Metric discrepancy results for geometric progressions perturbed by irrational rotations

Katusi Fukuyama, Shinya Mori, and Yusuke Tanabe

the date of receipt and acceptance should be inserted later

Received: date / Accepted: date

Abstract For $\theta \in (-\infty, -1) \cup (1, \infty)$ and for almost every x, it is known that the sequence $\{\theta^k x\}$ is uniformly distributed modulo 1. The speed of convergence sensitively depends on algebraic nature of θ. In this paper we prove that such dependence vanishes if we perturb the sequence by adding irrational rotation $\{k\gamma\}$. The speed becomes identical with that of uniformly distributed i.i.d.

Keywords discrepancy, lacunary sequence, law of the iterated logarithm.

Mathematics Subject Classification (2000) Primary 11K38, 42A55, 60F15

1 Introduction

A sequence $\{x_k\}$ of real numbers is said to be uniformly distributed modulo one if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_{[a,b)}(\langle x_k \rangle) = b - a \quad \text{for all} \quad 0 \le a < b \le 1,$$

where $\langle x \rangle$ denotes the fractional part $x - [\,x\,]$ of real number x and $\mathbf{1}_{[a,b)}$ is the indicator function of $[\,a, b)$. It is equivalent to the convergence of the empirical distribution $\frac{1}{N}\sum_{k=1}^{N} \delta_{\langle x_k \rangle}$ of the sequence $\{x_k\}$ towards the uniform distribution, and the following

The first author is supported by JSPS KAKENHI 16K05204.

Department of Mathematics,
Kobe University,
Rokko, Kobe, 657-8501, Japan
K. Fukuyama
E-mail: fukuyama@math.kobe-u.ac.jp
S. Mori
E-mail: smori@math.kobe-u.ac.jp
Y. Tanabe
E-mail: ytanabe@math.kobe-u.ac.jp

discrepancies are used to measure the speed of convergence;

$$D_N\{x_k\} = \sup_{0\le a<b\le 1}\left|\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{[a,b)}(\langle x_k\rangle) - (b-a)\right|,$$

$$D_N^*\{x_k\} = \sup_{0\le a\le 1}\left|\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{[0,a)}(\langle x_k\rangle) - a\right|.$$

As to the arithmetic progression $\{k\gamma\}$, Bohl [3], Sierpiński [21] and Weyl [25] proved independently that it is uniformly distributed modulo 1 if γ is irrational. Khintchine [17] determined the speed of convergence of discrepancies for almost every γ. See also Kesten [16].

For a sequence $\{U_k\}$ of uniformly distributed i.i.d., Chung [4] and Smirnov [22] determined the speed of convergence by proving the law of the iterated logarithm

$$\varlimsup_{N\to\infty}\frac{ND_N\{U_k\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty}\frac{ND_N^*\{U_k\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad \text{a.s.}$$

Philipp [19] studied the discrepancies $D_N\{n_k x\}$ when $\{n_k\}$ satisfies the Hadamard's gap condition $n_{k+1}/n_k > q > 1$, and proved the bounded law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} \le \varlimsup_{N\to\infty}\frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} \le C(q) < \infty, \quad \text{a.e.} \quad x, \tag{1}$$

by modifying a technique developed by Takahashi [24].

For geometric progressions, the following exact law of the iterated logarithm was proved in [6,9]: For any $\theta \in (-\infty,-1)\cup(1,\infty)$ there exist constants $\varSigma_\theta$ and $\varSigma_\theta^*$ such that

$$\varlimsup_{N\to\infty}\frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \varSigma_\theta, \quad \varlimsup_{N\to\infty}\frac{ND_N^*\{\theta^k x\}}{\sqrt{2N\log\log N}} = \varSigma_\theta^*, \quad \text{a.e.} \quad x.$$

The concrete value of $\varSigma_\theta$ is determined in some how a complicated way as below (See [6,7,9,10]). One has $\varSigma_\theta = \varSigma_\theta^* = 1/2$ if θ satisfies

$$\theta^r \notin \mathbf{Q} \quad \text{for all} \quad r \in \mathbf{N}. \tag{2}$$

When $\theta^k \in \mathbf{Q}$ for some $k = 1, 2, \dots$, denote

$$\theta^r = p/q \quad (p\in\mathbf{Z}, q\in\mathbf{N}, (p,q)=1) \text{ where } r = \min\{k\in\mathbf{N} : \theta^k \in \mathbf{Q}\}. \tag{3}$$

In this case, $\varSigma_\theta$ is independent of r and is greater than 1/2, i.e., $\varSigma_\theta = \varSigma_{p/q} > 1/2$. Here $\varSigma_\theta = \varSigma_\theta^*$ holds if and only if p is positive.

If both p and q are odd, then

$$\varSigma_{p/q} = \frac{1}{2}\sqrt{\frac{|p|q+1}{|p|q-1}}.$$

If $p/q = \pm 2$, then

$$\varSigma_2 = \frac{1}{9}\sqrt{42} \quad \text{and} \quad \varSigma_{-2} = \frac{1}{49}\sqrt{910}. \tag{4}$$

When pq is even, for p/q with large modulus, or more precisely for p/q belonging to

$$\{p/q : 2\nmid p,\ 2\mid q,\ |p|/q \ge 9/4\} \cup \{p/q : 2\mid p,\ 2\nmid q,\ |p|/q \ge 4\} \cup \{\pm 13/6\},$$

the next general formula was proved in [14,15]:

$$\Sigma_{p/q} = \sqrt{\frac{(|p|q)^I+1}{(|p|q)^I-1} v\Big(\frac{|p|-q-1}{2(|p|-q)}\Big) + \frac{2(|p|q)^I}{(|p|q)^I-1} \sum_{m=1}^{I-1} \frac{1}{(|p|q)^m} v\Big(q^m \frac{|p|-q-1}{2(|p|-q)}\Big)},$$

where $I = \min\{n \in \mathbf{N} \mid q^n = \pm 1 \bmod |p|-q\}$ and $v(x) = \langle x\rangle(1-\langle x\rangle)$.

Note that the last formula does not necessary hold when modulus of p/q is small. Indeed, the above formula yields $\Sigma_2 = \Sigma_{-2} = \Sigma_{3/2} = \Sigma_{4/3} = 0$ by $v(0) = 0$, the actual values of Σ_2 and Σ_{-2} are as (4), and the actual values of $\Sigma_{3/2}$ and $\Sigma_{4/3}$ are as below (See [11,14]):

$$\Sigma_{3/2} = \frac{2}{665}\sqrt{\frac{30567145176261688966144563679087 3}{1031442479849053554617194905 5}}, \quad \Sigma_{4/3} = \frac{18}{7}\sqrt{\frac{117609}{2985983}}.$$

Besides, examples $\Sigma_{8/3}$, $\Sigma_{10/3}$, $\Sigma_{12/5}$, $\Sigma_{17/8}$, $\Sigma_{19/10}$, $\Sigma_{12/7}$, and $\Sigma_{8/5}$ can be found in [14].

Although the value of Σ_θ, the constant determining the speed of convergence toward the uniform distribution, sensitively depends on algebraic nature of θ, such dependence miraculously vanishes if we perturb the sequence $\{\theta^k x\}$ by adding an irrational rotation $\{k\gamma\}$. Actually the law of the iterated logarithm for discrepancies of $\{\theta^k x + k\gamma\}$ coincides completely with that of the uniformly distributed i.i.d. Now we are in a position to state our main result.

Theorem 1 *If $|\theta| > 1$ and $\gamma \notin \mathbf{Q}$, then*

$$\varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x + k\gamma\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\{\theta^k x + k\gamma\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad a.e.\ x.$$

Moreover if θ satisfies (2), then it holds for every $\gamma \in \mathbf{R}$

After having this result, it is natural to ask whether this phenomenon occurs for every sequence $\{n_k\}$ satisfying the Hadamard's gap condition, i.e., if

$$\varlimsup_{N\to\infty} \frac{ND_N\{n_k x + k\gamma\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N^*\{n_k x + k\gamma\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad \text{a.e. } x$$

holds or not. Unfortunately the answer is negative as the following counter example shows.

Theorem 2 *There exist a strictly increasing sequence $\{m(k)\}$ of positive integers, an irrational number γ, and real numbers $\Sigma > \Sigma^* > 1/2$ such that*

$$\varlimsup_{N\to\infty} \frac{ND_N\{3^{m(k)}x + k\gamma\}}{\sqrt{2N\log\log N}} = \Sigma \quad and \quad \varlimsup_{N\to\infty} \frac{ND_N^*\{3^{m(k)}x + k\gamma\}}{\sqrt{2N\log\log N}} = \Sigma^*, \quad a.e. \quad x.$$

We also have the complex version of Theorem 1. For a sequence $\{z_k\}$ of complex numbers, define the discrepancy by

$$D_N\{z_k\} = \sup_{0\le a<a'\le 1,\, 0\le b<b'\le 1} \left| \frac{1}{N}\sum_{k=1}^N \mathbf{1}_{[a,b)}(\langle \Re z_k\rangle)\mathbf{1}_{[a',b')}(\langle \Im z_k\rangle) - (b-a)(b'-a') \right|,$$

$$D_N^*\{z_k\} = \sup_{0\le a\le 1, 0\le a'\le 1} \left| \frac{1}{N}\sum_{k=1}^N \mathbf{1}_{[0,a)}(\langle \Re z_k\rangle)\mathbf{1}_{[0,a')}(\langle \Im z_k\rangle) - aa' \right|.$$

In [12], when a complex number θ satisfies $|\theta| > 1$, the law of the iterated logarithm for discrepancies of $\{\theta^k z\}$ was proved. For $\{\theta^k z + k\gamma\}$, we have the following result which can be proved in the same way as Theorem 1 and the main theorem of [12].

Theorem 3 *If $\Re\gamma$ and $\Im\gamma$ are linearly independent over $\mathbf{Q}$, if $\theta \in \mathbf{C}$, and if $|\theta| > 1$, then*

$$\overline{\lim_{N\to\infty}} \frac{ND_N\{\theta^k z + k\gamma\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N^*\{\theta^k z + k\gamma\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad a.e.\ z \in \mathbf{C}.$$

Moreover if θ satisfies $\theta^n \notin \mathbf{Q}\big[\sqrt{-1}\big]$ for all $n = 1, 2, \dots,$ then it holds for every $\gamma \in \mathbf{C}$.

Before closing introduction, we make a remark. We consider any sequence $\{n_k\}$ of real numbers and a uniformly distributed i.i.d. $\{\xi_k\}$ on the probability space $(0,1)$ equipped with the Borel σ-field and the Lebesgue measure. Then for any $x \in (0,1)$, the sequence $\{\langle n_k x + \xi_k(\cdot)\rangle\}$ is again a uniformly distributed i.i.d., and by Chung-Smirnov result, we have

$$\overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x + \xi_k(y)\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_k x + \xi_k(y)\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \tag{5}$$

for almost every $y \in (0,1)$. By applying Fubini's theorem, we have (5) for almost every $x \in (0,1)$, for almost every $y \in (0,1)$. It means that for any sequence $\{n_k\}$, there exist sequences $\{a_k\}$ such that

$$\overline{\lim_{N\to\infty}} \frac{ND_N\{n_k x + a_k\}}{\sqrt{2N\log\log N}} = \overline{\lim_{N\to\infty}} \frac{ND_N^*\{n_k x + a_k\}}{\sqrt{2N\log\log N}} = \frac{1}{2}, \quad \text{a.e. } x.$$

By this argument, we cannot determine concrete sequence $\{a_k\}$ such that the abobe law of the iterated logarithm holds. Our result asserts that the sequence $\{k\gamma\}$ is such a concrete example. It is interesting that the discrepancies of the sequence $\{k\gamma\}$ behaves in completely different manner as compared to $\{\xi_k\}$, i.e., $ND_N\{k\gamma\} = O\big((\log N)^{1+\varepsilon}\big)$ for almost every γ for any $\epsilon > 0$ ([17]). We thank the anonymous referee for this valueable remark.

2 Preliminary

Put $e_x(\nu) = \exp(2\pi\sqrt{-1}\,\nu x)$. In this section f denotes a trigonometric polynomial with degree d and period 1 satisfying $\int_0^1 f = 0$ whose Fourier expansion is given by

$$f(x) = \sum_{\nu=1}^{d} (a_\nu \cos 2\pi\nu x + b_\nu \sin 2\pi\nu x) = \sum_{1\le|\nu|\le d} c_\nu e_x(\nu). \tag{6}$$

Lemma 4 *Let f be a function given by (6), $\{\gamma_k\}$ be an arbitrary sequence of real numbers, A be any real number, and $\{n_k\}$ be a sequence of real numbers satisfying the generalized Hadamard's gap condition*

$$|n_1| \ge 1, \qquad |n_{k+1}/n_k| > q > 1 \quad (k = 1, 2, \dots). \tag{7}$$

Then there exists a constant C_q depending only on q such that

$$\int_A^{A+1} \Big(\sum_{k=M+1}^{M+N} f(n_k x + \gamma_k) \Big)^4 dx \le C_q \Big(\sum_{\nu=1}^{d} (|a_\nu| + |b_\nu|) \Big)^4 N^2.$$

Proof Clearly it is enough to prove for $A = 0$. By applying the triangle inequality, we see that the L^4-norm of $\sum_{k=M+1}^{M+N} f(n_k \cdot + \gamma_k)$ is bounded from above by

$$\sum_{\nu=1}^{d} \left(|a_\nu| \Big\| \sum_{k=M+1}^{M+N} \cos 2\pi\nu(n_k \cdot + \gamma_k) \Big\|_4 + |b_\nu| \Big\| \sum_{k=M+1}^{M+N} \sin 2\pi\nu(n_k \cdot + \gamma_k) \Big\|_4 \right)$$

When $\{n_k\}$ is a non-negative sequence satisfying Hadamard's gap condition, we proved (Lemma 1 (1) of [5]) that there exists a constant C_q depending only on q such that

$$\int_0^1 \left(\sum_{j=1}^{\infty} (\alpha_j \cos 2\pi n_j x + \beta_j \sin 2\pi n_j x) \right)^4 dx \le C_q \left(\sum_{j=1}^{\infty} (\alpha_j^2 + \beta_j^2) \right)^2. \tag{8}$$

For general $\{n_k\}$, by noting the relation

$$\alpha_j \cos 2\pi n_j x + \beta_j \sin 2\pi n_j x = \alpha_j \cos 2\pi |n_j| x + (\pm\beta_j) \sin 2\pi |n_j| x,$$

we see that (8) is valid under the generalized Hadamard's gap condition. Hence

$$\Big\| \sum_{k=M+1}^{M+N} \cos 2\pi\nu(n_k \cdot + \gamma_k) \Big\|_4^4 \vee \Big\| \sum_{k=M+1}^{M+N} \sin 2\pi\nu(n_k \cdot + \gamma_k) \Big\|_4^4 \le C_q N^2.$$

By combining these, we have the conclusion. □

Lemma 5 *Let g be a bounded measurable function with period 1 satisfying $\int_0^1 g = 0$. For all $a < b$ and $\lambda \ne 0$, we have*

$$\left| \int_a^b g(\lambda x)\, dx \right| \le \frac{\|g\|_\infty}{|\lambda|}.$$

Proof It is enough to prove by assuming $\lambda > 0$. By changing variables $y = \lambda x$, the integral is expressed as below and the above inequality is proved:

$$\frac{1}{\lambda} \int_{\lambda a}^{\lambda b} g(y)\, dy = \frac{1}{\lambda} \sum_{k=0}^{[\lambda(b-a)]-1} \int_{\lambda a + k}^{\lambda a + k + 1} g(y)\, dy + \frac{1}{\lambda} \int_{\lambda a + [\lambda(b-a)]}^{\lambda b} g(y)\, dy. \quad \square$$

For a measurable function g, we define *the mean value* $\int_{\mathbf{R}} g(x)\, \mu_R(dx)$ by

$$\int_{\mathbf{R}} g(x)\, \mu_R(dx) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(x)\, dx$$

if the limit on the right hand side exists. If g is bounded and if the mean value of g exists, then

$$\int_{\mathbf{R}} g(x+\gamma)\, \mu_R(dx) = \int_{\mathbf{R}} g(x)\, \mu_R(dx) \quad (\gamma \in \mathbf{R}), \tag{9}$$

$$\int_{\mathbf{R}} g(\Theta x)\, \mu_R(dx) = \int_{\mathbf{R}} g(x)\, \mu_R(dx) \quad (\Theta \ne 0). \tag{10}$$

For trigonometric polynomials g and h with period 1 satisfying $\int_0^1 g = 0$, we have

$$\int_{\mathbf{R}} g(\Theta x)h(x)\,\mu_R(dx) = 0, \tag{11}$$

$$\int_{\mathbf{R}} g((P/Q)x)h(x)\,\mu_R(dx) = \int_{\mathbf{R}} g(Px)h(Qx)\,\mu_R(dx) = \int_0^1 g(Px)h(Qx)\,dx, \tag{12}$$

for $\Theta \notin \mathbf{Q}$ and $P, Q \in \mathbf{Z}^\times$.

Lemma 6 *Let* $|\theta| > 1$ *and* $\gamma \notin \mathbf{Q}$. *For a trigonometric polynomial* f *with period 1 satisfying* $\int_0^1 f = 0$, *we have*

$$\lim_{N\to\infty} \sup_{M\in\mathbf{N}} \left| \int_{\mathbf{R}} \left(\frac{1}{\sqrt{N}} \sum_{k=M+1}^{M+N} f(\theta^k x + k\gamma) \right)^2 \mu_R(dx) - \|f\|_2^2 \right| = 0, \tag{13}$$

where $\|f\|_2^2 = \int_0^1 f^2(x)\,dx$. *In case when* (2) *is satisfied,* (13) *is valid for all* $\gamma \in \mathbf{R}$.

Proof First we apply (10) and expand the square in the following way

$$\begin{aligned}
&\int_{\mathbf{R}} \left(\frac{1}{\sqrt{N}} \sum_{k=M+1}^{M+N} f(\theta^k x + k\gamma) \right)^2 \mu_R(dx) = \frac{1}{N} \sum_{k=1}^{N} \int_{\mathbf{R}} f^2(\theta^k x + (k+M)\gamma)\,\mu_R(dx) \\
&\qquad + 2 \sum_{l=1}^{N-1} \frac{1}{N} \sum_{k=1}^{N-l} \int_{\mathbf{R}} f(\theta^k x + (k+M)\gamma) f(\theta^{k+l} x + (k+M+l)\gamma)\,\mu_R(dx) \\
&=: J_1(M,N) + J_2(M,N).
\end{aligned}$$

By applying (10) and (9), we have

$$J_1(M,N) = \int_{\mathbf{R}} f^2(x)\,\mu_R(dx) = \int_0^1 f^2(x)\,dx.$$

By applying (10) we see that the summand in $J_2(M,N)$ equals to

$$\int_{\mathbf{R}} f(x + (k+M)\gamma) f(\theta^l x + (k+M+l)\gamma)\,\mu_R(dx), \tag{14}$$

and (11) shows that it equals to zero if $\theta^l \notin \mathbf{Q}$. Hence (13) is proved when (2) is satisfied.

Secondly we assume (3). Then (14) equals to zero if l is not a multiple of r. By putting $l = rm$ and applying (12) we see that $J_2(M,N)$ equals to

$$\begin{aligned}
&2 \sum_{1\le m\le (N-1)/r} \frac{1}{N} \sum_{k=1}^{N-l} \int_0^1 f(q^m x + (k+M)\gamma) f(p^m x + (k+M+rm)\gamma)\,dx \\
&= 2 \sum_{m,\nu,\nu'} c_\nu c_{\nu'} e_\gamma((\nu+\nu')M + \nu' rm) \left(\int_0^1 e_x(\nu q^m + \nu' p^m)\,dx \right) \frac{1}{N} \sum_{k=1}^{N-l} e_\gamma((\nu+\nu')k).
\end{aligned}$$

Hence we have the next estimate where the right hand side is independent of M:

$$|J_2(M,N)| \le 2 \sum_{m,\nu,\nu'} |c_\nu c_{\nu'}| \left| \int_0^1 e_x(\nu q^m + \nu' p^m)\,dx \right| \left| \frac{1}{N} \sum_{k=1}^{N-l} e_\gamma((\nu+\nu')k) \right|.$$

Since ν and ν' vary in a finite set, we have $\nu q^m + \nu' p^m \neq 0$ and $\int_0^1 e_x(\nu q^m + \nu' p^m)\,dx = 0$ except for finitely many m. Hence the summation for (m, ν, ν') is a finite sum. When $\nu q^m + \nu' p^m = 0$ we have $\nu + \nu' \neq 0$ which implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N-l} e_\gamma((\nu+\nu')k) = 0,$$

since $\gamma \notin \mathbf{Q}$. Hence (13) is proved. □

Lemma 7 *Put* $C = \min\big(\{\langle \log_{|\theta|} \nu - \log_{|\theta|} \nu' \rangle^* \mid \nu, \nu' = 1, \dots, d, \} \setminus \{0\}\big) \in (0, 1/2]$ *and* $D = (|\theta|^C - 1) \wedge 2 > 0$*, where* $\langle x \rangle^* = \min_{n\in\mathbf{Z}} |x - n|$*. Then for any* $L \in \mathbf{N}$*, we have*

$$|\theta^k \nu + \theta^l \nu'| \geq D|\theta|^L \quad \textit{if} \quad k, l \geq L,\ |\nu|, |\nu'| \leq d,\ \theta^k\nu + \theta^l \nu' \neq 0. \tag{15}$$

Proof If $\theta^k \nu \theta^l \nu' > 0$, then $|\theta^k \nu + \theta^l \nu'| = |\theta^k \nu| + |\theta^l \nu'| \geq 2|\theta|^L \geq D|\theta|^L$. Assume that $\theta^k \nu \theta^l \nu' < 0$. By $\theta^k \nu + \theta^l \nu' \neq 0$, we have $|\theta^k \nu| \neq |\theta^l \nu'|$. When $\log_{|\theta|} |\nu| - \log_{|\theta|} |\nu'| \notin \mathbf{Z}$, we have

$$\begin{aligned} \big|\log_{|\theta|} |\theta^k \nu| - \log_{|\theta|} |\theta^l \nu'|\big| &= \big|(k-l) + (\log_{|\theta|} |\nu| - \log_{|\theta|} |\nu'|)\big| \\ &\geq \langle \log_{|\theta|} |\nu| - \log_{|\theta|} |\nu'| \rangle^* \geq C. \end{aligned}$$

If $\log_{|\theta|} |\nu| - \log_{|\theta|} |\nu'| \in \mathbf{Z}$, by $|\theta^k \nu| \neq |\theta^l \nu'|$ we see that $\log_{|\theta|} |\theta^k \nu| - \log_{|\theta|} |\theta^l \nu'|$ is a non-zero integer and $\big|\log_{|\theta|} |\theta^k \nu| - \log_{|\theta|} |\theta^l \nu'|\big| \geq 1 \geq C$. When $|\theta^k \nu| > |\theta^l \nu'|$, we have $|\theta^k \nu / \theta^l \nu'| \geq |\theta|^C$ and hence $|\theta^k \nu + \theta^l \nu'| = |\theta^k \nu| - |\theta^l \nu'| \geq (|\theta|^C - 1)|\theta^l \nu'| \geq D\theta^L$. When $|\theta^k \nu| \leq |\theta^l \nu'|$, we have $|\theta^l \nu' / \theta^k \nu| \geq |\theta|^C$ and hence $|\theta^k \nu + \theta^l \nu'| = |\theta^l \nu'| - |\theta^k \nu| \geq (|\theta|^C - 1)|\theta^k \nu| \geq D|\theta|^L$. □

3 Martingale Approximation

In this section, we prove the following law of the iterated logarithm.

Proposition 8 *Let* f *be a trigonometric polynomial with period 1 satisfying* $\int_0^1 f = 0$*. For a real number* θ *with* $|\theta| > 1$ *and* $\gamma \notin \mathbf{Q}$*, we have*

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} f(\theta^k x + k\gamma) = \|f\|_2, \quad \textit{a.e. } x. \tag{16}$$

The proof is by martingale approximation, which is a simplification of the proof given by Aistleitner [1] and originated with Berkes [2] and Philipp-Stout [20].

Although we can prove (16) on $[\,A, A+1)$ for any A, we prove it on $[\,0, 1)$ for simplicity.

We divide $\mathbf{N}$ into consecutive blocks $\Delta_1', \Delta_1, \Delta_2', \Delta_2, \dots$ satisfying ${}^\#\Delta_i' = [\,9 \log_{|\theta|} i\,]$ and ${}^\#\Delta_i = i$. By putting $i^- = \min \Delta_i$ and $i^+ = \max \Delta_i$, we have

$$|\theta^{i^-}/\theta^{(i-1)^+}| \geq |\theta|^{9\log_{|\theta|} i} = i^9.$$

We denote $\mu(i) = [\log_2 i^4 |\theta^{i^+}|] + 1$ and introduce a σ-field $\mathcal{F}_i$ on $[0,1)$ defined by

$$\mathcal{F}_i = \sigma\{[j2^{-\mu(i)}, (j+1)2^{-\mu(i)}) \mid j = 0, \dots, 2^{\mu(i)} - 1\}.$$

Note that $i^4|\theta^{i^+}| \le 2^{\mu(i)} \le 2i^4|\theta^{i^+}|$. Set

$$T_i(x) = \sum_{k \in \Delta_i} f(\theta^k x + k\gamma), \quad T'_i(x) = \sum_{k \in \Delta'_i} f(\theta^k x + k\gamma), \quad Y_i = E(T_i \mid \mathcal{F}_i) - E(T_i \mid \mathcal{F}_{i-1}).$$

Clearly $\{Y_i, \mathcal{F}_i\}$ forms a martingale difference sequence. We prove

$$\|Y_i - T_i\|_\infty \le (\|f'\|_\infty + 2\|f\|_\infty)/i^3, \tag{17}$$

$$\|Y_i^2 - T_i^2\|_\infty \le 3\|f\|_\infty(\|f'\|_\infty + 2\|f\|_\infty)/i^2, \tag{18}$$

$$\|Y_i^4 - T_i^4\|_\infty \le 15\|f\|_\infty^3(\|f'\|_\infty + 2\|f\|_\infty). \tag{19}$$

For $k \in \Delta_i$ and $x \in I = [j2^{-\mu(i)}, (j+1)2^{-\mu(i)}) \in \mathcal{F}_i$, we have

$$\begin{aligned}
&|f(\theta^k x + k\gamma) - E(f(\theta^k \cdot + k\gamma) \mid \mathcal{F}_i)| = \left| |I|^{-1} \int_I (f(\theta^k x + k\gamma) - f(\theta^k y + k\gamma))\, dy \right| \\
&\le \max_{y \in I} |f(\theta^k x + k\gamma) - f(\theta^k y + k\gamma))| \le \|f'\|_\infty |\theta^k| 2^{-\mu(i)} \le \|f'\|_\infty |\theta^k| / |\theta^{i^+}| i^4 \\
&\le \|f'\|_\infty / i^4,
\end{aligned}$$

and hence we have

$$|T_i - E(T_i \mid \mathcal{F}_i)| \le \|f'\|_\infty {}^\# \Delta_i / i^4 = \|f'\|_\infty / i^3.$$

On $J = [j2^{-\mu(i-1)}, (j+1)2^{-\mu(i-1)}) \in \mathcal{F}_{i-1}$, by Lemma 5 we have

$$\begin{aligned}
&|E(f(\theta^k \cdot + k\gamma) \mid \mathcal{F}_{i-1})| = \left| |J|^{-1} \int_J f(\theta^k y + k\gamma)\, dy \right| \\
&\quad \le \|f(\cdot + k\gamma)\|_\infty 2^{\mu(i-1)} / |\theta^k| \le \|f\|_\infty 2(i-1)^4 |\theta^{(i-1)^+} / \theta^{i^-}| \le 2\|f\|_\infty / i^5,
\end{aligned}$$

and hence we have

$$|E(T_i \mid \mathcal{F}_{i-1})| \le 2\|f\|_\infty {}^\# \Delta_i / i^5 = 2\|f\|_\infty / i^4,$$

which shows (17).

By $\|T_i\|_\infty \le i\|f\|_\infty$, we have $\|E(T_i \mid \mathcal{F}_i)\|_\infty$, $\|E(T_i \mid \mathcal{F}_{i-1})\|_\infty \le i\|f\|_\infty$, which imply $\|Y_i\|_\infty \le 2i\|f\|_\infty$ and $\|Y_i + T_i\|_\infty \le 3i\|f\|_\infty$. Similarly we have $\|Y_i^2 + T_i^2\|_\infty \le 5i^2\|f\|_\infty^2$. By applying these to $\|Y_i^2 - T_i^2\|_\infty \le \|Y_i - T_i\|_\infty \|Y_i + T_i\|_\infty$ and $\|Y_i^4 - T_i^4\|_\infty \le \|Y_i^2 - T_i^2\|_\infty \|Y_i^2 + T_i^2\|_\infty$, we have (18) and (19).

Expanding T_i^2 into a trigonometric polynomial, $v_i = \int_{\mathbf{R}} T_i^2(x) \mu_R(dx)$ gives its constant term. Hence $T_i^2 - v_i$ has the trigonometric polynomial expansion

$$\sum_{k,l \in \Delta_i} \sum_{\nu,\nu' : \nu\theta^k + \nu'\theta^l \ne 0} c_\nu c_{\nu'} \exp\Big(2\pi\sqrt{-1}\, ((\nu\theta^k + \nu'\theta^l)x + (\nu k + \nu l)\gamma)\Big).$$

By applying Lemma 7 and Lemma 5, we have

$$|E(T_i^2 - v_i \mid \mathcal{F}_{i-1})| \le \Big(\sum_\nu |c_\nu|\Big)^2 i^2 \frac{2^{\mu(i-1)}}{D|\theta^{i^-}|} = O\Big(i^2 (i-1)^4 \frac{\theta^{(i-1)^+}}{\theta^{i^-}}\Big) = O\Big(\frac{1}{i^3}\Big).$$

By putting $\beta_M = \sum_{i=1}^M v_i$, we have

$$\left\| \sum_{i=1}^{M} E(T_i^2 \mid \mathcal{F}_{i-1}) - \beta_M \right\|_\infty = O(1). \tag{20}$$

Denote $V_M = \sum_{i=1}^M E(Y_i^2 \mid \mathcal{F}_{i-1})$. By (18), we see

$$\left\| \sum_{i=1}^{M} \big(E(Y_i^2 \mid \mathcal{F}_{i-1}) - E(T_i^2 \mid \mathcal{F}_{i-1}) \big) \right\|_\infty = O(1) \quad \text{and} \quad \big\| V_M - \beta_M \big\|_\infty = O(1). \tag{21}$$

Denote $l_M = M(M+1)/2$. By Lemma 6 we have $v_i \sim i\|f\|_2^2$ and

$$\beta_M \sim l_M \|f\|_2^2. \tag{22}$$

By these we see that there exists M_0 such that $V_M \geq l_M\|f\|_2^2/2$ for $M \geq M_0$. By Lemma 4 we have $ET_i^4 = O(i^2)$, and by (19) we have $EY_i^4 = O(i^2)$.

We use the next theorem by Monrad-Philipp [18], which is a version of Strassen's theorem [23].

Theorem 9 *Suppose that a square integrable martingale difference sequence* $\{Y_i, \mathcal{F}_i\}$ *satisfies*

$$V_M = \sum_{i=1}^{M} E(Y_i^2 \mid \mathcal{F}_{i-1}) \to \infty \ \textit{a.s.} \quad \textit{and} \quad \sum_{i=1}^{\infty} E\left(\frac{Y_i^2 \mathbf{1}_{\{Y_i^2 \geq \psi(V_i)\}}}{\psi(V_i)} \right) < \infty$$

for some non-decreasing function ψ *with* $\psi(x) \to \infty$ *(*$x \to \infty$*) such that* $\psi(x)(\log x)^\alpha/x$ *is non-increasing for some* $\alpha > 50$*. If there exists a uniformly distributed random variable* U *which is independent of* $\{Y_n\}$*, there exists a standard normal i.i.d.* $\{Z_i\}$ *such that*

$$\sum_{i \geq 1} Y_i \mathbf{1}_{\{V_i \leq t\}} = \sum_{i \leq t} Z_i + o\big(t^{1/2}(\psi(t)/t)^{1/50}\big), \quad (t \to \infty) \quad \textit{a.s.}$$

To apply this theorem, we prepare a probability space $(\Omega, \mathcal{G}, P)$ on which a uniformly distributed random variable U is defined. We regard the product $[0,1) \times \Omega$ as a probability space in natural manner and consider a filtration $\{\mathcal{F}_n \otimes \mathcal{G}\}$. We here abuse notation and denote by E the expectation on $[0,1) \times \Omega$, denote $\mathcal{F}_n \otimes \mathcal{G}$ simply by $\mathcal{F}_n$, and regard $\{T_n\}$, $\{Y_n\}$ and U as defined on $[0,1) \times \Omega$. Then $\{Y_i, \mathcal{F}_n\}$ is a martingale difference sequence which is independent of U, and the estimates so far we have proved are still valid.

Hence by putting $\psi(x) = x/(\log x)^{51}$, we have

$$\sum_{i \geq M_0} E\left(\frac{Y_i^2 \mathbf{1}_{\{Y_i^2 \geq \psi(V_i)\}}}{\psi(V_i)} \right) \leq \sum_i \frac{EY_i^4}{\psi^2(l_i\|f\|_2^2/2)} = O\left(\sum_i \frac{i^2 (\log l_i)^{102}}{l_i^2} \right) < \infty.$$

By (21) and $V_M - V_{M-1} = \beta_M - \beta_{M-1} + O(1) = v_M + O(1) \to \infty$, we have $V_{M-1} < \beta_M < V_{M+1}$ for large M. Hence $V_i \leq \beta_M$ is equivalent to $i \leq M-1$ or $i \leq M$. By $\|Y_i\|_\infty = O(i)$ we have

$$\sum_{i \geq 1} Y_i \mathbf{1}_{\{V_i \leq \beta_M\}} = \sum_{k=1}^{M} Y_k + O(M) = \sum_{k=1}^{M} Y_k + o(\phi_{l_M}),$$

where $\phi_x = \sqrt{2x \log\log x}$.

By (22) we have $\beta_M = O(l_M)$. By applying Theorem 9 and putting $t = \beta_M$, we have

$$\sum_{k=1}^{M} Y_k = \sum_{i\geq 1} Y_i \mathbf{1}_{\{V_i \leq \beta_M\}} + o(\phi_{l_M}) = \sum_{i\leq \beta_M} Z_i + o(\phi_{l_M}), \quad \text{a.s.}$$

Hence by the law of the iterated logarithm we have

$$\overline{\lim_{M\to\infty}}\, \phi_{l_M}^{-1} \Big| \sum_{k=1}^{M} Y_k \Big| = \|f\|_2, \tag{23}$$

for almost every $(x, \omega) \in [\,0, 1) \times \Omega$. By Fubini's theorem, there exists an ω such that (23) holds for almost every x. Since the left hand side of (23) is independent of ω, we see that (23) holds for almost every x. By noting (17), we have

$$\overline{\lim_{M\to\infty}}\, \phi_{l_M}^{-1} \Big| \sum_{k=1}^{M} T_k \Big| = \overline{\lim_{M\to\infty}}\, \phi_{l_M}^{-1} \Big| \sum_{k=1}^{M} Y_k \Big| = \|f\|_2, \quad \text{a.e.} \quad x \tag{24}$$

By applying (4), $\sum_{i=1}^{M} [\,9 \log_{|\theta|} i\,] = O(M \log M)$, and Beppo-Levi theorem, we have

$$E\bigg(\Big| M^{-7/8} \sum_{k=1}^{M} T_k' \Big|^4 \bigg) = O\big(M^{-3/2} (\log M)^2\big) \quad \text{and} \quad \sum_{M=1}^{\infty} \Big| M^{-7/8} \sum_{k=1}^{M} T_k' \Big|^4 < \infty \quad \text{a.e.} \quad x,$$

and hence

$$\Big| \sum_{k=1}^{M} T_k' \Big| = o(M^{7/8}) \quad \text{and} \quad \overline{\lim_{M\to\infty}}\, \phi_{l_M}^{-1} \Big| \sum_{k=1}^{M} T_k' \Big| = 0, \quad \text{a.e.} \quad x.$$

Therefore, by $M^+ = l_M + \sum_{i=1}^{M} [\,9 \log_{|\theta|} i\,] \sim l_M$, and (24), we have

$$\overline{\lim_{M\to\infty}}\, \phi_{M^+}^{-1} \Big| \sum_{i=1}^{M} \sum_{k \in \Delta_i' \cup \Delta_i} f(\theta^k x) \Big| = \|f\|_2, \quad \text{a.e.} \quad x.$$

Moreover we have $\sum_{k\in \Delta_M' \cup \Delta_M} \|f(\theta^k \cdot\,)\|_\infty = o(\phi_{M^+})$ and (16).

4 LIL for discrepancies

To begin with, for a, b, $x \in \mathbf{R}$ satisfying $0 \leq b - a \leq 1$, we put

$$\mathbf{I}_{a,b}(x) = \sum_{n\in\mathbf{Z}} \mathbf{1}_{[a,b)}(x+n) \quad \text{and} \quad \widetilde{\mathbf{I}}_{a,b}(x) = \mathbf{I}_{a,b}(x) - (b-a).$$

We see $\widetilde{\mathbf{I}}_{a,b} = \mathbf{1}_{[\langle a \rangle, \langle b \rangle)} - (b-a)$ if $\langle a \rangle \leq \langle b \rangle$, and $\widetilde{\mathbf{I}}_{a,b} = -\mathbf{1}_{[\langle b \rangle, \langle a \rangle)} + (b-a)$ otherwise. Therefore we have

$$D_N\{x_k\} = \sup_{a,b:\, 0\leq b-a\leq 1} \frac{1}{N} \Big| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{a,b}(x_k) \Big| \quad \text{and} \quad D_N^*\{x_k\} = \sup_{0\leq a\leq 1} \frac{1}{N} \Big| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{0,a}(x_k) \Big|.$$

By this expression of $D_N\{x_k\}$, we can verify

$$D_N\{x_k+t\} = D_N\{x_k\}, \text{ and hence } D_N\{(2k-1)\gamma\}_{k>M} = D_N\{(2k-1)\gamma\}_{k\geq 1}. \quad (25)$$

We can easily prove the proposition below by following the proof given in [13] in which the case when $\gamma_k \equiv 0$ is proved. Recall that $\phi_x = \sqrt{2x\log\log x}$.

Proposition 10 *Let $\{n_k\}$ be a sequence of real numbers satisfying the generalized Hadamard's gap condition (7), let $\{\gamma_k\}$ be a sequence of real numbers, and ϖ be a permutation of $\mathbf{N}$, i.e., a bijection $\mathbf{N}\to\mathbf{N}$. Then for a dense countable $S\subset[0,1)$, we have*

$$\varlimsup_{N\to\infty} \phi_N^{-1} N D_N\{n_{\varpi(k)}x+\gamma_{\varpi(k)}\} = \sup_{S\ni a<b\in S} \varlimsup_{N\to\infty} \phi_N^{-1}\left|\sum_{k=1}^N \widetilde{\mathbf{I}}_{a,b}(n_{\varpi(k)}x+\gamma_{\varpi(k)})\right|,$$

$$\varlimsup_{N\to\infty} \phi_N^{-1} N D_N^*\{n_{\varpi(k)}x+\gamma_{\varpi(k)}\} = \sup_{a\in S} \varlimsup_{N\to\infty} \phi_N^{-1}\left|\sum_{k=1}^N \widetilde{\mathbf{I}}_{0,a}(n_{\varpi(k)}x+\gamma_{\varpi(k)})\right|,$$

for almost every $x\in\mathbf{R}$. Denote the d-th subsum of the Fourier series of $\widetilde{\mathbf{I}}_{a,b}$ by $\widetilde{\mathbf{I}}_{a,b;d}$. Then

$$\varlimsup_{N\to\infty} \phi_N^{-1}\left|\sum_{k=1}^N \widetilde{\mathbf{I}}_{a,b}(n_{\varpi(k)}x+\gamma_{\varpi(k)})\right| = \lim_{d\to\infty}\varlimsup_{N\to\infty} \phi_N^{-1}\left|\sum_{k=1}^N \widetilde{\mathbf{I}}_{a,b;d}(n_{\varpi(k)}x+\gamma_{\varpi(k)})\right|$$

for almost every $x\in\mathbf{R}$.

By noting $\|\widetilde{\mathbf{I}}_{a,b;d}\|_2 \to \|\widetilde{\mathbf{I}}_{a,b}\|_2 = \sqrt{(b-a)(1-(b-a))}$ as $d\to\infty$, by applying (16) for $\widetilde{\mathbf{I}}_{a,b;d}$, and by using Proposition 10, we have

$$\varlimsup_{N\to\infty} \phi^{-1}(N) N D_N\{n_k x + k\gamma\} = \sup_{S\ni a<b\in S} \sqrt{(b-a)(1-(b-a))} = 1/2, \quad \text{a.e.} \quad x.$$

Thus we have proved the first part of Theorem 1. The second part can be proved similarly.

5 Construction of a counter example

In [9], we have proved the following. For relatively prime integers μ and ν, and for $0\leq b-a\leq 1$, $0\leq b'-a'\leq 1$, we have

$$\int_0^1 \widetilde{\mathbf{I}}_{a,b}(\mu t)\widetilde{\mathbf{I}}_{a',b'}(\nu t)\,dt = \frac{\widetilde{V}(\langle\nu a\rangle,\langle\nu b\rangle,\langle\mu a'\rangle,\langle\mu b'\rangle)}{\mu\nu}. \quad (26)$$

(There are typographical errors in the expression of this formula in [9] and (26) is the right form.). Here the function $\widetilde{V}$ is defined as

$$V(\xi,x) = \xi\wedge x - \xi x \quad \text{and} \quad \widetilde{V}(\xi,\eta,x,y), = V(\xi,x)+V(\eta,y)-V(\eta,x)-V(\xi,y).$$

We already proved in [9] the second inequality of the next formula.

$$-V(\langle\eta-\xi\rangle,\langle\eta-\xi\rangle) \leq \widetilde{V}(\langle x\rangle,\langle y\rangle,\langle\xi\rangle,\langle\eta\rangle) \leq V(\langle\eta-\xi\rangle,\langle\eta-\xi\rangle). \quad (27)$$

The first inequality is proved in the following way by using the second inequality.

$$\begin{aligned}\widetilde{V}(\langle x\rangle,\langle y\rangle,\langle \xi\rangle,\langle \eta\rangle) &= -\widetilde{V}(\langle x\rangle,\langle y\rangle,\langle \eta\rangle,\langle \xi\rangle) \geq -V(\langle \xi-\eta\rangle,\langle \xi-\eta\rangle)\\ &= -V(1-\langle \eta-\xi\rangle, 1-\langle \eta-\xi\rangle) = -V(\langle \eta-\xi\rangle,\langle \eta-\xi\rangle).\end{aligned}$$

Assume that $2\gamma \in (1/8, 7/8)$. If $\langle (2k-1)\gamma\rangle \in [\,0,1/8\,]$, then $\langle (2k+1)\gamma\rangle \in (1/8,1)$ and $\langle (2k+1)\gamma\rangle \notin [\,0,1/8\,]$. Put $K = \{k : \langle (2k-1)\gamma\rangle \in [\,0,1/8\,]\}$. We have proved that $k \in K$ implies $k+1 \notin K$. Define a sequence $\{m(k)\}$ of positive integers by $m(1) = 1$, $m(k+1) = m(k)+1$ for $k \in K$, and $m(k+1) = m(k)+k$ for $k \notin K$. Clearly we have $m(k)+k < m(k+2) < \cdots$.

When letting $d \to \infty$, by L^2-convergence $\widetilde{\mathbf{I}}_{a,b;d} \to \widetilde{\mathbf{I}}_{a,b}$, we have

$$\begin{aligned}\sigma^2(a,b;d) &:= \int_0^1 \widetilde{\mathbf{I}}^2_{a,b;d}(x)\,dx + 2\int_0^{1/8} dt \int_0^1 \widetilde{\mathbf{I}}_{a,b;d}(x)\widetilde{\mathbf{I}}_{a,b;d}(3x-t)\,dx.\\ &\to \int_0^1 \widetilde{\mathbf{I}}^2_{a,b}(x)\,dx + 2\int_0^{1/8} dt \int_0^1 \widetilde{\mathbf{I}}_{a,b}(x)\widetilde{\mathbf{I}}_{a+t,b+t}(3x)\,dx\\ &= \widetilde{V}(\langle a\rangle,\langle b\rangle,\langle a\rangle,\langle b\rangle) + \frac{2}{3}\int_0^{1/8} \widetilde{V}(\langle 3a\rangle,\langle 3b\rangle,\langle a+t\rangle,\langle b+t\rangle)\,dt =: \sigma^2(a,b)\end{aligned}$$

By (27) and $\widetilde{V}(\langle a\rangle,\langle b\rangle,\langle a\rangle,\langle b\rangle) = V(\langle b-a\rangle,\langle b-a\rangle)$, we have

$$\sigma^2(a,b) \geq V(\langle b-a\rangle,\langle b-a\rangle) - \frac{2}{3}\int_0^{1/8} V(\langle b-a\rangle,\langle b-a\rangle)\,dt.$$

Thus, for any $0 \leq a < b \leq 1$, we have $\sigma(a,b) > 0$ and hence $\sigma(a,b;d) > 0$ for large enough d. Here after we denote $\widetilde{\mathbf{I}}_{a,b;d}$ simply by f_d. Note that the frequencies of $f_d(3^{m(k)}x + k\gamma)$ belong to $[\,3^{m(k)}, d3^{m(k)}\,]$. For $k' > k > \log_3 d$, by $d3^{m(k)} < 3^{m(k)+k}$, we have

$$\int_0^1 f_d(3^{m(k)}x + k\gamma) f_d(3^{m(k')}x + k'\gamma)\,dx = 0$$

if $k+2 \leq k'$ or if $k \notin K$. Hence for $M \geq \log_3 d$, we see

$$\begin{aligned}I(M,N) &:= \int_{\mathbf{R}} \Bigl(\frac{1}{\sqrt{N}}\sum_{k=M+1}^{M+N} f_d(3^{m(k)}x + k\gamma)\Bigr)^2 \mu_R(dx)\\ &= \frac{1}{N}\sum_{k=M+1}^{M+N}\int_0^1 f_d^2(3^{m(k)}x + k\gamma)\,dx\\ &\quad + \frac{2}{N}\sum_{k=M+1}^{M+N-1}\int_0^1 f_d(3^{m(k)}x + k\gamma) f_d(3^{m(k+1)}x + (k+1)\gamma)\,dx\\ &= \int_0^1 f_d^2(x)\,dx + \frac{2}{N}\sum_{k=M+1}^{M+N-1} h((2k-1)\gamma),\end{aligned}$$

by changing variable $y = 3^{m(k)}x + k\gamma$, where $h(t) := \mathbf{1}_{[0,1/8]}(\langle t\rangle)\int_0^1 f_d(y)f_d(3y-t)\,dy$.

By noting $\int_0^1 f_d^2 + 2\int_0^1 h = \sigma^2(a,b;d)$ and by applying Koksma's inequality and (25) we have

$$|I(M,N) - \sigma^2(a,b;d)| \leq V(h)D_N\{(2k-1)\gamma\}_{k>M} = V(h)D_N\{(2k-1)\gamma\}_{k\geq 1} \to 0$$

and hence $I(M,N) \to \sigma^2(a,b;d)$ uniformly in $M \geq \log_3 d$. By applying triangle inequality, we see that $I(M,N) \to \sigma^2(a,b;d)$ holds for $M < \log_3 d$, and hence uniformly in $M \geq 1$.

In the same way as the proof of Proposition 8, we can prove

$$\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} f_d(\theta^{m(k)} x + k\gamma) = \sigma(a,b;d), \quad \text{a.e.}$$

As before, by applying Proposition 10, we can prove the equalities in Theorem 2 with

$$\Sigma = \sup_{S \ni a < b \in S} \sigma(a,b) \quad \text{and} \quad \Sigma^* = \sup_{b \in S} \sigma(0,b).$$

To calculate Σ^*, we investigate $\sigma^2(0,b)$. Firstly we consider the case $3/8 < b < 1/2$. For $t \in (0,1/8)$, we have $\langle t \rangle = t$, $\langle 3b \rangle = 3b-1$, $t < 1/8 < 3b-1$ and $3b-1 < b+t$. Hence

$$\widetilde{V}(\langle 0 \rangle, \langle 3b \rangle, \langle t \rangle, \langle b+t \rangle) = -t + (1-b)(3b-1)$$

and $\sigma^2(0,b) = -(5/4)b^2 + (4/3)b - 17/128$. We see that $\sigma^2(0,b)$ is increasing in $b \in (3/8,1/2)$.

Secondly we consider the case $1/2 \leq b < 2/3$. Since $b+t \leq 3b-1$ if and only if $t \leq 2b-1$, we have

$$\widetilde{V}(\langle 0 \rangle, \langle 3b \rangle, \langle t \rangle, \langle b+t \rangle) = \begin{cases} b - b(3b-1) & (t \leq 2b-1), \\ (3b-1-t) - b(3b-1) & (t \geq 2b-1). \end{cases}$$

In case $b < 9/16$, by noting $2b-1 \leq 1/8$ we have $\sigma^2(0,b) = -(31/12)b^2 + (8/3)b - 27/64$, which has maximum at $16/31 \in (1/2, 9/16)$ taking the value $\sigma^2(0,16/31) = 1585/5952$. In case $9/16 \leq b$, we have $\sigma^2(0,b) = -(5/4)b^2 + (7/6)b$, which is decreasing in $[\,9/16, 2/3)$.

By applying (27), we have $\sigma^2(0,b) \leq (13/12)b(1-b) =: h(b)$. Since we can verify $h(3/8) < 1585/5952$ and $h(2/3) < 1585/5952$, we can conclude $\sigma^2(0,b) \leq h(b) < \sigma^2(0,16/31)$ outside of $(3/8,2/3)$. Therefore we have proved $(\Sigma^*)^2 = 1585/5952$.

Finally, we calculate $\sigma^2(1/32,17/32)$. Since

$$\widetilde{V}(3/32, 19/32, 1/32+t, 17/32+t) = \begin{cases} t + 3/16 & (t \leq 1/16), \\ -t + 5/16 & (t \geq 1/16), \end{cases}$$

we have $\sigma^2(1/32,17/32) = 103/384$ and hence $\Sigma^2 \geq 103/384 > 1585/5952$. Therefore $\Sigma > \Sigma^*$.

References

1. C. Aistleitner, *On the law of the iterated logarithm for the discrepancy of lacunary sequences*, Trans. Amer. Math. Soc., **362** (2010) 5967-5982.
2. I. Berkes, *On the central limit theorem for lacunary trigonometric series*, Anal. Math., **4** (1978) 159–180.
3. P. Bohl, *Über ein in der Theorie der säkulären Störungen vorkommendes Problem*, Jour. reine angew. Math., **135** (1909) 189–283.
4. K. Chung, *An estimate concerning the Kolmogorov limit distribution,* Trans. Amer. Math. Soc., **67** (1949) 36-50.

5. K. Fukuyama, *The central limit theorem for Riesz-Raikov sums*, Prob. Theory Related Fields, **100** (1994), 57–75.
6. K. Fukuyama, *The law of the iterated logarithm for discrepancies of* $\{\theta^n x\}$, Acta Math. Hungar. **118** (2008) 155-170.
7. K. Fukuyama, *A central limit theorem and a metric discrepancy result for sequence with bounded gaps*, Dependence in probability, analysis and number theory, A volume in memory of Walter Philipp, Eds. I. Berkes, R. Bradley, H. Dehling, M. Peligrad, and R. Tichy, Kendrick press, 2010, pp. 233-246.
8. K. Fukuyama, *Metric discrepancy results for geometric progressions and variations*, Summer School on the Theory of Uniform Distribution, Ed. S. Akiyama, RIMS Kôkyûroku Bessatsu, **B29** (2012) 41–64, `http://hdl.handle.net/2433/196207`
9. K. Fukuyama, *Metric discrepancy results for alternating geometric progressions*, Monatsh. Math., **171** (2013) 33–63.
10. K. Fukuyama, *A metric discrepancy result for the sequence of powers of minus two*, Indag. Math. (NS), **25** (2014) 487-504.
11. K. Fukuyama, *A metric discrepancy result for geometric progression with ratio 3/2*, Adv. Studies Pure Math., (to appear), `https://arxiv.org/abs/1804.00085`
12. K. Fukuyama & N. Kuri, *The central limit theorem for complex Riesz-Raikov sums*, C. R. Math. Acad. Sci. Paris, **353** (2015) 749–753.
13. K. Fukuyama, and Y. Mitsuhata, *Bounded law of the iterated logarithm for discrepancies of permutations of lacunary sequences*, Summer School on the Theory of Uniform Distribution, RIMS Kôkyûroku Bessatsu, **B29** (2012) 65–88, `http://hdl.handle.net/2433/196206`.
14. K. Fukuyama, S. Sakaguchi, O. Shimabe, T. Toyoda, M. Tscheckl, *Metric discrepancy results for geometric progressions with small ratios*, Acta Math. Hungar., **155** (2018) 416-430, verbose edition `arxiv:1711.02839v1 [math.NT] 8 Nov 2017`.
15. K. Fukuyama & M. Yamashita, *Metric discrepancy results for geometric progressions with large ratios*, Monatsh. Math., **180** (2016) 713–730.
16. H. Kesten, *The discrepancy of random sequences* $\{kx\}$, Acta Arith., **10** (1964/1965) 183–213.
17. A. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*. Math. Ann. **92** (1924) 115–125.
18. D. Monrad, and W. Philipp, *Nearby variables with nearby conditional laws and a strong approximation theorem for Hilbert space valued martingales*, Probab. Theory rel. Fields, **88** (1991), 381–404.
19. W. Philipp, *Limit theorems for lacunary series and uniform distribution mod 1*, Acta Arith. **26** (1975) 241-251.
20. W. Philipp, and W. Stout, *Almost sure invariance principles for partial sums of weakly dependent random variables*, Memoirs Amer. Math. Soc., **161** (1975).
21. W. Sierpiński, *Sur la valeur asymptotique d'une certaine somme*, Bull. Int. Acad. Polon. Sci. (Carcovie) A, (1910) 9–11.
22. N. Smirnov, *Approximate variables from empirical data* (Russian), Uspehi. Mat. Nauk., **10** (1944) 36–50.
23. V. Strassen, *Almost sure behavior of sums of independent random variables and martingales*, Fifth Berkeley Symp. Math. Stat. Prob. Vol II, Part I, (1967), 315-343.
24. S. Takahashi, *An asymptotic property of a gap sequence*, Proc. Japan Acad. **38**, (1962) 101–104.
25. H. Weyl, *Über die Gibbssche Erscheinung und verwandte Konvergenzphänomene*, Rend. Circ. Mat. Palermo, **30** (1910) 377–407.