SOME LIMIT THEOREMS FOR
WEAKLY MULTIPLICATIVE SYSTEMS.

Katusi Fukuyama

Institute of Mathematics
University of Tsukuba

0. Introduction and results. In this note we will treat functional central limit theorem, Strassen's law of the iterated logarithms and mean central limit theorem for weakly multiplicative systems. A sequence $\{\xi_n\}$ of random variables is called a multiplicative system if

$$E\big(\xi_{i_1}\dots\xi_{i_r}\big)=0 \qquad (r\in\mathbf{N}\quad i_1<\dots<i_r).$$

To extend the notion of multiplicative system, we must prepare some notation. Let $B_r=(b_{i_1,\dots,i_r})_{i_1<\dots<i_r}$ be an infinite dimensional vector having $b_{i_1,\dots,i_r}=E\big(\xi_{i_1}\dots\xi_{i_r}\big)$ as its components and $\|B_r\|_\delta$ be its $\ell_\delta-$norm i.e. $\|B_r\|_\delta=\left(\sum_{i_1<\dots<i_r}|b_{i_1\dots i_r}|^\delta\right)^{\frac{1}{\delta}}$. Upper bound part of the law of the iterated logarithms for weakly multiplicative system was proved by Móricz [8].

Theorem A. *Let $\{\xi_n\}$ be a sequence of random variables satisfying*

$$|\xi_n|\le K \qquad (n\in\mathbf{N}), \tag{0.1}$$

$$\|B_r\|_2<\infty \qquad (r\in\mathbf{N}) \qquad \textit{and} \qquad \limsup_{n\to\infty}\|B_r\|_2^{\frac{1}{r}}=\tilde{B}<\infty \tag{0.2}$$

Then,

$$\limsup_{n\to\infty}\frac{S_n}{\sqrt{2(K^2+\tilde{B}^2)A_n^2\log\log A_n^2}}\le 1 \qquad a.s.$$

where $S_n=a_1\xi_1+\dots+a_n\xi_n$ and $A_n^2=a_1^2+\dots+a_n^2\to\infty$ as $n\to\infty$.

Berkes [1] proved Strassen's law of the iterated logarithms for weakly multiplicative systems satisfying (0.1),

$$\sum_{r=1}^\infty\|B_r\|_1<\infty \qquad \text{and} \qquad \sum_{r=1}^\infty\|B'_r\|_1<\infty$$

where B'_r is a vector defined in the same way as b_r using $\{\xi_n^2 - 1\}$ instead of $\{\xi_n\}$, i.e.

$$B'_r = \big(b'_{i_1,\dots,i_r}\big)_{i_1<\dots<i_r} \quad \text{and} \quad b'_{i_1,\dots,i_r} = E\big((\xi_{i_1}^2 - 1)\dots(\xi_{i_r}^2 - 1)\big).$$

We prove limit theorems under conditions much weaker than those of Berkes'. First we state functional central limit theorem.

We define $\mathbf{C}[0,1]-$valued random variables X_n by

$$X_n\left(\frac{A_j^2}{A_n^2}\right) = \frac{S_j}{A_n} \quad \text{and is linear in} \left[\frac{A_j^2}{A_n^2}, \frac{A_{j+1}^2}{A_n^2}\right] \quad (j = 0, \dots, n)$$

where $S_n = a_1\xi_1 + \dots + a_n\xi_n$.

Theorem 1. *Let $\{\xi_n\}$ be a sequence of random variables satisfying (0.1),*

$$\sup_{r\in\mathbf{N}} \|B_r\|_\delta^{\frac{1}{r}} = B < \infty \qquad \textit{for some} \quad \delta \in [1, 2) \tag{0.3}$$

and either

$$\lim_{\substack{i+j\to\infty \\ i\neq j}} E\left((\xi_i^2 - 1)(\xi_j^2 - 1)\right) = 0 \tag{0.4}$$

or

$$E\big((\xi_i^2 - 1)(\xi_j^2 - 1)\big) \le \beta_{|i-j|} \qquad \textit{for some sequence } \{\beta_j\} \textit{ with } \sum_{n=0}^{\infty} \beta_n < \infty. \tag{0.5}$$

Let $\{a_n\}$ be a sequence of real numbers satisfying

$$A_n^2 = a_1^2 + \dots + a_n^2 \to \infty \quad \textit{and} \quad a_n = o\left(A_n\right) \quad \textit{as} \quad n \to \infty. \tag{0.6}$$

Then the distribution of X_n converges weakly on $\mathbf{C}[0,1]$ to the Wiener measure as $n \to \infty$.

Next we state Strassen's law of the iterated logarithms. This is an extention of the results of Berkes.

Theorem 2.

(a) Let $\{\xi_n\}$ satisfy (0.1) and (0.3), and let $\{a_n\}$ satisfy

$$A_n^2 \to \infty \qquad \textit{and} \qquad a_n^2 = o\left(\frac{A_n^2}{\log\log A_n^2}\right) \qquad \textit{as} \quad n \to \infty. \tag{0.7}$$

Then

$$\left\{\frac{X_n}{\sqrt{2\log\log A_n^2}}\right\} \textit{ is relatively compact in } \mathbf{C}[0,1] \quad \textit{a.s.}$$

(b) Let $\{\xi_n\}$ satisfy (0.1), (0.3) and

$$\sup_{r\in\mathbf{N}} \|B'_r\|_2^{\frac{1}{r}} < \infty. \tag{0.8}$$

Let $\{a_n\}$ satisfy

$$A_n^2 \to \infty \quad \textit{and} \quad a_n^2 = o\left(\frac{A_n^2}{(\log\log A_n)^{\frac{\delta}{(2-\delta)}}}\right) \quad \textit{as} \quad n \to \infty. \tag{0.9}$$

Then

$$\left\{ \textit{Cluster of } \left\{ \frac{X_n}{\sqrt{2\log\log A_n^2}} \right\} \right\} \subset K \quad \textit{a.s.}$$

(c) Moreover if we suppose

$$A_n^2 \to \infty \quad \textit{and} \quad a_n = o\left(A_n^{1-\gamma}\right) \quad \textit{as} \quad n \to \infty \quad \textit{for some} \quad \gamma > 0, \tag{0.10}$$

then we have

$$\left\{ \textit{Cluster of } \left\{ \frac{X_n}{\sqrt{2\log\log A_n^2}} \right\} \right\} = K \quad \textit{a.s.},$$

where

$$K = \left\{x \in C[0,1]; x(0) = 0,\ x \textit{ is absolutely continuous and } \int_0^1 \left(\frac{dx}{dt}\right)^2 dx \le 1\right\}.$$

Finally we state mean central limit theorem. We define a sequence $\{C_n\}$ of positive numbers by

$$C_n = \frac{\max_{i\le n} |a_i|}{A_n}.$$

Under the condition (0.6), $\lim_{n\to\infty} C_n = 0$ holds. Let F_n be a distribution function of S_n and G be that of standard Gaussian distribution.

Theorem 3. *Under the conditions (0.1), (0.3), (0.6) and (0.8), there exsists positive constant L such that*

$$\|F_n - G\|_\infty \le LC_n^{\frac{4}{3}\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge\frac{1}{4}}$$

$$\|F_n - G\|_1 \le LC_n^{\frac{8}{5}\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge\frac{2}{7}}.$$

Mean central limit theorem for ESMS was proved by Paditz-Šarachmetov [11].

1. Proof of Theorem 1. To prove functional central limit theorem, it is sufficient to prove weak convergence of finite dimensional distributions and tightness of $\{X_n\}$. Tightness is easy to derive from the following lemma due to Móricz [8]. For details of the proof of tightness, see Oodaira [10] or Fukuyama [2].

Lemma B. *Under the conditions (0.1) and (0.2),*

$$E\exp(\lambda S_n) \le C\exp\left(\frac{1}{2}\left(K^2+\tilde{B}^2+1\right)\lambda^2 A_n^2\right) \tag{1.1}$$

and

$$P\{|S_n|\ge y\} \le 2C\exp\left(-\frac{y^2}{2(K^2+\tilde{B}^2+1)A_n^2}\right) \tag{1.2}$$

for all $\lambda\in\mathbf{R}$ and $y>0$.

Next we proceed to the other part of the proof. Here we prove only 1-dimensional case instead of multidimensional case using the following theorem due to McLeish [7], but it is easy to extend this theorem to the case of multidimensional distributions.

Theorem C. *Let $\{\zeta_{n,j}; 1\le j\le k_n\}$ be a given triangular array of random variables and put $T_n=\prod_{j\le k_n}(1+it\zeta_{n.j})$. Suppose for all $t\in\mathbf{R}$,*

$$(a)\quad ET_n\to 1,\qquad (b)\quad \{T_n\}\ \textit{is uniformly integrable},$$
$$(c)\quad \sum_{j\le k_n}\zeta_{n,j}^2\to 1\quad \textit{i.p.}\quad \textit{and}\ (d)\quad \max_{j\le k_n}|\zeta_{n,j}|\to 0\quad \textit{i.p.}\quad \textit{as}\quad n\to\infty.$$

Then the distribution of $\sum_{j\le k_n}\zeta_{n,j}$ converges weakly to the standard Gaussian distribution.

To apply this theorem to our case, we put $k_n=n$ and $\zeta_{n,j}=\dfrac{a_j}{A_n}\xi_j$. Then (a) and (d) is trivial because of (0.1) and (0.6). To verify (a), we prove more general lemma for convinience of the later use.

Lemma 1. *We assume (0.3). Let $\{\Lambda_n\}$ be a sequence of positive numbers satifying that*

$$\Lambda_n\le\frac{1}{2B}C_n^{-2\left(\frac{1}{\delta}-\frac{1}{2}\right)}\qquad (n\in\mathbf{N}).$$

Let $\{\lambda_{n,j}\}$ be a triangular array of complex numbers satisfying $|\lambda_{n,j}|\le\Lambda_n$. Then

$$\left|E\prod_{j=1}^n\left(1+\lambda_{n,j}\frac{a_j}{A_n}\xi_j\right)-1-\sum_{j=1}^n\lambda_{n,j}\frac{a_j}{A_n}b_j\right|\le 2B^2\Lambda_n^2C_n^{4\left(\frac{1}{\delta}-\frac{1}{2}\right)} \tag{1.3}$$

and

$$\left| E \prod_{j=1}^{n} \left(1 + \lambda_{n,j} \frac{a_j}{A_n} \xi_j \right) - 1 \right| \leq 2B\Lambda_n C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)} \tag{1.4}$$

Proof. First we prove (1.3) in case $\delta \in (1,2)$. Let ϵ be the dual of δ.

$$\begin{aligned}
&\left| E \prod_{j=1}^{n} \left(1 + \lambda_{n,j} \frac{a_j}{A_n} \xi_j \right) - 1 - \sum_{j=1}^{n} \lambda_{n,j} \frac{a_j}{A_n} b_j \right| \\
&\leq \sum_{r=2}^{n} \sum_{j_1 < \cdots < j_r \leq n} (2B)^r \left| \lambda_{n,j_1} \ldots \lambda_{n,j_r} \right| A_n^{-r} \left| a_{j_1} \ldots a_{j_r} \right| (2B)^{-r} \left| b_{j_1,\ldots,j_r} \right|,
\end{aligned}$$

using Hölder's inequality,

$$\begin{aligned}
&\leq \left(\sum_{r=2}^{n} (2B)^{r\epsilon} \sum_{j_1 < \cdots < j_r \leq n} \left| \lambda_{n,j_1} \ldots \lambda_{n,j_r} A_n^{-r} a_{j_1} \ldots a_{j_r} \right|^{\epsilon} \right)^{\frac{1}{\epsilon}} \\
&\qquad \times \left(\sum_{r=2}^{n} (2B)^{-r\delta} \sum_{j_1 < \cdots < j_r \leq n} \left| b_{j_i,\ldots,j_r} \right|^{\delta} \right)^{\frac{1}{\delta}} \\
&\leq \left(\sum_{r=2}^{n} (2B\Lambda_n C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)})^{r\epsilon} \sum_{j_1 < \cdots < j_r \leq n} A_n^{-2r} a_{j_1}^2 \ldots a_{j_r}^2 \right)^{\frac{1}{\epsilon}} \left(\sum_{r=2}^{n} \left(\frac{\|B\|_{\delta}^{\frac{1}{r}}}{2B} \right)^{r\delta} \right)^{\frac{1}{\delta}}.
\end{aligned}$$

Since $2B\lambda_n C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)} \leq 1$,

$$\begin{aligned}
&\leq \left((2B\Lambda_n C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)})^{2\epsilon} \sum_{r=2}^{n} \sum_{j_1 < \cdots < j_r \leq n} A_n^{-2r} a_{j_1}^2 \ldots a_{j_r}^2 \right)^{\frac{1}{\epsilon}} \left(\sum_{r=2}^{\infty} 2^{-r\delta} \right)^{\frac{1}{\delta}} \\
&\leq (2B\Lambda_n C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)})^2 \left(\prod_{j=1}^{n} \left(1 + \frac{a_j^2}{A_n^2} \right) - 2 \right)^{\frac{1}{\epsilon}} \left(\frac{2^{-2\delta}}{1 - 2^{-\delta}} \right)^{\frac{1}{\delta}} \\
&\leq 2B^2 \Lambda_n^2 C_n^{4\left(\frac{1}{\delta}-\frac{1}{2}\right)}.
\end{aligned}$$

Thus we have proved (1.3), and it is clear that

$$\leq B\Lambda_n C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)}$$

and

$$\left|\sum_{j=1}^{n}\lambda_{n,j}\frac{a_j}{A_n}b_j\right| \le \Lambda_n\left(\sum_{j=1}^{n}\left|A_n^{-1}a_j\right|^{\epsilon}\right)^{\frac{1}{\epsilon}}\left(\sum_{j=1}^{n}|b_j|^{\delta}\right)^{\frac{1}{\delta}}$$
$$\le \Lambda_n C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)}\left(\frac{1}{A_n^2}\sum_{j=1}^{n}a_j^2\right)^{\frac{1}{\epsilon}}\|B_1\|_{\delta}$$
$$\le B\Lambda_n C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)}.$$

These two estimates imply (1.4). We omit the case $\delta = 1$. (Cf. Fukuyama [3])

If we put $\lambda_{n,j} = it$ and $\Lambda_n = t$ in lemma 1, (1.4) implies (a) because of $\lim_{n\to\infty} C_n = 0$. Finally we verify (c) under (0.4). It is sufficient to show that $\frac{1}{A_n^2}\sum_{j=1}^{n} a_j^2\xi_j^2$ converges to 1 in L_2.

$$E\left(\frac{1}{A_n^2}\sum_{j=1}^{n}a_j^2\xi_j^2 - 1\right)^2$$
$$= \frac{1}{A_n^4}\sum_{j=1}^{n}a_j^4 E\left((\xi_j^2-1)^2\right) + \frac{2}{A_n^4}\sum_{1\le i<j\le n}a_i^2a_j^2 E\left((\xi_i^2-1)(\xi_j^2-1)\right)$$

The first term tends to 0 because of (0.1) and the second term also because of (0.4). We omit the proof of the case that (0.5) is assumed. See Stout [15], theorem 3.7.2 or Fukuyama [3].

2. Proof of Theorem 2. Relative compactness of the sequence $\left\{\frac{X_n}{\sqrt{2\log\log A_n^2}}\right\}$ is derived from (1.2) and theorem 1 of Móricz [9] using Ascoli-Arzera theorem and Borel-Cantelli lemma. For the details, refer Fukuyama [2]. For the proof of (b) and (c), we use the following theorem due to Kuelbs [6].

Theorem D. *Assume that*

$$\left\{\frac{X_n}{\sqrt{2\log\log A_n^2}}\right\} \text{ is relatively compact in } \mathbf{C}[0,1] \text{ a.s.}$$

and, for all signed measure ν *with bounded variation on* $[0,1]$,

$$\limsup_{n\to\infty}\frac{\int_0^1 X_n(t)\,\nu(dt)}{\sqrt{2\log\log A_n^2}} \le K_{\nu,1} \qquad a.s. \tag{2.1}$$

holds. Then we have

$$\left\{ \textit{Cluster of} \left\{ \frac{X_n}{\sqrt{2\log\log A_n^2}} \right\} \right\} \subset K \quad a.s..$$

Furthermore, suppose that

$$\limsup_{n\to\infty} \frac{\int_0^1 X_n(t)\,\nu(dt)}{\sqrt{2\log\log A_n^2}} = K_{\nu,1} \qquad a.s. \tag{2.2}$$

holds. Then we have

$$\left\{ \textit{Cluster of} \left\{ \frac{X_n}{\sqrt{2\log\log A_n^2}} \right\} \right\} = K \quad a.s.,$$

where

$$K_{\nu,\theta}^2 = E\left(\left(\int_0^1 W(t\wedge\theta^{-1})\,\nu(dt) \right)^2 \right) = \int_0^{\theta^{-1}} (\nu[x,1])^2\,dx$$

($W(t)$ denotes the standard Wiener process)

First we prepare some notation. Put

$$\phi_{n,j} = \begin{cases} 0 & \text{if} \quad t \in \left[0, \frac{A_{j-1}^2}{A_n^2}\right], \\ \textit{linear} & \text{if} \quad t \in \left[\frac{A_{j-1}^2}{A_n^2}, \frac{A_j^2}{A_n^2}\right], \\ 1 & \text{otherwise}, \end{cases}$$

$$c_{n,j} = \int_0^1 \phi_{n,j}\,\nu(dt) \qquad \text{and} \qquad A_{\nu,n}^2 = \sum_{j=1}^n (a_j c_{n,j})^2 .$$

Fix $\theta > 1$ and we take $p(r) \in \mathbf{N}$ satisfying $A_{p(r)}^2 \le \theta^r < A_{p(r)+1}^2$. We have

$$\lim_{n\to\infty} \frac{A_{\nu,n}^2}{A_n^2} = K_{\nu,1}^2. \tag{2.3}$$

It is a consequence of

$$\lim_{n\to\infty} E\left(\left(\int_0^1 Y_n(t)\,\nu(dt) \right)^2 \right) = E\left(\left(\int_0^1 W(t)\,\nu(dt) \right)^2 \right)$$

which follows from functional central limit theorem and uniform integrability of $\{Y_n\}$ where Y_n is a $\mathbf{C}[0,1]-$valued random variables defined in the same way as X_n using Rademacher sequence $\{r_n\}$ instead of $\{\xi_n\}$. Next we prove

$$\frac{1}{A_{p(r)}^2} \sum_{j=1}^{p(r)} \left(a_j c_{p(r),j} \xi_j\right)^2 \to K_{\nu,1}^2 \qquad \text{a.s.} \qquad \text{as} \quad r\to\infty \tag{2.4}$$

Since $|c_{n,j}| \le N$, (0.9) and (2.3) imply

$$\left(a_j c_{p(r),j}\right) = o\left(\frac{A_{\nu,n}^2}{(\log\log A_n)^{\frac{\delta}{(2-\delta)}}}\right) \quad \text{as} \quad n \to \infty.$$

Using (0.8) and (1.2), we have

$$\sum_{r=1}^{\infty} P\left\{\left|\frac{1}{A_{\nu,p(r)}^2}\sum_{j=1}^{p(r)}\left(a_j c_{p(r),j}\right)^2\left(\xi_j^2-1\right)\right| \ge \sqrt{\frac{2(K^4+B'^2+2)}{H_{p(r)}}}\right\}$$
$$\le \sum_{r=1}^{\infty} 2C\exp\left(-4\log\log\theta^r\right) < \infty$$

where

$$H_n = \frac{A_{\nu,n}^2}{\left(\max\limits_{j\le n}|a_j c_{n,j}|\right)\log\log A_n^2} \to \infty \qquad \text{as} \quad n\to\infty.$$

By Borel-Cantelli lemma, we have (2.4).
Putting $\lambda_{n,j} = c_{n,j}\sqrt{2\log\log A_n^2}$ and $\Lambda_n = N\sqrt{2\log\log A_n^2}$ in lemma 1, we have

$$\left|E\prod_{j=1}^{n}\left(1+\frac{c_{n,j}a_j\xi_j}{A_n}\sqrt{2\log\log A_n^2}\right)\right| \le L \qquad (n\in\mathbf{N})$$

for some $L>0$.
Now we can prove (2.1) using the method of Takahashi [16]. Put $\mu_n = K_{\nu,1}^{-1}\sqrt{2\log\log A_n^2}$.
Making use of $e^x \le (1+x)\exp\left(\frac{x^2}{2}+|x|^3\right) \quad (|x|\le\frac{1}{2})$, for large enough r,

$$E\left(\exp\left(\frac{\mu_{p(r)}}{A_{p(r)}}\sum_{j=1}^{p(r)}a_j c_{p(r),j}\xi_j - \frac{\mu_{p(r)}^2}{2A_{p(r)}^2}\sum_{j=1}^{p(r)}\left(a_j c_{p(r),j}\xi_j\right)^2 - (1+2\epsilon)\frac{K_{\nu,1}^2\mu_{p(r)}^2}{2}\right)\right)$$
$$\le E\prod_{j=1}^{n}\left(1+\frac{c_{n,j}a_j\xi_j}{A_n}\sqrt{2\log\log A_n^2}\right)$$
$$\exp\left(\frac{\mu_{p(r)}^3K^3}{A_{p(r)}^3}\sum_{j=1}^{p(r)}\left|a_j c_{p(r),j}\right|^3 - (1+2\epsilon)\frac{K_{\nu,1}^2\mu_{p(r)}^2}{2}\right)$$
$$= L\exp\left(o(1)\log\log A_{p(r)}^2 - (1+2\epsilon)\log\log A_{p(r)}^2\right)$$
$$\le K' r^{-1-\epsilon}.$$

Thus, by Beppo-Levi's theorem,

$$\lim_{r\to\infty}\mu_{p(r)}^2\left(\frac{1}{\mu_{p(r)}}\int_0^1 X_{p(r)}(t)\,\nu(dt) - (1+\epsilon)K_{\nu,1}^2\right) = -\infty.$$

This implies

$$\limsup_{r\to\infty} \frac{\int_0^1 X_{p(r)}(t)\,\nu(dt)}{\sqrt{2\log\log A_{p(r)}^2}} \le K_{\nu,1} \qquad \text{a.s.}$$

For given n, take r satisfying $p(r) < n \le p(r+1)$. Then

$$\begin{aligned}
&\frac{\int_0^1 X_n(t)\,\nu(dt)}{\sqrt{2\log\log A_n^2}} - \frac{\int_0^1 X_{p(r)}(t)\,\nu(dt)}{\sqrt{2\log\log A_{p(r)}^2}} \\
&\qquad = \frac{\int_0^1 \left(X_n(t) - X_{p(r)}(t)\right)\,\nu(dt)}{\sqrt{2\log\log A_n^2}} \\
&\qquad\qquad - \left(\frac{1}{\sqrt{2\log\log A_n^2}} - \frac{1}{\sqrt{2\log\log A_{p(r)}^2}}\right)\int_0^1 X_{p(r)}(t)\,\nu(dt) \\
&\qquad = I_1 + I_2.
\end{aligned}$$

Obviously $I_2 \to 0$ as $\theta \to 1$ and

$$\begin{aligned}
|I_1| \le \frac{1}{\sqrt{2\log\log A_{p(r-1)}^2}} &\left(\frac{A_{p(r)}}{A_n}\left|\int_0^1 \left(X_{p(r)}\left(\frac{A_n^2 t}{A_{p(r)}^2}\right) - X_{p(r)}(t)\right)\,\nu(dt)\right|\right. \\
&\qquad\qquad \left. + \left(\frac{A_{p(r)}}{A_n} - 1\right)\left|\int_0^1 X_{p(r)}(t)\,\nu(dt)\right|\right) \\
\to 0 \qquad \text{as} \qquad \theta \to 1,&
\end{aligned}$$

because of the equicontinuity. Since we have verified (2.1), the proof of (b) is completed.

Now we proceed to the proof of (c). We prove it in a similar way as Révész [13]. First we prepare some notation.

$$\begin{aligned}
&Z_n = \sum_{j=1}^{p(n)} a_j c_{p(n+1),j}\xi_j, \qquad\qquad D_n^2 = \sum_{j=p(n)+1}^{p(n+1)} \left(a_j c_{p(n+1),j}\right)^2 \\
&\alpha_n = \prod_{j=p(n)+1}^{p(n+1)} \left(1 + \frac{ita_j c_{p(n+1),j}\xi_j}{D_n}\right), \\
&\beta_n = \frac{1}{D_n^2}\sum_{j=p(n)+1}^{p(n+1)} \left(a_j c_{p(n+1),j}\xi_j\right)^2, \quad \eta_n = \frac{1}{D_n}\sum_{j=p(n)+1}^{p(n+1)} a_j c_{p(n+1),j}\xi_j, \\
&\phi_{n,m}(s,t) = E\left(\exp(is\eta_n + it\eta_{n+m})\right), \\
&F_{n,m}(x,y) = P\left\{\eta_n < x\,,\, \eta_{n+m} < y\right\}.
\end{aligned}$$

Since $\lim_{n\to\infty} \dfrac{D_n^2}{A_{p(n+1)}^2} = K_{\nu,1}^2 - K_{\nu,\theta}^2$, we have

$$\frac{1}{D_n^2} \max_{j\le p(n+1)} |a_j| \le LC_{p(n+1)} \qquad \text{for some} \quad L>0.$$

First we prove the next lemma.

Lemma 2. *There exist positive constants L_1 and L_2 such that*

$$\left|\phi_{n,m}(s,t) - \exp\left(-\frac{1}{2}(s^2+t^2)\right)\right| \le L_1 C_{p(r+1)}^{4\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge 1} \left(|t|^3+|s|^3+t^2+s^2\right)$$

if $|s|,|t| \le L_3 C_{p(r+1)}^{-2\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge\frac{1}{3}}$.

Proof. We use the following expansion formula.

$$e^x = (1+x)\exp\left(\frac{x^2}{2}+r(x)\right) \qquad \text{and} \qquad |r(x)|\le |x|^3 \quad \text{if} \quad |x|\le \frac{1}{2}.$$

We put

$$R_n(t) = \sum_{j=p(n)+1}^{p(n+1)} r\left(\frac{1}{D_n} t a_j c_{p(n+1),j}\xi_j\right).$$

If $\left|\frac{1}{D_n} t a_j c_{p(n+1),j}\xi_j\right| \le LNK|t|C_{p(r+1)} \le \frac{1}{2}$, we have

$$|R_n(t)| \le \frac{|t|^3K^3}{D_n^3}\sum_{j=p(n)+1}^{p(n+1)} |a_j c_{p(n+1),j}|^3 \le |t|^3K^3LNC_{p(r+1)}.$$

Using above expansion formula,

$$\begin{aligned}
&\left|\phi_{n,m}(s,t) - \exp\left(-\frac{1}{2}(s^2+t^2)\right)\right| \\
&\le \left|E\alpha_n(s)\alpha_{n+m}(t)\left(\exp\Big(-\frac{1}{2}(s^2\beta_n+t^2\beta_{n+m})\right.\right. \\
&\qquad \left.\left. +R_n(s)+R_{n+m}(t)\Big) - \exp\left(-\frac{1}{2}(s^2+t^2)\right)\right)\right| \\
&\qquad + |E\alpha_n(s)\alpha_{n+m}(t)-1|\exp\left(-\frac{1}{2}(s^2+t^2)\right).
\end{aligned}$$

Since $|\alpha_n(s)| \le \exp\left(\frac{1}{2}s^2\beta_n^2\right)$,

$$\begin{aligned}
&\le E\left|\exp\left(\frac{1}{2}\left(s^2(\beta_n-1)+t^2(\beta_{n+m}-1)\right)\right)-1\right| \\
&\qquad + E\left|\exp(R_n+R_{n+m})-1\right| + |E\alpha_n(s)\alpha_{n+m}(t)-1|
\end{aligned}$$

$$= E_1 + E_2 + E_3$$

Using $|e^x - 1| \le |x|e^{|x|}$,

$$\begin{aligned} E_1 \le& \frac{1}{2} E\left(\left|\left(s^2(\beta_n - 1) + t^2(\beta_{n+m} - 1)\right)\right|\right. \\ &\qquad \left.\times \exp\left(\left|\left(s^2(\beta_n - 1) + t^2(\beta_{n+m} - 1)\right)/2\right|\right)\right) \\ \le& \frac{1}{2}\left(s^2 E^{\frac{1}{2}}\left(\beta_n - 1\right)^2 + t^2 E^{\frac{1}{2}}\left(\beta_{n+m} - 1\right)^2\right) \\ &\qquad \times E^{\frac{1}{2}} \exp\left(\left|s^2(\beta_n - 1) + t^2(\beta_{n+m} - 1)\right|\right). \end{aligned}$$

By (0.3),

$$\begin{aligned} E\left(\beta_n^2 - 1\right)^2 \le& \frac{(K^2+1)^2}{D_n^4} \sum_{j=p(n)+1}^{p(n+1)} (a_j c_{p(n+1),j})^4 \\ &+ 2\frac{1}{D_n^4} \sum_{p(n)<i<j\le p(n+1)} (a_i c_{p(n+1),i})^2 (a_j c_{p(n+1),j})^2 b'_{i,j} \\ \le& \, 2(K^4 + \tilde{B}) N^2 L^2 C_{p(r+1)}^2 \end{aligned}$$

Using (1.1),

$$\begin{aligned} &E \exp\left(\left|s^2(\beta_n - 1) + t^2(\beta_{n+m} - 1)\right|\right) \\ &\le 2C \exp\left((K^4 + \tilde{B}^2 + 1)\left(\frac{s^4}{D_n^4} \sum_{j=p(n)+1}^{p(n+1)} (a_j c_{p(n+1),j})^4\right.\right. \\ &\qquad\qquad \left.\left. + \frac{t^4}{D_{n+m}^4} \sum_{j=p(n+m)+1}^{p(n+m+1)} (a_j c_{p(n+m),j})^4\right)\right) \\ &\le 2C \exp\left((K^4 + \tilde{B}^2 + 1) L^2 N^2 C_{p(n+1)}^2 (s^2 + t^2)\right). \end{aligned}$$

It is bounded because of the conditions on s and t. Thus

$$E_1 \le L_2 C_{p(n+1)}(t^2 + s^2).$$

Similarly we have

$$E_2 \le L_1 C_{p(n+1)}(|t|^3 + |s|^3).$$

In the same way as the proof of lemma 1, we have

$$E_3 \le L_2 C_{p(n+1)}^{4\left(\frac{1}{\delta} - \frac{1}{2}\right)} st \le L_2 C_{p(n+1)}^{4\left(\frac{1}{\delta} - \frac{1}{2}\right)} (t^2 + s^2).$$

Thus we have proved the lemma.

Next we use the following theorem due to Sadikova [14].

Theorem E. *Let $F(x,y)$ and $G(x,y)$ be two dimensional distribution functions and suppose that G has a bounded density function. Denote the corresponding characteristic functions by $f(s,t)$ and $g(s,t)$ and put*

$$\tilde{f}(s,t) = f(s,t) - f(s,0)f(0,t)$$
$$\tilde{g}(s,t) = g(s,t) - g(s,0)g(0,t).$$

Then

$$\sup_{x,y} |F(x,y) - G(x,y)| \leq C_1 \int_{-T}^{T} \int_{-T}^{T} \left| \frac{\tilde{f}(s,t) - \tilde{g}(s,t)}{st} \right| dsdt$$
$$+ C_2 \int_{-T}^{T} \left| \frac{f(s,0) - g(s,0)}{s} \right| ds + C_3 \int_{-T}^{T} \left| \frac{f(0,t) - g(0,t)}{t} \right| dt + \frac{C_4}{T}$$

for any $T > 0$ where C_1, C_2, C_3 and C_4 are positive constants.

Using this theorem, we have

$$\left| F_{n,m}(x,y) - \frac{1}{2\pi} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left(-\frac{u^2+v^2}{2}\right) dudv \right| \leq \frac{a}{\theta^{hn}}$$

for some $a > 0$ and $h > 0$. Setting $A_n = \left\{ \eta_n \geq \sqrt{(2-\epsilon)\log\log D_n^2} \right\}$ and using following generalized second Borel-Cantelli lemma (Cf. Rényi [12]),we have

$$\eta_n \geq \sqrt{(2-\epsilon)\log\log D_n^2} \qquad \text{i.o} \qquad \text{a.s.}$$

For details, see Révész [13].

Theorem F. *Suppose that the events $A_1, A_2, \dots$ satisfy*

$$\sum_{n=1}^{\infty} P(A_n) = \infty \qquad \textit{and} \qquad \liminf_{n\to\infty} \frac{\sum_{j=1}^{n}\sum_{k=1}^{n} P(A_j \cap A_k)}{\left(\sum_{j=1}^{n} P(A_j)\right)^2} = 1.$$

Then we have

$$P\left(\limsup_{n\to\infty} A_n\right) = 1.$$

In a similar way as the proof of (b), we can prove

$$\frac{\sum_{j=1}^{p(n)} a_j c_{p(n+1),j}\xi_j}{\sqrt{(2+\epsilon)A_{p(n+1)}^2 \log\log A_{p(n+1)}^2}} \leq K_{\nu,\theta} \qquad \text{f.e.} \quad \text{a.s.}$$

and now we have proved

$$\frac{\sum_{j=p(n)+1}^{p(n+1)} a_j c_{p(n+1),j}\xi_j}{\sqrt{(2-\epsilon)A_{p(n+1)}^2 \log\log A_{p(n+1)}^2}} \leq \sqrt{K_{\nu,1}^2 - K_{\nu,\theta}^2} \qquad \text{i.o.} \quad \text{a.s.}$$

Thus for any $\epsilon > 0$ and $\theta > 1$,

$$\sum_{j=1}^{p(n+1)} a_j c_{p(n+1),j}\xi_j \geq \left(\sqrt{(2-\epsilon)(K_{\nu,1}^2 - K_{\nu,\theta}^2)} - \sqrt{(2+\epsilon)K_{\nu,\theta}^2}\right) \times \sqrt{A_{p(n+1)}^2 \log\log A_{p(n+1)}^2} \qquad \text{i.o.} \quad \text{a.s.}$$

But

$$\sqrt{(2-\epsilon)(K_{\nu,1}^2 - K_{\nu,\theta}^2)} - \sqrt{(2+\epsilon)K_{\nu,\theta}^2} \to K_{\nu,1} \quad \text{as} \quad \theta \to \infty \quad \text{and} \quad \epsilon \to 0.$$

This implies (c).

3. Proof of theorem 3. In this section, z denotes a complex number satisfying $|\operatorname{Im} z| \leq 1$. Let G_n denote the distribution function of $N(M_n, 1)$ where $M_n = \frac{1}{A_n}\sum_{j=1}^{n} a_j b_j$, $\hat{F}_n$ and $\hat{G}_n$ denote corresponding Fourier-Stieltjes transform of F_n and G_n,

$$T_n(z) = \prod_{j=1}^{n}\left(1 + \frac{iza_j}{A_n}\right), \qquad U_n = \sum_{j=1}^{n} a_j^2\xi_j^2,$$
$$V_n = \sum_{j=1}^{n} a_j^2(\xi_j^2 - 1) \qquad \text{and} \qquad R_n = \sum_{j=1}^{n} r\left(\frac{iza_j\xi_j}{A_n}\right).$$

First we prove a lemma.

Lemma 3. *There exist positive constants L_4 and L_5 such that*

$$|\hat{F}_n(z) - \hat{G}_n(z)| \leq L_4\left(C_n|z|^3 + C_n^{4\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge 1}|z|^2\right)$$

if $|\operatorname{Im} z| \leq 1$ and $|z| \leq C_n^{-\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge\frac{1}{3}}$.

Proof. Using expansion formula,

$$|\hat{F}_n(z) - \hat{G}_n(z)|$$

$$\leq \left| ET_n \left(\exp\left(-\frac{z^2}{2} U_n + R_n \right) \right) - \exp\left(-\frac{z^2}{2} \right) \right|$$
$$+ \left| (ET_n - \exp(izM_n)) \exp\left(-\frac{z^2}{2} \right) \right|$$
$$= E_4 + E_5$$

Since $|T_n|^2 \leq \exp\left(|z|^2 - \frac{2(\operatorname{Im} z) S_n}{A_n} \right)$,

$$E_4 \leq E \left| \exp\left(\frac{1}{2}(|z|^2 - z^2) U_n - (\operatorname{Im} z) \frac{S_n}{A_n} \right) \right.$$
$$\left. \times \left(\exp(R_n) - \exp\left(\frac{z^2}{2} V_n \right) \right) \right|$$
$$\leq E \exp\left((\operatorname{Im} z)^2 V_n - (\operatorname{Im} z) \frac{S_n}{A_n} + (\operatorname{Im} z)^2 \right)$$
$$\times \left(|R_n| \exp(|R_n|) + \left| \frac{z^2 V_n}{2} \right| \exp\left(\left| \frac{z^2 V_n}{2} \right| \right) \right)$$
$$\leq E^{\frac{1}{8}} \exp\left(8(\operatorname{Im} z)^2 V_n \right) E^{\frac{1}{8}} \exp\left(-8(\operatorname{Im} z) \frac{S_n}{A_n} \right)$$
$$\times \left(E^{\frac{1}{2}} |R_n|^2 E^{\frac{1}{8}} \exp(|8R_n|) + \frac{|z|^2}{2} E^{\frac{1}{2}} V_n^2 E^{\frac{1}{8}} \exp\left(4|z^2 V_n| \right) \right)$$

estimating in a similar way as the proof of lemma 2, we have

$$\leq L_4 C_n (|z|^3 + |z|^2)$$

if $|z| \leq L_5 C_n^{-\frac{1}{3}}$ and $|\operatorname{Im} z| \leq 1$.

Next we estimate E_5.

$$E_5 \leq e \left| ET_n - 1 - izM_n \right| + e \left| \exp(izM_n) - 1 - izM_n \right|$$

By lemma 1, first term is less than $L_4 C_n^{4\left(\frac{1}{\delta} - \frac{1}{2}\right)} |z|^2$ if $|z| \leq L_5 C_n^{-2\left(\frac{1}{\delta} - \frac{1}{2}\right)}$. Since $|M_n| \leq C_n^{2\left(\frac{1}{\delta} - \frac{1}{2}\right)} \|B_i\|_\delta$, second term is also less than $L_4 C_n^{4\left(\frac{1}{\delta} - \frac{1}{2}\right)} |z|^2$ if $|z| \leq L_5 C_n^{-2\left(\frac{1}{\delta} - \frac{1}{2}\right)}$. Thus we have proved the lemma.

Let C_t be a circle in $\mathbf{C}$ with center $t \in \mathbf{R}$ and radius $\frac{1}{2} L_2 \wedge 1$. Since $\hat{F}_n(z)$ and $\hat{G}_n(z)$ are entire functions, using lemma 3,

$$\left| \frac{d}{dt} \frac{\hat{F}_n(t) - \hat{G}_n(t)}{t} \right| = \left| \frac{1}{2\pi i} \int_{C_t} \frac{\hat{F}_n(\zeta) - \hat{G}_n(\zeta)}{(\zeta - t)^2 \zeta} \, d\zeta \right|$$
$$\leq L_6 \left(C_n t^2 + C_n^{4\left(\frac{1}{\delta} - \frac{1}{2}\right) \wedge 1} (|t| + 1) \right)$$
$$\text{if} \qquad |t| \leq \frac{L_5}{2} C_n^{-2\left(\frac{1}{\delta} - \frac{1}{2}\right) \wedge \frac{1}{3}}.$$

Now we proceed to the proof of the theorem. We use the next theorem by Essen.(Cf.[4]).

Theorem G. *Let F and G are distribution functions. Then for some constants $C_i \quad (i = 1, \dots, 5)$,*

$$\begin{aligned}
\|F-G\|_\infty &\le C_1 \int_{-T}^{T} \left| \frac{\hat{F}(t)-\hat{G}(t)}{t} \right| dt + \frac{C_2}{T} \\
\|F-G\|_1 &\le C_3 \left(\int_{-T}^{T} \left| \frac{d}{dt} \frac{\hat{F}(t)-\hat{G}(t)}{t} \right|^2 dt \right)^{\frac{1}{2}} \\
&\quad + C_4 \left(\int_{-T}^{T} \left| \frac{d}{dt} \frac{\hat{F}(t)-\hat{G}(t)}{t} \right|^2 dt \right)^{\frac{1}{2}} \left(1+\frac{1}{T}\right) + \frac{C_5}{T}
\end{aligned}$$

Using this theorem, we have

$$\|F_n - G_n\|_\infty \le L\left(C_n T^3 + C_n^{4\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge 1} T^2 + \frac{1}{T} \right)$$

and

$$\|F_n - G_n\|_1 \le L\left(C_n T^{\frac{5}{2}} + C_n^{4\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge 1} T^{\frac{3}{2}} + C_n^{4\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge 1} T^{\frac{1}{2}} + \frac{1}{T} \right).$$

Putting $T = C_n^{-\frac{4}{3}\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge\frac{1}{4}}$ or $T = C_n^{-\frac{5}{8}\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge\frac{2}{7}}$ respectively,we have

$$\|F_n - G_n\|_\infty \le L C_n^{\frac{4}{3}\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge\frac{1}{4}}$$

and

$$\|F_n - G_n\|_1 \le L C_n^{\frac{5}{8}\left(\frac{1}{\delta}-\frac{1}{2}\right)\wedge\frac{2}{7}}.$$

Using the estimate of M_n, we have

$$\|G_n - G\|_\infty, \quad \|G_n - G\|_1 \le L C_n^{2\left(\frac{1}{\delta}-\frac{1}{2}\right)}.$$

Combining these estimstes, we can obtain the final resuls.

References

[1] I.Berkes, *On Strassen's version of the log log law for multiplicative systems*, Studia Sci. Math. Hungar. **8** (1973), 425–431.

[2] K.Fukuyama, *Some limit theorems of almost periodic function systems under the relative maesure*, J. Math. Kyoto Univ. **28** (1988), 557–577.

[3] ———, *Functional central limit theorem and Strassen's law of the iterated logarithms for weakly multiplicative systems*, J. Math. Kyoto Univ. **30** (1990), 625–635.
[4] I.A.Ibragimov-Yu.V.Linik, *Independent and stationary sequence of random variables*, Groningen, 1971.
[5] N.Kôno, *Functional central limit theorem and log log law for multiplicative systems*, Acta Math. Hungar. **52** (1988), 233–238.
[6] J.Kuelbs, *A strong convergence theorem for Banach space valued random variables*, Ann. Probab. **4** (1976), 744–771.
[7] D.L.McLeish, *Dependent central limit theorem and invariance principles*, Ann. Probab. **2** (1976), 620–628.
[8] F.Móricz, *The law of the iterated logarithm and related results for weakly multiplicative systems*, Anal. Math. **2** (1976), 211–229.
[9] ———, *Exponential estimates for the maximum of partial sums. I*, Acta Math. Acad. Sci. Hungar. **33** (1979), 159–167.
[10] H.Oodaira, *On Strassen's version of the iterated logarithm for Gaussian processes*, Z. Wahr. Geb. **21** (1972), 289–299.
[11] L.Paditz–Š.Šarachmetov, *A mean central limit theorem for multiplicative systems*, Math. Nachr. **139** (1988), 87–94.
[12] A.Rényi, *Probability theory*, Académiai Kiadó, 1970.
[13] P.Révész, *A new law of the iterated logarithm for multiplicative systems*, Acta Sci. Math. (Szeged) **34** (1973), 557–564.
[14] S.M.Sadikova, *On two dimensional analogue of Essen with application to the central limit theorem*, Theor. Probab. Appl. **11** (1966), 325–335.
[15] W.F.Stout, *Almost sure convergence*, Academic Press, 1977.
[16] S.Takahashi, *Notes on the law of the iterated logarithm*, Studia Sci. Math. Hungar. **7** (1972), 21–24.