ON THE CENTRAL LIMIT THEOREM FOR SOME GAP SERIES

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0. Introduction.

In this paper, we shall be concerned with the central limit theorem for gap series $\sum a_k f(\beta_k \omega)$ where f is an α-Lipschitz continuous function $(0 < \alpha \le 1)$ with period 2π satisfying

$$\int_0^{2\pi} f(x)\,dx = 0 \qquad \text{and} \qquad \int_0^{2\pi} f^2(x)\,dx = 2\pi,$$

and the sequence $\{\beta_k\}$ of positive numbers has *large gaps,* i.e.

$$\beta_{k+1}/\beta_k \to \infty \quad \text{as} \quad k \to \infty.$$

As to this problem, Kac [5] noticed the following result.

Theorem A. *Let $\Omega = [0, 2\pi]$ and P be the normalized Lebesgue measure on Ω. If a sequence $\{n_k\}$ of positive integers has large gaps and a sequence $\{a_k\}$ of real numbers satisfies*

$$A_n^2 = a_1^2 + \cdots + a_n^2 \to \infty \quad \textit{and} \quad a_n = o(A_n) \qquad \textit{as} \qquad n \to \infty, \tag{0.1}$$

then the distribution of $A_n^{-1} \sum_{k=1}^n a_k f(n_k \omega)$ converges weakly to the standard normal distribution.

There are various extensions of this result. For example, Takahashi [8] extended the class of functions of f and Berkes [1] studied the case in which $\{n_k\}$ have smaller gaps. Our aim is to extend Theorem A to the case in which $\Omega = \mathbf{R}$ and the probability measure P on Ω is not necessarily absolutely continuous with respect to the Lebesgue measure but may be singular measure satisfying one of the following conditions:

$$|\widehat{P}(u)| = O(|u|^{-\rho/2}) \quad \text{as} \quad u \to \infty, \tag{0.2}$$

$$P\{[\omega, \omega + h]\} \le M h^{\rho} \quad (\omega \in \Omega,\ h > 0). \tag{0.3}$$

Here M and ρ is positive constants and $\widehat{P}$ is the characteristic function of P. One can find some important and interesting examples of probability measures satisfying above conditions in [7], [11] and [12].

Let us now state our results.

Theorem 1. *Let P satisfy (0.2), $\{a_k\}$ satisfy (0.1), $\{\gamma_k\}$ be arbitrary and $\{\beta_k\}$ have large gaps. Then the distribution of $A_n^{-1}\sum_{k=1}^n a_k f(\beta_k\omega+\gamma_k)$ converges weakly to the standard normal distribution.*

Theorem 2. *Let P satisfy (0.3), $\{a_k\}$ satisfy (0.1), $\{\gamma_k\}$ be arbitrary and $\{\beta_k\}$ have large gaps. Then, for almost all $t\in\mathbf{R}$ with respect to the Lebesgue measure, the distribution of $A_n^{-1}\sum_{k=1}^n a_k f(t\beta_k\omega+\gamma_k)$ converges weakly to the standard normal distribution.*

Theorem 3. *Let P satisfy (0.3), $\{a_k\}$ satisfy (0.1), $\{\gamma_k\}$ be arbitrary and $\{\phi(k)\}$ satisfy*

$$\phi(1)>0 \qquad \textit{and} \qquad \phi(k+1)-\phi(k)\to\infty \quad \textit{as} \quad k\to\infty. \tag{0.4}$$

Then, for almost all $x>1$ with respect to the Lebesgue measure, the distribution of $A_n^{-1}\sum_{k=1}^n a_k f(x^{\phi(k)}\omega+\gamma_k)$ converges weakly to the standard normal distribution.

Remark. Under the condition (0.4), the sequence $\{x^{\phi(k)}\}$ has large gaps if $x>1$.

In the special case $f(x)=\sqrt{2}\cos x$, the gap series is called a lacunary trigonometric series. As to the lacunary trigonometric series, the corresponding results to our theorems were first proved by Takahashi [10], [9] and Kaufman [6] respectively, assuming Hadamard's gap condition:

$$\beta_{k+1}/\beta_k\geq q>1.$$

Recently the author [2] extended these results to the case of weaker gap conditions and the author [4] also proved functional law of the iterated logarithm both for lacunary trigonometric and gap series assuming the same conditions on P.

1. Approximation by trigonometric polynomials.

We shall first prove Theorem 1. In this section we shall approximate the gap series by some series of trigonometric polynomials, by modifying the method of Takahashi.

Let L denote the absolute constant which may change line by line.

Let us put $f(t)=\sum_{j=1}^\infty d_j\sqrt{2}\cos(jt+\gamma_j')$ and $\sigma_n(t)=\sum_{j\leq n} d_j\sqrt{2}\cos(jt+\gamma_j')$. Since f is α-Lipschitz continuous, classical results in Fourier analysis claim

$$\|f-\sigma_n\|_\infty\leq Ln^{-\alpha}\log n, \tag{1.1}$$

$$\sum_{j=n}^\infty d_j^2\leq L^2 n^{-2\alpha}, \tag{1.2}$$

$$\|f\|_\infty,\ \|\sigma_n\|_\infty\leq L \qquad (n\in\mathbf{N}). \tag{1.3}$$

Using (1.1) we can easily prove

$$\left|\sum_{k=1}^n a_k\left(f(\beta_k\omega+\gamma_k)-\sigma_{k^{1/\alpha}}(\beta_k\omega+\gamma_k)\right)\right|=o(A_n) \qquad \text{as} \quad n\to\infty.$$

Thus it is sufficient to prove the central limit theorem for $\{\sigma_{k^{1/\alpha}}(\beta_k\omega+\gamma_k)\}$. Without loss of generality, we may assume

$$\beta_1\geq 1, \quad \beta_{k+1}/\beta_k\geq 4\psi(k), \quad 2\geq\psi(k+1)/\psi(k)\geq 1 \quad \text{and} \quad \lim_{k\to\infty}\psi(k)=\infty,$$

for some sequence $\{\psi(k)\}$. We shall here prove the following convergence in probability:

$$\text{(1.4)} \qquad \frac{1}{A_n}\sum_{k=1}^{n} a_k \left(\sigma_{k^{1/\alpha}}(\beta_k\omega+\gamma_k)-\sigma_{\psi^{\rho/4}(k)}(\beta_k\omega+\gamma_k)\right)\to 0 \quad \text{as} \quad n\to\infty.$$

By expanding into Fourier series and putting $\gamma''_{k,j}=j\gamma_k+\gamma'_j$, we have

$$\begin{aligned}
&E\left(\sum_{k=1}^{n} a_k \left(\sigma_{k^{1/\alpha}}(\beta_k\omega+\gamma_k)-\sigma_{\psi^{\rho/4}(k)}(\beta_k\omega+\gamma_k)\right)\right)^2\\
&\quad\le \sum_{k\le n} a_k^2 \sum_{\psi^{\rho/4}(k)<j\le k^{1/\alpha}} d_j^2\\
&\qquad + 2\sum_{\epsilon=\pm1}\sum_{1\le k_1\le k_2\le n}|a_{k_1}a_{k_2}| \sum_{\substack{\psi^{\rho/4}(k_q)<j_q\le k_q^{1/\alpha}\\ (q=1,2)\\ |\beta_{k_2}j_2+\epsilon\beta_{k_1}j_1|\ge\beta_{k_2}}} |d_{j_1}d_{j_2}|\\
&\qquad\qquad\times\left|E\cos\left(\{\beta_{k_2}j_2+\epsilon\beta_{k_1}j_1\}\omega+\{\gamma''_{k_2,j_2}+\epsilon\gamma''_{k_1,j_1}\}\right)\right|\\
&\qquad + 2\sum_{1\le k_1\le k_2\le n}|a_{k_1}a_{k_2}| \sum_{\substack{\psi^{\rho/4}(k_q)<j_q\le k_q^{1/\alpha}\\ (q=1,2)\\ \beta_{k_2}\ge 2\beta_{k_1}k_1^{1/\alpha}}} |d_{j_1}d_{j_2}|\\
&\qquad\qquad\times\left|E\cos\left(\{\beta_{k_2}j_2-\beta_{k_1}j_1\}\omega+\{\gamma''_{k_2,j_2}-\gamma''_{k_1,j_1}\}\right)\right|\\
&\qquad + 2\sum_{1\le k_1\le k_2\le n}|a_{k_1}a_{k_2}| \sum_{\substack{\psi^{\rho/4}(k_q)<j_q\le k_q^{1/\alpha}\\ (q=1,2)\\ \beta_{k_2}<2\beta_{k_1}k_1^{1/\alpha}\\ |\beta_{k_2}j_2+\epsilon\beta_{k_1}j_1|\ge\beta_{k_1}/2}} |d_{j_1}d_{j_2}|\\
&\qquad\qquad\times\left|E\cos\left(\{\beta_{k_2}j_2-\beta_{k_1}j_1\}\omega+\{\gamma''_{k_2,j_2}-\gamma''_{k_1,j_1}\}\right)\right|\\
&\qquad + 2\sum_{1\le k_1< k_2\le n}|a_{k_1}a_{k_2}| \sum_{\substack{\psi^{\rho/4}(k_q)<j_q\le k_q^{1/\alpha}\\ (q=1,2)\\ |\beta_{k_2}j_2+\epsilon\beta_{k_1}j_1|<\beta_{k_1}/2}} |d_{j_1}d_{j_2}|\\
&\qquad\qquad\times\left|E\cos\left(\{\beta_{k_2}j_2-\beta_{k_1}j_1\}\omega+\{\gamma''_{k_2,j_2}-\gamma''_{k_1,j_1}\}\right)\right|\\
&\quad=\sum\nolimits_1+2\sum\nolimits_2+2\sum\nolimits_3+2\sum\nolimits_4+2\sum\nolimits_5 \qquad \text{(say).}
\end{aligned}$$

Since $\psi(k)$ diverges to infinity, trivially we have $\sum_1 = o(A_n^2)$. Using the estimate $|E\cos(\beta\omega+\gamma)|\le|\widehat{P}(\beta)|$, (0.2) and $\beta_{k_2}\ge\sqrt{\beta_{k_1}\beta_{k_2}}$, we obtain

$$\begin{aligned}
\sum\nolimits_2 \le L&\left(\sum_{1\le k_1\le k_2\le n} a_{k_1}^2a_{k_2}^2 \sum_{\substack{\psi^{\rho/4}(k_q)<j_q\le k_q^{1/\alpha}\\ (q=1,2)}} d_{j_1}^2d_{j_2}^2\right)^{1/2}\\
&\times\left(\sum_{1\le k_1\le k_2\le n}\ \sum_{\substack{\psi^{\rho/4}(k_q)<j_q\le k_q^{1/\alpha}\\ (q=1,2)}} (\beta_{k_1}\beta_{k_2})^{-\rho/2}\right)^{1/2}
\end{aligned}$$

$$\leq L\left(\sum_{1\leq k\leq n} a_k^2 \sum_{\psi^{\rho/4}(k)<j\leq k^{1/\alpha}} d_j^2\right)\left(\sum_{1\leq k\leq n}\sum_{\psi^{\rho/4}(k)<j\leq k^{1/\alpha}} \beta_k^{-\rho/2}\right)$$
$$= o(A_n^2)\sum_{k\leq n} k^{1/\alpha}2^{-k\rho/2} = o(A_n^2).$$

In $\sum_3$, we have $\beta_{k_2}j_2-\beta_{k_1}j_1\geq \beta_{k_2}-\beta_{k_1}k_1^{1/\alpha}\geq \beta_{k_2}/2\geq\sqrt{\beta_{k_1}\beta_{k_2}}$. Thus we can prove $\sum_3 = o(A_n^2)$ in the same way as the proof of $\sum_2 = o(A_n^2)$. In $\sum_4$, we have

$$\beta_{k_2}j_2-\beta_{k_1}j_1\geq \beta_{k_1}/2\geq k_1^{-1/\alpha}\beta_{k_2}/4\geq k_1^{-1/\alpha}\sqrt{\beta_{k_1}\beta_{k_2}}/2.$$

Thus we get

$$\sum\nolimits_4 = o(A_n^2)\sum_{k\leq n}\sum_{\psi^{\rho/4}(k)<j\leq k^{1/\alpha}} k^{1/\alpha}(\beta_k)^{-\rho/2} = o(A_n^2)\sum_{k\leq n} k^{2/\alpha}2^{-k\rho/2} = o(A_n^2).$$

In $\sum_5$ we have $|\beta_{k_2}j_2/\beta_{k_1}-j_1|<1/2$. From this, introducing the notation $[x]^*=\{n\in \mathbf{N}:|x-n|<1/2\}$, we have

$$\sum\nolimits_5 \leq \sum_{1\leq k_1<k_2\leq n}|a_{k_1}a_{k_2}|\left(\sum_{\psi(k_2)<j_2\leq k_2^{1/\alpha}} d_{j_2}^2\right)^{1/2}\left(\sum_{\psi(k_2)<j_2\leq k_2^{1/\alpha}} d_{[\beta_{k_2}j_2/\beta_{k_1}]^*}^2\right)^{1/2}$$
$$\leq \sum_{1\leq k_1<k_2\leq n}|a_{k_1}a_{k_2}|\left(\sum_{\psi(k_1)<j} d_j^2\right)^{1/2}\left(\sum_{\beta_{k_2}/\beta_{k_1}\leq j} d_j^2\right)^{1/2}.$$

Here, using (1.2) and $\beta_{k_2}/\beta_{k_1}\geq 2^{k_2-k_1}$, we get

$$\sum\nolimits_5 \leq L\sum_{1\leq k_1<k_2\leq n}|a_{k_1}a_{k_2}|\left(\sum_{\psi(k_1)<j} d_j^2\right)^{1/2}2^{-\alpha(k_2-k_1)}$$
$$= L\sum_{i=1}^{n-1}2^{-i\alpha}\sum_{k=1}^{n-i}|a_k a_{i+k}|\left(\sum_{\psi(k)<j} d_j^2\right)^{1/2}$$
$$\leq L\sum_{i=1}^{n-1}2^{-i\alpha}\left(\sum_{k=1}^{n-i}a_k^2\sum_{\psi(k)<j} d_j^2\right)^{1/2}\left(\sum_{k=1}^{n-i}a_{i+k}^2\right)^{1/2}$$
$$\leq L\left(\sum_{k=1}^{n}a_k^2\sum_{\psi(k)<j} d_j^2\right)^{1/2}\left(\sum_{k=1}^{n}a_k^2\right)^{1/2} = o(A_n^2).$$

Thus (1.4) is verified.

By (1.4), it is sufficient to prove the central limit theorem for $\{\sigma_{\psi^{\rho/4}(k)}(\beta_k\omega+\gamma_k)\}$.

2. Method of weakly multiplicative systems.

To prove the central limit theorem, we shall use the method of weakly multiplicative systems.

For the sequence $\{\xi_k\}$ of random variables, put

$$b_{k_1,\dots,k_r} = E\xi_{k_1}\dots\xi_{k_r} \qquad \text{and} \qquad \|B_r\| = \sum_{k_1<\dots<k_r} |b_{k_1,\dots,k_r}|.$$

If $\sup_{r\in\mathbf{N}} \|B_r\|^{1/r} < \infty$, we say that $\{\xi_k\}$ is a weakly multiplicative system.

We use the following theorem in [3].

Theorem B. *Let $\{\xi_n\}$ be a weakly multiplicative system in the above sense satisfying*

$$\|\xi_n\|_\infty \le B \qquad (n \in \mathbf{N}), \tag{2.1}$$

$$\lim_{\substack{i+j\to\infty\\ i\ne j}} E\left((\xi_i^2-1)(\xi_j^2-1)\right) = 0, \tag{2.2}$$

and $\{a_n\}$ satisfy (0.1). Then the distributions of $\{A_n^{-1}\sum_{k=1}^n a_k\xi_k\}$ converges weakly to the standard normal distribution.

Put $\xi_k = \sigma_{\psi^{\rho/4}(k)}(\beta_k\omega+\gamma_k)$. We get (2.1) by (1.3). Now let us verify the bound of $\|B_r\|$ and (2.2). Usual calculation yields

$$\begin{aligned}
|b_{k_1,\dots,k_r}| &= \sum_{\substack{j_q\le\psi(k_q)\\(q=1,\dots,r)}} \sqrt{2}^r |E\cos(\beta_{i_r}j_r\omega+\gamma''_{i_r,j_r})\dots\cos(\beta_{i_1}j_r\omega+\gamma''_{i_1,j_1})| \\
&\le L \sum_{\substack{j_q\le\psi^{\rho/4}(k_q)\\(q=1,\dots,r)}} \left(\beta_{k_r}j_r-\beta_{k_{r-1}}j_{r-1}-\dots-\beta_{k_1}j_1\right)^{-\rho/2} \\
&\le L\left(\psi(k_r)\dots\psi(k_1)\right)^{\rho/4}\left(\beta_{k_r}-\beta_{k_{r-1}}\psi(k_{r-1})-\dots-\beta_{k_1}\psi(k_1)\right)^{-\rho/2}.
\end{aligned}$$

Using the condition on $\{\beta_k\}$ and $\{\psi_k\}$, we have

$$\begin{aligned}
\beta_{k_r}-\beta_{k_{r-1}}\psi(k_{r-1})-\dots-\beta_{k_1}\psi(k_1) &\ge \beta_{k_r}(1-4^{-1}-\dots-4^{-k_r}) \\
&\ge L4^{k_r}\psi(k_r-1)\dots\psi(1).
\end{aligned}$$

Thus we have $|b_{k_1,\dots,k_r}| \le L2^{-k_r\rho}$, and this yields

$$\|B_r\| \le L\sum_{k_1<\dots<k_r} 2^{-k_r\rho} \le L\frac{1}{(r-1)!}\sum_{k=r}^{\infty} k^{r-1}2^{-k\rho} \le L^r.$$

It implies that $\{\xi_k\}$ is a weakly multiplicative system.

Next let us verify (2.2). We have

$$\left| E\left(\xi_k^2 - \sum_{j\le\psi^{\rho/4}(k)} d_j^2\right)\right| \le 2\sum_{j_1\le j_2\le\psi^{\rho/4}(k)} \beta_k^{-\rho/4} \le L2^{-\rho k},$$

and in case $k_1 < k_2$ we obtain

$$\begin{aligned}
&\Bigg|E\Bigg(\xi_{k_1}^2 - \sum_{j_1 \le \psi^{\rho/4}(k_1)} d_{j_1}^2\Bigg)\Bigg(\xi_{k_2}^2 - \sum_{j_2 \le \psi^{\rho/4}(k_2)} d_{j_2}^2\Bigg)\Bigg| \\
&\quad \le 4 \sum_{\substack{j_q \le i_q \le \psi^{\rho/4}(k_q) \\ (q=1,2)}} |d_{j_1} d_{i_1} d_{j_2} d_{i_2}| \sum_{\substack{\epsilon_q = \pm 1\ (q=1,2) \\ \epsilon = \pm 1 \\ \epsilon \ne -1 \text{ if } j_q = i_q}} \\
&\qquad\qquad |E \cos\big(\{\beta_{k_2}(i_2 + \epsilon_2 j_2) + \epsilon \beta_{k_1}(i_1 + \epsilon_1 j_1)\} \\
&\qquad\qquad\qquad + \{\gamma''_{k_2,i_2} + \epsilon_2 \gamma''_{k_2,j_2} + \epsilon \gamma''_{k_1,i_1} + \epsilon\epsilon_1 \gamma''_{k_1,j_1}\}\big)\big| \\
&\quad \le L \sum_{\substack{j_q \le i_q \le \psi^{\rho/4}(k_q) \\ (q=1,2)}} (\beta_{k_2})^{-\rho/2} \le L 2^{-\rho k_2}.
\end{aligned}$$

By these estimates we have

$$\begin{aligned}
&\big|E(\xi_{k_1}^2 - 1)(\xi_{k_2}^2 - 1)\big| \\
&\quad \le \Bigg|E\Bigg(\xi_{k_1}^2 - \sum_{j_1 \le \psi^{\rho/4}(k_1)} d_{j_1}^2\Bigg)\Bigg(\xi_{k_2}^2 - \sum_{j_2 \le \psi^{\rho/4}(k_2)} d_{j_2}^2\Bigg)\Bigg| \\
&\qquad + \Bigg|\sum_{j_1 \le \psi^{\rho/4}(k_1)} d_{j_1}^2 E\Bigg(\xi_{k_2}^2 - \sum_{j_2 \le \psi^{\rho/4}(k_2)} d_{j_2}^2\Bigg)\Bigg| \\
&\qquad + \Bigg|\sum_{j_2 \le \psi^{\rho/4}(k_2)} d_{j_2}^2 E\Bigg(\xi_{k_1}^2 - \sum_{j_1 \le \psi^{\rho/4}(k_1)} d_{j_1}^2\Bigg)\Bigg| + \sum_{j_1 \le \psi^{\rho/4}(k_1)} d_{j_1}^2 \sum_{j_2 \le \psi^{\rho/4}(k_2)} d_{j_2}^2 \\
&\quad \le L\left(2^{-\rho(k_1 \vee k_2)} + \psi^{-\rho\alpha/4}(k_1) 2^{-\rho k_2} + \psi^{-\rho\alpha/4}(k_2) 2^{-\rho k_1} + \psi^{-\rho\alpha/4}(k_1)\psi^{-\rho\alpha/4}(k_2)\right) \\
&\quad \longrightarrow 0 \qquad \text{as} \qquad k_1 + k_2 \to \infty.
\end{aligned}$$

This completes the proof.

3. Lemmas used in the proofs of Theorems 2 and 3.

In the proofs of Theorems 2 and 3 we use the following lemmas due to Takahashi [9] and Kaufman [6], respectively.

Lemma C. *Let P satisfy (0.3). Then there exists a constant D depending only on ρ and M satisfying*

$$\int_v^{v+1} |\widehat{P}(ut)|\, dt \le D|u|^{-\rho/2} \qquad (u, v \in \mathbf{R}).$$

Lemma D. *Let P satisfy (0.3). There exists a constant D depending only on ρ and M, such that for any $g \in C^2[v, v+1]$ satisfying $\min_{x\in[v,v+1]} g''(x) > 0$,*

$$\int_v^{v+1} |\widehat{P}(g(x))| dx \le D\Big(\min_{x\in[v,v+1]} g''(x)\Big)^{-\rho/(2+4\rho)}.$$

Using these lemmas and replacing the all estimates of $|\widehat{P}(\beta)|$ appearing in the proof of Theorem 1 by $\int_v^{v+1} |\widehat{P}(t\beta)|\, dt$ or $\int_v^{v+1} |\widehat{P}(p(x))|\, dx$ where $p(x)$ is polynomials, we can prove Theorems 2 and 3. For details, see [3].

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