THE CENTRAL LIMIT THEOREM FOR NON-CONVENTIONAL AVERAGE

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1. Introduction

A typical result that we are to present here is

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\big(f(\theta^n\cdot)g(\theta^{n^2}\cdot)-m\big)\xrightarrow{\mathcal{D}}\mathrm{N}(0,v)\qquad(N\to\infty),$$

where $\theta>1$, and f and g are smooth functions with period 1.

This study was motivated by the polynomial ergodic theorem below:

Theorem 1. *If the transform T on a probability space is weakly mixing, and p_1, $\dots$, p_K are polynomials with $p_k(\mathbf{N})\subset\mathbf{N}$, $p_k(\infty)=\infty$, and $(p_{k+1}-p_k)(\infty)=\infty$, then*

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{k=1}^{K}f_k(T^{p_k(n)}\cdot)\xrightarrow{L^2}\prod_{k=1}^{K}Ef_k\qquad(N\to\infty),$$

for any bounded measurable functions f_1, $\dots$, f_K.

This ergodic theorem for non-conventional average was proved by V. Bergelson [2], and the pointwise convergence for special cases were proved by J. Bourgain [6]. Earlier than these works, limiting behavior of the average of this type was studied by H. Furstenberg, Y. Katznelson & D. Ornstein [12] to give an ergodic theoretical proof of Szeméredi's theorem. For further results, we refer the reader to H. Furstenberg & B. Weiss [11]. The term 'non-conventional average' is due to them.

We give much simpler proof of the polynomial ergodic theorem for a transform $\omega\mapsto\theta\omega$ on $\mathbf{R}$, where $\theta>1$. When θ is not an integer, iteration of this transform can not be regarded as iteration of some transform on $[\,0,1\,]$ or other finite intervals, and hence the next theorem is not included in Theorem 1.

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Theorem 2. *Let $p_1, \ldots, p_K$ be polynomials with $p_k(\infty) = \infty$ and $(p_{k+1} - p_k)(\infty) = \infty$, and $f_1, \ldots, f_K$ be functions on $\mathbf{R}$ with period 1 satisfying $f_k \in L^{2K}\big([\,0,1\,], dx\big)$. Then we have*

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{k=1}^{K} f_k(\theta^{p_k(n)}\,\cdot\,) \to \prod_{k=1}^{K}\int_0^1 f_k(x)\,dx \qquad (N\to\infty),$$

in $L^2\big([\,0,1\,], dx\big)$-sense.

Having the weak law of large numbers, it is natural to ask whether the central limit theorem holds or not. In this note, we give an answer to this question.

Before stating our results, we introduce the results for $K = 1$. When p_1 is linear, our problem is reduced to the following central limit theorem for Riesz-Raikov sums.

Theorem 3. *Let $\theta > 1$ and let f be a locally square integrable function on $\mathbf{R}$ with period 1 satisfying the following L^2-Dini condition:*

$$\int_0^1 \frac{\omega_2(y)}{y}\,dy < \infty \quad \textit{where} \quad \omega_2(\delta) = \sup_{|h|\le\delta}\left(\int_0^1 \big|f(x+h) - f(x)\big|^2\,dx\right)^{1/2}. \tag{1.1}$$

Then under any probability measure on $[\,0,1\,]$ which is absolutely continuous with respect to the Lebesgue measure, we have the following convergence in law:

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\widetilde{f}(\theta^n\,\cdot\,) \xrightarrow{\mathcal{D}} \mathrm{N}(0,v) \qquad (N\to\infty),$$

where $\widetilde{f} = f - \int_0^1 f(x)\,dx$. The limiting variance v is determined as follows: $v = \int_0^1 \widetilde{f}(x)^2\,dx$ if

$$\theta^n \notin \mathbf{Q} \quad \textit{for all} \quad n\in\mathbf{N},$$

and if else

$$v = \int_0^1 \widetilde{f}(x)^2\,dx + 2\sum_{n=1}^{\infty}\int_0^1 \widetilde{f}(q^n x)\widetilde{f}(r^n x)\,dx,$$

where

$$l = \min\{\, n\in\mathbf{N} \mid \theta^n \in \mathbf{Q}\,\}, \quad \theta^l = q/r, \quad \textit{and} \quad q, r\in\mathbf{N}.$$

This theorem was initially proved by M. Kac [15] and I. Ibragimov [14] in case when θ is an integer, and after extensive studies by S. Takahashi [23], I. Berkes [3], [4], and R. Kaufman [16], it was established as above by B. Petit [17] and the author [9].

On the other hand, in case $\deg p_1 \ge 2$, we have $\theta^{p_1(n+1)}/\theta^{p_1(n)} \to \infty$, which implies that this problem reduced to the special case of the following central limit theorem for gap series with 'large gaps'.

Theorem 4. *Let f be as in Theorem 3 and $\{\beta_n\}$ be a sequence of positive numbers satisfying*

$$\beta_{n+1}/\beta_n \to \infty \qquad (n \to \infty).$$

Then under any absolutely continuous probability measure on $[\,0,1\,]$, we have

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \widetilde{f}(\beta_n \,\cdot\,) \xrightarrow{\mathcal{D}} \mathrm{N}(0,v) \qquad (N \to \infty),$$

where $\widetilde{f} = f - \int_0^1 f(x)\,dx$ and $v = \int_0^1 \widetilde{f}(x)^2\,dx$.

This result is due to S. Takahashi [23] and I. Berkes [3], [4]. Actually, Takahashi proved the above theorem by assuming that every β_n is an integer, and Berkes removed this condition by assuming a stronger regularity condition on f. But this version can be easily proved by the method of Berkes, and also by some modification of the method of Takahashi, which is similar to the proof found in [9].

Let us now state the results for $K \ge 2$. Assuming that $f_1, \dots, f_K$ are centered, i.e.,

$$\int_0^1 f_k(x)\,dx = 0 \qquad (k = 1, \dots, K), \tag{1.2}$$

the author [10] have proved the following result:

Theorem 5. *Let $K \ge 2$ and $\theta > 1$. Let $p_1, \dots, p_K$ be as in Theorem 2, and let functions $f_1, \dots, f_K$ on $\mathbf{R}$ with period 1 satisfy* (1.1), (1.2) *and*

$$\int_0^1 \bigl|f_k(x)\bigr|^{2K-2}\,dx < \infty \qquad (k = 1, \dots, K). \tag{1.3}$$

Then, under any probability measure on $\mathbf{R}$ which is absolutely continuous with respect to the Lebesgue measure, we have

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \prod_{k=1}^{K} f_k(\theta^{p_k(n)} \,\cdot\,) \xrightarrow{\mathcal{D}} \mathrm{N}(0,v) \qquad (N \to \infty).$$

The limiting variance v is determined as follows:

1. If $\max_k \deg p_k \ge 2$, then

$$v = \prod_{k=1}^{K} \int_0^1 f_k^2(x)\,dx. \tag{1.4}$$

2. *When all p_k are linear, let us put $p_k(x) = a_k x + b_k$.*
   a. *If the condition*
   $$\theta^{a_k n} \notin \mathbf{Q} \quad \text{for all} \quad n \in \mathbf{N}$$
   is satisfied for at lease one of $k = 1, \ldots, K$, then v is given by (1.4).
   b. *If else, let us take the smallest $n \geq 1$ satisfying $\theta^{a_k n} \in \mathbf{Q}$ for all k, and write $\theta^{a_k n} = q_k / r_k$ by using q_k, $r_k \in \mathbf{N}$. Then*
   $$v = \prod_{k=1}^{K} \int_0^1 f_k^2(x)\,dx + 2 \sum_{n=1}^{\infty} \prod_{k=1}^{K} \int_0^1 f_k(q_k^n x) f_k(r_k^n x)\,dx.$$

In the case when f_k are not centered, we could not give a complete answer as above. Actually, we could not prove any results in case when some of the p_i's have the same degree.

Theorem 6. *Let $K \geq 2$ and $\theta > 1$. Let polynomials $p_1, \ldots, p_K$ satisfy $p_k(\infty) = \infty$ and*
$$\deg p_1 < \cdots < \deg p_K, \tag{1.5}$$
and functions $f_1, \ldots, f_K$ on $\mathbf{R}$ with period 1 satisfy L^2-Dini condition (1.1) and (1.3). Set $m_k = \int_0^1 f_k(x)\,dx$ and let a be the coefficient of linear term of p_1. Then
$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} \Bigl(\prod_{k=1}^{K} f_k(\theta^{p_k(n)} \cdot\,) - \prod_{k=1}^{K} m_k \Bigr) \xrightarrow{\mathcal{D}} \mathrm{N}(0, v) \qquad (N \to \infty),$$
where
$$v = \prod_{k=1}^{K} \int_0^1 f_k^2(x)\,dx - \prod_{k=1}^{K} m_k^2,$$
if $\deg p_1 > 1$ or if $\theta^{an} \notin \mathbf{Q}$ for all $n \in \mathbf{N}$. If else
$$v = \prod_{k=1}^{K} \int_0^1 f_k^2(x)\,dx - \prod_{k=1}^{K} m_k^2 + 2 \prod_{k=2}^{K} m_k^2 \sum_{n=1}^{\infty} \int_0^1 \widetilde{f}_1(q^n x) \widetilde{f}_1(r^n x)\,dx,$$
where $\widetilde{f}_1 = f_1 - \int_0^1 f_1(x)\,dx$,
$$l = \min\{\, n \in \mathbf{N} \mid \theta^{an} \in \mathbf{Q} \,\}, \quad \theta^{al} = q/r, \quad \text{and} \quad q, r \in \mathbf{N}.$$

2. Polynomial ergodic theorem for Riesz-Raikov sums.

We use the notation $\|f\|_\infty = \operatorname{ess\,sup}|f(x)|$, $\|f\|_p = \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p}$, $m_k = \int_0^1 f_k(x)\,dx$, $\widetilde{f_k} = f_k - m_k$, and $v_k = \left\|\widetilde{f_k}\right\|_2^2$. Let $s_{f,I}$ denote the I-th subsum of the Fourier series of f. Let us put

$$h_\lambda(x) = \left(\frac{\sin \lambda x}{\lambda x}\right)^2 \quad \text{and} \quad h(x) = \frac{h_{1/2}(x) + h_{1/2\sqrt{2}}(x)}{2\pi\bigl(1+\sqrt{2}\,\bigr)}.$$

Then h is a positive integrable analytic function on $\mathbf{R}$ which satisfies $\widehat{h}(u) = 0$ $(|u| > 1)$, $|\widehat{h}(u)| \le 1$ $(u \in \mathbf{R})$, and $\int_{\mathbf{R}} h(u)\,du = 1$. (Cf. L. Breiman [8] pp.218).

Let μ_0 be a probability measure on $\mathbf{R}$ whose density is h. Since h is positive and continuous on $[\,0,1\,]$, $L^2(\mathbf{R},\mu_0)$-convergence implies $L^2([\,0,1\,],dx)$-convergence.

First let us prove Theorem 2 in the case when $f_1, \ldots, f_K$ are trigonometric polynomials whose degrees are less than I. Let us decompose the product as follows.

$$\begin{aligned} \prod_{k=1}^K f_k(\theta^{p_k(n)}\,\cdot\,) &= \sum_{\kappa=1}^K \prod_{k=1}^{\kappa-1} f_k(\theta^{p_k(n)}\,\cdot\,)\widetilde{f}_\kappa(\theta^{p_\kappa(n)}\,\cdot\,) \prod_{k=\kappa+1}^K m_k + \prod_{k=1}^K m_k \\ &= \sum_{\kappa=1}^K U_{\kappa,n}(\,\cdot\,) \prod_{k=\kappa+1}^K m_k + \prod_{k=1}^K m_k, \qquad \text{(say.)} \end{aligned} \tag{2.1}$$

Thus it is sufficient to prove $L^2(\mathbf{R},\mu_0)$-convergence $\frac{1}{N}\sum_{n=1}^N U_{\kappa,n} \to 0$, for each $\kappa = 1, \ldots, K$.

Since any frequency appears in the trigonometric polynomial expansion of $U_{\kappa,n}$ is of the form

$$\theta^{p_\kappa(n)} i_\kappa + \theta^{p_{\kappa-1}(n)} i_{\kappa-1} + \cdots + \theta^{p_1(n)} i_1, \quad (|i_1|, \ldots, |i_\kappa| \le I),$$

and because of $\theta^{p_k(n)} = o\bigl(\theta^{p_\kappa(n)}\bigr)$ $(k < \kappa,\ n \to \infty)$, for large n, modulus of any frequency of $U_{\kappa,n}$ belongs to $J_n = (\theta^{p_\kappa(n)}/2, 2I\theta^{p_\kappa(n)})$. Since p_κ is a polynomial diverges to infinity, there exists $c > 0$ such that $p_\kappa(n+1) - p_\kappa(n) \ge c$ for large n. Thus there exists n_0, $L \in \mathbf{N}$ such that, for $n \ge n_0$ and $l \ge L$, J_n and J_{n+l} are disjoint and separated at least 1. Therefore $U_{\kappa,n}$ and $U_{\kappa,n+l}$ are orthogonal in $L^2(\mathbf{R},\mu_0)$. Since $\|U_{\kappa,n}\|_\infty$ is bounded, we see that $\frac{1}{N}\sum_{n=1}^N U_{\kappa,n}$ converges to 0 in $L^2(\mathbf{R},\mu_0)$-sense.

Next, we prove the general case. Clearly, we have

$$\begin{aligned}&\Big\|\frac{1}{N}\sum_{n=1}^{N}\prod_{k=1}^{K} f_k(\theta^{p_k(n)}\,\cdot\,) - \frac{1}{N}\sum_{n=1}^{N}\prod_{k=1}^{K} s_{f_k,I}(\theta^{p_k(n)}\,\cdot\,)\Big\|_{L^2(\mu_0)}\\&\quad\le \frac{1}{N}\sum_{\kappa=1}^{K}\sum_{n=1}^{N}\Big\|\prod_{k=1}^{\kappa-1} f_k(\theta^{p_k(n)}\,\cdot\,)(f_\kappa - s_{f_\kappa,I})(\theta^{p_\kappa(n)}\,\cdot\,)\prod_{k=\kappa+1}^{K} s_{f_k,I}(\theta^{p_k(n)}\,\cdot\,)\Big\|_{L^2(\mu_0)}.\end{aligned}$$

Note that $\|F(\varphi\,\cdot\,)\|_{L^{2l}(\mu_0)} = \|F\|_{2l}$ $(\varphi > 1)$ is easily verified for F with period 1. It is known that if $F \in L^p\big([0,1], dx\big)$, $\|F - s_{F,I}\|_p \to 0$ and $\|s_{F,I}\|_p \le C\|F\|_p$ hold for an absolute constant C. (Cf. A. Zygmund [24].) Using Hölder's inequality and these relations in turn, we see that the summand in the above sum is estimated by

$$\begin{aligned}&\prod_{k=1}^{\kappa-1}\|f_k(\theta^{p_k(n)}\,\cdot\,)\|_{L^{2K}(\mu_0)}\|(f_\kappa - s_{f_\kappa,I})(\theta^{p_\kappa(n)}\,\cdot\,)\|_{L^{2K}(\mu_0)}\prod_{k=\kappa+1}^{K}\|s_{f_k,I}(\theta^{p_k(n)}\,\cdot\,)\|_{L^{2K}(\mu_0)}\\&\quad= \prod_{k=1}^{\kappa-1}\|f_k\|_{2K}\|f_\kappa - s_{f_\kappa,I}\|_{2K}\prod_{k=\kappa+1}^{K}\|s_{f_k,I}\|_{2K}\\&\quad\le C^{K-\kappa}\prod_{k=1}^{\kappa-1}\|f_k\|_{2K}\|f_\kappa - s_{f_\kappa,I}\|_{2K}\prod_{k=\kappa+1}^{K}\|f_k\|_{2K} \to 0 \qquad (I \to \infty),\end{aligned}$$

and thereby have the conclusion.

3. The central limit theorem for trigonometric polynomials.

In this section we prove Theorem 6 in case when $f_1, \ldots, f_K$ are trigonometric polynomials whose degrees are less than I. We may assume that p_1 is linear, i.e., $p_1(n) = an + b$. Actually, if $\deg p_1 > 1$, by putting $p_0(n) = n$ and $f_0 = 1$ the following argument yields the desired result because of $\int f_0^2 = 1$ and $\widetilde{f_0} = 0$.

For $u \in \mathbf{R}$, let μ_u be a $\mathbf{C}$-valued measure on $\mathbf{R}$ whose density is $e^{\sqrt{-1}\,ux}h(x)$. Clearly it satisfies $\big|\int_{\mathbf{R}} f(x)\,\mu_u(dx)\big| \le \int_{\mathbf{R}}\big|f(x)\big|\,\mu_0(dx)$ and

$$\int_{\mathbf{R}} e^{\sqrt{-1}\,\lambda x}\,\mu_u(dx) = 0 \quad \text{if} \quad |\lambda| \ge U = |u| + 1.$$

The next two lemmas make our calculation simple. The idea of Lemma 1 goes back to P. Hartman [13], and that of Lemma 2 originated to P. Révész [19] and R. Salem & A. Zygmund [21]. Proofs are found in [10].

Lemma 1. *If a sequence $\{X_N\}$ of real functions on* $\mathbf{R}$ *satisfies*

$$\int_{\mathbf{R}} e^{\sqrt{-1}\,tX_N(x)}\,\mu_u(dx) \to e^{-t^2v/2}\widehat{h}(u) \qquad (t \in \mathbf{R},\ N \to \infty), \tag{3.1}$$

for all $u \in \mathbf{R}$, then under any probability measure on $\mathbf{R}$ which is absolutely continuous with respect to the Lebesgue measure,

$$X_N \xrightarrow{\mathcal{D}} \mathrm{N}(0, v) \qquad (N \to \infty).$$

Lemma 2. *Let $u \in \mathbf{R}$ and $\{\xi_{m,N}\}_{1 \le m \le M_N,\, N \ge 1}$ be an array of functions on $\mathbf{R}$. If*

$$B_N = \sup_{1 \le m \le M_N} \|\xi_{m,N}\|_\infty \to 0 \quad (N \to \infty), \tag{3.2}$$

$$\int_{\mathbf{R}} \xi_{m_1,N} \ldots \xi_{m_r,N}\, d\mu_u = 0 \quad (N \in \mathbf{N},\ r \in \mathbf{N},\ m_0 \le m_1 < \cdots < m_r), \tag{3.3}$$

$$V_N = \sum_{m=m_0}^{M_N} \xi_{m,N}^2 \longrightarrow v \quad \text{in measure } \mu_0 \quad (N \to \infty), \tag{3.4}$$

$$B_0 = \sup_{N \ge 1} \|V_N\|_\infty < \infty, \tag{3.5}$$

are satisfied for some m_0, then (3.1) holds for $X_N = \sum_{m=1}^{M_N} \xi_{m,N}$.

First, let us put $\xi_{\kappa,N,n} = U_{\kappa,n} \prod_{k=\kappa+1}^{K} m_k/\sqrt{N}$ for $\kappa \ge 2$. By (2.1) we have

$$X_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \prod_{k=1}^{K} \widetilde{f}_k(\theta^{p_k(n)}\,\cdot\,) = \sum_{\kappa=2}^{K} \sum_{n=1}^{N} \xi_{\kappa,N,n} + \frac{1}{\sqrt{N}} \prod_{k=2}^{K} m_k \sum_{n=1}^{N} \widetilde{f}_1(\theta^{p_1(n)}\,\cdot\,).$$

We define $\xi_{1,N,n}$ as follows. Since the set $\{\, j\theta^{p_1(n)} \mid j \in \mathbf{Z}, 0 < |j| \le I, n \in \mathbf{N}\,\}$ is symmetric with respect to 0, we can write it as a sequence $\ldots,\ -\lambda_2,\ -\lambda_1,\ \lambda_1,\ \lambda_2,\ \ldots$ in increasing order. Note that $\lambda_{i+1}/\lambda_i > q > 1$ holds for some q. Let us take $i_0 > I$ such that

$$1 - \frac{1}{q} - \frac{1}{q^{i_0+1}} - \frac{1}{q^{2i_0+1}} - \cdots = 1 - \frac{1}{q(1 - 1/q^{i_0})} \ge \frac{1}{2}\Big(1 - \frac{1}{q}\Big) \tag{3.6}$$

$$1 + \frac{1}{q} + \frac{1}{q^{i_0+1}} + \frac{1}{q^{2i_0+1}} + \cdots = 1 + \frac{1}{q(1 - 1/q^{i_0})} \le 2 \tag{3.7}$$

$$1 - \frac{1}{q} - \frac{2}{q^{i_0+1}} \ge \frac{1}{2}\Big(1 - \frac{1}{q}\Big) \tag{3.8}$$

Let us take $j_0 > I$ as $\frac{1}{2}(1-\frac{1}{q})\lambda_{j_0} \geq U$. Let us put $\frac{1}{\sqrt{N}} \prod_{k=2}^{K} m_k \sum_{n=1}^{N} \widetilde{f}_1(\theta^{p_1(n)}x) = \sum_{i\in\mathbf{Z}} c_{N,i} \exp(2\pi\sqrt{-1}\,\lambda_i x)$, and define

$$\xi_{1,N,1} = \sum_{|i|=1}^{j_0} c_{N,i} \exp(2\pi\sqrt{-1}\,\lambda_i x), \quad \text{and} \quad \xi_{1,N,n} = \sum_{|i|=i_0 n+j_0+1}^{i_0(n+1)+j_0} c_{N,i} \exp(2\pi\sqrt{-1}\,\lambda_i x)$$

for $n \geq 2$. Since there are at most IN many positive frequencies, we have

$$\frac{1}{\sqrt{N}} \prod_{k=2}^{K} m_k \sum_{n=1}^{N} \widetilde{f}_1(\theta^{p_1(n)}x) = \sum_{n=1}^{N} \xi_{1,N,n} \quad \text{and hence} \quad X_N = \sum_{\kappa=1}^{K} \sum_{n=1}^{N} \xi_{\kappa,N,n}.$$

As we have explained in the proof of Theorem 1, when $\kappa \geq 2$, modulus of any frequency of $\xi_{\kappa,N,n}$ belongs to $(\theta^{p_\kappa(n)}/2, 2I\theta^{p_\kappa(n)})$. Let us set $n(1,N) = 2$, and take $n(\kappa,N) \in \mathbf{N}$ $(\kappa \geq 2)$ such that

$$\frac{\theta^{p_\kappa(n+1)}/2}{2I\theta^{p_\kappa(n)}} \geq 3 \quad (n \geq n(\kappa,N)), \qquad \frac{\theta^{p_{\kappa+1}(n(\kappa,N))}/2}{2I\theta^{p_\kappa(N)}} \geq 3, \tag{3.9}$$

$$\theta^{p_\kappa(n(\kappa,N))} \geq 2U, \quad \text{and} \quad \lim_{N\to\infty} \frac{n(\kappa,N)}{N} = 0 \tag{3.10}$$

Now let us verify the conditions of Lemma 2 for the sequence

$$\{\, \xi_{\kappa,N,n} \mid n(\kappa,N) \leq n \leq N, 1 \leq \kappa \leq K \,\}. \tag{3.11}$$

Let us first verify (3.3). Let us take arbitrary subset of (3.11), and take an arbitrary positive frequency ϕ_i of each function of $\kappa \geq 2$, and λ_i of each function of $\kappa = 1$. By the definition of i_0 and j_0, thanks to (3.6) and (3.7), we have

$$2I\theta^{p_1(N)} \geq 2\lambda_r \geq \lambda_r \pm \lambda_{r-1} \pm \cdots \pm \lambda_1 \geq \frac{1}{2}\Big(1 - \frac{1}{q}\Big)\lambda_r \geq U.$$

(3.9) assures that ϕ_1, ϕ_2, ...satisfies the Hadamard's gap condition with $q = 3$. Therefore we can verify (3.3) by

$$\begin{aligned} &\phi_\rho \pm \phi_{\rho-1} \pm \cdots \pm \phi_1 \pm \lambda_r \pm \lambda_{r-1} \pm \cdots \pm \lambda_1 \\ &\geq \phi_\rho - \phi_{\rho-1} - \cdots - \phi_1 - 2I\theta^{p_1(N)} \\ &\geq \phi_\rho\Big(1 - \frac{1}{3} - \cdots - \frac{1}{3^{\rho-1}} - \frac{1}{3^\rho}\Big) \\ &\geq \phi_\rho/2 \geq U. \end{aligned}$$

Next, let us verify the condition (3.4). Recall that Theorem 1 is originally proved as $L^2(\mu_0)$-convergence. In the case when $\kappa \geq 2$, by applying this, we have

$$\sum_{n=n(\kappa,N)}^{N} \xi_{\kappa,N,n}^2 \to \prod_{k=1}^{\kappa-1} \|f_k\|_2^2 \|\widetilde{f}_\kappa\|_2^2 \prod_{k=\kappa+1}^{K} m_k^2 \qquad (N \to \infty),$$

in $L^2(\mu_0)$-sense, and hence in measure μ_0. To calculate the case when $\kappa = 1$, we introduce the following notation. Denote

$$\int_{\mathbf{R}} f(x)\, \mu_R(dx) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx$$

whenever the right hand side exists. The right hand side is only a symbolic expression and does not mean integral with respect to the measure μ_R. Let us put $\beta_{N,n} = \int_{\mathbf{R}} \xi_{1,N,n}^2(x)\, \mu_R(dx)$, where it is well defined since $\xi_{1,N,n}^2$ is an almost periodic function (Cf. Besicovich [5]). Note that $\{\xi_{1,N,2n}^2 - \beta_{N,2n}\}$ and $\{\xi_{1,N,2n+1}^2 - \beta_{N,2n+1}\}$ are both orthogonal sequence under μ_0. Actually if $n - m \geq 2$, if $\pm\lambda_i \pm \lambda_j$ is a frequency of $\xi_{1,N,n}^2 - \beta_{N,n}$, and if $\pm\lambda_k \pm \lambda_r$ is a frequency of $\xi_{1,N,m}^2 - \beta_{N,m}$, then we have

$$|\pm\lambda_i \pm \lambda_j \pm \lambda_k \pm \lambda_r| \geq \lambda_i\Big(1 - \frac{1}{q} - \frac{2}{q^{i_0+1}}\Big) \geq \lambda_i \frac{1}{2}\Big(1 - \frac{1}{q}\Big) \geq U \geq 1.$$

By definition, there exists a constant does not depend on n and N such that $\|\xi_{1,N,n}\|_\infty \leq C/\sqrt{N}$. Thus we have $\big(\xi_{1,N,n}^2 - \beta_{N,n}\big)^2 \leq (2C^2)/N^2$ and hence

$$\sum_{n=1}^{N} \big(\xi_{1,N,n}^2 - \beta_{N,n}\big) \overset{L^2(\mu_0)}{\longrightarrow} 0.$$

By the orthogonality of $\xi_{1,N,n}$, we have

$$\begin{aligned}
\sum_{n=1}^{N} \beta_{N,n} &= \int_{\mathbf{R}} \Big(\sum_{n=1}^{N} \xi_{1,N,n}\Big)^2 \mu_R(dx) = \frac{1}{N}\prod_{k=2}^{K} m_k^2 \int_{\mathbf{R}} \Big(\sum_{n=1}^{N} \widetilde{f}_1(\theta^{an+b}x)\Big)^2 \mu_R(dx) \\
&= \frac{1}{N}\prod_{k=2}^{K} m_k^2 \sum_{n=1}^{N}\sum_{m=1}^{N} \int_{\mathbf{R}} \widetilde{f}_1(\theta^{a(n-m)}x)\widetilde{f}_1(x)\, \mu_R(dx) \\
&= \prod_{k=2}^{K} m_k^2 \Big(\int_{\mathbf{R}} \widetilde{f}_1(x)^2\, \mu_R(dx) + \frac{2(N-n)}{N} \sum_{n=1}^{N-1} \int_{\mathbf{R}} \widetilde{f}_1(\theta^{an}x)\widetilde{f}_1(x)\, \mu_R(dx)\Big) \\
&\to \prod_{k=2}^{K} m_k^2 \Big(\int_{\mathbf{R}} \widetilde{f}_1(x)^2\, \mu_R(dx) + 2\sum_{n=1}^{\infty} \int_{\mathbf{R}} \widetilde{f}_1(\theta^{an}x)\widetilde{f}_1(x)\, \mu_R(dx)\Big) \quad (N \to \infty)
\end{aligned}$$

Here the last convergence holds since f_1 is a trigonometric polynomial and the summand equals to 0 except for finitely many n. If $\theta^{an} \notin \mathbf{Q}$, then the summand equals to 0. Thus in case $\theta^{an} \notin \mathbf{Q}$ for all $n \in \mathbf{N}$, the sum reduced to $\int_{\mathbf{R}} \widetilde{f_1}(x)^2 \, \mu_R(dx) = \|\widetilde{f_1}\|_2^2$. If $\theta^{al} = q/r$, where l is a least n such that $\theta^{an} \in \mathbf{Q}$, the series equals to the sum of

$$\int_{\mathbf{R}} \widetilde{f_1}\big((q/r)^n x\big)\widetilde{f_1}(x) \, \mu_R(dx) = \int_{\mathbf{R}} \widetilde{f_1}(q^n x)\widetilde{f_1}(r^n x) \, \mu_R(dx) = \int_0^1 \widetilde{f_1}(q^n x)\widetilde{f_1}(r^n x) \, dx.$$

Combining these, we can verify (3.4).

Since $f_1, \ldots, f_K$ are trigonometric polynomials, condition (3.2) is verified as $B_N = O\big(1/\sqrt{N}\,\big)$ and by this, (3.5) is clear. By Lemma 2, we have (3.1) for the sum of (3.11). Because of the following estimate, (3.1) is also valid for X_N and hence the central limit theorem holds for X_N:

$$\begin{aligned}\Big\| \sum_{\kappa=1}^{K} \sum_{n=n(\kappa,N)}^{N} \xi_{N,n,\kappa} - \sum_{\kappa=1}^{K}\sum_{n=1}^{N} \xi_{N,n,\kappa} \Big\|_{L^2(\mu_0)} &\le \sum_{\kappa=1}^{K} \Big(\sum_{n=1}^{n(\kappa,N)-1} \|\xi_{N,n,\kappa}\|_\infty \Big)^{1/2} \\ &= O\bigg(\sum_{\kappa=1}^{K} \sqrt{\frac{n(\kappa,N)}{N}} \bigg) = o(1).\end{aligned}$$

4. The central limit theorem for L^2-Dini continuous functions.

In this section we complete the proof of Theorem 6. It is known that L^2-Dini condition (1.1) is equivalent to

$$\sum_{n=1}^{\infty} \big\| f - s_{f,2^n} \big\|_2 < \infty,$$

which implies an estimate of Fourier approximation

$$\big\| f - s_{f,n} \big\|_2^2 = o\big(1/\log n\big) \quad (n \to \infty).$$

The first equivalence can be proved inequalities (3.3) of pp. 241 of Zygmund [24] and (2.6) of pp. 160 of Bari [1], while the proof of the second implication can be found at the end of [10].

Assuming $K \ge 2$ and that $f_1, \ldots, f_K$ are centered, the author [10] have proved that

$$\lim_{I\to\infty} \limsup_{N\to\infty} \int_{\mathbf{R}} \big| X_N(x) - X_N^{(I)}(x) \big| \, \mu_0(dx) = 0 \tag{4.1}$$

where $X_N^{(I)}(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \prod_{k=1}^K s_{f_k,I}(\theta^{p_k(n)}x)$. In the proof of (4.1), $f_1, \ldots, f_{K-1}$ need not to be centered, and we see that it is valid only under the condition $\int_0^1 f_K(x)\,dx = 0$.

Let us now decompose $X_N^{(I)}$ in the same way as (2.1), and define $U_{\kappa,n}^{(I)}$ naturally. Then we have

$$X_N - X_N^{(I)} = \sum_{\kappa=1}^K \frac{1}{\sqrt{N}} \sum_{n=1}^N \big(U_{\kappa,n} - U_{\kappa,n}^{(I)}\big) \prod_{k=\kappa+1}^K m_k.$$

By using (4.1), if $\kappa \geq 2$, we have

$$\lim_{I\to\infty} \limsup_{N\to\infty} \int_{\mathbf{R}} \left| \frac{1}{\sqrt{N}} \sum_{n=1}^N \big(U_{\kappa,n} - U_{\kappa,n}^{(I)}\big) \right| \mu_0(dx) = 0. \tag{4.2}$$

In case $\kappa = 1$, (4.2) follows from (1) of Lemma 1 of [9]. Combining these, we have (4.1).

If we put the limping variance of $X_N^{(I)}$ by $v^{(I)}$, then we can easily prove that $v^{(I)} \to v$. (Cf. 65pp of [9].) Thus we can derive the central limit for X_N by standard argument.

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