seiberg-Witten Curve for E-String Theory Revisited*, Adv. Theor. Math. Phys., **7** (2003), 419-455.

[7] *Seiberg-Witten Curve for the E-String Theory*, JHEP05(2002)058.

[8] O.J.Ganor, D.R.Morrison and N.Seiberg, *Branes, Calabi-Yau Spaces, and Troidal Compactification of the $N = 1$ Six-Dimensional E_8 Theory*, hep-th/9610251.

[9] T.Hauer and A.Iqbal, *Del Pezzo Surfaces and Affine 7-brane Backgrounds*, JHEP 0001 043 (2000).

[10] M.Jimbo and H.Sakai, *q-analog of the sixth Painlevé equation* Lett. Math. Phys. 38 (1996)145–154.

[11] K.Kajiwara, T.Masuda, M.Noumi, Y.Ohta and Y.Yamada, *Cubic Pencils and Painlevé Hamiltonians*, Funkcial. Ekvac. **48** (2005) 147–160.

[12] K.Kajiwara, T.Masuda, M.Noumi, Y.Ohta and Y.Yamada, ${}_{10}E_9$ *solution to the elliptic Painlevé equation*, J. Phys. **A36** (2003) L263–L272.

[13] K.Kajiwara, M.Noumi and Y.Yamada, *Discrete dynamical systems with $W(A_{m-1}^{(1)} \times A_{n-1}^{(1)})$ symmetry*, Lett. Math. Phys. **60** (2002) 211–219.

[14] A.N.Kirillov, *Introduction to Tropical Combinatorics*, Physics and Combinatorics 2000 (Proceedings of the Nagoya 2000 International Workshop on Physics and Combinatorics, Nagoya University, August 21-26, 2000), 82-149, World Scientific, Singapore, 2001

[15] A.Kuniba, M.Okado, T.Takagi and Y.Yamada, *Geometric crystal and tropical R for $D_n^{(1)}$*, Int. Math. Res. Not. **48** (2003) 2565–2620.

[16] A.Kuniba, M.Okado, T.Takagi and Y.Yamada, *Tropical R and tau functions*, Comm. Math. Phys. **245** (2004) 491–517.

[17] Ju.I.Manin, *The Tate height of points on an abelian variety. Its variants and applications*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), AMS Transl. (2) **59**(1966),82-110.

[18] G.Mikhalkin, *Amoebas of algebraic varieties and tropical geometry*, In Different faces of geometry, Int. Math. Ser. (N. Y.), pages 257 Kluwer/Plenum, New York, 2004.

[19] G.Mikhalkin, *Tropical Geometry and its applications*, arXiv:math.AG/0601041.

[20] K.Mohri, *Exceptional String: Instanton Expansions and Seiberg-Witten Curve*, hep-th/0110121.

[21] S.Mizoguchi and Y.Yamada, $W(E_{10})$ *symmetry, M-theory and Painlevé equations*, Phys. Lett. **B537** (2002) 130–140.

[22] M.Noumi and Y.Yamada, *Affine Weyl group, Discrete dynamical systems and Painlevé equations*, Commun. Math. Phys. 199 (1998) 281.

[23] M.Noumi and Y.Yamada, *Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions*, Adv. Stud. Pure Math. **40** (2004) 371–442.

[24] H.Sakai, *Rational surfaces with affine root systems and geometry of the Painlevé equations*, Commun. Math. Phys. **220** (2001) 165-221.

[25] N.Seiberg and E.Witten, *Monopoles, Duality and Chiral Symmetry Breaking in* $N = 2$ *Supersymmetric QCD*, Nuclear Phys., **B431** (1994), 484-550.

[26] N.Seiberg, *Five Dimensional SUSY Field Theories, Non-trivial Fixed Points and String Dynamics*, hep-th/9608111.

[27] T.Tokihiro, D.Takahashi, J.Matsukidaira and J.Satsuma, *From soliton equations to integrable cellular automata through a limiting procedure*, Phys. Rev. Lett. 76(1996), 3247-3250.

[28] T.Tsuda, *Tropical Weyl group action via point configurations and* τ*-functions of the* q*-Painlevé equations*, Lett. Math. Phys. **77** (2006)21–30.
T.Tsuda and T.Takenawa, *Tropical representation of Weyl groups associated with certain rational varieties*, arXiv:math/0607661.

[29] Y.Yamada, *A birational representation of Weyl group, combinatorial R matrix and discrete Toda equation*, in Physics and Combinatorics 2000, Proceedings of the Nagoya International Workshop (Eds. A.N. Kirillov and N. Liskova), pp. 305-319, World Scientific, 2001.

[30] Y.Yamada and S.-K.Yang, *Affine 7-brane Backgrounds and Five-Dimensional* E_N *Theories on* S^1, Nuclear Physics **B566** 642 (2000).

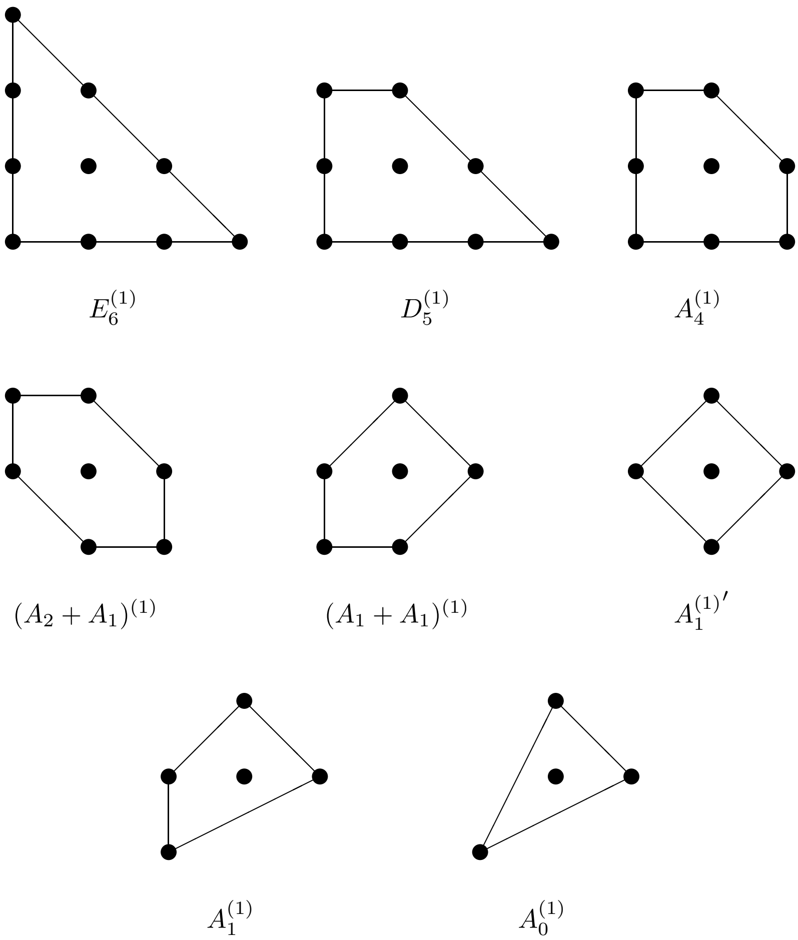


Figure 1: Newton polygons

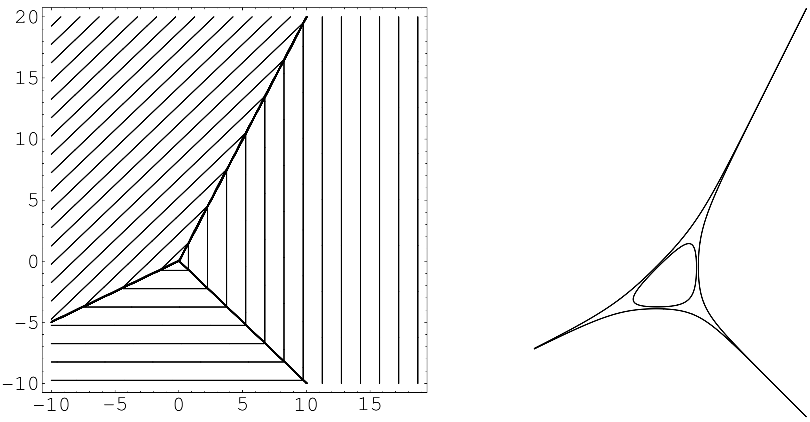


Figure 2: $A_0^{(1)}$ amoeba

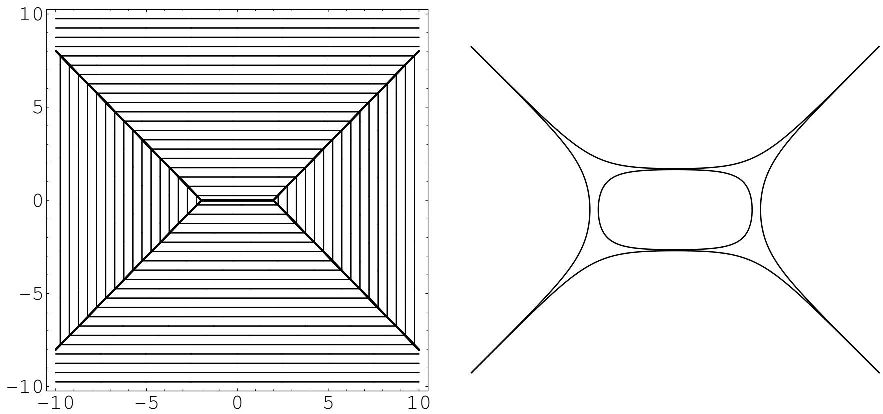


Figure 3: $A_1^{(1)'}$ amoeba : $(a_0, a_1) = (2, -2)$

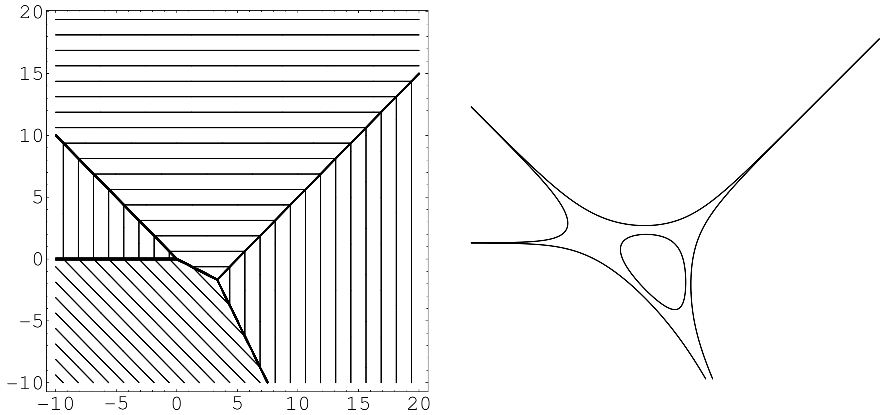


Figure 4: $A_1^{(1)}$ amoeba : $(a_0, a_1) = (-5, 5)$

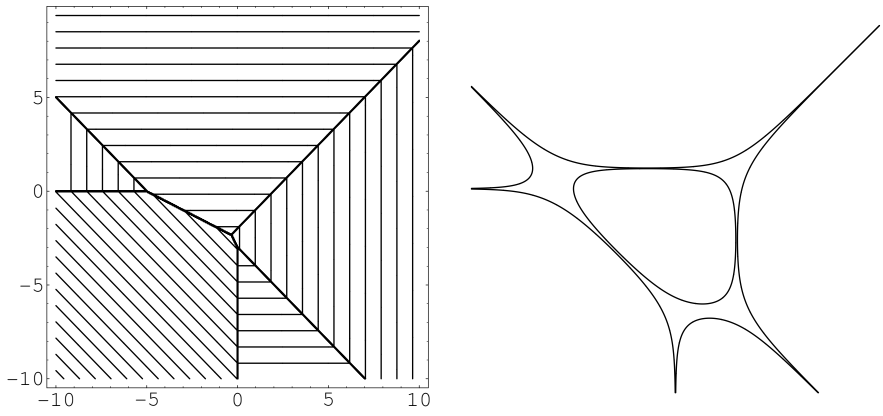


Figure 5: $(A_1 + A_1)^{(1)}$ amoeba : $(a_0, a_1, b) = (2, -1, 3)$

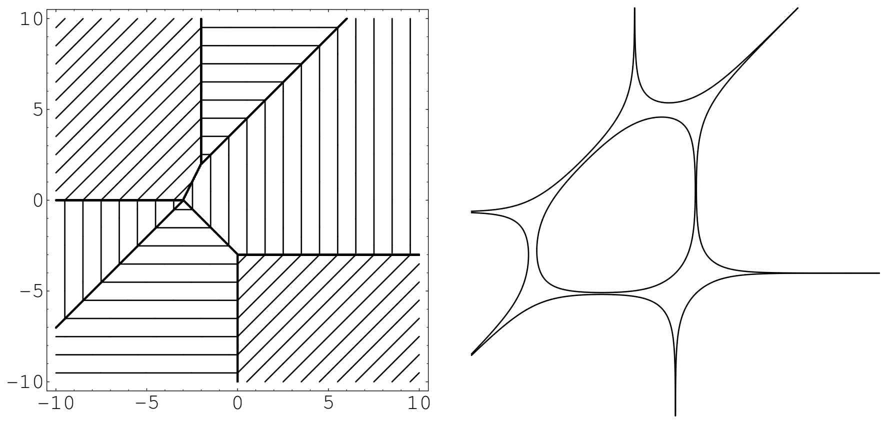


Figure 6: $(A_2 + A_1)^{(1)}$ amoeba : $(a_0, a_1, a_2, b_0, b_1) = (1, 2, -3, -4, 4)$

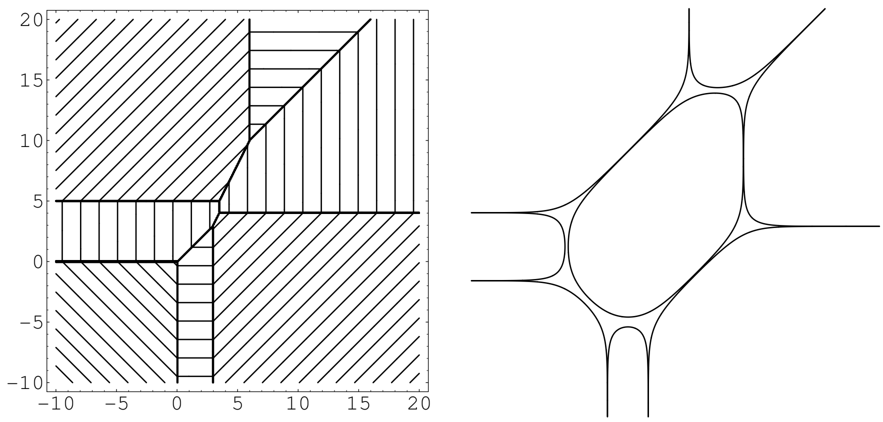


Figure 7: $A_4^{(1)}$ amoeba : $(b_0, \cdots, b_4) = (4, -5, 4, 3, -6)$

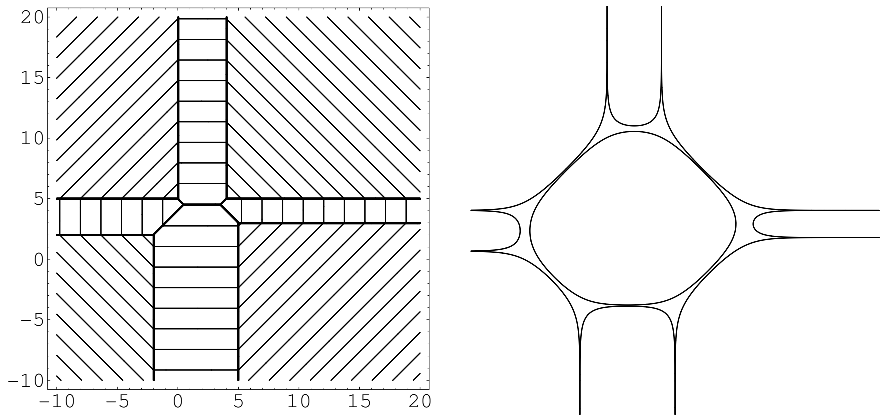


Figure 8: $D_5^{(1)}$ amoeba $(\mathbb{P}^1 \times \mathbb{P}^1)$: $(b_0, \cdots, b_5) = (3, 1, 4, -3, -2, -3)$

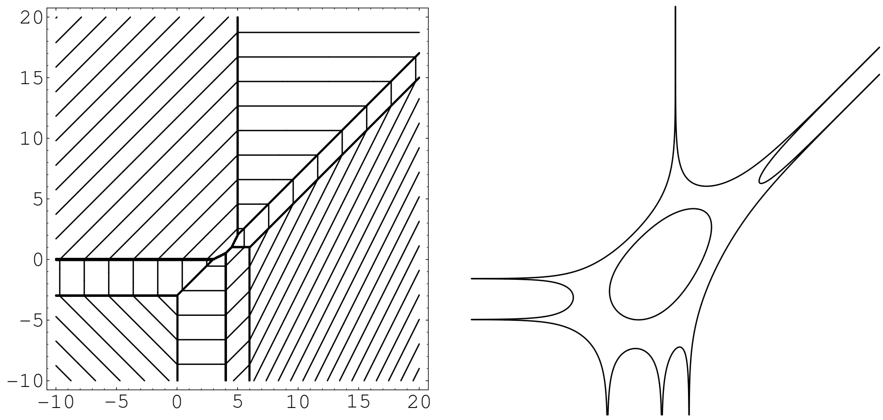


Figure 9: $D_5^{(1)}$ amoeba $(\mathbb{P}^2)$: $(b_0, \cdots, b_5) = (-4, 2, 4, -3, -2, 3)$

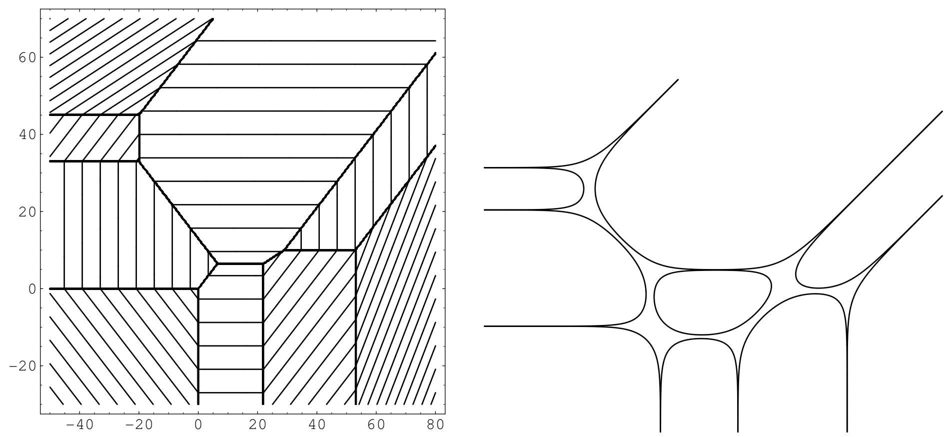


Figure 10: $E_6^{(1)}$ amoeba : $(b_0, \cdots, b_6) = (12, 31, 22, -43, 24, 84, -45)$

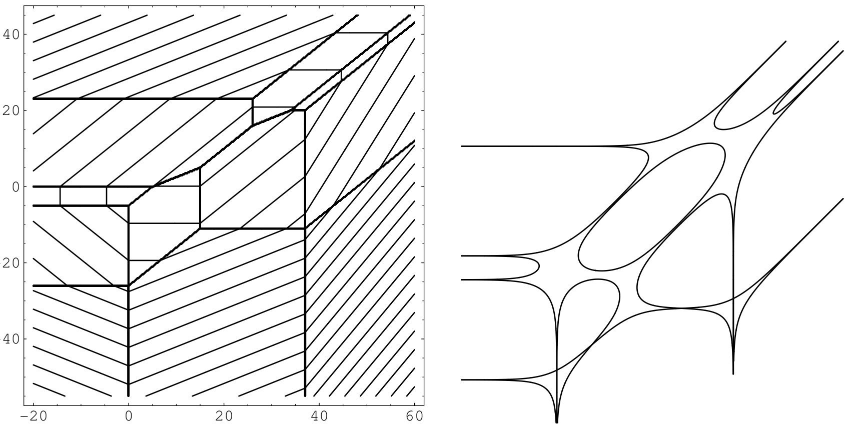


Figure 11: $E_7^{(1)}$ amoeba : $(b_0, \cdots, b_7) = (37, 21, 28, -23, -17, 14, -11, -34)$

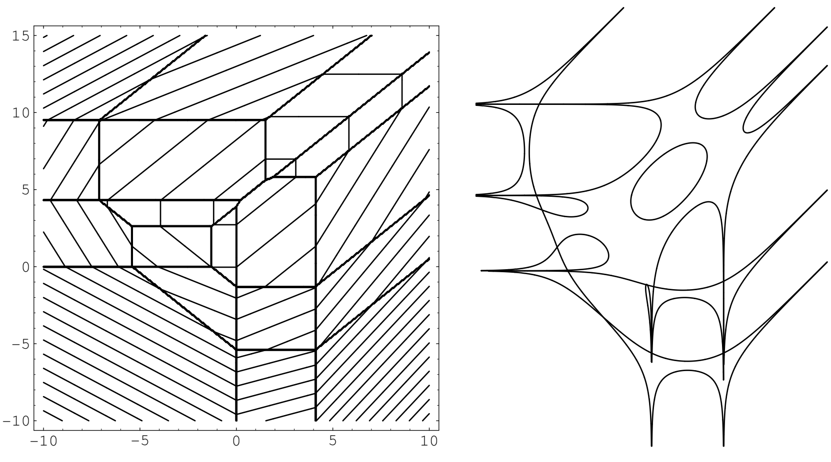


Figure 12: $E_8^{(1)}$ amoeba : $(b_0, \cdots, b_8) = (41, -52, -43, -54, -41, 112, 63, -41, 127)/10$