The law of the iterated logarithm for the discrepancy of perturbed geometric progressions.

Katusi Fukuyama

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Abstract We investigate the asymptotic distribution of perturbed geometric progression $\{\theta^k x + \gamma_k\}$ given by $\theta \in (-\infty, -1) \cup (1, \infty)$ and $\gamma_1, \gamma_2, \ldots \in \mathbf{R}$. We prove that the discrepancy obeys the law of the iterated logarithm with limsup constant sensitively depending on θ and $\{\gamma_k\}$.

Keywords discrepancy · lacunary sequence · law of the iterated logarithm

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1 Introduction

For a sequence $\{x_k\}$ of real numbers, we define its discrepancy by

$$D_N(\{x_k\}) = \sup_{0 \le a \le b \le 1} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_{[a,b)}(\langle x_k \rangle) - (b-a) \right|;$$

where $\langle x \rangle$ denotes the fractional part $x - [\,x\,]$ of x, and $\mathbf{1}_{[a,b)}$ denotes the indicator function of $[\,a, b)$.

For a sequence $\{U_k\}$ of independent random variables uniformly distributed over the unit interval, the celebrated Chung-Smirnov [3,17] result asserts the law of the iterated logarithm for the discrepancy:

$$\overline{\lim_{N\to\infty}} \frac{N D_N(\{U_k\})}{\sqrt{2N \log\log N}} = \frac{1}{2} \quad \text{a.s.}$$

Philipp [15] modified Takahashi's method [19] and proved the bounded law of the iterated logarithm for discrepancies of $\{n_k x\}$ by assuming the Hadamard gap condition

$$n_{k+1}/n_k > q > 1. \tag{1}$$

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Department of mathematics,
Kobe University,
Rokko, Kobe, 657-8501, Japan
E-mail: fukuyama@math.kobe-u.ac.jp

For geometric progressions $\{\theta^k x\}$, the author [4,6] proved the exact law of the iterated logarithm below. For any real number $\theta \in (-\infty,-1) \cup (1,\infty)$, there exists a constant Σ_θ such that

$$\varlimsup_{N\to\infty} \frac{ND_N(\{\theta^k x\})}{\sqrt{2N\log\log N}} = \Sigma_\theta \quad \text{a.e.} \tag{2}$$

We have $\Sigma_\theta = 1/2$ if θ satisfies the condition

$$\theta^r \notin \mathbf{Q} \quad \text{for all} \quad r \in \mathbf{N}. \tag{3}$$

When $\theta^k \in \mathbf{Q}$ for some $k = 1, 2, \dots$, denote

$$r = \min\{k \in \mathbf{N} : \theta^k \in \mathbf{Q}\} \quad \text{and} \quad \theta^r = p/q \quad (p \in \mathbf{Z}, q \in \mathbf{N}, \gcd(p,q) = 1). \tag{4}$$

Then the value of Σ_θ is independent of r and is determined by p and q, i.e., $\Sigma_\theta = \Sigma_{p/q}$.

The concrete values of $\Sigma_{p/q}$ are determined in the following cases:

1. [4] pq is odd and positive. [6] pq is odd and negative.
2. [12] p is odd, q is even, and $|p/q| \geq 9/4$. p is even, q is odd, and $|p/q| \geq 4$.
3. [4] $p/q = 2$. [7] $p/q = -2$.
4. [11] $p/q = \pm 13/6$, 4/3, 8/3, 10/3, 12/5, 17/8, 19/10, 12/7, 8/5, [8] $p/q = 3/2$.

In this paper we investigate the sequence $\{\theta^k x + \gamma_k\}$ and prove the exact law of the iterated logarithm. The limsup constant sensitively depends on θ and $\{\gamma_k\}$, and it takes various values. The result is similar to the case we studied in [9]. Now we are in a position to state our theorem.

Theorem 1 *Suppose that* $\theta \in (-\infty,-1) \cup (1,\infty)$ *and that* $\{\gamma_k\} \in \mathbf{R}^{\mathbf{N}}$*, where* $\mathbf{R}^{\mathbf{N}}$ *denotes the set of all sequences of real numbers.*

1. *We have*
$$\varlimsup_{N\to\infty} \frac{ND_N(\{\theta^k x + \gamma_k\})}{\sqrt{2N\log\log N}} = \Sigma_{\theta,\{\gamma_k\}} \quad \textit{a.e.} \tag{5}$$
for some constant $\Sigma_{\theta,\{\gamma_k\}} \in [\,1/2, \Sigma_{|\theta|}\,]$ *depending on* θ *and* $\{\gamma_k\}$.
2. *We have*
$$\{\, \Sigma_{\theta,\{\gamma_k\}} \mid \{\gamma_k\} \in \mathbf{R}^{\mathbf{N}} \,\} = [\,1/2, \Sigma_{|\theta|}\,]. \tag{6}$$

In our previous paper [13], we proved (5) with $\Sigma_{\theta,\{k\gamma\}} = 1/2$ when $\gamma \notin \mathbf{Q}$.

2 Preliminary

We define functions by $V(a,b) = a \wedge b - ab$ and

$$\widetilde{V}(a,b,c,d) = V(a,c) + V(b,d) - V(b,c) - V(a,d)$$

for a, b, c, $d \in [0,1)$, where $a \wedge b = \min\{a,b\}$. Clearly we have $\widetilde{V}(x,y,\xi,\eta) = \widetilde{V}(\xi,\eta,x,y) = -\widetilde{V}(y,x,\xi,\eta) = -\widetilde{V}(x,y,\eta,\xi)$ and we can verify (See [7])

$$\widetilde{V}(\langle -\eta\rangle, \langle -\xi\rangle, \langle -y\rangle, \langle -x\rangle) = \widetilde{V}(\langle \xi\rangle, \langle \eta\rangle, \langle x\rangle, \langle y\rangle), \tag{7}$$

$$\widetilde{V}(\langle \xi + c\rangle, \langle \eta + c\rangle, \langle x + c\rangle, \langle y + c\rangle) = \widetilde{V}(\langle \xi\rangle, \langle \eta\rangle, \langle x\rangle, \langle y\rangle). \tag{8}$$

For a, b, $x \in \mathbf{R}$ satisfying $0 \le b-a \le 1$, we put $\mathbf{I}_{a,b}(x) = \sum_{n\in\mathbf{Z}} \mathbf{1}_{[a,b)}(x+n)$ and $\widetilde{\mathbf{I}}_{a,b}(x) = \mathbf{I}_{a,b}(x) - (b-a)$. If $0 \le a \le b \le 1$, we see $\mathbf{I}_{a,b}(x) = \mathbf{1}_{[a,b)}(\langle x\rangle)$. In [6], we have proved the following. For relatively prime positive integers μ and ν, we have

$$\int_0^1 \widetilde{\mathbf{I}}_{a,b}(\mu x)\widetilde{\mathbf{I}}_{c,d}(\nu x)\,dx = \frac{\widetilde{V}(\langle \nu a\rangle, \langle \nu b\rangle, \langle \mu c\rangle, \langle \mu d\rangle)}{\mu\nu} \tag{9}$$

for $0 \le b-a \le 1$, $0 \le d-c \le 1$. (There are typographical errors in the expression of this formula in [6] and (9) is the right form.) We here prepare two lemmas.

Lemma 1 *For a, b, c, $d \in \mathbf{R}$, then we have*

$$\widetilde{V}\big(\langle a\rangle, \langle b\rangle, \langle c\rangle, \langle d\rangle\big) \le \widetilde{V}\big(0, \langle b-a\rangle, 0, \langle d-c\rangle\big). \tag{10}$$

Proof We may assume that $0 \le b-a \le 1$ and $0 \le d-c \le 1$ because it can be realized by adding some integers to b and d. We use (9) with $\mu = \nu = 1$. Note that

$$\int_0^1 \widetilde{\mathbf{I}}_{a,b}(x)\widetilde{\mathbf{I}}_{c,d}(x)\,dx = \int_0^1 \mathbf{I}_{a,b}(x)\mathbf{I}_{c,d}(x)\,dx - (b-a)(d-c).$$

By $\mathbf{I}_{a,b}(x)\mathbf{I}_{c,d}(x) \le \mathbf{I}_{a,b}(x)$, $\mathbf{I}_{c,d}(x)$, we have

$$\int_0^1 \mathbf{I}_{a,b}(x)\mathbf{I}_{c,d}(x)\,dx \le \left(\int_0^1 \mathbf{I}_{a,b}(x)\,dx\right) \wedge \left(\int_0^1 \mathbf{I}_{c,d}(x)\,dx\right) = (b-a)\wedge(d-c),$$

and hence we have the conclusion. □

Lemma 2 *If μ and ν are relatively prime integers, then we have*

$$\int_0^1 \widetilde{\mathbf{I}}_{a,b}(\mu x+\gamma)\widetilde{\mathbf{I}}_{c,d}(\nu x+\delta)\,dx \le \int_0^1 \widetilde{\mathbf{I}}_{0,b-a}(|\mu|x)\widetilde{\mathbf{I}}_{0,d-c}(|\nu|x)\,dx \tag{11}$$

for $0 \le b-a \le 1$, $0 \le d-c \le 1$, γ, $\delta \in \mathbf{R}$.

Remark 1 By (9), we see that the right hand side of (11) is non-negative.

Remark 2 By the last argument in the proof below and the formula (7), we can see that (9) holds also in case when one or both of μ and ν is negative.

Proof By noting the next formula we see that it is enough to prove for $\gamma = \delta = 0$:

$$\int_0^1 \widetilde{\mathbf{I}}_{a,b}(\mu x+\gamma)\widetilde{\mathbf{I}}_{c,d}(\nu x+\delta)\,dx = \int_0^1 \widetilde{\mathbf{I}}_{a-\gamma,b-\gamma}(\mu x)\widetilde{\mathbf{I}}_{c-\delta,d-\delta}(\nu x)\,dx.$$

When $\mu > 0$ and $\nu > 0$, it is a direct conclusion of (9) and (10). We now show the result assuming $\mu > 0$ and $\nu < 0$. The other cases can be proved similarly. By applying (9), we have

$$\begin{aligned}\int_0^1 \widetilde{\mathbf{I}}_{a,b}(\mu x)\widetilde{\mathbf{I}}_{c,d}(\nu x)\,dx &= \int_0^1 \widetilde{\mathbf{I}}_{a,b}(\mu x)\widetilde{\mathbf{I}}_{-d,-c}(-\nu x)\,dx \\ &= \widetilde{V}(\langle -\nu a\rangle, \langle -\nu b\rangle, \langle -\mu d\rangle, \langle -\mu c\rangle)/\mu(-\nu).\end{aligned}$$

By applying (10), we have the conclusion. □

For a bounded measurable function G, we define *the mean value* by

$$\int_{\mathbf{R}} G(x)\,\mu_R(dx) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} G(x)\,dx$$

if the limit on the right hand side exists.

For trigonometric polynomials g and h with period 1 with $\int_0^1 g = 0$ and $\int_0^1 h = 0$, we have

$$\int_{\mathbf{R}} g(\Theta x)h(x)\,\mu_R(dx) = 0$$

if $\Theta \notin \mathbf{Q}$, and

$$\int_{\mathbf{R}} g((P/Q)x)h(x)\,\mu_R(dx) = \int_{\mathbf{R}} g(Px)h(Qx)\,\mu_R(dx) = \int_0^1 g(Px)h(Qx)\,dx$$

if P and Q are non-zero integers.

For a function f of bounded variation over the unit interval with $f(x+1) = f(x)$ and $\int_0^1 f(x)\,dx = 0$, put

$$\sigma^2(f,\theta) = \begin{cases} \int_0^1 f^2(x)\,dx & \text{in case when } \theta \text{ satisfies (3),} \\ \int_0^1 f^2(x)\,dx + 2\sum_{l=1}^{\infty}\int_0^1 f(p^l x)f(q^l x)\,dx & \text{in case when } \theta \text{ satisfies (4).} \end{cases}$$

In [4], we proved for $\theta > 1$ that $\sigma(\widetilde{\mathbf{I}}_{0,a},\theta)$ is continuous in a and

$$\Sigma_\theta = \max_{0\le a\le 1} \sigma(\widetilde{\mathbf{I}}_{0,a},\theta). \tag{12}$$

We denote the d-th subsum of the Fourier series of $\widetilde{\mathbf{I}}_{a,b}$ by $\widetilde{\mathbf{I}}_{a,b;d}$, and denote $\widetilde{\mathbf{I}}_{a,b}$ by $\widetilde{\mathbf{I}}_{a,b;\infty}$. Clearly $\widetilde{\mathbf{I}}_{a,b}(x+\gamma) = \widetilde{\mathbf{I}}_{a-\gamma,b-\gamma}(x)$, $\widetilde{\mathbf{I}}_{a,b;d}(x+\gamma) = \widetilde{\mathbf{I}}_{a-\gamma,b-\gamma;d}(x)$ and

$$\int_{\mathbf{R}} \widetilde{\mathbf{I}}^2_{a,b;d}(\theta^k x+\gamma_k)\mu_R(dx) = \int_0^1 \widetilde{\mathbf{I}}^2_{a,b;d}(x+\gamma_k)\,dx = \int_0^1 \widetilde{\mathbf{I}}^2_{0,b-a;d}(x)\,dx.$$

If θ satisfies (4), then

$$\int_{\mathbf{R}} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x+\gamma_k)\widetilde{\mathbf{I}}_{a,b;d}(\theta^{k+l} x+\gamma_{k+l})\mu_R(dx) = 0 \tag{13}$$

if $r \nmid l$, and

$$\begin{aligned} &\int_{\mathbf{R}} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x+\gamma_k)\widetilde{\mathbf{I}}_{a,b;d}(\theta^{k+lr} x+\gamma_{k+lr})\mu_R(dx) \\ &= \int_0^1 \widetilde{\mathbf{I}}_{a,b;d}(q^l x+\gamma_k)\widetilde{\mathbf{I}}_{a,b;d}(p^l x+\gamma_{k+lr})\,dx \quad (l = 1,2,\dots). \end{aligned} \tag{14}$$

Hence, for a set Δ of finitely many consecutive positive integers, we have

$$\begin{aligned} \int_{\mathbf{R}} \Big(\sum_{k\in\Delta} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x+\gamma_k)\Big)^2 \mu_R(dx) &= ({}^{\#}\Delta)\int_0^1 \widetilde{\mathbf{I}}^2_{a,b;d}(x)\,dx \\ &+ 2\sum_{1\le l\le ({}^{\#}\Delta-1)/r}\ \sum_{k\in\Delta:k+lr\in\Delta} \int_0^1 \widetilde{\mathbf{I}}_{a,b;d}(q^l x+\gamma_k)\widetilde{\mathbf{I}}_{a,b;d}(p^l x+\gamma_{k+lr})\,dx. \end{aligned}$$

Denoting the right hand side of the above formula by $\Psi(a,b;d,\theta,\{\gamma_k\},\Delta)$, we estimate the difference between $\Psi(a,b;d,\theta,\{\gamma_k\},\Delta)$ and $\Psi(a,b;\infty,\theta,\{\gamma_k\},\Delta)$. Put

$$\rho(\theta,d) = \sqrt{2}\,(1+2\log_{|p/q|} d)/\big(\pi\sqrt{d}\big) + d^{-\log_{|p/q|}|pq|}/\big(2(1-1/|pq|)\big).$$

Lemma 3 *For $0 \le b-a \le 1$ and $d = 1, 2, \dots,$ we have*

$$\left|\Psi(a,b;\infty,\theta,\{\gamma_k\},\Delta) - \Psi(a,b;d,\theta,\{\gamma_k\},\Delta)\right| \le (^\#\Delta)\rho(\theta,d), \tag{15}$$

and

$$\int_{\mathbf{R}} \Big(\sum_{k\in\Delta} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x+\gamma_k)\Big)^2 \mu_R(dx) \le (^\#\Delta)\big(\sigma^2\big(\widetilde{\mathbf{I}}_{0,b-a},|\theta|\big)+\rho(\theta,d)\big) \tag{16}$$

if θ satisfies (4)*, and*

$$\int_{\mathbf{R}} \Big(\sum_{k\in\Delta} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x+\gamma_k)\Big)^2 \mu_R(dx) = (^\#\Delta)\sigma^2\big(\widetilde{\mathbf{I}}_{0,b-a;d},\theta\big) \tag{17}$$

if θ satisfies (3)*.*

Proof If θ satisfies (3), then we have (13) for $l = 1, 2, \dots,$ and thereby we have

$$\int_{\mathbf{R}} \Big(\sum_{k\in\Delta} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x+\gamma_k)\Big)^2 \mu_R(dx) = (^\#\Delta)\int_0^1 \widetilde{\mathbf{I}}^2_{a,b;d}(x)\,dx,$$

which shows (17).

Since the absolute values of the frequencies of $\widetilde{\mathbf{I}}_{a,b;d}(q^l x+\gamma_k)$ belong to $[\,q^l, dq^l\,]$ and those of $\widetilde{\mathbf{I}}_{a,b;d}(p^l x+\gamma_{k+lr})$ belong to $[\,|p|^l, d|p|^l\,]$, (14) vanishes if $dq^l < |p|^l$ or $l > \log_{|p/q|} d$. For $l > \log_{|p/q|} d$, we use the estimate

$$\begin{aligned}&\left|\int_0^1 \widetilde{\mathbf{I}}_{a,b}(q^l x+\gamma_k)\widetilde{\mathbf{I}}_{a,b}(p^l x+\gamma_{k+lr})\,dx\right|\\ &= \frac{1}{|pq|^l}\left|\widetilde{V}\big(\langle p^l(a-\gamma_k)\rangle,\langle p^l(b-\gamma_k)\rangle,\langle q^l(a-\gamma_{k+lr})\rangle,\langle q^l(b-\gamma_{k+lr})\rangle\big)\right| \le \frac{1}{4|pq|^l}.\end{aligned}$$

Since we have $\|\widetilde{\mathbf{I}}_{a,b;d}(q^l\cdot+\gamma_k)\|_2 = \|\widetilde{\mathbf{I}}_{a,b;d}(p^l\cdot+\gamma_{k+lr})\|_2 = \|\widetilde{\mathbf{I}}_{a,b;d}\|_2 \le \|\widetilde{\mathbf{I}}_{a,b}\|_2 \le 1/2$, and

$$\|\widetilde{\mathbf{I}}_{a,b;d} - \widetilde{\mathbf{I}}_{a,b}\|_2 \le \Big(\sum_{n:|n|>d}\frac{1}{(\pi n)^2}\Big)^{1/2} \le \frac{\sqrt{2}}{\pi\sqrt{d}},$$

we can prove the estimate below for $l \le \log_{|p/q|} d$:

$$\begin{aligned}&\left|\int_0^1 \widetilde{\mathbf{I}}_{a,b}(q^l x+\gamma_k)\widetilde{\mathbf{I}}_{a,b}(p^l x+\gamma_{k+lr})\,dx - \int_0^1 \widetilde{\mathbf{I}}_{a,b;d}(q^l x+\gamma_k)\widetilde{\mathbf{I}}_{a,b;d}(p^l x+\gamma_{k+lr})\,dx\right|\\ &\le \big\|\widetilde{\mathbf{I}}_{a,b}\big\|_2\big\|\widetilde{\mathbf{I}}_{a,b}-\widetilde{\mathbf{I}}_{a,b;d}\big\|_2 + \big\|\widetilde{\mathbf{I}}_{a,b}-\widetilde{\mathbf{I}}_{a,b;d}\big\|_2\big\|\widetilde{\mathbf{I}}_{a,b;d}\big\|_2 \le \sqrt{2}/\pi\sqrt{d}.\end{aligned}$$

Thereby we can bound $\left|\Psi(a,b;\infty,\theta,\{\gamma_k\},\Delta) - \Psi(a,b;d,\theta,\{\gamma_k\},\Delta)\right|$ by

$$(^\#\Delta)\frac{\sqrt{2}\,(1+2\log_{|p/q|} d)}{\pi\sqrt{d}} + 2(^\#\Delta)\sum_{l>\log_{|p/q|} d}\frac{1}{4|pq|^l} \le (^\#\Delta)\rho(\theta,d).$$

Hence (15) is proved.

By applying the estimate (11), we see

$$\Psi(a,b;\infty,\theta,\{\gamma_k\},\Delta) \le \Psi(0,b-a;\infty,|\theta|,\{0\},\Delta) \le (^\#\Delta)\sigma^2\big(\widetilde{\mathbf{I}}_{0,b-a},|\theta|\big),$$

where the last inequality is by non-negativity of $\int_0^1 \widetilde{\mathbf{I}}_{0,b-a}(q^l x)\widetilde{\mathbf{I}}_{0,b-a}(|p|^l x)\,dx$ (See Remark 1). This together with (15) proves (16) □

3 Martingale Approximation

In this section, we prove the following law of the iterated logarithm.

Proposition 1 *For a real number θ with $|\theta| > 1$, a sequence $\{\gamma_k\}$ of real numbers, $0 \le b - a \le 1$, and a positive integer d, we have*

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left| \sum_{k=1}^{N} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x + \gamma_k) \right| = \sigma(\widetilde{\mathbf{I}}_{a,b;d}, \{\gamma_k\}, \theta) \quad a.e. \tag{18}$$

for some constant $\sigma(\widetilde{\mathbf{I}}_{a,b;d}, \{\gamma_k\}, \theta)$. It holds that

$$0 \le \sigma(\widetilde{\mathbf{I}}_{a,b;d}, \{\gamma_k\}, \theta) \le \left(\sigma^2\left(\widetilde{\mathbf{I}}_{0,b-a}, |\theta|\right) + \rho(\theta, d)\right)^{1/2} \tag{19}$$

if θ satisfies (4)*, and that*

$$\sigma(\widetilde{\mathbf{I}}_{a,b;d}, \{\gamma_k\}, \theta) = \sigma(\widetilde{\mathbf{I}}_{0,b-a;d}, \theta) \tag{20}$$

if θ satisfies (3)*,*

Proof The proof is by martingale approximation due to Aistleitner [1], Berkes [2] and Philipp-Stout [16]. We denote $\left(\sigma^2\left(\widetilde{\mathbf{I}}_{0,b-a}, |\theta|\right) + \rho(\theta, d)\right)^{1/2}$ by κ. We divide $\mathbf{N}$ into consecutive blocks Δ_1', Δ_1, Δ_2', Δ_2, ... satisfying $^\#\Delta_i' = [\,9\log_{|\theta|} i\,]$ and $^\#\Delta_i = i$. Denote $i^- = \min \Delta_i$ and $i^+ = \max \Delta_i$. We denote $\mu(i) = [\,\log_2 i^4 |\theta^{i^+}|\,] + 1$ and introduce a σ-field $\mathcal{F}_i$ on $[\,0, 1)$ defined by

$$\mathcal{F}_i = \sigma\{[\,j2^{-\mu(i)}, (j+1)2^{-\mu(i)}) \mid j = 0, \dots, 2^{\mu(i)} - 1\}.$$

Set

$$T_i(x) = \sum_{k\in\Delta_i} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x + \gamma_k), \quad T_i'(x) = \sum_{k\in\Delta_i'} \widetilde{\mathbf{I}}_{a,b;d}(\theta^k x + \gamma_k), \quad \text{and}$$
$$Y_i = E(T_i \mid \mathcal{F}_i) - E(T_i \mid \mathcal{F}_{i-1}).$$

Clearly $\{Y_i, \mathcal{F}_i\}$ forms a martingale difference sequence. Denote $v_i = \int_{\mathbf{R}} T_i^2(x)\mu_R(dx)$, $\beta_M = \sum_{i=1}^{M} v_i$, and $V_M = \sum_{i=1}^{M} E(Y_i^2 \mid \mathcal{F}_{i-1})$. In the same way as the proof in [13], we can prove

$$\begin{aligned} \|Y_i - T_i\|_\infty &\le (\|\widetilde{\mathbf{I}}'_{a,b;d}\|_\infty + 2\|\widetilde{\mathbf{I}}_{a,b;d}\|_\infty)/i^3, \\ EY_i^4 = O(i^2), &\quad \text{and} \quad \left\|V_M - \beta_M\right\|_\infty = O(1). \end{aligned} \tag{21}$$

Denote $l_M = M(M+1)/2$. By (16), we have $v_i = \Psi(a, b; d, \theta, \{\gamma_k\}, \Delta_i) \le \kappa^2 i$ and $\beta_M \le \kappa^2 l_M$ if (4) is satisfied, and $v_i = \sigma^2(\widetilde{\mathbf{I}}_{0,b-a;d}, \theta)i$ and $\beta_M = \sigma^2(\widetilde{\mathbf{I}}_{0,b-a;d}, \theta)l_M$ if (3) is satisfied.

We use the next theorem by Monrad-Philipp [14], which is a version of Strassen's theorem [18]. To apply the theorem, we modify our function Y_i by adding independent random variables in the following way to ensure that the conditional second moment are not too small. We prepare another probability space $(\Omega, \mathcal{G}, P)$ on which a sequence of independent random variables U, ξ_1, ξ_2, ... is defined. Here we assume that the law of U is the uniform distribution over unit interval, and that $P(\xi_i = 1) = P(\xi_i = -1) =$

1/2. We take a product of this probability space and $[0,1)$ on which our martingale difference is defined. We define a filtration $\{\widehat{\mathcal{F}}_i\}$ on this space by $\widehat{\mathcal{F}}_i = \mathcal{F}\otimes\sigma(\Xi_1,\dots,\Xi_i)$ and define a martingale difference by $\widehat{Y}_i = Y_i + \varepsilon\Xi_i$ where $\Xi_i = \sum_{k\in\Delta_i}\xi_k$ and ε is an arbitrary positive number. Put $\widehat{\beta}_M = \beta_M + \varepsilon^2 l_M$ and $\widehat{V}_M = \sum_{i=1}^M E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1})$. We have

$$\varepsilon^2 l_M \le \widehat{\beta}_M \le (\kappa^2+\varepsilon^2) l_M \quad \text{or} \quad \widehat{\beta}_M = (\sigma^2(\widetilde{\mathbf{I}}_{0,b-a;d},\theta)+\varepsilon^2) l_M, \tag{22}$$

$$E\widehat{Y}_i^4 = O(i^2). \tag{23}$$

$$\bigl\|\widehat{V}_M - \widehat{\beta}_M\bigr\|_\infty = O(1), \tag{24}$$

according as (4) or (3) is satisfied.

Theorem 2 (Monrad-Philipp [14]) *Suppose that a square integrable martingale difference sequence* $\{Y_i,\mathcal{F}_i\}$ *satisfies*

$$V_M = \sum_{i=1}^M E(Y_i^2 \mid \mathcal{F}_{i-1}) \to \infty \ \text{a.s.} \quad \text{and} \quad \sum_{i=1}^\infty E\biggl(\frac{Y_i^2 \mathbf{1}_{\{Y_i^2\ge\psi(V_i)\}}}{\psi(V_i)}\biggr) < \infty$$

for some non-decreasing function ψ *with* $\psi(x)\to\infty$ $(x\to\infty)$ *such that* $\psi(x)(\log x)^\alpha/x$ *is non-increasing for some* $\alpha > 50$. *If there exists a uniformly distributed random variable* U *which is independent of* $\{Y_n\}$, *there exists a standard normal i.i.d.* $\{Z_i\}$ *such that*

$$\sum_{i\ge1} Y_i \mathbf{1}_{\{V_i\le t\}} = \sum_{i\le t} Z_i + o\bigl(t^{1/2}(\psi(t)/t)^{1/50}\bigr) \quad (t\to\infty) \quad \text{a.s.}$$

By putting $\psi(x) = x/(\log x)^{51}$ and by noting (23), we have

$$\sum_{i\ge M_0} E\biggl(\frac{\widehat{Y}_i^2 \mathbf{1}_{\{\widehat{Y}_i^2\ge\psi(\widehat{V}_i)\}}}{\psi(\widehat{V}_i)}\biggr) \le \sum_i \frac{E\widehat{Y}_i^4}{\psi^2(\varepsilon^2 l_i)} = O\biggl(\sum_i \frac{i^2(\log l_i)^{102}}{l_i^2}\biggr) < \infty.$$

By (24) and $\widehat{V}_M - \widehat{V}_{M-1} = \widehat{\beta}_M - \widehat{\beta}_{M-1} + O(1) \ge \varepsilon^2 M + O(1) \to \infty$, we have $\widehat{V}_{M-1} < \widehat{\beta}_M < \widehat{V}_{M+1}$ for large M. Hence $\widehat{V}_i \le \widehat{\beta}_M$ is equivalent to $i \le M-1$ or $i \le M$. By $\|\widehat{Y}_i\|_\infty = O(i)$ we have

$$\sum_{i\ge1} \widehat{Y}_i \mathbf{1}_{\{\widehat{V}_i\le\widehat{\beta}_M\}} = \sum_{i=1}^M \widehat{Y}_i + O(M) = \sum_{i=1}^M \widehat{Y}_i + o(\phi_{l_M}),$$

where $\phi_x = \sqrt{2x\log\log x}$.

By applying Theorem 2 and putting $t = \widehat{\beta}_M$, we have

$$\sum_{i=1}^M \widehat{Y}_i = \sum_{i\ge1} \widehat{Y}_i \mathbf{1}_{\{\widehat{V}_i\le\widehat{\beta}_M\}} + o(\phi_{l_M}) = \sum_{i\le\widehat{\beta}_M} Z_i + o(\phi_{l_M}) \quad \text{a.s.}$$

If (3) is satisfied, we have

$$\varlimsup_{M\to\infty} \phi_{l_M}^{-1}\biggl|\sum_{i=1}^M \widehat{Y}_i\biggr| = \varlimsup_{M\to\infty} (\phi_{l_M}^{-1}\phi_{\widehat{\beta}_M})\phi_{\widehat{\beta}_M}^{-1}\biggl|\sum_{i\le\widehat{\beta}_M} Z_i\biggr| = (\sigma^2(\widetilde{\mathbf{I}}_{0,b-a;d},\theta)+\varepsilon^2)^{1/2} \quad \text{a.s.}$$

By

$$\left|\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{i=1}^{M}Y_i\right|-\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{i=1}^{M}\widehat{Y}_i\right|\right|\le\varepsilon\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{i=1}^{M}\Xi_i\right|=\varepsilon\quad\text{a.s.},\tag{25}$$

we see that

$$\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{i=1}^{M}Y_i\right|=\sigma(\widetilde{\mathbf{I}}_{0,b-a;d},\theta)\tag{26}$$

a.s, i.e., for almost every $(x,\omega)\in[0,1)\times\Omega$. By Fubini's theorem, there exists an ω such that (26) holds for almost every x. Since the left hand side of (26) is independent of ω, we see that (26) holds for almost every x. By noting (21), we have

$$\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{k=1}^{M}T_k\right|=\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{i=1}^{M}Y_i\right|=\sigma(\widetilde{\mathbf{I}}_{0,b-a;d},\theta)\quad\text{a.e.}$$

In the same way as the proof given in [13], we can prove $M^+\sim l_M$,

$$\left|\sum_{k=1}^{M}T_k'\right|=o(M^{7/8})\quad\text{a.e.}\quad\text{and}\sum_{k\in\Delta_M'\cup\Delta_M}\|\widetilde{\mathbf{I}}_{a,b;d}(\theta^k\cdot+\gamma_k)\|_\infty=o(\phi_{l_M}).$$

By these we have (18) with (20).

If (4) is satisfied, then we have

$$\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{i=1}^{M}\widehat{Y}_i\right|=\varlimsup_{M\to\infty}(\phi_{l_M}^{-1}\phi_{\widehat{\beta}_M})\left|\sum_{i\le\widehat{\beta}_M}Z_i\right|\le\sqrt{\kappa^2+\varepsilon^2},\quad\text{a.s.}\tag{27}$$

By zero-one law, we see that the limsup in (27) is constant a.s. and we denote it by σ_ε. By (25), we have

$$\left|\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{i=1}^{M}Y_i\right|-\sigma_\varepsilon\right|\le\varepsilon\quad\text{a.s.}$$

Hence the difference between the essential supremum and the essential infimum of above limsup is less than 2ε. It means that limsup is constant a.s. By denoting the constant by $\sigma(\widetilde{\mathbf{I}}_{a,b;d},\{\gamma_k\},\theta)$, we have $\sigma(\widetilde{\mathbf{I}}_{a,b;d},\{\gamma_k\},\theta)\le\sigma_\varepsilon+\varepsilon\le\sqrt{\kappa^2+\varepsilon^2}+\varepsilon$. Since $\varepsilon>0$ is arbitrary, we have (19). By applying Fubini's theorem as before, and by noting (21), we have

$$\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{k=1}^{M}T_k\right|=\varlimsup_{M\to\infty}\phi_{l_M}^{-1}\left|\sum_{k=1}^{M}Y_k\right|=\sigma(\widetilde{\mathbf{I}}_{a,b},\{\gamma_k\},\theta)\quad\text{a.e.}\tag{28}$$

As before, we have (18) with (19).

4 LIL for discrepancies

We can easily prove the proposition below by following the proof given in [10] in which the case when $\gamma_k \equiv 0$ is proved. Recall that $\phi_x = \phi(x) = \sqrt{2x \log\log x}$.

Proposition 2 *Let $\{n_k\}$ be a sequence of real numbers such that $\{|n_k|\}$ satisfies the Hadamard gap condition (1), and let $\{\gamma_k\}$ be a sequence of real numbers. Then for a dense countable subset S of $[0,1]$, we have*

$$\varlimsup_{N\to\infty} \phi_N^{-1} N D_N(\{n_k x + \gamma_k\}) = \sup_{S\ni a<b\in S} \varlimsup_{N\to\infty} \phi_N^{-1} \left| \sum_{k=1}^N \widetilde{\mathbf{I}}_{a,b}(n_k x + \gamma_k) \right| \quad a.e.$$

For $0 \le a < b \le 1$, we have

$$\varlimsup_{N\to\infty} \phi_N^{-1} \left| \sum_{k=1}^N \widetilde{\mathbf{I}}_{a,b}(n_k x + \gamma_k) \right| = \lim_{d\to\infty} \varlimsup_{N\to\infty} \phi_N^{-1} \left| \sum_{k=1}^N \widetilde{\mathbf{I}}_{a,b;d}(n_k x + \gamma_k) \right| \quad a.e.$$

By applying this, we have

$$\Sigma_{\theta,\{\gamma_k\}} = \varlimsup_{N\to\infty} \phi_N^{-1} N D_N(\{n_k x + \gamma_k\}) = \sup_{S\ni a<b\in S} \lim_{d\to\infty} \sigma(\widetilde{\mathbf{I}}_{a,b;d}, \{\gamma_k\}, \theta) \quad \text{a.e.}$$

When (4) is satisfied, by (19), $\lim_d \rho(\theta, d) = 0$, (12) and continuity, we see

$$\Sigma_{\theta,\{\gamma_k\}} \le \sup_{S\ni a<b\in S} \sigma(\widetilde{\mathbf{I}}_{0,b-a}, |\theta|) = \Sigma_{|\theta|},$$

If (3) is satisfied, by (17) and $\lim_d \sigma(\widetilde{\mathbf{I}}_{0,b-a;d}, |\theta|) = \sigma(\widetilde{\mathbf{I}}_{0,b-a}, |\theta|)$, we see

$$\Sigma_{\theta,\{\gamma_k\}} = \sup_{S\ni a<b\in S} \sigma(\widetilde{\mathbf{I}}_{0,b-a}, \theta) = \sup_{S\ni a<b\in S} \sqrt{(b-a)(1-(b-a))} = 1/2.$$

Here after, we assume (4) and prove $\Sigma_{\theta,\{\gamma_k\}} \ge 1/2$. First, we prove

$$\int_0^1 da \int_0^1 \widetilde{\mathbf{I}}_{a,a+1/2}(q^l x + \gamma_k) \widetilde{\mathbf{I}}_{a,a+1/2}(p^l x + \gamma_{k+lr})\, dx = \begin{cases} 1/4 & l = 0, \\ 0 & l > 0. \end{cases}$$

The case $l = 0$ is clear and we assume $l > 0$. By (9) the left hand side equals to the integral on the unit interval with respect to a of

$$\widetilde{V}\big(\langle p^l(a-\gamma_k)\rangle, \langle p^l(a+\tfrac{1}{2}-\gamma_k)\rangle, \langle q^l(a-\gamma_{k+lr})\rangle, \langle q^l(a+\tfrac{1}{2}-\gamma_{k+lr})\rangle\big)/(pq)^l. \quad (29)$$

If p is even, then $\langle p^l(a+\frac{1}{2}-\gamma_k)\rangle = \langle p^l(a-\gamma_k)\rangle$ and we see that the value of (29) equals to zero. If q is even, then $\langle q^l(a+\frac{1}{2}-\gamma_{k+lr})\rangle = \langle q^l(a-\gamma_{k+lr})\rangle$ and again the value of (29) equals to zero. We assume p and q are both odd, and prove that the integral of (29) equals to zero. Denote the numerator of (29) by $W(a)$, $1/2(p^l - q^l)$ by c, and $q^l c$ by d. By (8), we can easily see that $W(a+c)$ equals to

$$\begin{aligned} &\widetilde{V}\big(\langle p^l(a+\tfrac{1}{2}-\gamma_k)+d\rangle, \langle p^l(a-\gamma_k)+d\rangle, \langle q^l(a-\gamma_{k+lr})+d\rangle, \langle q^l(a+\tfrac{1}{2}-\gamma_{k+lr})+d\rangle\big) \\ &= \widetilde{V}\big(\langle p^l(a+\tfrac{1}{2}-\gamma_k)\rangle, \langle p^l(a-\gamma_k)\rangle, \langle q^l(a-\gamma_{k+lr})\rangle, \langle q^l(a+\tfrac{1}{2}-\gamma_{k+lr})\rangle\big) = -W(a). \end{aligned}$$

Hence the integral of $W(a)$ on an interval with length $1/(p^l - q^l)$ equals to zero, and thereby the integral on the unit interval also equals to zero.

Let m be an arbitrary integer greater than 1. We use blocks Δ'_1, Δ_1, ... defined in the last section. Take I satisfying

$$l_I \le m^N < l_{I+1} \tag{30}$$

and denote it by I_N. By $I_N^2/2 < l_{I_N} \le m^N < l_{I_N+1} < (I_N+2)^2/2$, we see $I_N = \sqrt{2}\, m^{N/2} + O(1)$,

$$l_{I_N} = m^N + O(m^{N/2}) \quad \text{and} \quad l^{\circ}_{I_N} := l_{I_N} - l_{I_{N-1}} \sim (1-1/m)m^N. \tag{31}$$

We have

$$\int_0^1 \Bigl(\sum_{i=I_{N-1}+1}^{I_N} \Psi(a, a+\tfrac{1}{2}; \infty, \theta, \{\gamma_k\}, \Delta_i) \Bigr) da = \frac{1}{4} \sum_{i=I_{N-1}+1}^{I_N} {}^{\#}\Delta_i = \frac{1}{4} l^{\circ}_{I_N}.$$

Because $\widetilde{V}(\langle x\rangle, \langle y\rangle, \langle z\rangle, \langle w\rangle)$ is continuous with respect to $(x,y,z,w) \in \mathbf{R}^4$, $W(a)$ and $\Psi(a, a+\frac{1}{2}; \infty, \theta, \{\gamma_k\}, \Delta)$ are also continuous with respect to a. Therefore, we can find a_N such that

$$\sum_{i=I_{N-1}+1}^{I_N} \Psi(a_N, a_N+\tfrac{1}{2}; \infty, \theta, \{\gamma_k\}, \Delta_i) = \frac{1}{4} l^{\circ}_{I_N}.$$

Take $\frac{1}{4} > \varepsilon > 0$ arbitrarily and take d large enough to have $0 < \rho(\theta, d) < \varepsilon$. By (15), we have

$$\Bigl(\frac{1}{4} - \varepsilon\Bigr) l^{\circ}_{I_N} \le \sum_{i=I_{N-1}+1}^{I_N} \Psi(a_N, a_N+\tfrac{1}{2}; d, \theta, \{\gamma_k\}, \Delta_i) \le \Bigl(\frac{1}{4} + \varepsilon\Bigr) l^{\circ}_{I_N}. \tag{32}$$

Recall that $I_N^+ = \max \Delta_{I_N}$. For $k \in (I_{N-1}^+, I_N^+]$, put $\nu(k) = N$. Putting

$$T_i(x) = \sum_{k\in\Delta_i} \widetilde{\mathbf{I}}_{a_{\nu(k)}, a_{\nu(k)}+1/2; d}(\theta^k x + \gamma_k), \text{ and } T'_i(x) = \sum_{k\in\Delta'_i} \widetilde{\mathbf{I}}_{a_{\nu(k)}, a_{\nu(k)}+1/2; d}(\theta^k x + \gamma_k),$$

we define $\mathcal{F}_i$, Y_i, v_i, β_M, and V_M as before. By (32), we see

$$\bigl(\tfrac{1}{4} - \varepsilon\bigr) l^{\circ}_{I_N} \le \beta_{I_N} - \beta_{I_{N-1}} \le \bigl(\tfrac{1}{4} + \varepsilon\bigr) l^{\circ}_{I_N} \quad \text{and} \quad \beta_{I_N} \ge \bigl(\tfrac{1}{4} - \varepsilon\bigr) l_{I_N} \to \infty. \tag{33}$$

Because we have $T_i(x) = \sum_{k\in\Delta_i} \widetilde{\mathbf{I}}_{0,1/2;d}(\theta^k x + \gamma_k - a_{\nu(k)})$, we can apply the argument of the last section (Here, there is no need to add Ξ_i terms because of (33)) and prove the existence of standard normal i.i.d. $\{Z_i\}$ such that

$$\sum_{i=1}^{M} Y_i = \sum_{i\le\beta_M} Z_i + o(\phi_{l_M}) \quad \text{a.s.}$$

Hence we have

$$\sum_{i=I_{N-1}+1}^{I_N} Y_i = \sum_{i\in(\beta_{I_{N-1}}, \beta_{I_N}]} Z_i + o(\phi_{l_{I_N}}) \quad \text{a.s.} \tag{34}$$

Put $\beta^\circ_{I_N} = {}^\#\big(\mathbf{Z}\cap(\beta_{I_{N-1}},\beta_{I_N}]\big) = \beta_{I_{N-1}} - \beta_{I_{N-1}} + O(1)$. We have

$$C(-\varepsilon,m)m^N \sim (\tfrac{1}{4}-\varepsilon)l^\circ_{I_N} + O(1) \le \beta^\circ_{I_N} \le (\tfrac{1}{4}+\varepsilon)l^\circ_{I_N} + O(1) \sim C(+\varepsilon,m)m^N, \quad (35)$$

as $N\to\infty$ where $C(\pm\varepsilon,m) = (\frac{1}{4}\pm\varepsilon)(1-\frac{1}{m})$, and thereby

$$P\Big(\Big|\sum_{i\in(\beta_{I_{N-1}},\beta_{I_N}]} Z_i\Big| \ge \sqrt{(2-\varepsilon)\beta^\circ_{I_N}\log\log\beta^\circ_{I_N}}\Big) = \sqrt{\frac{2}{\pi}}\int_{\sqrt{(2-\varepsilon)\log\log\beta^\circ_{I_N}}}^\infty e^{-x^2/2}\,dx$$

$$\ge \sqrt{\frac{2}{\pi(2-\varepsilon)\log\log\beta^\circ_{I_N}}}\exp(-(1-\tfrac{\varepsilon}{2})\log\log\beta^\circ_{I_N}) \sim \sqrt{\frac{2}{\pi(2-\varepsilon)\log N}}N^{-1+\varepsilon/2}.$$

By applying the second Borel-Cantelli Lemma, we have

$$\Big|\sum_{i\in(\beta_{I_{N-1}},\beta_{I_N}]} Z_i\Big| \ge \sqrt{(2-\varepsilon)\beta^\circ_{I_N}\log\log\beta^\circ_{I_N}} \quad \text{i.o.} \quad \text{a.s.}$$

Note that $I_N^+ = \sum_{i=1}^{I_N}({}^\#\Delta_i + {}^\#\Delta_i') = I_N(I_N+1)/2 + O(I_N\log I_N) \sim I_N^2/2 \sim m^N$. By applying (34) and (35), we have

$$\overline{\lim_{N\to\infty}}\,\phi^{-1}(I_N^+)\Big|\sum_{i=I_{N-1}+1}^{I_N} T_i\Big| \ge \sqrt{(1-\tfrac{\varepsilon}{2})(\tfrac{1}{4}-\varepsilon)(1-\tfrac{1}{m})} \quad \text{a.e.}$$

As the proof given in [13], we can prove

$$\overline{\lim_{N\to\infty}}\,\phi^{-1}(I_N^+)\Big|\sum_{i=I_{N-1}+1}^{I_N} T_i'\Big| = 0 \quad \text{a.e.}$$

By combining these, we have

$$\overline{\lim_{N\to\infty}}\,\phi^{-1}(I_N^+)\Big|\sum_{k=I_{N-1}^++1}^{I_N^+} \widetilde{\mathbf{I}}_{a_N,a_N+1/2;d}(\theta^k x+\gamma_k)\Big| \ge \sqrt{(1-\tfrac{\varepsilon}{2})(\tfrac{1}{4}-\varepsilon)(1-\tfrac{1}{m})} \quad \text{a.e.} \tag{36}$$

We use the following theorem, which is implicitly included in [15]. For the proof, see [10] or [5].

Theorem 3 *Let f be a real valued function with*

$$f(x+1)=f(x), \quad \int_0^1 f(x)\,dx, \quad \|f\|_2^2 = \int_0^1 f^2(x)\,dx < \infty, \quad |\widehat{f}(n)| \le \frac{C}{|n|}.$$

Assume that $\{|n_k|\}$ satisfies the Hadamard gap condition (1)*. Then there exist an absolute constant $\delta>0$ and a constant C' depending only on C and q such that*

$$\overline{\lim_{N\to\infty}}\,\phi^{-1}(N)\Big|\sum_{k=1}^N f(n_k x)\Big| \le C'\|f\|_2^\delta \quad \textit{a.e.} \tag{37}$$

Take d large enough to have $\|\widetilde{\mathbf{I}}_{a_N,a_N+1/2} - \widetilde{\mathbf{I}}_{a_N,a_N+1/2;d}\|_2^\delta < \varepsilon/C'$. By applying the above theorem, we have

$$\overline{\lim_{N\to\infty}}\, \phi^{-1}(I_N^+)\left|\sum_{k=1}^{I_N^+}(\widetilde{\mathbf{I}}_{a_{\nu(k)},a_{\nu(k)}+1/2} - \widetilde{\mathbf{I}}_{a_{\nu(k)},a_{\nu(k)}+1/2;d})(\theta^k x + \gamma_k)\right| < \varepsilon \quad \text{a.e.}$$

Because the above inequality is still valid if we replace I_N^+ by I_{N-1}^+, noting $\phi^{-1}(I_{N-1}^+) < \phi^{-1}(I_N^+)$, we have

$$\overline{\lim_{N\to\infty}}\, \phi^{-1}(I_N^+)\left|\sum_{k=I_{N-1}^+ +1}^{I_N^+}(\widetilde{\mathbf{I}}_{a_N,a_N+1/2} - \widetilde{\mathbf{I}}_{a_N,a_N+1/2;d})(\theta^k x + \gamma_k)\right| < 2\varepsilon \quad \text{a.e.}$$

This together with (36), we have

$$\overline{\lim_{N\to\infty}}\, \phi^{-1}(I_N^+)\left|\sum_{k=I_{N-1}^+ +1}^{I_N^+}\widetilde{\mathbf{I}}_{a_N,a_N+1/2}(\theta^k x+\gamma_k)\right| \geq \sqrt{(1-\tfrac{\varepsilon}{2})(\tfrac{1}{4}-\varepsilon)(1-\tfrac{1}{m})} - 2\varepsilon \quad \text{a.e.}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\overline{\lim_{N\to\infty}}\, \phi^{-1}(I_N^+) \sup_{0\leq a\leq b\leq 1}\left|\sum_{k=I_{N-1}^+ +1}^{I_N^+}\widetilde{\mathbf{I}}_{a,b}(\theta^k x + \gamma_k)\right| \geq \frac{1}{2}\sqrt{1-\tfrac{1}{m}} \quad \text{a.e.}$$

On the other hand, by (5) and $I_N^+/I_{N-1}^+ \sim m$, we have

$$\overline{\lim_{N\to\infty}}\, \phi^{-1}(I_N^+) \sup_{0\leq a\leq b\leq 1}\left|\sum_{k=1}^{I_{N-1}^+}\widetilde{\mathbf{I}}_{a,b}(\theta^k x + \gamma_k)\right| \leq \frac{\Sigma_{|\theta|}}{\sqrt{m}} \quad \text{a.e.}$$

These imply

$$\overline{\lim_{N\to\infty}}\, \phi^{-1}(I_N^+) \sup_{0\leq a\leq b\leq 1}\left|\sum_{k=1}^{I_N^+}\widetilde{\mathbf{I}}_{a,b}(\theta^k x + \gamma_k)\right| \geq \frac{1}{2}\sqrt{1-\tfrac{1}{m}} - \frac{\Sigma_{|\theta|}}{\sqrt{m}} \quad \text{a.e.}$$

The left hand side is $\overline{\lim}_N\, \phi^{-1}(I_N^+) I_N^+ D_{I_N^+}(\{\theta^k x + \gamma_k\})$ which is less than or equal to $\overline{\lim}_N\, \phi^{-1}(N) N D_N(\{\theta^k x + \gamma_k\})$. Hence we have

$$\Sigma_{\theta,\{\gamma_k\}} = \overline{\lim_{N\to\infty}}\, \phi^{-1}(N) N D_N(\{\theta^k x + \gamma_k\}) \geq \frac{1}{2}\sqrt{1-\tfrac{1}{m}} - \frac{\Sigma_{|\theta|}}{\sqrt{m}} \quad \text{a.e.}$$

By letting $m \to \infty$, we can prove $\Sigma_{\theta,\{\gamma_k\}} \geq \frac{1}{2}$.

5 The proof of the second part of the theorem

When θ satisfies (3), there is nothing to be proved. Hence we assume (4).

By (12), there exists a c such that $\sigma(\widetilde{\mathbf{I}}_{0,c},|\theta|)=\Sigma_{|\theta|}$. We define a sequence $\{\gamma_k\}$ in the following way. Put $\gamma_k\equiv 0$ if $p>0$, and $\gamma_k=(1+(-1)^{k+1})c$ if $p<0$. Note that r is odd if $p<0$.

Let $\{\Gamma_k^{(\eta)}\}_{k\in\mathbf{N}}$ be an i.i.d. whose distribution is defined by

$$P(\Gamma_k^{(\eta)}\in dt)=\frac{1}{2\eta}\mathbf{1}_{[-\eta,\eta]}(t)\,dt\quad(\eta>0),\quad\text{and}\quad P(\Gamma_k^{(0)}=0)=1.$$

We consider the discrepancy of the sequence $\{\theta^k x+\gamma_k+\Gamma_k\}$. Note that

$$\frac{1}{l_M}\sum_{i=1}^{M}\Psi(a,b;d,\theta,\{\gamma_k+\Gamma_k^{(\eta)}\},\Delta_i)=\int_0^1\widetilde{\mathbf{I}}_{a,b;d}^2(x)\,dx+2\sum_{l=1}^{\infty}A_{a,b;d,\eta}(M,l),$$

where $A_{a,b;d,\eta}(M,l)$ is given by

$$\frac{1}{l_M}\sum_{i=lr+1}^{M}\sum_{k=i^-}^{i^+-lr}\int_0^1\widetilde{\mathbf{I}}_{a,b;d}(q^lx+\gamma_k+\Gamma_k^{(\eta)})\widetilde{\mathbf{I}}_{a,b;d}(p^lx+\gamma_{k+lr}+\Gamma_{k+lr}^{(\eta)})\,dx.$$

Since we have

$$\sum_{i=lr+1}^{M}\sum_{k=i^-}^{i^+-lr}1=\frac{1}{2}((M-lr)(M-lr+1))\vee 0\sim l_M\quad(M\to\infty),$$

and since $\{(\Gamma_k^{(\eta)},\Gamma_{k+lr}^{(\eta)})\}_{k\in\mathbf{N}}\}$ is an lr-dependent sequence of identically distributed random variables, by the law of large numbers, we have

$$E_{a,b;d,\eta}(l):=\lim_{M\to\infty}A_{a,b;d,\eta}(M,l)=\begin{cases}E_{a,b;d,\eta}^{0,0}(l) & \text{if } p>0,\\ \frac{1}{2}(E_{a,b;d,\eta}^{0,0}(l)+E_{a,b;d,\eta}^{c,c}(l)) & \text{if } p<0,\ 2\mid l,\\ \frac{1}{2}(E_{a,b;d,\eta}^{0,c}(l)+E_{a,b;d,\eta}^{c,0}(l)) & \text{if } p<0,\ 2\nmid l,\end{cases}$$

with probability 1, where

$$E_{a,b;d,\eta}^{s,t}(l)=E\int_0^1\widetilde{\mathbf{I}}_{a,b;d}(q^lx+s+\Gamma_1^{(\eta)})\widetilde{\mathbf{I}}_{a,b;d}(p^lx+t+\Gamma_2^{(\eta)})\,dx.$$

Because we have

$$\begin{aligned}&\int_0^1\widetilde{\mathbf{I}}_{a,b;d}(q^lx+\gamma_k+\Gamma_k^{(\eta)})\widetilde{\mathbf{I}}_{a,b;d}(p^lx+\gamma_{k+lr}+\Gamma_{k+lr}^{(\eta)})\,dx\\&=\sum_{1\le|\lambda|\le d/|p|^l}\widehat{\widetilde{\mathbf{I}}}_{a,b}(\lambda p^l)\widehat{\widetilde{\mathbf{I}}}_{a,b}(-\lambda q^l)\exp\Big(2\pi\sqrt{-1}\,\lambda\big(p^l(\gamma_k+\Gamma_k^{(\eta)})-q^l(\gamma_{k+lr}+\Gamma_{k+lr}^{(\eta)})\big)\Big),\end{aligned}$$

and

$$\widehat{\widetilde{\mathbf{I}}}_{a,b}(\lambda)=\big(e^{-2\pi\sqrt{-1}\,a\lambda}-e^{-2\pi\sqrt{-1}\,b\lambda}\big)\big/2\pi\sqrt{-1}\,\lambda,$$

we have

$$|A_{a,b;d,\eta}(M,l)|\le 1/3|p|^lq^l.$$

Since the right hand side is independent of M and summable in l, by applying dominated convergence theorem for series, we have

$$\frac{1}{l_M}\sum_{i=1}^{M}\Psi(a,b;d,\theta,\{\gamma_k+\Gamma_k^{(\eta)}\},\Delta_i)\to\int_0^1\widetilde{\mathbf{I}}_{a,b;d}^2(x)\,dx+2\sum_{l=1}^{\infty}E_{a,b;d,\eta}(l)=:\sigma_{a,b;d,\theta,\eta}^2,$$

almost surely as $M\to\infty$. For $\{\theta^k x+\gamma_k+\Gamma_k^{(\eta)}\}$, we apply the proof of Proposition 1 and have $\beta_M\sim\sigma_{a,b;d,\theta,\eta}^2 l_M$. As the derivation of (26), we have

$$\overline{\lim_{M\to\infty}}\,\phi_{l_M}^{-1}\left|\sum_{i=1}^{M}Y_i\right|=\sigma_{a,b;d,\theta,\eta}\quad\text{a.e.}$$

with probability one. As before we can prove

$$\overline{\lim_{N\to\infty}}\,\phi_N^{-1}\left|\sum_{k=1}^{N}\widetilde{\mathbf{I}}_{a,b;d}(\theta^k x+\gamma_k+\Gamma_k^{(\eta)})\right|=\sigma_{a,b;d,\theta,\eta}\quad\text{a.e.}$$

with probability one.

Because $E_{a,b;d,\eta}(l)\to E_{a,b;\infty,\eta}(l)$, as $d\to\infty$, and $\sum_l 1/3|p|^l q^l$ is a majorizing series as before, by dominated convergence theorem again, we have

$$\sigma_{a,b;\theta,\eta}^2=\lim_{d\to\infty}\sigma_{a,b;d,\theta,\eta}^2=\int_0^1\widetilde{\mathbf{I}}_{a,b}^2(x)\,dx+2\sum_{l=1}^{\infty}E_{a,b;\infty,\eta}(l).$$

Hence by applying Proposition 2, we have

$$\overline{\lim_{N\to\infty}}\,\phi_N^{-1}\left|\sum_{k=1}^{N}\widetilde{\mathbf{I}}_{a,b}(\theta^k x+\gamma_k+\Gamma_k^{(\eta)})\right|=\sigma_{a,b;\theta,\eta}\quad\text{a.e.}$$

with probability one. By applying Proposition 2 again, we have

$$\overline{\lim_{N\to\infty}}\,\phi_N^{-1}ND_N(\{\theta^k x+\gamma_k+\Gamma_k^{(\eta)}\})=\sup_{S\ni a\le b\in S}\sigma_{a,b;\theta,\eta}=:\Sigma_{\theta,\eta}\quad\text{a.e.}$$

with probability one.

Note that the characteristic function $\widehat{\Gamma}_k^{(\eta)}$ of $\Gamma_k^{(\eta)}$ satisfies

$$\widehat{\Gamma}_k^{(\eta)}(2\pi\nu)=\sin 2\pi\nu\eta/2\pi\nu\eta\quad(\eta>0),\quad\text{and}\quad\widehat{\Gamma}_k^{(0)}(2\pi\nu)=1.$$

Here $\Gamma_k^{(\eta)}(2\pi\nu)$ is bounded and continuous in η. We see that $\sigma_{a,b;\theta,\eta}^2$ is expanded as

$$\frac{1}{2}(b-a)(1-(b-a))+2\sum_{l,\lambda}\widehat{\widetilde{\mathbf{I}}}_{a,b}(\lambda p^l)\widehat{\widetilde{\mathbf{I}}}_{a,b}(-\lambda q^l)\psi(l,\lambda)\widehat{\Gamma}_1^{(\eta)}(2\pi\lambda p^l)\widehat{\Gamma}_2^{(\eta)}(-2\pi\lambda q^l),$$

where

$$\psi(l,\lambda)=\begin{cases}1 & \text{if } p>0,\\ \frac{1}{2}\left(1+\exp(2\pi\sqrt{-1}\,\lambda(p^l-q^l)c)\right) & \text{if } p<0 \text{ and } 2\mid l,\\ \frac{1}{2}\left(\exp(2\pi\sqrt{-1}\,\lambda p^l c)+\exp(-2\pi\sqrt{-1}\,\lambda q^l c)\right) & \text{if } p<0 \text{ and } 2\nmid l.\end{cases}$$

Here the absolute value of the (l,λ)-th term of the last series is bounded by $1/\pi^2\lambda^2|p|^l q^l$, which is independent of (a,b,η) and is summable in l and λ. Because each term is continuous in $(a,b,\eta)\in[0,1]^3$, we see that the series is also uniformly continuous in $(a,b,\eta)\in[0,1]^3$. Hence we see that

$$\Sigma_{\theta,\eta}=\sup_{S\ni a\le b\in S}\sigma_{a,b;\theta,\eta}=\max_{0\le a\le b\le 1}\sigma_{a,b;\theta,\eta}$$

is continuous in η. Since $\Gamma_1^{(1/2)}$ is uniformly distributed over the unit interval, we have $E\widetilde{\mathbf{I}}_{a,b}(q^l x+\Gamma_1^{(1/2)})=0$, $E_{a,b;\theta,1/2}(l)=0$, $\sigma^2_{a,b;\theta,1/2}=(b-a)(1-(b-a))$, and $\Sigma_{\theta,1/2}=1/2$ in turn. Nextly, consider the case $\eta=0$. We first prove

$$E_{0,c;\infty,0}(l)=\int_0^1\widetilde{\mathbf{I}}_{0,c}(q^l x)\widetilde{\mathbf{I}}_{0,c}(|p|^l x)\,dx. \tag{38}$$

Since it is clear when $p>0$, we prove in case $p<0$. Note that $\widetilde{\mathbf{I}}_{0,c}(-Ax+c)=\widetilde{\mathbf{I}}_{-c,0}(-Ax)=\widetilde{\mathbf{I}}_{0,c}(Ax)$ a.e. For odd l, by applying $\widetilde{\mathbf{I}}_{0,c}(p^l x)=\widetilde{\mathbf{I}}_{0,c}(|p|^l x+c)$ and replacing x by $-x$, we have

$$E^{c,0}_{0,c;\infty,0}(l)=\int_0^1\widetilde{\mathbf{I}}_{0,c}(q^l x+c)\widetilde{\mathbf{I}}_{0,c}(|p|^l x+c)\,dx=\int_0^1\widetilde{\mathbf{I}}_{0,c}(-q^l x+c)\widetilde{\mathbf{I}}_{0,c}(-|p|^l x+c)\,dx$$

which equals to the right hand side of (38). By applying $\widetilde{\mathbf{I}}_{0,c}(p^l x+c)=\widetilde{\mathbf{I}}_{0,c}(|p|^l x)$, we also have

$$E^{0,c}_{0,c;\infty,0}(l)=\int_0^1\widetilde{\mathbf{I}}_{0,c}(q^l x)\widetilde{\mathbf{I}}_{0,c}(p^l x+c)\,dx=\int_0^1\widetilde{\mathbf{I}}_{0,c}(q^l x)\widetilde{\mathbf{I}}_{0,c}(|p|^l x)\,dx,$$

and hence we have proved (38) for odd l. When l is even, we have

$$E^{c,c}_{0,c;\infty,0}(l)=\int_0^1\widetilde{\mathbf{I}}_{0,c}(q^l x+c)\widetilde{\mathbf{I}}_{0,c}(|p|^l x+c)\,dx=\int_0^1\widetilde{\mathbf{I}}_{0,c}(-q^l x+c)\widetilde{\mathbf{I}}_{0,c}(-|p|^l x+c)\,dx$$

which equals to the right hand side of (38) and we have proved (38).

It shows $\Sigma_{\theta,0}\ge\sigma_{0,c;\theta,0}=\sigma(\widetilde{\mathbf{I}}_{0,c},|\theta|)=\Sigma_{|\theta|}$. On the other hand, by the first part of our theorem, we have $\Sigma_{\theta,0}\le\Sigma_{|\theta|}$, and thereby $\Sigma_{\theta,0}=\Sigma_{|\theta|}$. Because of continuity, $\Sigma_{\theta,\eta}$ takes all values between $1/2$ and $\Sigma_{|\theta|}$.

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