A remark to the g-CLT result by Morgenthaler

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Abstract. Morgenthaler proved that any bounded orthogonal sequence has a subsequence which obey the g-CLT. In this note, it is proved that for any non-negative integrable function g, there exists an orthogonal sequence which obey the g-CLT.

0. Introduction

In this note, we are concerned with limit theorems on the Lebesgue probability space, i.e., the probability space consists of unit interval $[0,1]$ and Lebesgue measure λ on it. We denote by EX the expectation of X and by L^p the L^p-space both on $[0,1]$ with respect to λ.

Let us first introduce the notion of variance mixture of normal distribution. Suppose that a function g satisfy

$$g(t) \geq 0 \quad \text{and} \quad \int_0^1 g(t)\,dt = 1. \tag{0.0}$$

We denote by $\frac{1}{\lambda(S)}\int_S N_{0,g(t)}(\,\cdot\,)\,dt$ the variance variance mixture on S of normal distribution with variance function g, that is the probability measure μ on $\mathbf{R}$ given by

$$\mu(A) = \frac{1}{\lambda(S)}\int_S N_{0,g(t)}(A)\,dt.$$

We say that a sequence $\{\phi_k\}$ of random variables on Lebesgue probability space obeys the g-CLT if the convergence in law

$$\frac{1}{A_n}\sum_{k=1}^n a_k\phi_k \xrightarrow{\mathscr{D}} \frac{1}{\lambda(S)}\int_S N_{0,g(t)}(\,\cdot\,)\,dt \quad \text{as} \quad n\to\infty$$

holds on probability space $\big(S, dt/\lambda(S)\big)$ for any $S\subset[0,1]$ with $\lambda(S)>0$, and for any sequence $\{a_n\}$ of real numbers satisfying

$$A_n = \left(\sum_{k=1}^n a_k^2\right)^{1/2} \to \infty \quad \text{and} \quad a_n = o(A_n) \quad \text{as} \quad n\to\infty. \tag{0.1}$$

Morgenthaler [0] proved that for any orthonormal sequence $\{\phi_k\}$, there exists a subsequence $\{\phi_{n_k}\}$ obeying the g-CLT for some bounded function g satisfying (0.0).

In this note we prove that for any g with (0.0), there exists an orthonormal sequence obeying the g-CLT. Moreover we investigate integrability or boundedness of the sequence.

Theorem 1. *Suppose that g satisfies (1.2). Then there exists an orthonormal sequence $\{\phi_k\}$ on $[0,1]$ obeying the g-CLT. We can take $\{\phi_k\}$ having the following properties, according as the summability condition on g.*

(1) *$\{\phi_k\}$ can be taken to be multiplicatively orthogonal, i.e., to satisfy*

$$E(\phi_{k_1}\dots\phi_{k_r}) = 0 \quad \textit{for} \quad r\in\mathbf{N} \quad \textit{and} \quad k_1<\dots<k_r.$$

(2) *In case $g\in L^p$ for some $p\geq 1$, $\{\phi_k\}$ can be taken to be bounded with respect to the L^{2p}-norm.*

(3) *In case g is a bounded function, $\{\phi_k\}$ can be taken to be uniformly bounded sequence.*

For uniformly bounded multiplicatively orthogonal sequence, ordinary central limit theorem is proved assuming the orthogonality of $\{\phi_k^2 - E\phi_k^2\}$. The above theorem prove that multiplicatively orthogonal sequence may obey the g-CLT holds if we drops the orthogonality of $\{\phi_k^2 - E\phi_k^2\}$.

2. Integrable variance function

Let s_k be k-th sub-sum of the Fourier series of $\sqrt{g}$. We have

$$\|s_k\|_2 \to \|\sqrt{g}\,\|_2 = 1, \quad \text{and} \quad \|s_k\|_{2p} \to \|\sqrt{g}\,\|_{2p}.$$

We prove that $\phi_k(\omega) = \sqrt{2}\,\|s_k\|_2^{-1} s_k(\omega)\cos 2\pi 3^k\omega$, is multiplicatively orthogonal and obey the g-CLT.

Let us put $n_k := 3^k + k$. It is clear that $\{n_k\}$ has Hadamard's gaps, i.e., $n_{j+1}/n_j > q > 1$ $(j\in\mathbf{N})$. Let us denote by $\mathrm{Spec}\,(f)$ the set of all the frequencies appearing in the Fourier series of f, and by $|\,\mathrm{Spec}\,(f)|$ the set $\{\,|n| \mid n\in \mathrm{Spec}\,(f)\,\}$. Note that

$$|\,\mathrm{Spec}\,(\phi_k)| \subset [\,3^k - k, 3^k + k\,] \subset (n_{k-1}, n_k\,].$$

If $r\in\mathbf{N}$, $k_1<\dots<k_r$, and $i_j \in |\,\mathrm{Spec}\,(\phi_{k_j})|$,

$$\begin{aligned} i_r \pm\dots\pm i_1 &\geq (3^{k_r}-k_r) - (3^{k_{r-1}} + k_{r-1}) - \dots - (3^{k_1}+k_1) \\ &\geq (3^{k_r}-k_r) - (3^{k_r-1} + k_r - 1) - \dots - (3^0+0) \\ &= \frac{3^{k_r}+1}{2} - \frac{k_r(k_r+1)}{2} \\ &> 1 \end{aligned}$$

Since any element of $|\operatorname{Spec}(\phi_{k_1}\dots\phi_{k_r})|$ can be written as $\pm i_r \pm \cdots \pm i_1$ with $i_j \in |\operatorname{Spec}(\phi_{k_j})|$, it holds that $0 \notin |\operatorname{Spec}(\phi_{k_1}\dots\phi_{k_r})|$, and hence $\{\phi_k\}$ is multiplicatively orthogonal. If $g \in L^p$, it is clear that $\|\phi_k\|_{2p} \le \sqrt{2}\,\|s_k\|_{2p} \le C\|g\|_p$.

Nextly let us prove the g-CLT. Take arbitrarily a sequence $\{a_n\}$ satisfying (0.1). We can take a sequence $\{m(k)\}$ of integers such that

$$m(k) \le k, \quad m(k) \uparrow \infty, \quad \text{and} \quad |a_k| \sup_{\omega\in[\,0,1\,]} |s_{m(k)}(\omega)| = o(A_k) \quad \text{as} \quad k \to \infty.$$

Let us put $\psi_k(\omega) = \sqrt{2}\, s_{m(k)}(\omega) \cos 2\pi 3^k \omega$.

We here prove the following central limit theorem for $\{\psi_k\}$:

$$\frac{1}{A_n}\sum_{k=1}^{n} a_k\psi_k \xrightarrow{\mathscr{D}} \frac{1}{\lambda(S)}\int_S N_{0,g(t)}(\,\cdot\,)\,dt \quad \text{as} \quad n \to \infty. \tag{1.0}$$

We use the next theorem due to Takahashi [1].

Theorem T. *Suppose that a sequence $\{n_j\}$ of integers had Hadamard's gaps, and a sequence $\{\Delta_k\}$ of trigonometric polynomials satisfies the following conditions:*

$$|\operatorname{Spec}(\Delta_k)| \subset (n_k, n_{k+1}\,] \quad \textit{for} \quad k \in \mathbf{N}; \tag{1.1}$$

$$C_n^2 = \sum_{k=1}^{n} \|\Delta_k\|_2^2 \to \infty \quad \textit{as} \quad n \to \infty; \tag{1.2}$$

$$\sup_{\omega\in[\,0,1\,]} |\Delta_k(\omega)| = o(C_k) \quad \textit{as} \quad k \to \infty; \tag{1.3}$$

$$\frac{1}{C_n^2}\sum_{k=1}^{n}(\Delta_k^2 + 2\Delta_k\Delta_{k+1}) \xrightarrow{L^1} g \quad \textit{as} \quad n \to \infty. \tag{1.4}$$

Then

$$\frac{1}{C_n}\sum_{k=1}^{n} \Delta_k \xrightarrow{\mathscr{D}} \frac{1}{\lambda(S)}\int_S N_{0,g(t)}(\,\cdot\,)\,dt \quad \textit{as} \quad n \to \infty$$

holds on probability space $\big(S, dt/\lambda(S)\big)$ for any $S \subset [\,0,1\,]$ with $\lambda(S) > 0$.

Let us put $\Delta_k = a_k\psi_k$. It is clear that (1.1) and the estimate below hold:

$$\sup_{\omega\in[\,0,1\,]} |\Delta_k(\omega)| = o(A_k) \quad \text{as} \quad k \to \infty.$$

Because of the expression $\Delta_k^2 - a_k^2 s_{m(k)}^2 = a_k^2 s_{m(k)}^2 \cos 2\pi 2\cdot 3^k\omega$, we have

$$|\operatorname{Spec}(\Delta_k^2 - a_k^2 s_{m(k)}^2)| \subset [\,2\cdot 3^k - 2k,\ 2\cdot 3^k + 2k\,] \subset (2n_k, 2n_{k+1}\,],$$

and hence $\{\Delta_k^2 - a_k^2 s_{m(k)}^2\}$ is orthogonal. Especially, $0 \notin |\operatorname{Spec}(\Delta_k^2 - a_k^2 s_{m(k)}^2)|$ implies

$$E\Delta_k^2 = a_k^2 E s_{m(k)}^2, \quad \text{and hence} \quad \frac{C_n^2}{A_n^2} = \frac{1}{A_n^2}\sum_{k=1}^n E\Delta_k^2 = \frac{1}{A_n^2}\sum_{k=1}^n a_k^2 E s_{m(k)}^2 \to 1.$$

Therefore we have (1.2) and (1.3).

Because of $\Delta_k \Delta_{k+1} = a_k a_{k+1}(\cos 2\pi 2\cdot 3^k\omega + \cos 4\pi 2\cdot 3^k\omega)s_{m(k)}^2(\omega)$, we have

$$|\operatorname{Spec}(\Delta_k\Delta_{k+1})| \subset [\,2\cdot 3^k - 2k,\ 4\cdot 3^k + 2k\,]$$

and thereby $\{\Delta_k\Delta_{k+1}\}$ is orthogonal.

To prove (1.4), we divide the left hand-side into three parts:

$$\begin{aligned}\sum_{k=1}^n (\Delta_k^2 + 2\Delta_k\Delta_{k+1}) &= \sum_{k=1}^n a_k^2 s_{m(k)}^2 + \sum_{k=1}^n (\Delta_k^2 - a_k^2 s_{m(k)}^2) + \sum_{k=1}^n 2\Delta_k\Delta_{k+1} \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3.\end{aligned}$$

Because of $s_{m(k)} \to \sqrt{g}$ in L^2, it is clear that $s_{m(k)}^2 \to g$ in L^1, and hence

$$\frac{\Sigma_1}{A_n^2} \xrightarrow{L^1} g \quad \text{as} \quad n \to \infty.$$

By the orthogonality, the condition (1.3), and $A_n^2/A_{n+1}^2 = 1 - a_{n+1}^2/A_{n+1}^2 \to 1$, we have the estimates below:

$$\begin{aligned}\big(E|\Sigma_2|\big)^2 &\le E\Sigma_2^2 = \sum_{k=1}^n E\Delta_k^4 = \sum_{k=1}^n o(A_k^2)E\Delta_k^2 = o(A_n^4);\\ \big(E|\Sigma_3|\big)^2 &\le E\Sigma_3^2 = \sum_{k=1}^n E\Delta_k^2\Delta_{k+1}^2 = \sum_{k=1}^n o(A_k^2)E\Delta_k^2 = o(A_{n+1}^2A_n^2) = o(A_n^4).\end{aligned}$$

These imply (1.4) and thereby the proof of (1.0) is over.

$\{\phi_k - \psi_k\}$ is orthogonal, since $|\operatorname{Spec}(\phi_k - \psi_k)| \subset |\operatorname{Spec}(\phi_k)|$. Thus we have

$$E\left(\frac{1}{A_n}\sum_{k=1}^n a_k(\phi_k - \psi_k)\right)^2 = \frac{1}{A_n^2}\sum_{k=1}^n a_k^2 E(\phi_k - \psi_k)^2 \to 0,$$

since $\phi_k,\, \psi_k \to \sqrt{g}$ in L^2. This and (1.0) prove the g-CLT for $\{\phi_k\}$.

2. Bounded variance function

Let σ_k be k-th Cesaro sum of the Fourier series of $\sqrt{g}$. Since $\sqrt{g}$ is bounded, it holds that

$$\sigma_k \to \sqrt{g} \text{ a.e.}, \quad \|\sigma_k\|_2 \to \|\sqrt{g}\,\|_2 = 1, \quad \text{and} \quad \sup_{k\in\mathbf{N}} \sup_{\omega\in[\,0,1\,]} |\sigma_k(\omega)| < \infty.$$

Thus, if we put $\phi_k(\omega) = \sqrt{2}\,\|\sigma_k\|_2^{-1}\sigma_k(\omega)\cos 2\pi 3^k\omega$,

$$\sup_{k\in\mathbf{N}} \sup_{\omega\in[\,0,1\,]} |\sigma_k(\omega)| < \infty.$$

In this case, the proof can be carried out almost in the same way as before.

References

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