METRIC DISCREPANCY RESULTS FOR GEOMETRIC PROGRESSIONS WITH SMALL RATIOS

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Abstract. Although the law of the iterated logarithm for discrepancies of geometric progression is proved, the constants appearing there are not concretely evaluated for small ratios. We consider the case when ratio is a power root of small rational number, and make a conjecture on the concrete value of the constant. We give several examples which are consistent with the conjecture.

1. Introduction

A sequence $\{x_k\}$ of real numbers is said to be uniformly distributed mod 1 if

$$\lim_{N\to\infty}\frac{1}{N}\#\{k\le N \mid \langle x_k\rangle \in [\,a,b)\} = b-a \quad \text{for all} \quad 0\le a<b\le 1,$$

where $\langle x\rangle$ denotes the fractional part $x-[\,x\,]$ of x. Since the convergence is uniform in a and b, we use the following discrepancy $D_N(\{x_k\})$ to measure the speed of convergence:

$$D_N(\{x_k\}) = \sup_{0\le a<b\le 1}\left|\frac{1}{N}\#\{k\le N \mid \langle x_k\rangle\in[\,a,b)\}-(b-a)\right|.$$

For geometric progressions $\{\theta^k x\}$ with $|\theta|>1$, we can prove the law of the iterated logarithm in exact form as below and determine the speed of convergence toward the uniform distribution.

$$\varlimsup_{N\to\infty}\frac{ND_N(\{\theta^k x\})}{\sqrt{2N\log\log N}} = \Sigma_\theta \quad \text{a.e. } x,$$

where $\Sigma_\theta \ge 1/2$ is a constant determined by θ. The case $\theta>1$ was proved in [1] and the case $\theta<-1$ in [3].

Before this result, Philipp [8] applied the method of Takahashi [9] and proved that the limsup above is bounded from below and above by positive constants if we replace θ^k by n_k satisfying $n_{k+1}/n_k \ge q > 1$.

We are interested in the concrete value of Σ_θ because it indicates the speed of convergence.

When $\theta^k \notin \mathbf{Q}$ for all $k=1, 2, \ldots,$ then

$$\Sigma_\theta = \frac{1}{2}. \tag{1}$$

When $\theta^k\in\mathbf{Q}$ for some $k=1, 2, \ldots,$ denote $r=\min\{k\in\mathbf{N} : \theta^k\in\mathbf{Q}\}$ and $\theta^r = p/q$ by $p\in\mathbf{Z}$ and $q\in\mathbf{N}$ with $\gcd(p,q)=1$. We first see that Σ_θ is independent of r and and is determined only by p and q, i.e.,

$$\Sigma_\theta = \Sigma_{p/q}. \tag{2}$$

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We also have the estimate

$$\frac{1}{2} < \Sigma_{p/q} \le \Sigma_{|p|/q} \le \frac{1}{2}\sqrt{\frac{|p|q+1}{|p|q-1}}. \tag{3}$$

If both p and q are odd, then

$$\Sigma_{p/q} = \frac{1}{2}\sqrt{\frac{|p|q+1}{|p|q-1}}. \tag{4}$$

If $|p| \ge 4$ is even and $q = 1$, then

$$\Sigma_p = \frac{1}{2}\sqrt{\frac{(|p|+1)|p|(|p|-2)}{(|p|-1)^3}} \tag{5}$$

If $p = 2$ and $q = 1$, then

$$\Sigma_2 = \frac{1}{9}\sqrt{42}. \tag{6}$$

By (1), (2), (3), and (6), we see that Σ_2 is largest among Σ_θ ($|\theta| > 1$) and that $\{2^k x\}$ is furthest from the uniform distribution a.e. These results are proved in [1, 2, 3]. Determining the concrete value of Σ_{-2} is rather hard work [4]:

$$\Sigma_{-2} = \frac{1}{49}\sqrt{910}. \tag{7}$$

When $q > 1$ and pq is even, evaluation of the concrete value of $\Sigma_{p/q}$ needs very delicate estimate. We succeeded in giving the closed formula below to have the concrete evaluation when p/q is large. If p is odd, q is even and $|p|/q \ge 9/4$, or if p is even, q is odd and $|p|/q \ge 4$, then [7]

$$\Sigma_{p/q} = \sqrt{\frac{(|p|q)^I+1}{(|p|q)^I-1} v\Big(\frac{|p|-q-1}{2(|p|-q)}\Big) + \frac{2(|p|q)^I}{(|p|q)^I-1}\sum_{m=1}^{I-1}\frac{1}{(|p|q)^m} v\Big(q^m \frac{|p|-q-1}{2(|p|-q)}\Big)}, \tag{8}$$

where $I = \min\{n \in \mathbf{N} \mid q^n = \pm 1 \bmod |p|-q\}$ and $v(x) = \langle x\rangle(1-\langle x\rangle)$.

Having these results, it is very natural to have a question if the formula (8) is valid when p/q is small. We already have counterexamples Σ_2 and Σ_{-2}, since $I = 1$ and the right hand side of the formula equals to 0 in these cases which is different from the actual values (6) and (7). In this note, we make a conjecture and give a few affirmative examples for that. Because the case when p/q is negative is very delicate and hard to be investigated, we restrict ourselves to the case of positive p/q.

If p/q is positive, we [1] have proved

$$\Sigma_\theta^2 = \sup_{0\le a\le 1} \sigma_\theta^2(a) = \sup_{0\le a\le 1}\Big(V(a,a) + 2\sum_{k=1}^{\infty}\frac{V(\langle p^k a\rangle, \langle q^k a\rangle)}{p^k q^k}\Big),$$

where $V(x,\xi) = x \wedge \xi - x\xi$.

The formula (8) is derived from

$$\Sigma_{p/q} = \sigma_{p/q}\Big(\frac{p-q-1}{2(p-q)}\Big). \tag{9}$$

This equation holds since the point $x = (p-q-1)/2(p-q)$ is the maximal point of the function $V(x,x) + 2V(\langle px\rangle, \langle qx\rangle)/pq$ and the remainder terms can be negligible when p/q is large.

When p/q is small, then the remainder terms grow and validity of the formula (9) cannot be expected. Actually we can find several p/q such that

$$\Sigma_{p/q} = \sigma_{p/q}\Big(\frac{n}{p^k - q^k}\Big) \tag{10}$$

holds for some $n \in \mathbf{N}$, and we say that p/q is of type k in this case. The formula (9) shows that p/q is of type I if pq is even and p/q is large.

Since we have

$$\Sigma_2 = \sigma_2\Big(\frac{1}{2^2 - 1^2}\Big),$$

we see that $2 = 2/1$ is of type II.

We here state our conjecture.

Conjecture 1. *There exists k and n such that (10) holds if pq is even.*

We are now in a position to state our result.

Theorem 1. *Ratio $13/6$ is under the threshold of validity of (8), it is of Type I and the concrete evaluation given by (8) is as follows:*

$$\Sigma_{13/6} = \sigma_{13/6}\Big(\frac{3}{13-6}\Big) = \frac{2}{7}\sqrt{\frac{237}{77}}.$$

Ratios $4/3$, $8/3$, $10/3$, $12/5$, $17/8$ are of Type II and the concrete evaluations are as follows:

$$\begin{aligned}
\Sigma_{4/3} &= \sigma_{4/3}\Big(\frac{3}{4^2 - 3^2}\Big) = \frac{18}{7}\sqrt{\frac{117609}{2985983}},\\
\Sigma_{8/3} &= \sigma_{8/3}\Big(\frac{24}{8^2 - 3^2}\Big) = \frac{2}{275}\sqrt{\frac{15766778926301268305131994422 2}{32159909742724829389686571}},\\
\Sigma_{10/3} &= \sigma_{10/3}\Big(\frac{40}{10^2 - 3^2}\Big) = \frac{6}{637}\sqrt{\frac{43479927170}{14877551}},\\
\Sigma_{12/5} &= \sigma_{12/5}\Big(\frac{55}{12^2 - 5^2}\Big)\\
\Sigma_{17/8} &= \sigma_{17/8}\Big(\frac{101}{17^2 - 8^2}\Big)
\end{aligned}$$

Ratio $19/10$ is of type III and the concrete evaluation is as follows:

$$\Sigma_{19/10} = \sigma_{19/10}\Big(\frac{2879}{19^3 - 10^3}\Big).$$

Ratio $12/7$ is of type IV and the concrete evaluation is as follows:

$$\begin{aligned}
\Sigma_{12/7} &= \sigma_{12/7}\Big(\frac{8717}{12^4 - 7^4}\Big)\\
&= \frac{1}{18335}\sqrt{\frac{1288914789424650371352900618359881195696318380071236938}{152301038780983553895924756542673273316819 59935}}.
\end{aligned}$$

Ratio $8/5$ is of type V:

$$\Sigma_{8/5} = \sigma_{8/5}\Big(\frac{13690}{8^5 - 5^5}\Big).$$

Ratio $3/2$ is of type VI and the concrete evaluation is as follows:

$$\Sigma_{3/2} = \sigma_{3/2}\Big(\frac{277}{3^6 - 2^6}\Big) = \frac{2}{665}\sqrt{\frac{3056714517626168896614456367908 73}{10314424798490535546171949055}}.$$

(Concrete evaluations of Σ_θ for $\theta = 12/5$, $17/8$, $19/10$ and $8/5$ can be found in [6]*.)*

Since the evaluation of $\Sigma_{3/2}$ contains very delicate calculation and the proof is lengthy, it will be proved in a separate paper [5].

By the above results, concrete values of $\Sigma_{p/2}$ and $\Sigma_{p/3}$ are completely determined.

2. Preliminary

We denote $\sigma_\theta^2(a)$ simply by $\sigma^2(a)$.

The inequality $0 \le V(x, y) \le 1/4$ (x, $y \in [0, 1)$) implies

$$\left|2\sum_{n=N+1}^{\infty}\frac{1}{(pq)^n}V\big(\langle p^n x\rangle, \langle q^n x\rangle\big)\right| \le \frac{1}{2}\sum_{n=N+1}^{\infty}\frac{1}{(pq)^n} = \frac{1}{2(pq-1)(pq)^N}. \tag{11}$$

If $x > y$, then we have $V(x, y) = y(1-x)$, $V_x(x, y) = -y < 0$, and $V_y(x, y) = 1 - x > 0$. If $x < y$, then we have $V(x, y) = x(1-y)$, $V_x(x, y) = 1 - y > 0$, and $V_y(x, y) = -x < 0$. Since one of $V_x(x, y)$ and $V_y(x, y)$ is positive and the other is negative, we see

$$\left|\frac{d}{dx}V\big(\langle p^n x\rangle, \langle q^n x\rangle\big)\right| = \left|p^n V_x\big(\langle p^n x\rangle, \langle q^n x\rangle\big) + q^n V_y\big(\langle p^n x\rangle, \langle q^n x\rangle\big)\right| \le p^n \quad \text{a.e.}$$

and

$$\left|2\sum_{n=N+1}^{\infty}\frac{1}{(pq)^n}\frac{d}{dx}V\big(\langle p^n x\rangle, \langle q^n x\rangle\big)\right| \le 2\sum_{n=N+1}^{\infty}\frac{1}{q^n} = \frac{2}{(q-1)q^N} \qquad \text{a.e.} \tag{12}$$

Actually, the function $V\big(\langle p^n x\rangle, \langle q^n x\rangle\big)$ can not to be differentiated on finitely many points, and the estimate above is valid outside of the countable dense set H. We can, however, argue in the following way. Since the series (11) and (12) are uniformly convergent outside of H, we see

$$\sigma^2(a) = \int_0^a \left(\frac{d}{dx}V(x, x) + 2\sum_{k=1}^{\infty}\frac{1}{p^k q^k}\frac{d}{dx}V(\langle p^k a\rangle, \langle q^k a\rangle)\right)dx.$$

It make us possible to conclude that $\sigma^2(a)$ increases in the interval in which

$$\frac{d}{dx}V(x, x) + 2\sum_{k=1}^{\infty}\frac{1}{p^k q^k}\frac{d}{dx}V(\langle p^k a\rangle, \langle q^k a\rangle) \ge 0 \quad \text{a.e.}$$

3. Type IV case, $\Sigma_{12/7}$

Put $c = \dfrac{8717}{12^4 - 7^4} = \dfrac{8717}{18335}$. By

$$12^{24} = 7^{24} = 1 \mod 12^4 - 7^4,$$

we have $V\big(\langle 7^{n+24k}c\rangle, \langle 12^{n+24k}c\rangle\big) = V\big(\langle 7^n c\rangle, \langle 12^n c\rangle\big)$ and

$$\begin{aligned}\sigma^2(c) &= V\big(\langle c\rangle, \langle c\rangle\big) + 2\frac{7^{24}12^{24}}{7^{24}12^{24}-1}\sum_{n=1}^{24}\frac{1}{7^n 12^n}V\big(\langle 7^n c\rangle, \langle 12^n c\rangle\big)\\ &= \frac{128891478942465037135290061835988119569631838007123693 8}{511993790768145290016004438395337817389483746370980537 5}.\end{aligned}$$

We divide $[0, 1/2)$ into $[0, 3/7)$, $[3/7, 68/12^2)$, $[68/12^2, 45/95)$, $[45/95, 820/12^3)$, $[820/12^3, 821/12^3)$, $[821/12^3, 9858/12^4)$, $[9858/12^4, c)$, $[c, 9859/12^4)$, $[9859/12^4, 822/12^3)$, $[822/12^3, 659/1385)$, $[659/1385, 69/12^2)$, $[69/12^2, 46/95)$, $[46/95, 1/2)$, and prove $\sigma^2(x) < \sigma^2(c)$ ($x \ne c$) on each.

3.1. $[\,0, 3/7)$ **part.** On $[\,0, 3/7)$, by applying (11) for $N = 0$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{1}{2 \cdot 83} - \sigma^2(c)\bigg|_{x=3/7} \\
&= -\frac{41252563376217772720292821795697340812639084418949979 9}{50175391495278238421568434962743106104169407144356092675 0} < 0.
\end{aligned}$$

3.2. $[\,3/7, 68/12^2)$ **and** $[\,46/95, 1/2)$ **parts.** On $[\,3/7, 1/2)$, we see $\langle 12x \rangle = 12x - 5$, $\langle 7x \rangle = 7x - 3$, $\langle 12x \rangle - \langle 7x \rangle = 5x - 2 > 0$, and

$$V\big(\langle 7x \rangle, \langle 12x \rangle\big) = (7x-3)(6-12x) \quad x \in [\,3/7, 1/2). \tag{13}$$

On $[\,3/7, 68/12^2)$, by applying (11) for $N = 1$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{2}{7 \cdot 12}(7x-3)(6-12x) + \frac{1}{2 \cdot 83 \cdot 7 \cdot 12} - \sigma^2(c)\bigg|_{x=68/12^2} \\
&= -1.906 \cdots \times 10^{-5} < 0,
\end{aligned}$$

since the bounding quadratic function on the right hand side has the axis of symmetry at $x = 10/21 \in (68/12^2, 46/95)$. (Exact values can be found in [6].) On $[\,46/95, 1/2)$, it also implies

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{2}{7 \cdot 12}(7x-3)(6-12x) + \frac{1}{2 \cdot 83 \cdot 7 \cdot 12} - \sigma^2(c)\bigg|_{x=46/95} \\
&= -1.647 \cdots \times 10^{-4} < 0.
\end{aligned}$$

3.3. $[\,69/12^2, 46/95)$ **part.** We have $\langle 12^2 x \rangle = 12^2 x - 69$, $\langle 7^2 x \rangle = 7^2 x - 23$, and $\langle 12^2 x \rangle - \langle 7^2 x \rangle = 95x - 46 < 0$. By applying (11) for $N = 2$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{2}{7 \cdot 12}(7x-3)(6-12x) + \frac{2}{7^2 \cdot 12^2}(12^2 x - 69)(24 - 7^2 x) \\
&\qquad + \frac{1}{2 \cdot 83 \cdot 7^2 \cdot 12^2} - \sigma^2(c)\bigg|_{x=5639/11760} \\
&= -6.868 \cdots \times 10^{-5} < 0,
\end{aligned}$$

since the quadratic function has the axis at $x = 5639/11760$.

3.4. $[\,68/12^2, 45/95)$**,** $[\,45/95, 820/12^3)$**, and** $[\,659/1385, 69/12^2)$ **parts.** On $[\,68/12^2, 69/12^2)$, we see $\langle 12^2 x \rangle = 12^2 x - 68$, $\langle 7^2 x \rangle = 7^2 x - 23$, $\langle 12^2 x \rangle - \langle 7^2 x \rangle = 95x - 45$ and

$$V\big(\langle 7^2 x \rangle, \langle 12^2 x \rangle\big) = \begin{cases} (12^2 x - 68)(24 - 7^2 x) & x \in [\,68/12^2, 45/95), \\ (7^2 x - 23)(69 - 12^2 x) & x \in [\,45/95, 69/12^2). \end{cases} \tag{14}$$

On $[\,68/12^2, 45/95)$, by applying (11) for $N = 2$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{2}{7 \cdot 12}(7x-3)(6-12x) + \frac{2}{7^2 \cdot 12^2}(12^2 x - 68)(24 - 7^2 x) \\
&\qquad + \frac{1}{2 \cdot 83 \cdot 7^2 \cdot 12^2} - \sigma^2(c)\bigg|_{x=45/95} \\
&= -1.442 \cdots \times 10^{-5} < 0,
\end{aligned}$$

since the quadratic function has the axis at $x = 4217/8820 > 45/95$.

On $[45/95, 820/12^3)$, by applying (11) for $N = 2$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{2}{7\cdot 12}(7x-3)(6-12x) + \frac{2}{7^2\cdot 12^2}(7^2x-23)(69-12^2x) \\
&\qquad + \frac{1}{2\cdot 83\cdot 7^2\cdot 12^2} - \sigma^2(c)\bigg|_{x=820/12^3} \\
&= -3.211\cdots\times 10^{-6} < 0,
\end{aligned}$$

since the quadratic function has the axis at $x = 5591/11760 \in (820/12^3, 659/1385)$.
On $[659/1385, 69/12^2)$ it also implies

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{2}{7\cdot 12}(7x-3)(6-12x) + \frac{2}{7^2\cdot 12^2}(7^2x-23)(69-12^2x) \\
&\qquad + \frac{1}{2\cdot 83\cdot 7^2\cdot 12^2} - \sigma^2(c)\bigg|_{x=659/1385} \\
&= -1.718\cdots\times 10^{-8} < 0.
\end{aligned}$$

3.5. $[820/12^3, 821/12^3)$**,** $[821/12^3, 9858/12^4)$**,** $[9859/12^4, 822/12^3)$**, and** $[822/12^3, 659/1385)$ **parts.** On $[820/12^3, 821/12^3)$ we have $\langle 12^3x\rangle = 12^3x - 820$, $\langle 7^3x\rangle = 7^3x - 162$, $\langle 12^3x\rangle - \langle 7^3x\rangle = 1385x - 658$, and

$$V\big(\langle 7^3x\rangle, \langle 12^3x\rangle\big) = \begin{cases} (12^3x-820)(163-7^3x) & x\in[820/12^3, 658/1385), \\ (7^3x-162)(821-12^3x) & x\in[658/1385, 821/12^3). \end{cases}$$

On $[820/12^3, 658/1385)$, by applying (11) for $N = 3$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{2}{7\cdot 12}(7x-3)(6-12x) + \frac{2}{7^2\cdot 12^2}(7^2x-23)(69-12^2x) \\
&\qquad + \frac{2}{7^3\cdot 12^3}(12^3x-820)(163-7^3x) + \frac{1}{2\cdot 83\cdot 7^3\cdot 12^3} - \sigma^2(c)\bigg|_{x=658/1385} \\
&= -5.302\cdots\times 10^{-7} < 0,
\end{aligned}$$

since the quadratic function has the axis at $x = 123241/259308 > 658/1385$.

On $[658/1385, 821/12^3)$, by applying (11) for $N = 3$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c) \\
&\le x(1-x) + \frac{2}{7\cdot 12}(7x-3)(6-12x) + \frac{2}{7^2\cdot 12^2}(7^2x-23)(69-12^2x) \\
&\qquad + \frac{2}{7^3\cdot 12^3}(7^3x-162)(821-12^3x) + \frac{1}{2\cdot 83\cdot 7^3\cdot 12^3} - \sigma^2(c)\bigg|_{x=658/1385} \\
&= -5.302\cdots\times 10^{-7} < 0,
\end{aligned}$$

since the quadratic function has the axis at $x = 1970471/4148928 < 658/1385$.

On $[\,821/12^3, 163/7^3)$, we see $\langle 12^3x\rangle = 12^3x - 821$, $\langle 7^3x\rangle = 7^3x - 162$, and $\langle 12^3x\rangle - \langle 7^3x\rangle = 1385x - 659 < 0$. By applying (11) for $N = 3$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c)\\
&\le x(1-x) + \frac{2}{7\cdot 12}(7x-3)(6-12x) + \frac{2}{7^2\cdot 12^2}(7^2x-23)(69-12^2x)\\
&\quad + \frac{2}{7^3\cdot 12^3}(12^3x-821)(163-7^3x) + \frac{1}{2\cdot 83\cdot 7^3\cdot 12^3} - \sigma^2(c)\Big|_{x=163/7^3}\\
&= -3.247\cdots\times 10^{-7} < 0,
\end{aligned}$$

since the quadratic function has the axis at $x = 1972199/4148928 > 163/7^3$.

On $[\,163/7^3, 822/12^3)$, we see $\langle 12^3x\rangle = 12^3x - 821$, $\langle 7^3x\rangle = 7^3x - 163$, $\langle 12^3x\rangle - \langle 7^3x\rangle = 1385x - 658 > 0$ by $658/1385 < 163/7^3$, and

$$V\big(\langle 7^3x\rangle, \langle 12^3x\rangle\big) = (7^3x-163)(822-12^3x) \quad x \in [\,163/7^3, 822/12^3). \tag{15}$$

On $[\,163/7^3, 9858/12^4)$, by applying (11) for $N = 3$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c)\\
&\le x(1-x) + \frac{2}{7\cdot 12}(7x-3)(6-12x) + \frac{2}{7^2\cdot 12^2}(7^2x-23)(69-12^2x)\\
&\quad + \frac{2}{7^3\cdot 12^3}(7^3x-163)(822-12^3x) + \frac{1}{2\cdot 83\cdot 7^3\cdot 12^3} - \sigma^2(c)\Big|_{x=9858/12^4}\\
&= -5.648\cdots\times 10^{-9} < 0,
\end{aligned}$$

since the quadratic function has the axis at $x = 328757/691488 \in [\,9858/12^4, 9859/12^4)$. On $[\,9859/12^4, 822/12^3)$, it also implies

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c)\\
&\le x(1-x) + \frac{2}{7\cdot 12}(7x-3)(6-12x) + \frac{2}{7^2\cdot 12^2}(7^2x-23)(69-12^2x)\\
&\quad + \frac{2}{7^3\cdot 12^3}(7^3x-163)(822-12^3x) + \frac{1}{2\cdot 83\cdot 7^3\cdot 12^3} - \sigma^2(c)\Big|_{x=9859/12^4}\\
&= -2.319\cdots\times 10^{-9} < 0.
\end{aligned}$$

On $[\,822/12^3, 659/1385)$, we see $\langle 12^3x\rangle = 12^3x - 822$, $\langle 7^3x\rangle = 7^3x - 163$, and $\langle 12^3x\rangle - \langle 7^3x\rangle = 1385x - 659 < 0$. By applying (11) for $N = 3$ we have

$$\begin{aligned}
&\sigma^2(x) - \sigma^2(c)\\
&\le x(1-x) + \frac{2}{7\cdot 12}(7x-3)(6-12x) + \frac{2}{7^2\cdot 12^2}(7^2x-23)(69-12^2x)\\
&\quad + \frac{2}{7^3\cdot 12^3}(12^3x-822)(164-7^3x) + \frac{1}{2\cdot 83\cdot 7^3\cdot 12^3} - \sigma^2(c)\Big|_{x=659/1385}\\
&= -3.136\cdots\times 10^{-7} < 0,
\end{aligned}$$

since the quadratic function has the axis at $x = 329045/691488 > 659/1385$.

3.6. $[\,9858/12^4, c)$ **and** $[\,c, 9859/12^4)$ **parts.** We see $\langle 12^4x\rangle = 12^4x - 9858$, $\langle 7^4x\rangle = 7^4x - 1141$, and

$$V\big(\langle 7^4x\rangle, \langle 12^4x\rangle\big) = \begin{cases} (12^4x-9858)(1142-7^4x) & x \in [\,9858/12^4, c),\\ (7^4x-1141)(9859-12^4x) & x \in [\,c, 9859/12^4). \end{cases}$$

On $[9858/12^4, c)$, by applying (12) for $N=4$ we have

$$\begin{aligned}
&\frac{d}{dx}\sigma^2(x)\\
&\geq \frac{d}{dx}\Big(x(1-x)+\frac{2}{7\cdot 12}(7x-3)(6-12x)+\frac{2}{7^2 12^2}(7^2x-23)(69-12^2x)\\
&\qquad +\frac{2}{7^3 12^3}(7^3x-163)(822-12^3x)+\frac{2}{7^4 12^4}(12^4x-9858)(1142-7^4x)\Big)\\
&\quad -\frac{2}{6\cdot 7^4}\\
&=-\frac{74680704x-35506607}{4148928}\geq -\frac{74680704c-35506607}{4148928}=\frac{21942577}{76070594880}>0.
\end{aligned}$$

On $[c, 9859/12^4)$, by applying (12) for $N=4$ we have

$$\begin{aligned}
&\frac{d}{dx}\sigma^2(x)\\
&\leq \frac{d}{dx}\Big(x(1-x)+\frac{2}{7\cdot 12}(7x-3)(6-12x)+\frac{2}{7^2 12^2}(7^2x-23)(69-12^2x)\\
&\qquad +\frac{2}{7^3 12^3}(7^3x-163)(822-12^3x)+\frac{2}{7^4 12^4}(7^4x-1141)(9859-12^4x)\Big)\\
&\quad +\frac{2}{6\cdot 7^4}\\
&=-\frac{448084224x-213028219}{24893568}\leq -\frac{448084224c-213028219}{24893568}=-\frac{77785243}{456423569280}\\
&<0.
\end{aligned}$$

4. Type I case, $\Sigma_{13/6}$

Put

$$c=\frac{3}{7}.$$

Because of $V(\langle 6^n c\rangle, \langle 13^n c\rangle)=V(c,c)$, we have

$$\sigma^2(c)=\frac{6\cdot 13+1}{6\cdot 13-1}V(c,c).$$

In the same way as the proof of the previous section, we divide interval into small pieces, and prove one of the following result to accomplish the proof.

$$\sigma^2(x)-\sigma^2(c)<0, \tag{16}$$

$$\frac{d}{dx}\sigma^2(x)>0, \tag{17}$$

$$\frac{d}{dx}\sigma^2(x)<0. \tag{18}$$

In this case, we divide in the following way, apply an inequality and prove the necessary result on each interval.

Interval	Applying inequality	Shown result
$[0, 5/13)$	(11) for $N=0$	(16)
$[5/13, c)$	(12) for $N=1$	(17)
$[c, 942/13^3)$	(12) for $N=3$	(18)
$[942/13^3, 943/13^3)$	(11) for $N=3$	(16)
$[943/13^3, 73/13^2)$, $[73/13^2, 58/133)$	(11) for $N=2$	(16)
$[58/133, 6/13)$, $[6/13, 1/2)$	(11) for $N=1$	(16)

Detailed calculation for this case can be found in [6].

5. Type II case, $\Sigma_{4/3}$

Put $c = \dfrac{3}{7}$. By $3^6 = 4^6 = 1 \bmod 7$, we see $V\big(\langle 3^{k+6j}c\rangle, \langle 4^{k+6j}c\rangle\big) = V\big(\langle 3^k c\rangle, \langle 4^k c\rangle\big)$ and

$$\sigma^2(c) = V(c,c) + \frac{2(3\cdot 4)^6}{(3\cdot 4)^6 - 1}\sum_{k=1}^{6}\frac{1}{(3\cdot 4)^k}V\big(\langle 3^k c\rangle, \langle 4^k c\rangle\big).$$

In this case, we divide in the following way, apply an inequality and prove the necessary result on each interval.

Interval	Applying inequality	Shown result
$[0, 1/4)$	(11) for $N = 0$	(16)
$[1/4, 1/3)$, $[1/3, 3/8)$	(11) for $N = 1$	(16)
$[3/8, 27/64)$	(11) for $N = 2$	(16)
$[27/64, c)$	(12) for $N = 3$	(17)
$[c, 16/37)$	(12) for $N = 3$	(18)
$[16/37, 7/16)$, $[7/16, 4/9)$, $[4/9, 1/2)$	(11) for $N = 2$	(16)

6. Type II case, $\Sigma_{8/3}$

Put $c = \dfrac{24}{55}$. By $3^{20} = 8^{20} = 1 \bmod 55$, we see $V\big(\langle 3^{k+20j}c\rangle, \langle 8^{k+20j}c\rangle\big) = V\big(\langle 3^k c\rangle, \langle 8^k c\rangle\big)$ and

$$\sigma^2(c) = V(c,c) + \frac{2(3\cdot 8)^{20}}{(3\cdot 8)^{20} - 1}\sum_{k=1}^{20}\frac{1}{(3\cdot 8)^k}V\big(\langle 3^k c\rangle, \langle 10^k c\rangle\big).$$

In this case, we divide in the following way, apply an inequality and prove the result on each interval.

Interval	Applying inequality	Shown result
$[0, 3/8)$	(11) for $N = 0$	(16)
$[3/8, 2/5)$, $[2/5, 27/8^2)$	(11) for $N = 1$	(16)
$[27/8^2, 223/8^3)$	(11) for $N = 2$	(16)
$[223/8^3, c)$	(12) for $N = 3$	(17)
$[c, 28/8^2)$	(12) for $N = 2$	(18)
$[28/8^2, 29/8^2)$, $[29/8^2, 30/8^2)$	(11) for $N = 2$	(16)
$[30/8^2, 1/2)$	(11) for $N = 1$	(16)

7. Type II case, $\Sigma_{10/3}$

Put $c = \dfrac{40}{91}$. By $3^6 = 10^6 = 1 \bmod 91$, we see $V\big(\langle 3^{k+6j}c\rangle, \langle 10^{k+6j}c\rangle\big) = V\big(\langle 3^k c\rangle, \langle 10^k c\rangle\big)$ and

$$\sigma^2(c) = V(c,c) + \frac{2(3\cdot 10)^6}{(3\cdot 10)^6 - 1}\sum_{k=1}^{6}\frac{1}{(3\cdot 10)^k}V\big(\langle 3^k c\rangle, \langle 10^k c\rangle\big).$$

In this case, we divide in the following way, apply an inequality and prove the necessary result on each interval.

Interval	Applying inequality	Result
$[0, 4/10)$	(11) for $N = 0$	(16)
$[4/10, 43/10^2)$	(11) for $N = 1$	(16)
$[43/10^2, 4389/10^4)$	(11) for $N = 2$	(16)
$[4389/10^4, 439/10^3)$, $[439/10^3, 4395/10^4)$	(11) for $N = 3$	(16)
$[4395/10^4, c)$	(12) for $N = 4$	(17)
$[c, 44/10^2)$	(12) for $N = 2$	(18)
$[44/10^2, 4/3^2)$, $[4/3^2, 45/10^2)$, $[45/10^2, 46/10^2)$	(11) for $N = 2$	(16)
$[46/10^2, 1/2)$	(11) for $N = 1$	(16)

8. Type II case, $\Sigma_{12/5}$

Put $c = \dfrac{55}{119}$. Since we have $12^{48} = 1$, $5^{48} = 1 \mod 119$, we have $\langle 12^{48k+n}c\rangle = \langle 12^n c\rangle$, $\langle 5^{48k+n}c\rangle = \langle 5^n c\rangle$, and

$$\sigma^2(c) = V(c, c) + 2\frac{12^{48}5^{48}}{12^{48}5^{48} - 1}\sum_{n=1}^{48}\frac{1}{12^n 5^n}V\big(\langle 12^n c\rangle, \langle 5^n c\rangle\big).$$

In this case, we divide in the following way, apply an inequality and prove the necessary result on each interval.

Interval	Applying inequality	Shown result
$[0, 3/7)$	(11) for $N = 0$	(16)
$[3/7, 66/12^2)$	(11) for $N = 1$	(16)
$[66/12^2, c)$	(12) for $N = 2$	(17)
$[c, 801/12^3)$	(12) for $N = 3$	(18)
$[801/12^3, 67/12^2)$	(12) for $N = 2$	(18)
$[67/12^2, 68/12^2)$	(11) for $N = 2$	(16)
$[68/12^2, 1/2)$	(11) for $N = 1$	(16)

9. Type II case, $\Sigma_{17/8}$

Put $c = \dfrac{101}{225}$. By $17^{20} = 1 \mod 225$ and $8^{20} = 1 \mod 225$, we have $\langle 17^{20k+n}c\rangle = \langle 17^n c\rangle$, $\langle 8^{20k+n}c\rangle = \langle 8^n c\rangle$, and

$$\sigma^2(c) = V(c, c) + 2\frac{17^{20}8^{20}}{17^{20}8^{20} - 1}\sum_{n=1}^{20}\frac{1}{17^n 8^n}V\big(\langle 17^n c\rangle, \langle 8^n c\rangle\big).$$

In this case, we divide in the following way, apply an inequality and prove the nessesary result on each interval.

Interval	Applying inequality	Shown result
$[0, 7/17)$	(11) for $N = 0$	(16)
$[7/17, 129/17^2)$	(11) for $N = 1$	(16)
$[129/17^2, 2205/17^3)$	(11) for $N = 2$	(16)
$[2205/17^3, c)$	(12) for $N = 3$	(17)
$[c, 130/17^2)$	(12) for $N = 2$	(18)
$[130/17^2, 29/8^2)$	(11) for $N = 2$	(16)
$[29/8^2, 8/17)$, $[8/17, 1/2)$	(11) for $N = 1$	(16)

10. Type III case, $\Sigma_{19/10}$

Put $c = \dfrac{2879}{19^3 - 10^3}$. Since we have $19^{30} = 1$, $10^{30} = 1 \mod 19^3 - 10^3$, we have $\langle 19^{30k+n}c \rangle = \langle 19^n c \rangle$, $\langle 10^{30k+n}c \rangle = \langle 10^n c \rangle$, and

$$\sigma^2(c) = V(c,c) + 2\frac{19^{30}10^{30}}{19^{30}10^{30}-1}\sum_{n=1}^{30}\frac{1}{19^n10^n}V\big(\langle 19^n c\rangle, \langle 10^n c\rangle\big).$$

In this case, we divide in the following way, apply an inequality and prove the nessesary result on each interval.

Interval	Applying inequality	Shown result
$[0, 8/19)$	(11) for $N = 0$	(16)
$[8/19, 9/19)$, $[9/19, 176/19^2)$	(11) for $N = 1$	(16)
$[176/19^2, 177/19^2)$, $[177/19^2, 128/261)$	(11) for $N = 2$	(16)
$[128/261, 3369/19^3)$	(11) for $N = 2$	(16)
$[3369/19^3, 3370/19^3)$	(11) for $N = 3$	(16)
$[3370/19^3, c)$	(12) for $N = 3$	(17)
$[c, 3371/19^3)$	(12) for $N = 3$	(18)
$[3371/19^3, 178/19^2)$	(11) for $N = 2$	(16)
$[178/19^2, 1/2)$	(11) for $N = 1$	(16)

11. Type V case, $\Sigma_{8/5}$

Put $c = \dfrac{13690}{8^5 - 5^5} = \frac{13690}{29643}$. By

$$8^{40} = 5^{40} = 1 \mod 8^5 - 5^5,$$

we have $V\big(\langle 5^{n+40k}c\rangle, \langle 8^{n+40k}c\rangle\big) = V\big(\langle 5^n c\rangle, \langle 8^n c\rangle\big)$ and

$$\sigma^2(c) = V\big(\langle c\rangle, \langle c\rangle\big) + 2\frac{5^{40}8^{40}}{5^{40}8^{40}-1}\sum_{n=1}^{40}\frac{1}{5^n8^n}V\big(\langle 5^n c\rangle, \langle 8^n c\rangle\big).$$

In this case, we divide in the following way, apply an inequality and prove the nessesary result on each interval.

Interval	Applying inequality	Shown result
$[0, 2/5)$	(11) for $N = 0$	(16)
$[2/5, 29/8^2)$	(11) for $N = 1$	(16)
$[29/8^2, 236/8^3)$	(11) for $N = 2$	(16)
$[236/8^3, 18/39)$, $[18/39, 1891/8^4)$	(11) for $N = 3$	(16)
$[1891/8^4, 15133/8^5)$	(11) for $N = 4$	(16)
$[15133/8^5, c)$	(12) for $N = 5$	(17)
$[c, 15134/8^5)$	(12) for $N = 5$	(18)
$[15134/8^5, 1892/8^4)$, $[1892/8^4, 1893/8^4)$	(11) for $N = 4$	(16)
$[1893/8^4, 179/387)$, $[179/387, 237/8^3)$	(11) for $N = 3$	(16)
$[237/8^3, 30/8^2)$, $[30/8^2, 12/5^2)$	(11) for $N = 2$	(16)
$[12/5^2, 1/2)$	(11) for $N = 1$	(16)

References

[1] K. Fukuyama, The law of the iterated logarithm for discrepancies of $\{\theta^n x\}$, *Acta Math. Hungar.*, **118** (2008), 155-170.

[2] K. Fukuyama, A central limit theorem and a metric discrepancy result for sequence with bounded gaps, *Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp*, Eds. I. Berkes, R. Bradley, H. Dehling, M. Peligrad, and R. Tichy, Kendrick press, 2010, pp. 233-246.

[3] K. Fukuyama, Metric discrepancy results for alternating geometric progressions, *Monatsh. Math.*, **171** (2013), 33–63.

[4] K. Fukuyama, A metric discrepancy result for the sequence of powers of minus two, *Indag. Math. (NS)*, **25** (2014), 487-504.
[5] K. Fukuyama, A metric discrepancy result for geometric progression with ratio 3/2, (preprint)
[6] K. Fukuyama, S. Sakaguchi, O. Shimabe, T. Toyoda, M. Tscheckl, Metric discrepancy results for geometric progressions with small ratios, arXiv:1711.02839v2 [math.NT] 8 Jan 2018.
[7] K. Fukuyama & M. Yamashita, Metric discrepancy results for geometric progressions with large ratios, *Monatsh. Math.*, **180** (2016), 713–730.
[8] W. Philipp, Limit theorems for lacunary series and uniform distribution mod 1, *Acta Arith.* **26** (1975), 241-251
[9] S. Takahashi, An asymptotic property of a gap sequence, *Proc. Japan Acad.*, **38** (1962), 101-104.

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