ON PERMUTATIONAL INVARIANCE OF THE METRIC DISCREPANCY RESULTS

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Abstract. Let $\{n_k\}$ be a sequence of non-zero real numbers. We prove that the law of the iterated logarithm for discrepancies of the sequence $\{n_k x\}$ is permutational invariant if $|n_{k+1}/n_k| \to \infty$ is satisfied.

1. Introduction

A sequence $\{x_k\}$ of real numbers is said to be uniformly distributed modulo one (u. d. mod 1) if ${}^{\#}\{k \le N \mid \langle x_k \rangle \in [\,a, b)\}/N \to b - a$ for all $[\,a, b) \subset [\,0, 1)$, where $\langle x \rangle$ denotes the fractional part $x - [\,x\,]$ of real number x. To measure the speed of convergence, we use the discrepancy

$$D_N\{x_k\} = \sup_{0 \le a < b \le 1} \left| \frac{1}{N} {}^{\#}\{k \le N \mid \langle x_k \rangle \in [\,a, b)\} - (b - a) \right|.$$

General theory of discrepancies can be found in [13]. It is known that an arithmetic progression $\{kx\}$ is u. d. mod 1 if and only if x is irrational, and a geometric progression $\{\theta^k x\}$ is u. d. mod 1 for almost every x if and only if $|\theta| > 1$.

Manifestly the sequence $\{U_k\}$ of independent random variables with uniform distribution $P(a \le U_k < b) = b - a$ $(0 \le a < b \le 1)$ is u. d. mod 1, almost surely. The speed is described by Chung-Smirnov law of the iterated logarithm

$$\varlimsup_{N \to \infty} \frac{N D_N\{U_k\}}{\sqrt{2N \log\log N}} = \frac{1}{2} \quad \text{a.s.} \tag{1}$$

In Nijenrode Lecture, Erdős [6] made a conjecture: If a sequence $\{n_k\}$ of positive integers satisfies the Hadamard gap condition $n_{k+1}/n_k \ge q > 1$, then $\{n_k x\}$ imitates the behaviour of $\{U_k\}$ and obeys the law of Chung-Smirnov type (1). After Takahashi's method [16], Philipp [14] solved the conjecture by showing

$$\frac{1}{4\sqrt{2}} \le \varlimsup_{N \to \infty} \frac{N D_N\{n_k x\}}{\sqrt{2N \log\log N}} \le \frac{1}{\sqrt{2}} \left(166 + \frac{664}{q^{1/2} - 1} \right) \quad \text{a.e. } x.$$

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By martingale approximation, Philipp [15] removed the assumption that n_k's are integers. Dhompongsa [5] proved the exact result of Chung-Smirnov type

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e. } x,$$

assuming $\log(n_{k+1}/n_k)/\log\log k \to \infty$. It was relaxed in [8] to $n_{k+1}/n_k \to \infty$.

Results for geometric progressions $\{\theta^k x\}$ are complicated. If $|\theta| \le 1$, it is not u. d. mod 1. When $|\theta| > 1$, it is proved that there exists a real number Σ_θ such that

$$\varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta \quad \text{a.e. } x.$$

(Cf. [7, 9, 10, 11]). If $\theta^r \notin \mathbf{Q}$ for any $r \in \mathbf{N}$, then $\Sigma_\theta = 1/2$, and otherwise $\Sigma_\theta > 1/2$. When θ satisfies $\theta^r \in \mathbf{Q}$ for some $r \in \mathbf{N}$, take the smallest $r \in \mathbf{N}$ with $\theta^r \in \mathbf{Q}$ and write $\theta^r = p/q$ by $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ with $\gcd(p,q) = 1$. If p and q are odd, $\Sigma_\theta = \frac{1}{2}\sqrt{(|p|q+1)/(|p|q-1)}$. If $q = 1$, then Σ_θ equals to $\frac{1}{2}\sqrt{(|p|+1)/(|p|-1)}$, $\frac{1}{2}\sqrt{(|p|+1)|p|(|p|-2)/(|p|-1)^3}$, $\frac{1}{9}\sqrt{42}$, or $\frac{1}{49}\sqrt{910}$, according as p is odd, $|p| \ge 4$ is even, $p = 2$, or $p = -2$. If $p = \pm 5$ and $q = 2$, we have $\Sigma_\theta = \frac{1}{9}\sqrt{22}$. The values of Σ_θ for other cases are so far unknown.

Although (1) is permutational invariant, we can prove the contrary.

Theorem 1 ([8]). *For an unbounded sequence $\{n_k\}$ of positive real numbers, there exists a permutation σ over $\mathbf{N}$ (i.e., a bijection $\mathbf{N} \to \mathbf{N}$) such that*

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_{\sigma(k)} x\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e. } x. \tag{2}$$

For $a = 2, 3, \ldots$, there exists a permutation σ such that

$$\varlimsup_{N\to\infty} \frac{ND_N\{2^{\sigma(k)} x\}}{\sqrt{2N\log\log N}} = \frac{1}{2}\sqrt{\frac{(2^a+1)2^a(2^a-2)}{(2^a-1)^3}} \quad \text{a.e. } x. \tag{3}$$

Aistleitner-Berkes-Tichy [1, 2, 3] studied the effect of permutation on metric discrepancy results. As a corollary, they derived the beautiful result below.

Theorem 2 (Aistleitner-Berkes-Tichy [3]). *If a sequence $\{n_k\}$ of positive integers satisfies $n_{k+1}/n_k \to \infty$, then (2) holds for any permutation σ over $\mathbf{N}$.*

It can be applied only for integers $\{n_k\}$, and we here prove it for real numbers.

Theorem 3. *If a sequence $\{n_k\}$ of non-zero real numbers satisfies the condition $|n_{k+1}/n_k| \to \infty$, then (2) holds for any permutation σ over $\mathbf{N}$.*

2. Proof

We first introduce the notion of weakly multiplicative system. It is said that a sequence $\{X_k\}$ of random variables is a weakly multiplicative system if it satisfies

$$\sum_{r=1}^{\infty} \sum_{k(1),\dots,k(r):\ k(1)<\dots<k(r)} \left| EX_{k(1)} \dots X_{k(r)} \right| < \infty. \tag{4}$$

We use the next law of the iterated logarithm for weakly multiplicative sysmtems.

Theorem 4 (Berkes [4]). *Let $\{X_k\}$ be a sequence of uniformly bounded random variables. If both of $\{X_k\}$ and $\{X_k^2 - 1\}$ are weakly multiplicative systems, then*

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} X_k = 1 \quad a.s. \tag{5}$$

Since (4) is permutational invariant, (5) remains valid if we replace X_k by $X_{\sigma(k)}$. Moreover, if $\{X_k\}$ satisfies (4), then $\{-X_k\}$ also. Therefore we have

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N \log\log N}} \left| \sum_{k=1}^{N} X_{\sigma(k)} \right| = 1 \quad \text{a.s.} \tag{6}$$

The next key lemma states a result on the probability space $\big([\,0,1\,), \mathcal{B}[\,0,1\,), dx\big)$.

Lemma 1. *Let f be a trigonometric polynomial satisfying $f(x+1) = f(x)$, $\int_0^1 f(x)\,dx = 0$ and $\|f\|_2^2 = \int_0^1 f^2(x)\,dx = 1$. For $\{n_k\}$ in Theorem 3, both of $\{f(n_k x)\}$ and $\{f^2(n_k x) - 1\}$ are uniformly bounded weakly multiplicative systems.*

Proof. Denote $f(x) = \sum_{j=1}^{J} a_j \cos(2\pi j x + \gamma_j)$ and $L = 1 + \sum_{j=1}^{J} |a_j| \ge 1$. Since $\{|n_k|\}$ is eventually increasing, and since $|n_k|$ diverges to infinity, we see that $|n_1|, \dots, |n_k| \le |n_k|$ holds for large k. Take K_0 such that

$$|n_{k+1}/n_k| \ge 6JL \quad (k \ge K_0) \quad \text{and} \quad \max\{|n_1|, \dots, |n_{K_0}|\} = |n_{K_0}|. \tag{7}$$

Put $K_1 = [\,K_0 + 2 + \log_{6JL} K_0\,] \ge K_0$ and $C = (6JL)^{-K_0} |n_{K_0}|/2 > 0$. We have

$$(6JL)^{k-K_0-1} \ge K_0 \quad (k \ge K_1). \tag{8}$$

By (7), we have $|n_k| = \big|(n_k/n_{k-1}) \dots (n_{K_0+1}/n_{K_0}) n_{K_0}\big| \ge 2C(6JL)^k$ for $k \ge K_0$. Suppose that $k(1) < \dots < k(r)$ and $k(r) \ge K_1$. By (7) and (8), we have

$$\begin{aligned} \sum_{i=1}^{r-1} |n_{k(i)}| &\le \sum_{i=1}^{k(r)-1} |n_i| \le \sum_{i=K_0+1}^{k(r)-1} |n_i| + K_0 |n_{K_0}| \\ &\le \sum_{i=K_0+1}^{k(r)-1} \frac{|n_{k(r)}|}{(6JL)^{k(r)-i}} + \frac{K_0 |n_{k(r)}|}{(6JL)^{k(r)-K_0}} \le |n_{k(r)}| \Big(\frac{2}{6JL} + \frac{1}{6JL} \Big) \le \frac{|n_{k(r)}|}{2J}. \end{aligned}$$

Hence, for $j_1, \dots, j_r \le J$, and $\varsigma_1, \dots, \varsigma_{r-1} = \pm 1$, $\varsigma_r = 1$, we have the estimate $\left|\sum_{i=1}^r \varsigma_i j_i n_{k(i)}\right| \ge |n_{k(r)}| - J\sum_{i=1}^{r-1} |n_{k(i)}| \ge |n_{k(r)}|/2 \ge C(6L)^{k(r)}$, and thereby $\left|\int_0^1 \cos\sum_{i=1}^r (2\pi\varsigma_i j_i n_{k(i)}x + \varsigma_i\gamma_i)\,dx\right| \le 1/C(6L)^{k(r)}$. Let $I_{k(1),\dots,k(r)}$ denotes

$$\int_0^1 f(n_{k(1)}x)\dots f(n_{k(r)}x)\,dx = \sum_{j_1}^{J}\cdots\sum_{j_r}^{J}\int_0^1\prod_{i=1}^r a_{j_i}\cos(2\pi j_i n_{k(i)}x + \gamma_i)\,dx.$$

$\prod_{i=1}^r \cos(2\pi j_i n_{k(i)}x + \gamma_i) = \frac{1}{2^{r-1}}\sum_{\varsigma_1,\dots,\varsigma_{r-1}=\pm 1}\cos\sum_{i=1}^r (2\pi\varsigma_i j_i n_{k(i)}x + \varsigma_i\gamma_i)$ gives $|I_{k(1),\dots,k(r)}| \le \left(\sum_{j=1}^J |a_j|\right)^r/C(6L)^{k(r)} \le L^r/C(6L)^{k(r)} \le 1/C6^{k(r)}$ when $k(1) < \dots < k(r)$ and $k(r) \ge K_1$. Note that we have

$$\sum_{r=1}^{\infty}\sum_{\substack{k(1),\dots,k(r):\\ k(1)<\dots<k(r)}} |I_{k(1),\dots,k(r)}| = \sum_{k=1}^{\infty}\sum_{r=1}^{k}\sum_{\substack{k(1),\dots,k(r-1):\\ k(1)<\dots<k(r-1)<k}} |I_{k(1),\dots,k(r-1),k}|$$

We divide the summation as $\sum_{k=1}^{\infty} = \sum_{k=1}^{K_1-1} + \sum_{k=K_1}^{\infty}$. The first is finite and the second is bounded by $\sum_{k=K_1}^{\infty}\sum_{r=1}^{k}\binom{k-1}{r-1}/C6^k \le \sum_k 2^{k-1}/C6^k < \infty$. □

We prove Theorem 3. For a trigonometric polynomial f satisfying $f(x+1) = f(x)$, $\int_0^1 f(x)\,dx = 0$ and $\int_0^1 f^2(x)\,dx > 0$, by applying Lemma 1, we see that $\{f(n_k x)/\|f\|_2\}$ and $\{(f(n_k x)/\|f\|_2)^2 - 1\}$ satisfy (4). By (6), we have

$$\overline{\lim_{N\to\infty}}\frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^N f(n_{\sigma(k)}x)\right| = \|f\|_2 \quad \text{a.e. } x. \tag{9}$$

The next result is convenient to deal with the discrepancies.

Theorem 5 ([12]). *Let $\{n_k\}$ be a sequence of non-zero real numbers satisfying $|n_{k+1}/n_k| \ge q > 1$, σ be a permutation over $\mathbf{N}$, and S denotes $\mathbf{Q}\cap[0,1)$. Then*

$$\overline{\lim_{N\to\infty}}\frac{ND_N\{n_{\sigma(k)}x\}}{\sqrt{2N\log\log N}} = \sup_{S\ni a<b\in S}\lim_{d\to\infty}\overline{\lim_{N\to\infty}}\frac{1}{\sqrt{2N\log\log N}}\left|\sum_{k=1}^N \widetilde{\mathbf{1}}_{a,b}^{(d)}(n_{\sigma(k)}x)\right| \text{ a.e.,}$$

where $\widetilde{\mathbf{1}}_{a,b}^{(d)}(x)$ denotes the d-th sum of the Fourier series of $\mathbf{1}_{[a,b)}(\langle x\rangle) - (b-a)$.

By changing values of the first finitely many terms of $\{n_k\}$, we may assume $|n_{k+1}/n_k| \ge 2$. By applying (9) for $f = \widetilde{\mathbf{1}}_{a,b}^{(d)}$, by $\lim_{d\to\infty}\left\|\widetilde{\mathbf{1}}_{a,b}^{(d)}\right\|_2 = \left\|\widetilde{\mathbf{1}}_{a,b}\right\|_2 \le \left\|\widetilde{\mathbf{1}}_{0,1/2}\right\|_2 = \frac{1}{2}$ and Theorem 5, we have (2).

References

[1] C. Aistleitner, I. Berkes, R. Tichy, *Lacunary sequences and permutations*, Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp, Eds. I. Berkes, R. Bradley, H. Dehling, M. Peligrad, and R. Tichy, Kendrick press, 2010, pp. 35–49.

[2] C. Aistleitner, I. Berkes, R. Tichy, *On the asymptotic behaviour of weakly lacunary series*, Proc. Amer. Math. Soc., **139** (2011) 2505–2517.

[3] C. Aistleitner, I. Berkes, R. Tichy, *On the law of the iterated logarithm for permuted lacunary sequences*, Proc. Steklov Inst. Math., **276** (2012) 3–20.

[4] I. Berkes, *On Strassen's version of the log log law for multiplicative systems*, Studia Sci. Math. Hungar., **8** (1973) 425–431.

[5] S. Dhompongsa, *Almost sure invariance principles for the empirical process of lacunary sequences.* Acta Math. Hungar. **49** (1987) 83–102.

[6] P. Erdős, *Problems and results on diophantine approximations*, Compos. Math., **16** (1964) 52-65.

[7] K. Fukuyama, *The law of the iterated logarithm for discrepancies of* $\{\theta^n x\}$, Acta Math. Hungar. **118** (2008) 155–170.

[8] K. Fukuyama, *The law of the iterated logarithm for the discrepancies of a permutation of* $\{n_k x\}$, Acta Math. Hungar. **123** (2009) 121–125.

[9] K. Fukuyama, *A central limit theorem and a metric discrepancy result for sequence with bounded gaps*, Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp, Eds. I. Berkes, R. Bradley, H. Dehling, M. Peligrad, and R. Tichy, Kendrick press, 2010, pp. 233-246.

[10] K. Fukuyama, *Metric discrepancy results for alternating geometric progressions*, Monatsh. Math., **171** (2013) 33–63.

[11] K. Fukuyama, *A metric discrepancy result for the sequence of powers of minus two*, Indag. Math. (NS), **25** (2014) 487-504.

[12] K. Fukuyama, Y. Mitsuhata, *Bounded law of the iterated logarithm for discrepancies of permutations of lacunary sequences*, Summer School on the Theory of Uniform Distribution, RIMS Kôkyûroku Bessatsu, **B29** (2012) 65–88.

[13] L. Kuipers, H. Niederreiter, Uniform distribution of sequences, Wiley-Interscience, New York, 1974; Dover, New York, 2006.

[14] W. Philipp, *Limit theorems for lacunary series and uniform distribution mod 1*, Acta Arith. **26** (1975) 241–251.

[15] W. Philipp, *A functional law of the iterated logarithm for empirical distribution functions of weakly dependent random variables.* Ann. Probab. **5** (1977) 319–350.

[16] S. Takahashi, *An asymptotic property of a gap sequence*, Proc. Japan Acad., **38** (1962), 101–104.

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