# Metric discrepancy results for geometric progressions with large ratios II

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**Abstract** We try to determine the threshold of the validity of the formula describing the limsup constant of the law of the iterated logarithm for the discrepancy of geometric progressions with large ratio.

**Keywords** discrepancy · lacunary sequence · law of the iterated logarithm

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For a sequence $\{x_k\}$ of real numbers, its discrepancy is defined by

$$D_N(\{x_k\}) = \sup_{0\le a\le b\le 1} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbf{1}_{[a,b)}(\langle x_k \rangle) - (b-a) \right|;$$

where $\langle x \rangle$ denotes the fractional part $x - [\,x\,]$ of $x$, and $\mathbf{1}_{[a,b)}$ denotes the indicator function of $[\,a,b)$.

For any $\theta \in (-\infty,-1)\cup(1,\infty)$, there exist a constant $\Sigma_\theta$ such that

$$\varlimsup_{N\to\infty} \frac{ND_N\{\theta^k x\}}{\sqrt{2N\log\log N}} = \Sigma_\theta, \quad \text{a.e.} \quad x.$$

Some of the concrete values of constants are determined as below ([6–12]). It holds that $\Sigma_\theta = 1/2$ if $\theta$ satisfies $\theta^k \notin \mathbf{Q}$ for all $k\in\mathbf{N}$. When $\theta^k \in \mathbf{Q}$ for some $k \in \mathbf{N}$, we denote

$$\theta^r = p/q \quad (p\in\mathbf{Z},\ q\in\mathbf{N},\ \gcd(p,q)=1) \quad \text{where} \quad r = \min\{k\in\mathbf{N} : \theta^k \in \mathbf{Q}\}.$$

In this case, $\Sigma_\theta$ is independent of $r$ and is greater than $1/2$, i.e., $\Sigma_\theta = \Sigma_{p/q} > 1/2$.

If both of $p$ and $q$ are odd, then

$$\Sigma_{p/q} = \frac{1}{2}\sqrt{\frac{|p|q+1}{|p|q-1}}.$$

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If $p/q = \pm 2$, then

$$\varSigma_2 = \frac{1}{9}\sqrt{42} \quad \text{and} \quad \varSigma_{-2} = \frac{1}{49}\sqrt{910}.$$

When $pq$ is even, for $p/q$ belonging to

$$\begin{aligned}\varTheta :=& \{p/q : 2 \nmid p,\ 2 \mid q,\ \gcd(p,q)=1,\ |p|/q \geq 9/4\} \\ &\cup \{p/q : 2 \mid p,\ 2 \nmid q,\ \gcd(p,q)=1,\ |p|/q \geq 4\} \cup \{\pm 13/6\},\end{aligned}$$

we [11,12] proved the next formula.

$$\varSigma_{p/q} = \sqrt{\frac{(|p|q)^I+1}{(|p|q)^I-1} v\Big(\frac{|p|-q-1}{2(|p|-q)}\Big) + \frac{2(|p|q)^I}{(|p|q)^I-1}\sum_{m=1}^{I-1}\frac{1}{(|p|q)^m} v\Big(q^m\frac{|p|-q-1}{2(|p|-q)}\Big)}, \quad (1)$$

where $I := \min\{n \in \mathbf{N} \mid q^n = \pm 1 \bmod |p|-q\}$ and $v(x) := \langle x \rangle (1-\langle x \rangle)$.

Generally, if positive $p/q$ satisfies (1), then $-p/q$ also satisfies (1) and $\varSigma_{-p/q}$ coincides with $\varSigma_{p/q}$. For positive $p/q$ for which (1) is false, we still have $\varSigma_{-p/q} \leq \varSigma_{p/q}$ but the equality does not necessarily hold as is known by the example $\varSigma_{-2} < \varSigma_2$. Since analysis of negative $p/q$ seems to be difficult, we restrict ourselves to the case $p/q > 0$.

The formula (1) is derived in the following way. For $a,\ a' \in [0,1)$, put $V(a,a') = a \wedge a' - aa'$ and

$$\sigma^2_{p/q}(a) = V(a,a) + 2\sum_{k=1}^{\infty}\frac{1}{(pq)^k} V(\langle p^k a\rangle, \langle q^k a\rangle). \quad (2)$$

We have

$$0 \leq V(a,a') \leq V(a,a) = v(a) \leq 1/4 = v(1/2), \quad (3)$$

and we see $V(a,a')$ together with $\sigma^2_{p/q}(a)$ is continuous with respect to its arguments. We [6] proved the following relation, which makes it possible to calculate the concrete value.

$$\varSigma_{p/q} = \sup_{0 \leq a \leq 1/2} \sigma_{p/q}(a). \quad (4)$$

We call that positive $p/q$ is of type 0 if $\varSigma_{p/q} = \sigma_{p/q}(1/2)$, and is of type $N$ ($N \geq 1$) if it is not of type $N'$ for $N' < N$ and there exists a positive integer $i$ with

$$\varSigma_{p/q} = \sigma_{p/q}(i/(p^N - q^N)).$$

If $2 \nmid pq$, then $\varSigma_{p/q} = \sigma_{p/q}(1/2)$ and is of type 0. When $2 \mid pq$, we have $\sigma_{p/q}(1/2) = 1/2$ and we [11] proved $\varSigma_{p/q} > 1/2$, and hence it is not of type 0. For positive $p/q \in \varTheta$ we see that (1) is derived from

$$\varSigma_{p/q} = \sigma_{p/q}((p-q-1)/2(p-q)). \quad (5)$$

and it is of type I.

We [11] have evaluated $\varSigma_{p/q}$ for the following $p/q$ of other types: 2, 4/3, 8/3, 10/3, 12/5, 17/8 are of type II, 19/10 is of type III, 12/7 is of type IV, and 8/5 is of type V. We [10] proved that 3/2 is of type VI. For $2 < p/q < 9/4$ with even $q \leq 8$, types are determined. 13/6 is of type I and 17/8 is of type II ([11]). For $2 < p/q < 4$ with odd $q \leq 3$, types are determined. 8/3 and 10/3 are of type II ([11]).

By these results, it is natural to aks if the formula (1) is always false when $p/q$ is small. We are now in a position to state our result, which almost gives positive answer to this question.

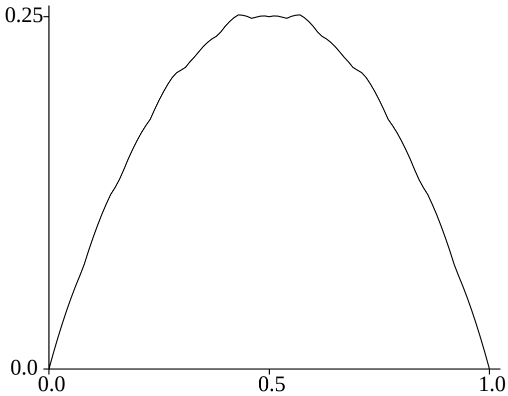


**Fig. 1** Graph of $\sigma^2_{13/6}(a)$. Maximal points are $a = 3/7$ and $a = 4/7$.

**Theorem 1** *Suppose that $p > 0$, $q > 0$, $2 \mid p$, $2 \nmid q$, and $\gcd(p, q) = 1$. The formula (1) holds if and only if $p/q \geq 4$.*

**Theorem 2** *Suppose that $p > 0$, $q > 0$, $2 \nmid p$, $2 \mid q$ and $\gcd(p, q) = 1$.*

1. *Suppose $p - q = 2 \mod 3$. The formula (1) holds if and only if*

$$p - \Big(1 + \frac{\sqrt{6}}{2}\Big)q > -\frac{21\sqrt{6} - 47}{38} =: t_1 = -0.1168\ldots$$

2. *Suppose $p - q = 1 \mod 3$. The formula (1) holds if and only if*

$$p - \Big(1 + \frac{\sqrt{6}}{2}\Big)q > -\frac{6\sqrt{6} - 8}{19} =: t_2 = -0.3523\ldots$$

3. *Suppose $p - q = 0 \mod 3$ and $p - \big(1 + \frac{\sqrt{6}}{2}\big)q > 0$. The formula (1) holds if and only if*

$$p - \Big(1 + \frac{\sqrt{6}}{2}\Big)q > \frac{39 - 15\sqrt{6}}{19} =: t_0 = 0.1188\ldots$$

4. *Suppose $p - q = 0 \mod 3$ and $p - \big(1 + \frac{\sqrt{6}}{2}\big)q < 0$. Put*

$$t_{3,-} = \frac{360\sqrt{6} - 1050}{19^2} = -0.4658\ldots,$$

$$t_{3,+} = -\frac{\sqrt{5}\sqrt[4]{6}(210 - 72\sqrt{6}) + 75\sqrt{6} - 309}{19^2} = -0.0209\ldots,$$

$$E = \{\, (p_j, q_j) \mid j \geq 0,\ 11 \nmid j, 7 \nmid (j - 6) \,\},$$

*where integers $p_j$ and $q_j$ are defined by*

$$q_j + \frac{p_j - q_j}{3}\sqrt{6} = \frac{1}{19^2}\Big((2 + 11\sqrt{6})(5 + 2\sqrt{6})^{171 + 18 \cdot 19j} - 720 - 350\sqrt{6}\Big).$$

*Then there exists a $C > 0$ such that (1) holds if*

$$t_{3,-} < p - \Big(1 + \frac{\sqrt{6}}{2}\Big)q < t_{3,+} - \frac{C}{q} \quad \text{and} \quad (p, q) \notin E,$$

*and does not hold if*

$$p-\Big(1+\frac{\sqrt{6}}{2}\Big)q<t_{3,-},\quad t_{3,+}<p-\Big(1+\frac{\sqrt{6}}{2}\Big)q<0\quad \textit{or}\quad (p,q)\in E.$$

We see that $p_j/q_j$ is located just right of $t_{3,-}$ and converges very rapidly to $t_{3,-}$. The set $E$ is the exceptional set of $t$ in the right neighborhood of $t_{3,-}$. As compared to this, we do not know anything about the exceptional set in the left neighborhood of $t_{3,+}$.

Later on, we assume $p>q\geq 4$ in the proof, since the results are already proved in cases $q=1$ (Cf. [6]), $q=2$ (Cf. [12,10]), and $q=3$ (Cf. [12,11]).

## 1 Preliminaries

We introduce notation, a part of which is used in [12]. Put $b_i:=\dfrac{i}{p-q}$, $c_1:=\dfrac{q-2}{2q}$, $c_2:=\dfrac{p-3}{2p}$, $c_3:=\dfrac{p-1}{2p}$, $c_4:=b_{(p-q-1)/2}$, $c_5:=\dfrac{q-1}{2q}$, $c_6:=\dfrac{p-2}{2p}$, and $c_7:=b_{(p-q-3)/2}$. When $p/q\geq 2$, one can verify $c_1<c_2<c_4<c_3$ and $c_5\leq c_6\leq c_4$.

By $p=q \mod p-q$, we have

$$\langle p^k b_i\rangle=\langle q^k b_i\rangle\quad\text{and}\quad V\big(\langle p^k b_i\rangle,\langle q^k b_i\rangle\big)=v(p^k b_i)=v(q^k b_i).\tag{6}$$

Put

$$c_8:=\frac{\lfloor p^2c_4\rfloor+1}{p^2}=c_4+\frac{1-\langle q^2c_4\rangle}{p^2},\quad c_9:=c_4+\frac{1-\langle q^2c_4\rangle}{q^2}<c_{11}:=c_4+\frac{2-\langle q^2c_4\rangle}{q^2},$$

where we used $\langle p^2c_4\rangle=\langle q^2c_4\rangle$. Clearly, $c_4<c_8<c_9$. We introduce

$$c_8<c_{10}:=c_4+\frac{2-\langle q^2c_4\rangle}{p^2}<c_{12}:=c_4+\frac{3-\langle q^2c_4\rangle}{p^2}<c_{14}:=c_4+\frac{4-\langle q^2c_4\rangle}{p^2},$$
$$c_8<c_{13}:=c_4+\frac{1}{p^2-q^2},\quad c_{15}:=c_4+\frac{1-\langle q^3c_4\rangle}{q^3},\quad c_{16}:=c_4+\frac{1}{p^3-q^3}.$$

Put $c_{20+j}:=c_4+\dfrac{j-\langle q^3c_4\rangle}{p^3}$ for $j\geq 1$. Clearly we have $c_{21}<c_{15}$.

When $p/q\geq\sqrt{3}$, we have $(2p^2-4q^2)/(p^2-q^2)\geq 1$ which implies $\langle q^2c_4\rangle<(2p^2-4q^2)/(p^2-q^2)$ or $c_{14}<c_{11}$.

Thanks to $v(-x)=v(x)$, we have $v(q^{m+nI}c_4)=v(\pm q^m c_4)=v(q^m c_4)$. Hence $\sigma_{p/q}(c_4)$ equals to the left hand side of (1).

We denote

$$X_N(a):=V\big(\langle a\rangle,\langle a\rangle\big)+2\sum_{k=1}^{N}\frac{1}{(pq)^k}V\big(\langle p^ka\rangle,\langle q^ka\rangle\big).$$

We denote the right derivative of $f(t)$ at $t=a$ by $D_+f(a)$. Note that $\langle p^ka\rangle=\langle q^ka\rangle=0$ does not hold for $k\geq 0$ and $0<a<1$. The function $g(a)=V\big(\langle p^ka\rangle,\langle q^ka\rangle\big)$ is continuous with respect to $a$, and is differentiable at $a$ if $\langle p^ka\rangle\neq 0$, $\langle q^ka\rangle\neq 0$, and $\langle p^ka\rangle\neq\langle q^ka\rangle$, If $\langle p^ka\rangle\leq\langle q^ka\rangle$, then $D_+g(a)=p^k-p^k\langle q^ka\rangle-q^k\langle p^ka\rangle=:h(a)$, and if $\langle p^ka\rangle>\langle q^ka\rangle$, $D_+g(a)=q^k-p^k\langle q^ka\rangle-q^k\langle p^ka\rangle=:i(a)$. Hence at differentiable point $a$, $D_+g(a)$ is decreasing. If $\langle q^k a\rangle = 0$, then $D_+g(a) = i(a)$ and $D_+g(a-0) := \lim_{b\to a-0} D_+g(b) = h(a)$, and hence it is decreasing at this point. If $\langle p^k a\rangle = \langle q^k a\rangle$, then $D_+g(a) = i(a)$ and $D_+g(a-0) = h(a)$, and hence it is also decreasing at this point. Therefore we can conclude that $D_+g(a)$ is decreasing if $\langle p^k a\rangle \neq 0$. Especially, we have

$$D_+V\big(\langle p^k a\rangle, \langle q^k a\rangle\big) = \begin{cases} q^k - q^k\langle p^k a\rangle & \text{if } \langle q^k a\rangle = 0, \\ p^k - p^k\langle q^k a\rangle & \text{if } \langle p^k a\rangle = 0, \\ q^k - (p^k+q^k)\langle q^k c_4\rangle & \text{if } a = c_4. \end{cases} \tag{7}$$

We also note that $(D_+)^2 g(a) = -2p^k q^k$. We see $\big|D_+V\big(\langle p^k a\rangle, \langle q^k a\rangle\big)\big| \le p^k$ and

$$\left|2\sum_{k=N}^{\infty} \frac{1}{(pq)^k} D_+V\big(\langle p^k a\rangle, \langle q^k a\rangle\big)\right| \le \frac{2}{q^{N-1}(q-1)}. \tag{8}$$

By denoting the series (2) by $\sum f_k$, we see $f_k$ is continuous, right differentiable, $\sum \|f_k\|_\infty < \infty$ and $\sum \|D_+f_k\|_\infty < \infty$, which implies that $\sum f_k$ is continuous and right differentiable with $D_+\sum f_k = \sum D_+f_k$. Note that $\sigma^2(a)$ is strictly increasing on the interval on which $D_+\sigma^2(a) > 0$ is satisfied. Also note that, if $D_+\sigma^2(a_0) > 0$ then there exists $\varepsilon > 0$ such that $\sigma^2(a) > \sigma^2(a_0)$ holds for $a \in (a_0, a_0+\varepsilon)$.

By denoting $Z_{N,\pm}(a) := D_+X_N(a) \pm \dfrac{2}{q^N(q-1)}$ and by noting (8), we have

$$Z_{N,-}(a) \le D_+\sigma^2(a) \le Z_{N,+}(a). \tag{9}$$

From now on, we simply denote $\sigma_{p/q}$ and $\Sigma_{p/q}$ by $\sigma$ and $\Sigma$.

**Lemma 1** *If $p/q > 1$, $2 \mid pq$, $\gcd(p,q) = 1$, and is of type I, i.e., $\Sigma = \sigma(b_i)$ for some $i \in \{0,\dots,p-q\}$, then* (5) *and* (1) *hold.*

*Proof* If $p/q \ge 4$, it is of type I and satisfies (1). The case $q = 1$ is also solved. Actually, (1) holds if $p \ge 4$, and $p/q = 2$ is of type II. Hence we assume $1 < p/q < 4$ and $q \ge 2$. Note that $\sigma^2(c_4) \ge v(c_4)$. For $i \neq (p-q\pm 1)/2$, by applying $v(b_i) \le v(c_7)$ and $v(p^k b_i) \le 1/4$, we have

$$\sigma^2(b_i) \le v(c_7) + \sum_{k=1}^{\infty} \frac{1}{2(pq)^k} = v(c_7) + \frac{1}{2}\frac{1}{pq-1}.$$

Thereby we have

$$\sigma^2(c_4) - \sigma^2(b_i) \ge \frac{q^2(-1+6(p/q)-(p/q)^2)-4}{2(p-q)^2(pq-1)} > \frac{4(q^2-1)}{2(p-q)^2(pq-1)} > 0.$$

Therefore we see that $\Sigma = \sigma(c_4)$ if it is of type I. □

**Lemma 2** *If $1 < p/q \le 2$, $2 \mid pq$, and $\gcd(p,q) = 1$, then $\sigma(c_4) < 1/2$ holds and* (1) *does not hold. It is not of type I.*

*Proof* By $\langle q^k c_4\rangle \in \{b_1,\dots,b_{p-q-1}\}$, we have $v(q^k c_4) \le v(c_4)$ and

$$\sigma^2(c_4) \le v(c_4) + 2\sum_{k=1}^{\infty} \frac{1}{(pq)^k} v(c_4) = \frac{1}{4} + \frac{(2p-q)(p-2q)-1}{4(p-q)^2(pq-1)} < \frac{1}{4}.$$

Since the right hand side of (1) equals to $\sigma(c_4)$, by noting $\sigma^2(1/2) = 1/4$, we see (1) is invalid in this case. The previous Lemma shows that it is not type I. □

## 2 Even/Odd case

**Lemma 3** *Suppose that $2 \mid p$, $2 \nmid q$, and $\gcd(p,q)=1$. If $1 \le q < p \le 4q-6$ then* (1) *does not hold.*

*Proof* By Lemma 2, we may assume $p/q > 2$. For $1/2 > a > c_4 \ge c_6 \ge c_5$, we have $\langle pa\rangle = pa-(p-2)/2 > \langle qa\rangle = qa-(q-1)/2$ and

$$X_1(a) = -3a^2 + \Big(3-\frac{1}{q}\Big)a + \frac{1}{2q} - \frac{1}{2}.$$

By applying (9), we have

$$D_+\sigma^2(a) \ge -6a+3-\frac{1}{q}-\Big(\frac{2}{q-1}-\frac{2}{q}\Big) = -6a+3+\frac{1}{q}-\frac{2}{q-1}.$$

Hence at $a=c_4$, we see

$$D_+\sigma^2(c_4) \ge \frac{3}{p-q}+\frac{1}{q}-\frac{2}{q-1}$$

The left hand side is decreasing in $p$, and equals to $2\big\{\frac{1}{2}\big(\frac{1}{q-2}+\frac{1}{q}\big)-\frac{1}{q-1}\big\} > 0$ when $p=4q-6$, where the last inequality is by convexity of $y=1/x$. Therefore we see $D_+\sigma^2(c_4) > 0$ and $c_4$ cannot be locally maximal when $p \le 4q-6$. □

The next lemma together with the last lemma completes the proof of Theorem 1.

**Lemma 4** *Suppose that $2 \nmid q \ge 3$. For $p=4q-2$ and $p=4q-4$, the formula* (1) *does not hold.*

*Proof* We may assume $q \ge 5$ since the case $q=3$ is already solved. By $4q-4 \le p \le 4q$ and $q \ge 5$, we have $p \ge 3q$, $p^2-q^2 \ge 8q^2$, $p/(p^2-q^2) \le 4q/8q^2 = 1/2q$ and $q/(p^2-q^2) \le q/8q^2 = 1/8q$ in turn. Since $c_4 = 1/2 - 1/2(p-q) \le 1/2$, we have

$$\langle pc_4\rangle = \langle qc_4\rangle = \frac{1}{2}-\frac{q/2}{p-q}, \quad \langle pc_{13}\rangle = \langle qc_4\rangle + \frac{p}{p^2-q^2}, \quad \text{and} \quad \langle qc_{13}\rangle = \langle qc_4\rangle + \frac{q}{p^2-q^2}.$$

Clearly $\langle pc_{13}\rangle > \langle qc_{13}\rangle$ and $V\big(\langle pc_{13}\rangle, \langle qc_{13}\rangle\big) = \langle qc_{13}\rangle\big(1-\langle pc_{13}\rangle\big)$. We investigate the fractional part of $q^2c_4$. In case $p=4q-4$, we have $q^2c_4 = 1/2 - q^2/2(3q-4) \mod 1$ and $q^2/2 = (q/6+2/9)(3q-4)+8/9$. Hence,

$$p^2c_4 = q^2c_4 = \frac{5-3q}{18} - \frac{8/9}{3q-4} \mod 1.$$

By noting $3q-4 \ge 11$ and $0 < \frac{8/9}{3q-4} < 1/9$, we have

$$\langle p^2c_4\rangle = \langle q^2c_4\rangle = \begin{cases} 1/9-(8/9)/(3q-4) & \text{if} \quad q=1 \mod 6, \\ 7/9-(8/9)/(3q-4) & \text{if} \quad q=3 \mod 6, \\ 4/9-(8/9)/(3q-4) & \text{if} \quad q=5 \mod 6. \end{cases}$$

By $p^2 = q^2 \mod p^2-q^2$, and $0 < \frac{q^2}{p^2-q^2} \le 1/8$, we have

$$\langle p^2c_{13}\rangle = \langle q^2c_{13}\rangle = \langle q^2c_4\rangle + \frac{q^2}{p^2-q^2}.$$

By applying (3), we have

$$\sigma^2(c_4) \le X_2(c_4) + \frac{1}{2(pq)^2(pq-1)} \quad \text{and} \quad \sigma^2(c_{13}) \ge X_2(c_{13}).$$

By denoting $\Upsilon := (\sigma^2(c_4) - \sigma^2(c_{13}))(96(q-1)^2q^2(3q-4)^2(5q-4)^2(4q^2-4q-1))$ when $p = 4q-4$, we have

$$\Upsilon \le \begin{cases} -(7040q^6 - 27008q^5 + 36605q^4 - 17344q^3 - 3136q^2 + 4608q - 768) < 0, \\ -(6080q^6 - 24000q^5 + 33773q^4 - 16832q^3 - 2880q^2 + 4608q - 768) < 0, \\ -(6560q^6 - 25504q^5 + 35189q^4 - 17088q^3 - 3008q^2 + 4608q - 768) < 0, \end{cases}$$

according as $q = 1$, $q = 3$, or $q = 5 \mod 6$. Thus we have verified that the formula (1) does not hold when $p = 4q-4$.

In case $p = 4q-2$, we have $q^2c_4 \equiv 1/2 - q^2/2(3q-2)$ and $q^2/2 = (q/6 + 1/9)(3q-2) + 2/9$. Hence we have

$$p^2c_4 \equiv q^2c_4 \equiv \frac{7-3q}{18} - \frac{2/9}{3q-2}.$$

By noting $3q-2 \ge 13$ and $0 < \frac{2/9}{3q-2} < 1/9$, we have

$$\langle p^2c_4 \rangle = \langle q^2c_4 \rangle = \begin{cases} 2/9 - (2/9)/(3q-2) & \text{if} \quad q \equiv 1 \mod 6, \\ 8/9 - (2/9)/(3q-2) & \text{if} \quad q \equiv 3 \mod 6, \\ 5/9 - (2/9)/(3q-2) & \text{if} \quad q \equiv 5 \mod 6. \end{cases}$$

By denoting $\Upsilon := (\sigma^2(c_4) - \sigma^2(c_{13}))(24q^2(2q-1)^2(3q-2)^2(5q-2)^2(4q^2-2q-1))$ when $p = 4q-2$, we have

$$\Upsilon \le \begin{cases} -(3040q^6 - 6064q^5 + 3109q^4 + 1056q^3 - 1504q^2 + 480q - 48) < 0, \\ -(2080q^6 - 4560q^5 + 2581q^4 + 928q^3 - 1440q^2 + 480q - 48) < 0, \\ -(2560q^6 - 5312q^5 + 2845q^4 + 992q^3 - 1472q^2 + 480q - 48) < 0, \end{cases}$$

according as $q = 1$, $q = 3$, or $q = 5 \mod 6$. Thus we have verified that the formula (1) does not hold when $p = 4q-2$. □

## 3 Odd/Even case

We assume $p$, $q > 0$, $2 \nmid p$, $2 \mid q$, and $\gcd(p,q) = 1$. We denote

$$\Gamma = 1 + \frac{\sqrt{6}}{2}, \quad \gamma = \frac{p}{q}, \quad t = p - \Gamma q, \quad r_1 = \frac{t^2}{2(p-q)}, \quad \text{and} \quad \tau = t - r_1.$$

Let us consider the case $q \le 6$. By our previous results together with the previous two lemmas, we see (1) holds if and only if $p/2 \ge 5/2$, $p/4 \ge 9/4$, or $p/6 \ge 13/6$. It coincides the assertion of Theorem 2, and hence we may prove it by assuming $q \ge 8$.

In [12], we proved the following: under the condition $p/q \ge 2$,

1. $\sigma(a) \le \sigma(c_4)$ holds for $a \in [\,0, c_1\,]$, $a \in [\,c_1, c_2\,]$, and $a \in [\,c_2, c_4\,]$.

2. $\sigma(a) \le \sigma(c_4)$ is valid for $a \in [\,c_4, c_3\,]$ if the inequality below is satisfied.

$$A_1(p,q) := 2p^3 - (4q+2)p^2 - (q^2-2q)p + 3q^2 > 0.$$

3. $\sigma(a) \le \sigma(c_4)$ is valid for $a \in [\,c_3, 1/2\,]$ if

$$A_2(p,q) := (8q^2-6)p^3 + (-16q^3+4q+24)p^2 + (-q^4+10q^2-24q-24)p + q^3 > 0.$$

4. We see $A_2(p,q) > \widehat{A}_2(p,q) := (8q^2-6)p^2 + (-16q^3+4q)p - q^4$, and that $\widehat{A}_2(p,q)$ is increasing in $p \in [\,2q, \infty)$.

Hence we only have to prove that $\sigma(c_4) = \max_{a\in[c_4,1/2]} \sigma(a)$ to prove $\Sigma = \sigma(c_4)$.

**Lemma 5** *We have $A_2(p,q) > 0$ for $p \ge \Gamma q - 1$ and $q \ge 8$.*

*Proof* It is enough to prove $\widehat{A}_2(p,q) > 0$. We see

$$\widehat{A}_2(\Gamma q - 1, q) = \big(3q^4 - 8\sqrt{6}\,q^3 - (4\sqrt{6}+3)q^2\big) + \big((6\sqrt{6}+8)q - 6\big).$$

The second part of the right hand side of the above formula is clearly positive. By $q \ge 8$, we see (The first part) $\ge \big(3 - 8\sqrt{6}/8 - (4\sqrt{6}+3)/8^2\big)q^4 > 0$. Hence we have $\widehat{A}_2(\Gamma q - 1, q) > 0$ for $q \ge 8$. We can easily verify $\Gamma q - 1 \ge 2q$ by $q \ge 8$. Hence by monotonicity, we have $\widehat{A}_2(p,q) \ge \widehat{A}_2(\Gamma q - 1, q) > 0$ for $p \ge \Gamma q - 1$ and $q \ge 8$. $\square$

**Lemma 6** *Suppose that $q \ge 8$. We have $A_1(p,q) > 0$ if $p > \Gamma q + \Gamma - 2$.*

*Proof* First we prove $A_1(p,q) > 0$ by assuming $p \ge \Gamma q + \frac{1}{2}$. By $2p - 2q - 1 \ge \sqrt{6}$, taking the square, we have $4p^2 - 8pq - 2q^2 \ge 4p - 4q - 1$. Since the left hand side is divisible by 4, we have $4p^2 - 8pq - 2q^2 \ge 4p - 4q$. By applying this, we have

$$\begin{aligned} A_1(p,q) &= (p-1)(2p^2 - 4pq - q^2 - 2q) + 2q^2 - 2q \\ &\ge (p-1)(2p-4q) + 2q^2 - 2q = 2p^2 - (4q+2)p + 2q^2 + 2q. \end{aligned}$$

Regarding the last formula as a quadratic function of $p$, the axis is located at $\frac{4q+2}{4} = q + \frac{1}{2} < 2q$. Because of $p \ge \Gamma q$, it is increasing in $p$. Hence we have

$$A_1(p,q) \ge 2p^2 - (4q+2)p + 2q^2 + 2q\,\Big|_{p=\Gamma+\frac{1}{2}} = 3q^2 - \frac{1}{2} > 0.$$

Next, we prove $A_1(p,q) > 0$ by assuming $\Gamma q + \Gamma - 2 < p < \Gamma q + \frac{1}{2}$. We have $p - q + 1 > \frac{\sqrt{6}}{2}(q+1)$ and hence $2(p-q+1)^2 > 3(q+1)^2$ or $2p^2 - 4pq - q^2 + 4p - 10q \ge 1$. Since $q$ is even, the left hand side can be divided by 4 and hence we have $2p^2 - 4pq - q^2 + 4p - 10q \ge 4$. Hence, as before, we have

$$A_1(p,q) \ge (p-1)(-4p+8q+4) + 2q^2 - 2q \ge (4\Gamma - 6)q - 1 > 0,$$

where the last formula is given by putting $p = \Gamma q + \frac{1}{2}$. $\square$

By these two lemmas, we see that the formula (1) holds if $p > \Gamma q + \Gamma - 2$.

Next we prove that the formula (1) does not hold if $p < \Gamma q - (\Gamma - 1)/2$. In $[\,c_2, c_3)$, we have $\langle qa\rangle = qa - \frac{q-2}{2}$ , $\langle pa\rangle = pa - \frac{p-3}{2}$, and in $[\,c_4, c_3)$, we have $\langle qa\rangle \le \langle pa\rangle$. Hence we have

$$X_1(a) = -3a^2 + \Big(3 - \frac{2}{q} - \frac{1}{p}\Big)a - \frac{1}{pq} + \frac{1}{q} + \frac{1}{2p} - \frac{1}{2}, \quad (a \in [\,c_4, c_3)), \qquad (10)$$

and it has axis at

$$a_1 = \frac{1}{6}\Big(3 - \frac{2}{q} - \frac{1}{p}\Big) = c_4 + \frac{-2p^2 + 4pq + q^2}{6pq(p-q)} = c_4 - \frac{2\tau}{3pq}. \tag{11}$$

If $a \in [\,c_4, c_3)$, because of $X_1(a) = -3(a - a_1)^2 + X_1(a_1)$, by putting $a = c_4$, we have

$$X_1(c_4) - X_1(a_1) = -3(a_1 - c_4)^2 \leq 0. \tag{12}$$

**Lemma 7** *If $p \leq \Gamma q - (\Gamma - 1)/2$, then the formula* (1) *does not hold.*

*Proof* We may assume $2q \leq p < \Gamma q$. Thereby we have $1 - \sqrt{6}/2 < p/q < 1 + \sqrt{6}/2$ and $2p^2 - 4pq - q^2 < 0$. We can verify $a_1 \in (c_4, c_3)$ by

$$a_1 - c_4 = \frac{-2p^2 + 4pq + q^2}{6pq(p-q)} > 0 \quad \text{and} \quad a_1 - c_3 = \frac{q-p}{3pq} < 0.$$

By (12), $\sigma^2(c_4) \leq X_1(c_4) + \frac{2}{pq(pq-1)} v(c_4)$, and $\sigma^2(a_1) \geq X_1(a_1)$, we have

$$\sigma^2(c_4) - \sigma^2(a_1) \leq \frac{-(pq-1)(-2p^2 + 4pq + q^2)^2 + 6pq((p-q)^2 - 1)}{12p^2q^2(p-q)^2(pq-1)} =: \frac{A}{B}.$$

By assumption on $p$ and $q$, we have $p - q < (q/2 - 1/4)\sqrt{6}$ and hence $(p-q)^2 < 6(q/2 - 1/4)^2$ which implies $8p^2 - 16pq - 4q^2 + 12q < 3$. Because the left hand side is divisible by 4, we have $8p^2 - 16pq - 4q^2 + 12q \leq 0$ or $2p^2 - 4pq - q^2 \leq -3q$. By this we have $(2p^2 - 4pq - q^2)^2 \geq 9q^2$ and

$$\begin{aligned} A &\leq -9q^2(pq-1) + 6pq((p-q)^2 - 1) = 3q(p(2p^2 - 4pq - q^2) + 3q - 2p) \\ &\leq 3q(-3pq + 3q - 2p) = 3q(-3(p-1)q - 2p) < 0. \end{aligned}$$

Hence we have $\sigma^2(c_4) < \sigma^2(a_1)$. □

Therefore we have to investigate $p/q$ satisfying $\Gamma q - (\Gamma - 1)/2 < p < \Gamma q + \Gamma - 2$. By Lemma 5, to prove $\Sigma = \sigma(c_4)$, it is enough to prove $\sigma(c_4) = \max_{c_4 \leq a \leq c_3} \sigma(a)$.

We here prove the assertion of our Theorem for $q \leq 18$. We find odd $p \in (\Gamma q - 1, \Gamma q + 1)$ and calculate $\gcd(p,q)$, $p - q$ mod 3, and $t$.

| $p/q$ | $\gcd(p,q)$ | $p-q$ mod 3 | $t$ | validity of (1) |
|---|---|---|---|---|
| 5/2 | 1 | 0 | 0.5505 | valid due to [12] or Lemma 6 |
| 9/4 | 1 | 2 | 0.1010 | valid due to [12] |
| 13/6 | 1 | 1 | −0.3484 | valid due to [11] |
| 17/8 | 1 | 0 | −0.7979 | invalid due to [11] or Lemma 7 |
| 23/10 | 1 | 1 | 0.7525 | valid due to [12] or Lemma 6 |
| 27/12 | 3 | 0 | 0.3030 | $p$ and $q$ are not relatively prime |
| 31/14 | 1 | 2 | −0.1464 | invalid as below |
| 35/16 | 1 | 1 | −0.5959 | invalid as below |
| 41/18 | 1 | 2 | 0.9545 | valid due to [12] or Lemma 6 |

We use the estimate (9). Evaluating the left hand side for $N = 3$, in case $p/q = 31/14$ and 35/16, we see $D_+\sigma^2(c_4) \geq Z_{3,-}(c_4) = \frac{1458155}{4516494346}$ if $p/q = 31/14$, and $D_+\sigma^2(c_4) \geq Z_{3,-}(c_4) = \frac{3375283}{1001011200}$ if $p/q = 35/16$. Hence we see (1) is invalid in both cases.

Since the table shows that the assersion holds for $q \le 18$, we assume $q \ge 20$ and

$$t \in \Big(-\frac{\Gamma-1}{2}, \Gamma-2\Big) \subset \Big(-\frac{\Gamma-1}{40}q, \frac{\Gamma-2}{20}q\Big).$$

From this, we can derive estimates below. We denote $\rho_{1,-} := 0$.

$$\begin{gathered}
\Gamma_\flat := \frac{39\Gamma+1}{40} < \frac{p}{q} < \frac{21\Gamma-2}{20} =: \Gamma_\sharp, \quad (\Gamma_\flat-1)q < p-q < (\Gamma_\sharp-1)q, \\
p > 20\Gamma_\flat, \quad \frac{\rho_{1,-}}{q} \le r_1 \le \frac{5(\Gamma-1)}{39q} =: \frac{\rho_{1,+}}{q} \le \frac{\Gamma-1}{156}, \quad -\frac{79}{156}(\Gamma-1) < \tau < \Gamma-2, \\
pq-1 = q^2\Big(\frac{p}{q}-\frac{1}{q^2}\Big) \ge \Big(\frac{39}{40}\Gamma+\frac{9}{400}\Big)q^2 \ge 20\Big(\frac{39}{40}\Gamma+\frac{9}{400}\Big)q =: 20\Gamma_\circ q,
\end{gathered} \tag{13}$$

Because of $c_4 = \frac{1}{2} - \frac{1}{2(p-q)}$ and $\frac{q^2}{2} \in \mathbf{N}$, we have $\langle q^2 c_4\rangle = \big\langle -\frac{q^2}{2(p-q)}\big\rangle$. By $q = \frac{2}{\sqrt{6}}\frac{\sqrt{6}\,q}{2} = \frac{2}{\sqrt{6}}(p-q-t)$, we see

$$\langle q^2 c_4\rangle = \Big\langle -\frac{1}{3}\Big((p-q)-2t+\frac{t^2}{p-q}\Big)\Big\rangle.$$

We now prove

$$\langle q^2 c_4\rangle = \frac{1}{3}i + \frac{2}{3}\tau, \tag{14}$$

where

$$i = \begin{cases} 2 & \text{if} \quad p-q=1 \mod 3 \quad \text{and} \quad t > -(\Gamma-1)/2 =: t_{2,-}, \\ 0 & \text{if} \quad p-q=0 \mod 3 \quad \text{and} \quad t > 0 =: t_{0,-}, \\ 3 & \text{if} \quad p-q=0 \mod 3 \quad \text{and} \quad t < t_{0,-}, \\ 1 & \text{if} \quad p-q=2 \mod 3 \quad \text{and} \quad t > -1/2 =: t_{1,-}, \\ 4 & \text{if} \quad p-q=2 \mod 3 \quad \text{and} \quad t < t_{1,-}. \end{cases}$$

By (13), we have $\frac{2}{3}+\frac{2}{3}\tau \in (0,1)$. Hence we see $i=2$ in case $p-q=1 \mod 3$.

In case $p-q=0 \mod 3$, by noting $\frac{2}{3}\tau \in (-1,1)$, it is proved by the fact that $\frac{2\tau}{3} = \frac{2}{3}t(1-\frac{t}{2(p-q)}) \ge 0$ if and only if $t \ge 0$.

We consider the case $p-q=2 \mod 3$. If $t \le -\frac{1}{2}$, then we have $\frac{1}{3}+\frac{2}{3}\tau < \frac{1}{3}+\frac{2}{3}t \le 0$ and $i=4$. Suppose $t > -\frac{1}{2}$. By Lemma 13 and (13), we have $0 < \frac{1}{2}+t = \frac{1}{2}(2p-2q+1-\sqrt{6}\,q)$. Hence we have $(2p-2q+1)^2-6q^2 \ge 1$, and we prove that the equality does not hold. Actually, if it holds, we have $(p-q)(2p-2q+2) = 3q^2$. By $p-q \ge q \ge 20$, we see $\gcd(p,q) = \gcd(p-q,q) > 1$, which contradicts with the assumption. Therefore we can apply Lemma 13 and have

$$\frac{1}{2}+t = \frac{1}{2}(2p-2q+1-\sqrt{6}\,q) \ge \frac{2\sqrt{2}-\sqrt{6}}{2q} > \frac{5(\Gamma-1)}{39q} \ge r_1.$$

It implies $\frac{1}{3}+\frac{2}{3}\tau = \frac{2}{3}(\frac{1}{2}+\tau) > 0$ and $i=1$.

**Lemma 8** *If*

$$c_3 > a > d_0 := c_4 - \frac{2\tau}{3pq} + \frac{1}{3q(q-1)},$$

*then* $D_+\sigma^2(a) < 0$. *We have* $c_{14} > d_0$.

*Proof* On $[c_4, c_3)$, by $X_1(a) = -3(a-a_1)^2 + X_1(a_1)$, we have $D_+X_1(a) = -6(a-a_1)$. Applying (9), we see $D_+\sigma^2(a) \le Z_{1,+}(a) \le -6(a-a_1) + \frac{2}{q(q-1)}$. Hence $D_+\sigma^2(a) < 0$ is derived from $a > a_1 + \frac{1}{3q(q-1)} = d_0$. By (14), we have $c_{14} \ge c_4 + \frac{1}{p^2}\big(4 - \frac{4}{3} - \frac{2\tau}{3}\big)$. By these together with (13) we have

$$p^2(c_{14} - d_0) \ge \frac{8}{3} + \left(\frac{p}{q} - 1\right)\frac{2\tau}{3} - \frac{p^2}{3q^2}\frac{q}{q-1}$$
$$\ge \frac{8}{3} - (\Gamma_\sharp - 1)\frac{2}{3}\frac{79}{156}(\Gamma - 1) - \frac{1}{3}\Gamma_\sharp^2\frac{20}{19} > 0. \quad \square$$

Hence, to prove (1), it is enough to prove $\sigma^2(c_4) \ge \sigma^2(a)$ for $a \in [c_4, c_{14}]$. Put

$$\psi_{j,k}(a) := X_1(a) + \frac{2}{p^2q^2}(p^2(a-c_4) + \langle q^2c_4\rangle - j)(1 - q^2(a-c_4) - \langle q^2c_4\rangle + k),$$
$$\Lambda_{j,k} := \big\{a \in [c_4, c_{14}] \bigm| [p^2a] = [p^2c_4] + j,\ [q^2a] = [q^2c_4] + k\big\}.$$

For $a \in \Lambda_{j,k}$, we see $\langle p^2a\rangle = p^2(a-c_4) + \langle q^2c_4\rangle - j$, $\langle q^2a\rangle = q^2(a-c_4) + \langle q^2c_4\rangle - k$,

$$X_2(a) = \psi_{j,k}(a) \wedge \psi_{j+1,k-1}(a) = \begin{cases} \psi_{j,k}(a) & \text{if } \langle p^2a\rangle \le \langle q^2a\rangle, \\ \psi_{j+1,k-1}(a) & \text{if } \langle p^2a\rangle \ge \langle q^2a\rangle, \end{cases}$$

where the second case is verified by

$$(q^2(a-c_4) + \langle q^2c_4\rangle - k)(1 - p^2(a-c_4) - \langle q^2c_4\rangle + j)$$
$$= (p^2(a-c_4) + \langle q^2c_4\rangle - (j+1))(1 - q^2(a-c_4) - \langle q^2c_4\rangle + (k-1)).$$

Since the axis of the second term of $\psi_{j,k}$ is located at $c_4 + \frac{(k+1)p^2 + jq^2 - (p^2+q^2)\langle q^2c_4\rangle}{2p^2q^2}$, the axis of $\psi_{j,k}$ located at

$$a_{j,k} := c_4 - \frac{2\tau}{5pq} + \frac{(k+1)p^2 + jq^2 - (p^2+q^2)\langle q^2c_4\rangle}{5p^2q^2}.$$

By $\psi_{j,k}(a) = -5(a-a_{j,k})^2 + \psi_{j,k}(a_{j,k})$, we see $\psi_{j,k}(c_4) - \psi_{j,k}(a_{j,k}) = -5(c_4 - a_{j,k})^2$.

Let $a \in \Lambda_{j,k}$. For $(\alpha,\beta) = (j,k)$ and $(j+1, k-1)$, we have

$$\sigma^2(a) \le \psi_{\alpha,\beta}(a) + \frac{1}{2(pq)^2(pq-1)} \le \psi_{\alpha,\beta}(a_{\alpha,\beta}) + \frac{1}{2(pq)^2(pq-1)}. \tag{15}$$

On the other hand we have

$$\psi_{j,k}(c_4) = X_1(c_4) + \frac{2}{p^2q^2}(\langle q^2c_4\rangle - j)(1 - \langle q^2c_4\rangle + k)$$
$$= X_2(c_4) - \frac{2}{p^2q^2}\Big((k+j)(1 - \langle q^2c_4\rangle) + k(j-1)\Big), \tag{16}$$
$$\psi_{j,k}(c_4) \le \sigma^2(c_4) - \frac{2}{p^2q^2}\Big((k+j)\Big(1 - \frac{i}{3} - \frac{2}{3}\tau\Big) + k(j-1)\Big).$$

By applying (14), we have

$$a_{j,k} - c_4 = \frac{1}{5pq}\big(-2\tau + h_{i,j,k}(\gamma,\tau)\big), \tag{17}$$

where

$$h_{i,j,k}(\gamma,\tau) := \frac{1}{\gamma}\Big((k+1)\gamma^2 + j - (\gamma^2+1)\Big(\frac{i}{3}+\frac{2}{3}\tau\Big)\Big).$$

On $[\,c_4, c_8)$, we have $(j,k) = (0,0)$ and $\langle q^2 a\rangle \leq \langle p^2 a\rangle$, and thereby $X_2(a) = \psi_{1,-1}(a) = -5(a-a_{1,-1})^2 + \psi_{1,-1}(a_{1,-1})$ and

$$D_+X_2(a) = -10(a-a_{1,-1}) \leq D_+X_2(c_4) = \frac{2}{pq}\big(-2\tau + h_{i,1,-1}(\gamma,\tau)\big).$$

We consider the case $i \leq 3$. Put

$$\widetilde{Z}_{2,\pm}(a) := D_+X_2(a) \pm \frac{2}{pq}\frac{20}{19}\varGamma_\sharp\frac{1}{q}.$$

Note that $Z_{2,\pm}(a)$ and $\widetilde{Z}_{2,\pm}(a)$ are both decreasing on $[\,c_4, c_8)$. By applying (9) and (13) we have $\widetilde{Z}_{2,-}(a) \leq Z_{2,-}(a) \leq D_+\sigma^2(a) \leq Z_{2,+}(a) \leq \widetilde{Z}_{2,+}(a)$, for $a \in [\,c_4, c_8)$. If $\widetilde{Z}_{2,+}(c_4) < 0$, then $D_+\sigma^2(a) \leq \widetilde{Z}_{2,+}(a) < 0$ for $a \in [\,c_4, c_8)$ and $c_4$ is maximal on $[\,c_4, c_8)$. On the other hand, if $\widetilde{Z}_{2,-}(c_4) > 0$, then $D_+\sigma^2(c_4) \geq \widetilde{Z}_{2,-}(c_4) > 0$ and $c_4$ cannot be locally maximal.

Put $\varXi_{i,q,\pm}$ as below. Note here that $W > 0$.

$$\varXi_{i,q,\pm} := \frac{1-(\gamma^2+1)\frac{i}{3} \pm \gamma\frac{20}{19}\varGamma_\sharp\frac{1}{q}}{2\gamma + (\gamma^2+1)\frac{2}{3}} + r_1 =: \frac{V}{W} + r_1.$$

By $\widetilde{Z}_{2,\pm}(c_4) = \frac{2}{pq\gamma}(V - W\tau)$, one can derive $\widetilde{Z}_{2,\pm}(c_4) \lesseqgtr 0$ from $\varXi_{i,q,\pm} \lesseqgtr t$.

Put $B_\varGamma := 2\varGamma_\flat + \big(\varGamma_\flat^2+1\big)\frac{2}{3}$ and $C_\varGamma := \big(2\varGamma_\sharp^2 + \frac{4}{3}\varGamma_\sharp\big)/B_\varGamma^2$. By (13) we have

$$0 > \frac{d}{d\gamma}\frac{1-(\gamma^2+1)\frac{i}{3}}{2\gamma+(\gamma^2+1)\frac{2}{3}} = -\frac{i(\gamma^2-1)\frac{2}{3} + 2 + \frac{4}{3}\gamma}{(2\gamma+(\gamma^2+1)\frac{2}{3})^2} \geq -C_\varGamma, \quad (i \leq 3).$$

By $\gamma - \varGamma = \frac{t}{q}$, we have

$$\frac{\rho_{2,-}}{q} := -\frac{C_\varGamma(\varGamma-2)}{q} \leq \frac{1-(\gamma^2+1)\frac{i}{3}}{2\gamma+(\gamma^2+1)\frac{2}{3}} - \frac{1-(\varGamma^2+1)\frac{i}{3}}{2\varGamma+(\varGamma^2+1)\frac{2}{3}} \leq \frac{C_\varGamma(\varGamma-1)}{2q} =: \frac{\rho_{2,+}}{q}.$$

Note that $t_i$ $(i \leq 2)$ is the solution of the equation $h_{i,1,-1}(\varGamma, t) = 2t$, which is given by

$$t_i = \frac{1-(\varGamma^2+1)\frac{i}{3}}{2\varGamma+(\varGamma^2+1)\frac{2}{3}} = \frac{39-15\sqrt{6}}{19} - \frac{31-9\sqrt{6}}{38}i.$$

By this we put $t_3 := -\frac{15+3\sqrt{6}}{38} = -0.5881$. We have

$$t_i + \frac{\widetilde{\rho}_{2,-}}{q} \leq \varXi_{i,q,\pm} \leq t_i + \frac{\widetilde{\rho}_{2,+}}{q}, \quad \text{where} \quad \widetilde{\rho}_{2,\pm} := \rho_{1,\pm} + \rho_{2,\pm} \pm \frac{\varGamma_\sharp^2}{B_\varGamma}\frac{20}{19}.$$

Here we see $\widetilde{\rho}_{2,-} = -0.6795\ldots$ and $\widetilde{\rho}_{2,+} = 0.9102\ldots$ It proves the following lemma.

**Lemma 9** *Suppose that $q \geq 20$, $i = 0, 1, 2, 3$, and $p - q = 2i \mod 3$. If*

$$t_{i,-} < p - \Gamma q < t_i + \frac{\widetilde{\rho}_{2,-}}{q},$$

*then $D_+\sigma^2(c_4) > 0$ and the formula (1) does not hold. If*

$$t_i + \frac{\widetilde{\rho}_{2,+}}{q} < p - \Gamma q < \Gamma - 2,$$

*then we have $D_+\sigma^2(a) < 0$ and $\sigma^2(c_4) > \sigma^2(a)$ for $a \in (c_4, c_8)$.*

We investigate the behaviour on $[\,c_8, c_{14}\,]$. We assume $i = 0, \ldots, 4$, $j = 1, \ldots, 4$, and $k = 0, 1, 2$. Note that $\frac{i}{3} + \frac{2}{3}\tau = \langle q^2 c_4 \rangle \in (0, 1)$. We estimate

$$\widetilde{h}_{i,j,k}(\gamma, \tau) := h_{i,j,k}(\gamma, \tau) - h_{i,j,k}(\Gamma, \tau).$$

By $\frac{\partial}{\partial\gamma} h_{i,j,k}(\gamma, \tau) = k + 1 - \frac{j}{\gamma^2} - \left(1 - \frac{1}{\gamma^2}\right)\left(\frac{i}{3} + \frac{2}{3}\tau\right)$, we have $\frac{\partial}{\partial\gamma} h_{i,j,k}(\gamma, \tau) \leq k + 1 - \frac{1}{\gamma^2}$ and $\frac{\partial}{\partial\gamma} h_{i,j,k}(\gamma, \tau) \geq \frac{1}{\gamma^2} - \frac{j}{\gamma^2} \geq \frac{1}{\gamma^2} - 1 - k$. Hence by $\left|\frac{\partial}{\partial\gamma} h_{i,j,k}(\gamma, \tau)\right| \leq k + 1 - \frac{1}{\gamma^2}$ and $|\gamma - \Gamma| = |t|/q \leq (\Gamma - 1)/2q$, we have

$$|\widetilde{h}_{i,j,k}(\gamma, \tau)| \leq \Big(k + 1 - \frac{1}{\Gamma_\sharp^2}\Big) \frac{\Gamma - 1}{2q} =: \frac{\rho_3(k)}{q}.$$

We put

$$\Delta := -2 - \frac{\Gamma^2 + 1}{\Gamma} \frac{2}{3} < 0, \qquad A := -\frac{\Delta^2}{5}, \qquad B_{i,j,k} := -\frac{2\Delta}{5} h_{i,j,k}(\Gamma, 0) - \frac{4}{3}(k + j),$$
$$C_{i,j,k} := -\frac{1}{5} h_{i,j,k}^2(\Gamma, 0) + 2(k + j)\Big(1 - \frac{i}{3}\Big) + 2k(j - 1),$$
$$\Psi_{i,j,k}(s, R) := As^2 + B_{i,j,k}s + C_{i,j,k} + R, \qquad \widetilde{\widetilde{h}}_{i,j,k}(\gamma, \tau) := \frac{1}{\Delta} \widetilde{h}_{i,j,k}(\gamma, \tau).$$

By (17) we see

$$a_{j,k} - c_4 = \frac{1}{5pq}\Big(\Delta\big(\tau + \widetilde{\widetilde{h}}_{i,j,k}(\gamma, \tau)\big) + h_{i,j,k}(\Gamma, 0)\Big).$$

For $a \in \Lambda_{j,k}$, by combining (15), (16) and $\psi_{j,k}(c_4) - \psi_{j,k}(a_{j,k}) = -5(c_4 - a_{j,k})^2$, we have

$$\begin{aligned}
&\sigma^2(c_4) - \sigma^2(a) \\
&\geq -5(c_4 - a_{j,k})^2 + \frac{2}{p^2q^2}\Big((k + j)\big(1 - \frac{i}{3} - \frac{2}{3}\tau\big) + k(j - 1)\Big) - \frac{1}{2(pq)^2(pq - 1)} \\
&= \frac{1}{p^2q^2}\Big(-\frac{1}{5}\big(\Delta\widetilde{t} + h_{i,j,k}(\Gamma, 0)\big)^2 + 2(k + j)\big(1 - \frac{i}{3} - \frac{2}{3}\widetilde{t}\big) + 2k(j - 1) + R^-_{i,j,k,q}\Big) \\
&= \frac{1}{p^2q^2}\Psi_{i,j,k}(\widetilde{t}, R^-_{i,j,k,q}),
\end{aligned}$$

where

$$\widetilde{t} := t + \widetilde{\widetilde{h}}_{i,j,k}(\gamma, \tau) - r_1, \quad \text{and} \quad R^{\pm}_{i,j,k,q} := \frac{4}{3}(k + j)\widetilde{\widetilde{h}}_{i,j,k}(\gamma, \tau) \pm \frac{1}{2(pq - 1)}. \tag{18}$$

For $a \in \Lambda_{j,k}$, we also have

$$\sigma^2(c_4) - \sigma^2(a) \geq \frac{1}{p^2q^2}\Psi_{i,j+1,k-1}(\widetilde{t}, R^-_{i,j+1,k-1,q}).$$

We introduce two functions $s_{i,j,k,\pm}$ and $\xi_{i,j,k}$ as below.

$$s_{i,j,k,\pm}(x) := \frac{1}{2A}\left(-B_{i,j,k} \mp \left(B^2_{i,j,k} - 4Ax\right)^{1/2}\right) \text{ and } \xi_{i,j,k}(x) := \left(B^2_{i,j,k} - 4Ax\right)^{-1/2}.$$

Note that $\pm s'_{i,j,k,\pm}(x) = \xi_{i,j,k}(x)$ is decreasing in $x$. Since $s_{i,j,k,-}(C_{i,j,k} + R) < s_{i,j,k,+}(C_{i,j,k} + R)$ are roots of $\Psi_{i,j,k}(s, R) = 0$, the inequality $\Psi_{i,j,k}(s, R) > 0$ holds if and only if

$$s_{i,j,k,-}(C_{i,j,k} + R) < s < s_{i,j,k,+}(C_{i,j,k} + R).$$

By denoting

$$\widetilde{s}_{i,j,k,\pm}(R, x) := s_{i,j,k,\pm}(C_{i,j,k}) \pm R\xi_{i,j,k}(C_{i,j,k} + x),$$

the mean value theorem claims $s_{i,j,k,\pm}(C_{i,j,k}+R) = \widetilde{s}_{i,j,k,\pm}(R, \vartheta R)$ for some $\vartheta \in (0, 1)$. Note that $R\xi_{i,j,k}(C_{i,j,k}) > R\xi_{i,j,k}(C_{i,j,k} + \vartheta R) > R\xi_{i,j,k}(C_{i,j,k} + R)$ provided that $R \neq 0$, $\xi_{i,j,k}(C_{i,j,k} + \vartheta R) \in \mathbf{R}$ and $\xi_{i,j,k}(C_{i,j,k}) \in \mathbf{R}$. Hence we have

$$\pm\widetilde{s}_{i,j,k,\pm}(R, R) < \pm s_{i,j,k,\pm}(C_{i,j,k} + R) < \pm\widetilde{s}_{i,j,k,\pm}(R, 0)$$

and it shows the following two implications.

$$\widetilde{s}_{i,j,k,-}(R, R) < s < \widetilde{s}_{i,j,k,+}(R, R) \implies \Psi_{i,j,k}(s, R) > 0, \tag{19}$$

$$s < \widetilde{s}_{i,j,k,-}(R, 0) \quad \text{or} \quad \widetilde{s}_{i,j,k,+}(R, 0) < s \implies \Psi_{i,j,k}(s, R) < 0. \tag{20}$$

**Lemma 10** *Let* $q \geq 20$ *and* $i = 0, 1, 2$. *If* $p - q = 2i \mod 3$ *and* $t_i < p - \Gamma q < \Gamma - 2$, *then* $\sigma^2(c_4) > \sigma^2(a)$ *for* $a \in [c_8, c_{14})$.

*Proof* For $a \in [c_8, c_{14})$, we see $a \in \Lambda_{j,k}$ for some $(j, k) = (1, 0)$, $(2, 0)$, $(3, 0)$, $(1, 1)$, $(2, 1)$, $(3, 1)$. We prove $\sigma^2(c_4) - \sigma^2(a) > 0$ by showing $\Psi_{i,j,k}(\widetilde{t}, R^-_{i,j,k,q}) > 0$ for $i \leq 2$ and $(j, k) = (1, 0)$, $(2, 0)$, $(3, 0)$, $(4, 0)$. By putting

$$\widetilde{R}^{\pm}_{i,j,k,q} := \left(\frac{4}{3}(k + j)\frac{\mp\rho_3(k)}{\Delta} \pm \frac{1}{40\Gamma_\circ}\right)\frac{1}{q},$$

we see $R^-_{i,j,k,q} \geq \widetilde{R}_{0,4,0,20} =: R^*$ and $\Psi_{i,j,k}(\widetilde{t}, R^-_{i,j,k,q}) \geq \Psi_{i,j,k}(\widetilde{t}, R^*)$. Hence we are to prove $\Psi_{i,j,k}(\widetilde{t}, R^*) > 0$ by proving the hypothesis in the implication (19) for $s = \widetilde{t}$ and $R = R^*$. By noting $-\frac{-\rho_3(k)/\Delta}{q} \leq t - \widetilde{t} \leq \frac{-\rho_3(k)/\Delta + \rho_{1,+}}{q}$, we see that it is sufficient to prove

$$\widetilde{s}_{i,j,k,-}(R^*, R^*) + \frac{-\rho_3(k)/\Delta + \rho_{1,+}}{q} < t < \widetilde{s}_{i,j,k,+}(R^*, R^*) - \frac{-\rho_3(k)/\Delta}{q} \tag{21}$$

for $i \leq 2$, $k = 0$, $1 \leq j \leq 4$ and $q = 20$. The intervals determined by this inequalities are, $(-0.1530, 1.1218)$, $(-0.5384, 1.2790)$, $(-0.8589, 1.3713)$ and $(-1.1461, 1.4302)$ in case $i = 0$, all of which include $(t_0, \Gamma - 2)$. In case $i = 1$, we have $(-0.2841, 0.7816)$, $(-0.6273, 0.8966)$, $(-0.9151, 0.9561)$ and $(-1.1745, 0.9873)$, all of which include $(t_1, \Gamma - 2)$. In case $i = 2$, we have $(-0.3890, 0.4153)$, $(-0.6804, 0.4783)$, $(-0.9276, 0.4974)$ and $(-1.1524, 0.4939)$, all of which include $(t_2, \Gamma - 2)$. $\square$

If $X_2(a_{j,k}) = \psi_{j,k}(a_{j,k})$, we have $\sigma^2(a_{j,k}) \geq \psi_{j,k}(a_{j,k})$. By $\sigma^2(c_4) \leq X_2(c_4) + \frac{1}{2p^2q^2(pq-1)}$ and (16) we have

$$\sigma^2(c_4) - \sigma^2(a_{j,k}) \leq \frac{1}{p^2q^2}\Psi_{i,j,k}(\widetilde{t}, R^+_{i,j,k,q}) \quad \text{if} \quad X_2(a_{j,k}) = \psi_{j,k}(a_{j,k}). \tag{22}$$

By $R^+_{i,j,k,q} \leq \widetilde{R}^+_{i,j,k,q}$ and $\Psi_{i,j,k}(\widetilde{t}, R^+_{i,j,k,q}) \leq \Psi_{i,j,k}(\widetilde{t}, \widetilde{R}^+_{i,j,k,q})$, we have

$$\sigma^2(c_4) - \sigma^2(a_{j,k}) \leq \frac{1}{p^2q^2}\Psi_{i,j,k}(\widetilde{t}, \widetilde{R}^+_{i,j,k,q}) \quad \text{if} \quad X_2(a_{j,k}) = \psi_{j,k}(a_{j,k}). \tag{23}$$

**Lemma 11** *If* $2 \nmid p$, $2 \mid q$, $q \geq 20$, $\gcd(p,q) = 1$, $p - q = 2 \mod 3$ *and* $t_{2,-} < p - \Gamma q < t_{1,-}$, *then the formula* (1) *dose not hold.*

*Proof* We have $i = 4$, $c_8 < c_9$ and we prove $c_9 < a_{2,0} < c_{10}$. By (17), we have $5pq(a_{2,0} - c_9) = -2\tau + \frac{1}{\gamma} + \left(-4\gamma + \frac{1}{\gamma}\right)\left(1 - \frac{4}{3} - \frac{2}{3}\tau\right)$. By differentiation, we see that the right hand side is increasing in $\tau$ and decreasing in $\gamma$. Hence it is bounded from below by $\frac{129\sqrt{6}-158}{156} > 0$ which is the value at $\tau = -\frac{79}{156}(\Gamma - 1)$ and $\gamma = \Gamma$. Hence we have $c_9 < a_{2,0}$. In the same way, we have $5pq(a_{2,0} - c_{10}) = -2\tau + \gamma - \frac{8}{\gamma} - \left(\gamma - \frac{4}{\gamma}\right)\left(\frac{4}{3} + \frac{2}{3}\tau\right)$, and we see that the right hand side is decreasing in $\tau$ and increasing in $\gamma$. Hence it is bounded from above by $-\frac{378\sqrt{6}-681}{468} < 0$ which is the value at $\gamma = \Gamma$ and $\tau = -\frac{79}{156}(\Gamma - 1)$. Hence we have $a_{2,0} < c_{10}$.

For $a \in (c_9, c_{10})$, we have $a \in \Lambda_{1,1}$. Here, $\langle q^2 a\rangle < \langle p^2 a\rangle$ is clear by $\langle q^2 c_9\rangle = 0 < \langle p^2 c_9\rangle$ and $\langle q^2 c_{10}\rangle < 1 = \langle p^2(c_{10} - 0)\rangle$. Therefore $X_2(a) = \psi_{2,0}(a)$. When $q \geq 20$, we see that $\Psi_{4,2,0}(s, \widetilde{R}^+_{4,2,0,q}) \leq \Psi_{4,2,0}(s, \widetilde{R}^+_{4,2,0,20}) < 0$ for all $s$. Actually, it is verified by the evaluation of the discriminant $B^2_{4,2,0} - 4A(C^2_{4,2,0} + \widetilde{R}^+_{4,2,0,20}) = -4.3821 < 0$. Because of $a_{2,0} \in (c_9, c_{10})$, by (23) we have $\sigma^2(c_4) < \sigma^2(a_{2,0})$ and conclude that (1) dose not hold. □

**Lemma 12** *If* $q \geq 20$ *and* $p - q = 0 \mod 3$, *there exists a constant* $C > 0$ *such that*

1. *if* $t_{3,-} < p - \Gamma q < t_{3,+} - \frac{C}{q}$ *and* $(p,q) \notin E$, *then* $\sigma^2(c_4) > \sigma^2(a)$ *for* $a \in (c_4, c_{14})$;
2. *if* $t_{2,-} < p - \Gamma q < t_{3,-}$, $t_{3,+} < p - \Gamma q < t_{0,-}$, *or* $(p,q) \in E$ *then the formula* (1) *does not hold.*

*Proof* We have $i = 3$. By Lemma 9 we see $\sigma^2(c_4) > \sigma^2(a)$ for $a \in (c_4, c_8)$ if $t > t_3 + \frac{\widetilde{\rho}_{2,+}}{20}$ and hence if $t > t_{3,-}$, since $t_3 + \frac{\widetilde{\rho}_{2,+}}{20} = -0.5504 < t_{3,-}$.

By $-\frac{\Gamma-1}{2} < t < 0$, we see $0 \leq r_1 = (-t)\frac{-t}{2(p-q)} \leq (-t)\frac{-t}{2(\Gamma_\flat-1)q} \leq \frac{\Gamma-1}{2}\frac{-t}{2(\Gamma_\flat-1)20} = \frac{-t}{78} < \frac{\Gamma-1}{156}$, $0 > t \geq \tau = t - r_1 \geq t - \frac{-t}{78} = \frac{79}{78}t$, and $t^2 \leq \tau t \leq \frac{79}{78}t^2$.

Note that $c_8 < c_9 < c_{12}$ when $q \geq 20$. Actually we already proved $c_8 < c_9$, and we here prove $c_9 < c_{12}$. It is equivalent to $\langle q^2 c_4\rangle > \frac{p^2-3q^2}{p^2-q^2}$. By (13), we have

$$\langle q^2 c_4\rangle - \frac{p^2 - 3q^2}{p^2 - q^2} = \frac{2}{3}\tau + \frac{2}{\gamma^2 - 1} \geq -\frac{2}{3}\frac{79}{156}(\Gamma - 1) + \frac{2}{\Gamma^2 - 1} > 0.$$

Hence for $a \in [\,c_8, c_{14})$, $a \in \Lambda_{j,k}$ can occur only for $(1,0)$, $(2,0)$, $(1,1)$, $(2,1)$, and $(3,1)$.

For $a \in \Lambda_{3,1}$, the interval given by (21) with $i = 3$, $(j,k) = (3,1)$, $q = 20$, $R^* = \widetilde{R}^-_{3,3,1,20}$ equals to $(-0.9198, 0.7448)$ which includes $(t_{3,-}, t_{3,+})$. Hence we see that $\sigma^2(c_4) > \sigma^2(a)$ holds if $a \in \Lambda_{3,1}$, $q \geq 20$, and $t_{3,-} < t < t_{3,+}$.

For $a \in \Lambda_{2,1}$, the interval given by (21) with $i = 3$, $(j,k) = (3,0)$, $q = 20$, $R^* = \widetilde{R}^-_{3,3,0,20}$ equals to $(-0.8300, -0.07153)$. And the interval given by (21) with $i = 3$,

$(j,k) = (2,1)$, $q = 20$, $R^* = \widetilde{R}^-_{3,2,1,20}$ equals $(-0.3913, 0.4445)$. Since the union of these two intervals includes $(t_{3,-}, t_{3,+})$, we see that $\sigma^2(c_4) > \sigma^2(a)$ holds if $a \in \Lambda_{2,1}$, $q \geq 20$, and $t_{3,-} < t < t_{3,+}$.

For $a \in \Lambda_{1,1} \cup \Lambda_{2,0}$, the interval given by (21) with $i = 3$, $(j,k) = (2,0)$, $q = 20$, $R^* = \widetilde{R}^-_{3,2,0,20}$ equals to $(-0.6489, -0.02434)$ which includes $t_{3,-}$ inside. By denoting the right end $\widetilde{s}_{3,2,0,-}(\widetilde{R}^-_{3,2,0,q}, \widetilde{R}^-_{3,2,0,q}) - \frac{-\rho_3(0)/\Delta}{q}$ by $t_{3,+,q}$, we see that $\sigma^2(c_4) > \sigma^2(a)$ holds if $a \in \Lambda_{1,1} \cup \Lambda_{2,0}$, $q \geq 20$, and $t_{3,-} < t < t_{3,+,q}$. Note that $t_{3,+,q} = t_{3,+} - \frac{0.3264}{q}$, which is proved by $s_{3,2,0,+}(C_{3,2,0}) = t_{3,+}$ and $0 > \widetilde{R}^-_{3,2,0,q}\xi_{3,2,0}(C_{3,2,0} + \widetilde{R}^-_{3,2,0,q}) - \frac{-\rho_3(0)/\Delta}{q} > \widetilde{R}^-_{3,2,0,q}\xi_{3,2,0}(C_{3,2,0} + \widetilde{R}^-_{3,2,0,20}) - \frac{-\rho_3(0)/\Delta}{q} = -\frac{0.3264}{q}$.

For $a \in \Lambda_{1,0}$, the interval given by (21) with $i = 3$, $(j,k) = (1,0)$, $q = 20$, $R^* = \widetilde{R}^-_{3,1,0,20}$ equals to $(-0.4444, -6.4189 \times 10^{-4})$ which includes $(-2/5, t_{3,+})$. Hence we see that $\sigma^2(c_4) > \sigma^2(a)$ holds if $a \in \Lambda_{2,1}$, $q \geq 20$, and $-2/5 \leq t \leq t_{3,+}$. By $\frac{\partial}{\partial\gamma} h_{3,1,0}(\gamma,\tau) = -\left(1 - \frac{1}{\gamma^2}\right)\frac{2}{3}\tau > 0$, we see $\widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) = \frac{1}{\Delta}\frac{\partial}{\partial\gamma} h_{3,1,0}(\widetilde{\gamma},\tau)(\gamma - \Gamma) = \frac{1}{-\Delta}\frac{2}{3}\left(1 - \frac{1}{\widetilde{\gamma}^2}\right)\frac{t\tau}{q} > 0$ for some $\widetilde{\gamma} \in (\gamma, \Gamma) \subset (\Gamma_\flat, \Gamma)$. Hence

$$\frac{1}{-\Delta}\frac{2}{3}\left(1 - \frac{1}{\Gamma_\flat^2}\right)\frac{t^2}{q} \leq \widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) \leq \frac{1}{-\Delta}\frac{2}{3}\left(1 - \frac{1}{\Gamma^2}\right)\frac{79}{78}\frac{t^2}{q}.$$

By assuming $t < -2/5$, we can verify the estimates below

$$R^-_{3,1,0,q} \geq \left(\frac{4}{3}\frac{1}{-\Delta}\frac{2}{3}\left(1 - \frac{1}{\Gamma_\flat^2}\right)\left(-\frac{2}{5}\right)^2 - \frac{1}{40\Gamma_\circ}\right)\frac{1}{q} = \frac{0.0183}{q} > 0,$$

$$R^-_{3,1,0,q} \leq \frac{4}{3}\widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) \leq \frac{4}{3}\frac{1}{-\Delta}\frac{2}{3}\left(1 - \frac{1}{\Gamma^2}\right)\frac{79}{78}\left(-\frac{\Gamma - 1}{2}\right)^2\frac{1}{20} =: R_\# = 0.003560.$$

Put $H_0 := -R^-_{3,1,0,q}\xi_{3,1,0}(C_{3,1,0} + R^-_{3,1,0,q}) - \widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) + r_1$ and note $s_{3,1,0,-}(C_{3,1,0}) = t_{3,-}$, $s_{3,1,0,+}(C_{3,1,0}) = 0$, and $-\frac{2}{5} + \frac{3}{4}R_\# < 0 < \widetilde{s}_{3,1,0,+}(R^-_{3,1,0,q}, R^-_{3,1,0,q})$. If $t_{3,-} + H_0 < t < -\frac{2}{5}$, then we have $\widetilde{s}_{3,1,0,-}(R^-_{3,1,0,q}, R^-_{3,1,0,q}) < \widetilde{t} < \widetilde{s}_{3,1,0,+}(R^-_{3,1,0,q}, R^-_{3,1,0,q})$, which implies $\Psi_{3,1,0}(\widetilde{t}, R^-_{3,1,0,q}) > 0$ and $\sigma^2(c_4) > \sigma^2(a)$. By $R^-_{3,1,0,q} > 0$, we see

$$\begin{aligned}
H_0 &\leq -\left(\frac{4}{3}\widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) - \frac{1}{2(pq-1)}\right)\xi_{3,1,0}(C_{3,1,0} + R_\#) - \widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) + r_1 \\
&= \left(-\frac{4}{3}\xi(C_{3,1,0} + R_\#) - 1\right)\widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) + r_1 + \frac{1}{2(pq-1)}\xi_{3,1,0}(C_{3,1,0} + R_\#) \\
&\leq \left(\left(-\frac{4}{3}\xi_{3,1,0}(C_{3,1,0} + R_\#) - 1\right)\frac{1}{-\Delta}\frac{2}{3}\left(1 - \frac{1}{\Gamma_\flat^2}\right) + \frac{1}{2(\Gamma_\flat - 1)}\right)\frac{t^2}{q} \\
&\quad + \frac{1}{40\Gamma_\circ}\xi_{3,1,0}(C_{3,1,0} + R_\#)\frac{1}{q} < \frac{0.03906}{q},
\end{aligned}$$

where the last bound is given by $t = t_{3,-}$ since the coefficient of $t^2$ equals to $0.1410 > 0$. By (34) of Lemma 15, there is no $(p,q) \notin E$ satisfying $t_{3,-} < p - q\Gamma < t_{3,-} + H_0$. It completes the proof of the first part of Lemma.

We prove the second part. First we prove $c_8 < c_{13} < c_{10} < c_9$ when $t_{2,-} \leq t < t_{3,-} + \frac{\Gamma - 1}{156} =: \widehat{t}_{3,-}$, and $c_8 < c_9 < c_{10}$ when $t_{3,+} < t < 0$. We assume $t_{2,-} \leq t < \widehat{t}_{3,-}$. Since $c_8 < c_{13}$ is trivial, we prove $c_{13} < c_{10}$. By noting $\tau < t < \widehat{t}_{3,-}$, we see

$$(p^2 - q^2)(c_{10} - c_{13}) = \left(1 - \frac{1}{\gamma^2}\right)\left(1 - \frac{2}{3}\tau\right) - 1 \geq \left(1 - \frac{1}{\Gamma_\flat^2}\right)\left(1 - \frac{2}{3}\widehat{t}_{3,-}\right) - 1 > 0.$$

We show that $c_{10} < c_9$ if $t \le \widehat{t}_{3,-}$, and that $c_9 < c_{10}$ if $t_{3,+} < t < 0$. Note that $c_{10} < c_9$ is equivalent to $\frac{p^2-2q^2}{p^2-q^2} > \langle q^2 c_4\rangle$. By $t \le \widehat{t}_{3,-}$, we have $\frac{p^2-2q^2}{p^2-q^2} - \langle q^2 c_4\rangle = -\frac{1}{\gamma^2-1} - \frac{2}{3}\tau \ge -\frac{1}{\Gamma_\flat^2-1} - \frac{2}{3}\widehat{t}_{3,-} > 0$, on the other hand, by $t_{3,+} < t < 0$, $\gamma < \Gamma$ and $\tau \ge \frac{79}{78}t_{3,+}$, we have $-\frac{1}{\gamma^2-1} - \frac{2}{3}\tau < -\frac{1}{\Gamma^2-1} - \frac{2}{3}\frac{79}{78}t_{3,+} < 0$.

Here we assume $t_{2,-} < t < \widehat{t}_{3,-}$ and $a \in [\,c_8, c_{13}\,]$. We see $a \in \Lambda_{1,0}$ and we can verify $\langle p^2 a\rangle < \langle q^2 a\rangle$ by $\langle p^2 c_8\rangle = 0$ and $\langle p^2 c_{13}\rangle = \langle q^2 c_{13}\rangle$. Therefore $X_2(a) = \psi_{1,0}(a)$. We verify $a_{1,0} \in [\,c_8, c_{13}\,]$. Let $l = 8$ or 10. We see

$$5pq(a_{j,0} - c_l) = \Delta\Big(\tau + \widetilde{\widetilde{h}}_{3,j,0}(\gamma,\tau)\Big) + h_{3,j,0}(\Gamma,0) - \frac{5}{\gamma}\Big(\frac{l-8}{2} - \frac{2}{3}\tau\Big). \qquad (24)$$

Note $\frac{l-8}{2} - \frac{2}{3}\tau > 0$ and $\Gamma_\flat < \gamma < \Gamma$. By $\Delta + \frac{10}{3\Gamma_\flat} < 0$, $h_{3,1,0}(\Gamma,0) = 0$, and $\tau \le t \le \widehat{t}_{3,-}$, we see $5pq(a_{1,0} - c_{10}) \ge \big(\Delta + \frac{10}{3\Gamma_\flat}\big)\tau + \widetilde{h}_{3,1,0}(\gamma,\tau) + h_{3,1,0}(\Gamma,0) \ge \big(\Delta + \frac{10}{3\Gamma_\flat}\big)\widehat{t}_{3,-} - \frac{\rho_3(0)}{20} > 0$ and $5pq(a_{1,0} - c_{13}) = \Delta\tau + \widetilde{h}_{3,1,0}(\gamma,\tau) + h_{3,1,0}(\Gamma,0) - \frac{5\gamma}{\gamma^2-1} \le \Delta\big(-\frac{79}{156}(\Gamma-1)\big) + \frac{\rho_3(0)}{20} - \frac{5\Gamma}{\Gamma^2-1} < 0$, since $-\frac{5\gamma}{\gamma^2-1}$ is increasing. Hence $X_2(a_{1,0}) = \psi_{1,0}(a_{1,0})$.

Put $H_1 := -R^+_{3,1,0,q}\xi_{3,1,0}(C_{3,1,0}) - \widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) + r_1$ and note $s_{3,1,0,-}(C_{3,1,0}) = t_{3,-}$. We have $H_1 = -2\widetilde{\widetilde{h}}_{3,1,0}(\gamma,\tau) + r_1 - \frac{3}{4}\frac{1}{2(pq-1)} \le \frac{\Gamma-1}{156}$. If $t < t_{3,-} + H_1 \le \widehat{t}_{3,-}$, then we have $\widetilde{t} < \widetilde{s}_{3,1,0,-}(R^+_{3,1,0,q}, 0)$, $\Psi_{3,1,0}(\widetilde{t}, R^+_{3,1,0,q}) < 0$ and $\sigma^2(c_4) < \sigma^2(a_{1,0})$. We see

$$H_1 \ge \left(-2\frac{1}{-\Delta}\frac{2}{3}\Big(1 - \frac{1}{\Gamma^2}\Big)\frac{79}{78} + \frac{1}{2(\Gamma-1)}\right)\frac{t^2}{q} - \frac{3}{8}\frac{1}{20\Gamma_\circ}\frac{1}{q} \ge \frac{0.01733}{q} > 0,$$

where the last evaluation is given by $t = \widehat{t}_{3,-}$. We see $t < t_{3,-} + \frac{0.01733}{q}$ implies $\sigma^2(c_4) < \sigma^2(a_{1,0})$. By (35) of Lemma 15, we see every $(p,q) \in E$ is located in this region and we have the conclusion.

We consider on $(t_{3,+}, 0)$. By $c_8 < c_9 < c_{10} < c_{14} < c_{11}$, we see $[\,c_9, c_{10}) \subset \Lambda_{1,1}$. By $\langle q^2 c_9\rangle = 0 < \langle p^2 c_9\rangle$ and $\langle q^2 c_{10}\rangle < 1 = \langle p^2(c_{10} - 0)\rangle$, we see $\langle q^2 a\rangle < \langle p^2 a\rangle$ and $X_2(a) = \psi_{2,0}(a)$ for $a \in [\,c_9, c_{10})$. We prove $a_{2,0} \in (c_9, c_{10})$ to have $X_2(a_{2,0}) = \psi_{2,0}(a_{2,0})$. We see

$$\begin{aligned} 5pq(a_{2,0} - c_9) &= \Delta\Big(\tau + \widetilde{\widetilde{h}}_{3,2,0}(\gamma,\tau)\Big) + h_{3,2,0}(\Gamma,0) + \frac{10}{3}\gamma\tau \\ &\ge \Big(\Delta + \frac{10}{3}\Gamma\Big)\Big(t_{3,+} - \frac{\rho_{1,+}}{20}\Big) - \frac{\rho_3(0)}{20} + h_{3,2,0}(\Gamma,0) > 0 \end{aligned}$$

since $\gamma < \Gamma$, $\Delta + \frac{10}{3}\Gamma > 0$, and $0 > \tau > t_{3,+} - \frac{\rho_{1,+}}{20}$. By $t \ge t_{3,+}$, (24), $\Delta + \frac{10}{3\Gamma_\flat} < 0$, and $\gamma > \Gamma_\flat$, we have

$$5pq(a_{2,0} - c_{10}) \le \Big(\Delta + \frac{10}{3\Gamma_\flat}\Big)\Big(t_{3,+} - \frac{\rho_{1,+}}{20}\Big) + \frac{\rho_3(0)}{20} + h_{3,2,0}(\Gamma,0) - \frac{5}{\Gamma_\flat} < 0.$$

Recall that $s_{3,2,0,+}(C_{3,2,0}) = t_{3,+}$. Let $t_{3,+} \le t < 0$ and $H_2 := R^+_{3,2,0,q}\xi_{3,2,0}(C_{3,2,0}) - \widetilde{\widetilde{h}}_{3,2,0}(\gamma,\tau) + r_1$. If we prove $t_{3,+} + H_2 < t$, then we have $\widetilde{s}_{3,2,0,+}(R^+_{3,2,0,q}, 0) < \widetilde{t}$, $\Psi_{3,2,0}(\widetilde{t}, \widetilde{R}^+_{3,2,0,q}) < 0$, and $\sigma^2(c_4) < \sigma^2(a_{2,0})$ in turn.

We divide into two cases, i.e., the case $\frac{t_{3,+}}{2} \le t < 0$ and the case $t_{3,+} \le t \le \frac{t_{3,+}}{2}$. If we prove $t_{3,+} + H_2 < \frac{t_{3,+}}{2}$, then we have $t_{3,+} + H_2 < t$ for all $t$ with $\frac{t_{3,+}}{2} \le t < 0$. We have

$$\frac{t_{3,+}}{2} + H_2 \le \frac{t_{3,+}}{2} + \Big\{ \widetilde{R}^{+}_{3,2,0,q} \xi_{3,2,0}(C_{3,2,0}) + \Big( \frac{-\rho_3(0)}{\Delta} + \rho_{1,+} \Big) \frac{1}{q} \Big\}.$$

Since the second term of the right hand side is decreasing in $q$, and is proved to be negative when $q = 48$, we see that $t_{3,+} + H_2 < \frac{t_{3,+}}{2}$ holds for $q \ge 48$. We also see that there is no even $20 < q < 48$ such that $t \in (\frac{t_{3,+}}{2}, 0)$. Therefore we see $\sigma^2(c_4) < \sigma^2(a_{2,0})$ holds if $q \ge 20$ and $t \in (\frac{t_{3,+}}{2}, 0)$.

In the other case, we see $\frac{\partial}{\partial\gamma} h_{3,2,0}(\gamma,\tau) = \big(\frac{2}{3}\tau - 1\big)\frac{1}{\gamma^2} - \frac{2}{3}\tau \le \big(\frac{2}{3}\tau - 1\big)\frac{1}{\Gamma^2} - \frac{2}{3}\tau$ since it is increasing in $\gamma$. By $\tau \ge \frac{79}{78}t$ and $\widetilde{h}_{3,2,0}(\gamma,\tau) = \frac{\partial}{\partial\gamma} h_{3,2,0}(\widetilde{\gamma},\tau)(\gamma - \Gamma)$ $(\widetilde{\gamma} \in (\gamma, \Gamma))$, we have $\widetilde{\widetilde{h}}_{3,2,0}(\gamma,\tau) \le \frac{1}{-\Delta}\big(\frac{1}{\Gamma^2} + \big(1 - \frac{1}{\Gamma^2}\big)\frac{2}{3}\frac{79}{78}t\big)\frac{t}{q}$. We see $H_2$ is less than

$$\frac{1}{-\Delta}\Big(\frac{1}{\Gamma^2} + \Big(1 - \frac{1}{\Gamma^2}\Big)\frac{2}{3}\frac{79}{78}t\Big)\Big(\frac{8}{3}\xi_{3,2,0}(C_{3,2,0}) - 1\Big)\frac{t}{q} + \frac{\xi_{3,2,0}(C_{3,2,0})}{2\Gamma_{\circ} q^2} + \frac{t^2}{2(\Gamma_{\flat} - 1)q}.$$

It is negative for $t = t_{3,+}$, $t = \frac{t_{3,+}}{2}$, and $q = 700$. Since it is a quadratic function of $t$ with positive leading coefficient, it is negative for all $t \in [\,t_{3,+}, \frac{t_{3,+}}{2}\,]$ and $q \ge 700$. For such $t$ and $q$, we see $H_2 < 0$, and $t_{3,+} + H_2 < t$ is valid.

For $20 < q < 700$, only $p/q = 703/316$ satisfies $2 \nmid p$, $\gcd(p,q) = 1$, $p - q = 0 \mod 3$, and $t_{3,+} < p - \Gamma q = t < \frac{t_{3,+}}{2}$. As to this, we calculate directly $\sigma^2(c_4) - \sigma^2(a_{2,0}) \le X_2(c_4) - X_2(a_{2,0}) + \frac{1}{2p^2q^2(pq-1)} = -6.047 \times 10^{-14} < 0$. □

## 4 Liouville type results for Diophantine approximation

In this section, we use the technique originated to Liouville [15].

**Lemma 13** *For positive integers $u$ and $v$ with $u^2 - 6v^2 > 1$, we have*

$$\frac{u}{v} - \sqrt{6} \ge \frac{2\sqrt{2} - \sqrt{6}}{v^2}.$$

*Proof* We assume $\frac{u}{v} - \sqrt{6} < 2\sqrt{2} - \sqrt{6}$. Then we have $\frac{u}{v} + \sqrt{6} < 2\sqrt{2} + \sqrt{6}$ and

$$\frac{u}{v} - \sqrt{6} = \frac{u^2 - 6v^2}{v^2(\frac{u}{v} + \sqrt{6})} > \frac{2}{(2\sqrt{2} + \sqrt{6})v^2} = \frac{2\sqrt{2} - \sqrt{6}}{v^2}. \quad \square$$

**Lemma 14** *Suppose that positive integers $p$ and $q$ satisfy $2 \nmid p$, $2 \mid q$, $\gcd(p,q) = 1$, $q \ge 20$, $i \in \{0,1,2\}$, and $p - q = 2i \mod 3$. If*

$$t_i + \frac{\widetilde{\rho}_{2,-}}{q} \le p - \Gamma q < t_i,$$

*then the formula (1) does not hold. If*

$$t_i < p - \Gamma q \le t_i + \frac{\widetilde{\rho}_{2,+}}{q},$$

*then the formula (1) holds.*

*Proof* We have the expression below:

$$t - t_i = \big((38p - 38q - u_i) - \sqrt{6}\,(19q - v_i)\big)/38, \quad \text{where} \quad u_i - \sqrt{6}\,v_i = 38t_i. \qquad (25)$$

By $p \le \Gamma q + \Gamma - 2$, we have

$$\frac{38p - 38q - u_i}{19q - v_i} \le 2\Gamma - 2 + \frac{(2\Gamma - 2)v_i + 38\Gamma - u_i - 76}{19 \cdot 20 - v_i} \le \sqrt{6} + \frac{62\Gamma - 116}{368},$$

where the left hand side is the value of the middle part with $i = 2$. By noting

$$t - t_i = \frac{(38p - 38q - u_i)^2 - 6(19q - v_i)^2}{38(19q - v_i)(\frac{38p-38q-u_i}{19q-v_i} + \sqrt{6})}$$

and by denoting the numerator by $\nu$, we see

$$t - t_i > \frac{\nu}{38 \cdot 19(2\sqrt{6} + \frac{62\Gamma-116}{368})}\frac{1}{q} \quad \text{or} \quad t - t_i < \frac{\nu}{38 \cdot 19 \cdot 2\sqrt{6}}\frac{1}{q}$$

according as $\nu > 0$ or $\nu < 0$, since $\frac{38p-38q-u_i}{19q-v_i} < \sqrt{6}$ in the latter case. Therefore it is sufficient to consider the cases

$$0 < \nu \le 38 \cdot 19\Big(2\sqrt{6} + \frac{62\Gamma - 116}{368}\Big)\widetilde{\rho}_{2,+} \quad \text{and} \quad 0 > \nu \ge 38 \cdot 19 \cdot 2\sqrt{6}\,\widetilde{\rho}_{2,-}.$$

Hence we consider the case $-2403 \le \nu \le 3258$.

For $0 < \varepsilon < 1$, we have $(6-\varepsilon)^{1/2} = \sqrt{6} - \frac{\varepsilon}{2\sqrt{6}} - \frac{\varepsilon^2}{8(6-\vartheta\varepsilon)^{3/2}} \ge \sqrt{6} - \frac{\varepsilon}{2\sqrt{6}} - \frac{\varepsilon^2}{40\sqrt{5}}$ and $(6+\varepsilon)^{1/2} \le \sqrt{6} + \frac{\varepsilon}{2\sqrt{6}}$. By $19q - v_i \ge 19 \cdot 20 - 30$ and $\left|\frac{\nu}{(19q-v_i)^2}\right| < 1$, we have

$$\frac{38p - 38q - u_i}{19q - v_i} = \Big(6 + \frac{\nu}{(19q - v_i)^2}\Big)^{1/2} \ge \sqrt{6} + \frac{\nu}{2\sqrt{6}\,(19q - v_i)^2} - \frac{\nu^2 \cdot}{40\sqrt{5}\,(19q - v_i)^4}$$

if $\nu < 0$, and

$$\frac{38p - 38q - u_i}{19q - v_i} \le \sqrt{6} + \frac{\nu}{2\sqrt{6}\,(19q - v_i)^2}$$

if $\nu > 0$. By $19q - v_i > 19q - 30 > 19q - 30\frac{q}{20} = \frac{35}{2}q$, we have

$$0 < \frac{1}{19q - v_i} - \frac{1}{19q} \le \frac{1}{19q - 30} - \frac{1}{19q} = \frac{30}{(19q - 30)19q} \le \frac{12}{7 \cdot 19q^2},$$

and hence

$$\begin{aligned} t - t_i &= \frac{19q - v_i}{38}\Big(\frac{38p - 38q - u_i}{19q - v_i} - \sqrt{6}\Big) \\ &\ge \frac{\nu}{38 \cdot 2\sqrt{6}\,(19q - v_i)} - \frac{\nu^2}{38 \cdot 40\sqrt{5}\,(19q - v_i)^3} \\ &\ge \frac{\nu}{38 \cdot 2\sqrt{6} \cdot 19q} + \frac{12\nu}{38 \cdot 2\sqrt{6} \cdot 7 \cdot 19q^2} - \frac{\nu^2}{38 \cdot 40\sqrt{5}\,(35/2)^3 q^3} > \frac{\nu}{4 \cdot 19^2\sqrt{6}\,q} - \frac{2}{q^2} \end{aligned}$$

if $\nu < 0$, and

$$t - t_i \le \frac{\nu}{38 \cdot 2\sqrt{6}\,(19q - v_i)} \le \frac{\nu}{38 \cdot 2\sqrt{6} \cdot 19q} + \frac{12\nu}{38 \cdot 2\sqrt{6} \cdot 7 \cdot 19q^2} < \frac{\nu}{4 \cdot 19^2\sqrt{6}\,q} + \frac{2}{q^2}$$

if $\nu > 0$. Hence

$$\left|q(t-t_i) - \frac{\nu}{4\cdot 19^2\sqrt{6}}\right| < \frac{2}{q}$$

and

$$|q(\gamma - \Gamma) - t_i| = |t - t_i| \le \frac{|\nu|}{4\cdot 19^2\sqrt{6}\, q} + \frac{2}{q^2} < \frac{2}{q}.$$

By $|t| \le \frac{\Gamma - 1}{2}$ and $|t + t_i| \le |t - t_i| + 2|t_i| < 1$, we see

$$|t^2 - t_i^2| \le \frac{2}{q}|t + t_i| < \frac{2}{q} \quad \text{and} \quad |\gamma - \Gamma| = \frac{|t|}{q} < \frac{\Gamma - 1}{2q} < \frac{1}{q}.$$

Therefore

$$\Gamma - \frac{1}{q} < \gamma \le \Gamma + \frac{1}{q} \quad \text{and} \quad 0 < \frac{1}{p-q} = \frac{1}{\gamma - 1}\frac{1}{q} \le \frac{1}{\Gamma - 1 - \frac{1}{20}}\frac{1}{q} < \frac{1}{q}.$$

By applying these, we see $|\gamma^3 - \Gamma^3| \le 3(\Gamma + \frac{1}{20})^2|\gamma - \Gamma| \le 3(\Gamma + \frac{1}{20})^2\frac{1}{q}$,

$$\left|\frac{1}{\gamma} - \frac{1}{\Gamma}\right| \le \frac{1}{\Gamma(\Gamma - \frac{1}{20})q} < \frac{1}{4q} < \frac{1}{q}, \quad \left|\frac{1}{\gamma^3} - \frac{1}{\Gamma^3}\right| \le \frac{3}{(\Gamma - \frac{1}{20})^4}\frac{1}{q} < \frac{1}{q},$$

$$\left|\frac{1}{\gamma^3 - 1} - \frac{1}{\Gamma^3 - 1}\right| \le \frac{3(\Gamma + \frac{1}{20})^2}{((\Gamma - \frac{1}{20})^3 - 1)(\Gamma^3 - 1)}\frac{1}{q} \le \frac{1}{q},$$

$$\left|q\Big(\frac{1}{\gamma} - \frac{1}{\Gamma}\Big) + \frac{t_i}{\Gamma^2}\right| \le \frac{|-\Gamma\big(q(\gamma - \Gamma) - t_i\big) + (\gamma - \Gamma)t_i|}{(\Gamma - \frac{1}{20})\Gamma^2} \le \frac{2\Gamma + 1}{(\Gamma - \frac{1}{20})\Gamma^2}\frac{1}{q} < \frac{1}{q},$$

and

$$\left|\frac{19q - v_i}{p - q} - \frac{38}{\sqrt{6}}\right| = \left|\frac{19 - v_i/q}{\gamma - 1} - \frac{19}{\Gamma - 1}\right| \le \frac{v_i}{(\Gamma - 1 - \frac{1}{20})q} + \frac{19|\Gamma - \gamma|}{(\Gamma - 1)(\Gamma - 1 - \frac{1}{20})} < \frac{39}{q}.$$

By putting

$$M_0(i, \nu) := \frac{\nu}{4\cdot 19^2\sqrt{6}} - \frac{t_i^2}{2(\Gamma - 1)},$$

we have

$$\Big|q(\tau - t_i) - M_0(i,\nu)\Big| \le \left|q(t - t_i) - \frac{\nu}{4\cdot 19^2\sqrt{6}}\right| + \frac{|t^2 - t_i^2|(\Gamma - 1) + t_i^2|\Gamma - \gamma|}{2(\gamma - 1)(\Gamma - 1)} < \frac{3}{q},$$

and hence, by $|M_0(i,\nu)| \le \frac{3258}{4\cdot 19^2\sqrt{6}} + \frac{t_2^2}{2(\Gamma - 1)} < 1$, we have

$$|\tau - t_i| \le |M_0(i,\nu)|\frac{1}{q} + \frac{3}{q^2} \le \frac{2}{q}, \quad \text{and} \quad |\tau| \le |t_i| + \frac{2}{q} < \frac{1}{2}.$$

By putting

$$M_1(i,\nu) := -\frac{t_i}{\Gamma^2} - \Big(t_i - \frac{t_i}{\Gamma^2}\Big)\Big(\frac{i}{3} + \frac{2}{3}t_i\Big) - \Big(\Gamma + \frac{1}{\Gamma}\Big)\frac{2}{3}M_0(i,\nu),$$

and by noting $0 < \frac{i}{3} + \frac{2}{3}t_i < 1$, we have $|M_1(i,\nu)| \le 3$ and

$$
\begin{aligned}
&|q\big(h_{i,1,-1}(\gamma,\tau) - h_{i,1,-1}(\Gamma,t_i)\big) - M_1(i,\nu)| \\
&\le \left| q\Big(\frac{1}{\gamma} - \frac{1}{\Gamma}\Big) + \frac{t_i}{\Gamma^2} \right| + \left| q(\gamma - \Gamma) + q\Big(\frac{1}{\gamma} - \frac{1}{\Gamma}\Big) - \Big(t_i - \frac{t_i}{\Gamma^2}\Big) \right| \left| \frac{i}{3} + \frac{2}{3}\tau \right| \\
&\quad + \left| t_i - \frac{t_i}{\Gamma^2} \right| \left| \frac{2}{3}\tau - \frac{2}{3}t_i \right| + \Big(\Gamma + \frac{1}{\Gamma}\Big)\frac{2}{3}\big|q(\tau - t_i) - M_0(i,\nu)\big| \\
&\le \left(1 + \frac{8}{3} + 2|t_2|\Big(1 - \frac{1}{\Gamma^2}\Big) + 2\Big(\Gamma + \frac{1}{\Gamma}\Big)\right)\frac{1}{q} < \frac{10}{q}.
\end{aligned}
$$

By noting $pqD_+X_2(c_4) = 2\big(-2\tau + h_{i,1,-1}(\gamma,\tau) - (-2t_i + h_{i,1,-1}(\Gamma,t_i))\big)$ and by putting

$$M_2(i,\nu) := \frac{1}{\Gamma}(-4M_0(i,\nu) + 2M_1(i,\nu)),$$

we have

$$
\begin{aligned}
&|q^3 D_+X_2(c_4) - M_2(i,\nu)| \le \frac{4}{\gamma}\big|q(\tau - t_i) - M_0(i,\nu)\big| \\
&\quad + \frac{2}{\gamma}\Big|q\big(h_{i,1,-1}(\gamma,\tau) - h_{i,1,-1}(\Gamma,t_i)\big) - M_1(i,\nu)\Big| + 2\left|\frac{1}{\Gamma} - \frac{1}{\gamma}\right||M_1(i,\nu)| \le \frac{21}{q}.
\end{aligned}
$$

This yields important estimate below:

$$q^3 D_+\sigma^2(c_4) \lesseqgtr q^3 D_+Z_{2,\pm}(c_4) \lesseqgtr M_2(i,\nu) \pm \frac{21}{q} \pm \frac{2q}{q-1}, \tag{26}$$

where $\pm$ stands for $+$ when $\lesseqgtr$ stands for $\le$, and $\pm$ stands for $-$ when $\lesseqgtr$ stands for $\ge$.

By the definition of $\nu$, we have $\nu = (38p - 38q - u_i)^2 - 6(19q - v_i)^2$. Here we can verify $19 \mid u_i^2 - 6v_i^2$, and hence $19 \mid \nu$. Therefore we have

$$\nu/19 = 4 \cdot 19(p-q)^2 - 4u_i(p-q) - 6 \cdot 19q^2 + 12v_iq + (u_i^2 - 6v_i^2)/19,$$

and thereby

$$\big(\nu - (u_i^2 - 6v_i^2)\big)/19 = -4u_i(p-q) + 12v_iq = (-4u_i)p + (4u_i + 12v_i)q \qquad \mathrm{mod}\ 19.$$

By applying the definition of $\nu$, we have

$$
\begin{aligned}
(19q)^3 &= (19q - v_i + v_i)^3 = (19q - v_i)^2\big((19q - v_i) + 3v_i\big) + v_i^2\big(3(19q - v_i) + v_i\big) \\
&= \Big(\frac{1}{6}(38(p-q) - u_i)^2 - \frac{\nu}{6}\Big)\big((19q - v_i) + 3v_i\big) + v_i^2\big(3(19q - v_i) + v_i\big) \\
&= \frac{2 \cdot 19^2(p-q)^2 - 2 \cdot 19u_i(p-q)}{3}\big((19q - v_i) + 3v_i\big) \\
&\quad + (19q - v_i)\Big(3v_i^2 + \frac{u_i^2 - \nu}{6}\Big) + \Big(v_i^3 + \frac{v_i(u_i^2 - \nu)}{2}\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
q^3c_4 = -\frac{(19q)^3}{2 \cdot 19^3(p-q)} &= -\Big(\frac{p-q}{3} - \frac{u_i}{3 \cdot 19}\Big)\Big(\frac{19q - v_i}{19} + \frac{3v_i}{19}\Big) \\
&\quad - \frac{19q - v_i}{p-q}\,\frac{18v_i^2 + u_i^2 - \nu}{12 \cdot 19^3} - \frac{2v_i^3 + v_i(u_i^2 - \nu)}{4 \cdot 19^3(p-q)} =: \mathrm{I} + \mathrm{II} + \mathrm{III}, \qquad \mathrm{mod}\ 1.
\end{aligned}
$$

Since $p-q=2i \mod 3$, $3 \mid v_i$, $3 \mid 38i-u_i$, we have

$$\mathrm{I}=-\Big(\frac{p-q-2i}{3}+\frac{(38i-u_i)/3}{19}\Big)\Big(q+\frac{2v_i}{19}\Big)=-q\frac{(38i-u_i)/3}{19}-\Big(\frac{p-q}{3}-\frac{u_i/3}{19}\Big)\frac{2v_i}{19}$$
$$=-\frac{1}{19}\Big(\frac{2v_i}{3}p+\frac{38i-u_i-2v_i}{3}q-\frac{2u_iv_i/3}{19}\Big) \mod 1.$$

We can verify $7(-4u_i)=\frac{2v_i}{3} \mod 19$ and $7(4u_i+12v_i)=\frac{38i-u_i-2v_i}{3} \mod 19$, and hence we see the mod 1 relation

$$\mathrm{I}=-\frac{1}{19}\Big(7\big((-4u_i)p+(4u_i+12v_i)q\big)-\frac{2u_iv_i/3}{19}\Big)=\frac{-7\big(\nu-(u_i^2-6v_i^2)\big)+2u_iv_i/3}{19^2}.$$

By $|18v_i^2+u_i^2-\nu|\le 18v_0^2+u_0^2+3258$, we have

$$\Big|\mathrm{II}+\frac{18v_i^2+u_i^2-\nu}{6\sqrt{6}\,19^2}\Big|=\Big|-\frac{19q-v_i}{p-q}+\frac{38}{\sqrt{6}}\Big|\frac{|18v_i^2+u_i^2-\nu|}{12\cdot 19^3}<\frac{13}{q},$$

and by $|2v_i^3+v_i(u_i^2-\nu)|\le 2v_0^3+v_0(u_0^2+3258)=334260$, we have $|\mathrm{III}|\le\frac{334260}{4\cdot 19^3(\Gamma_\flat-1)q}<\frac{11}{q}$. We here denote

$$M_3(i,\nu):=\Big\langle\frac{-7\big(\nu-(u_i^2-6v_i^2)\big)+2u_iv_i/3}{19^2}-\frac{18v_i^2+u_i^2-\nu}{6\sqrt{6}\,19^2}\Big\rangle.$$

By these we can conclude the following implication. If

$$\frac{24}{q}\le M_3(i,\nu)\le 1-\frac{24}{q} \tag{27}$$

holds, then we have

$$\big|\langle q^3c_4\rangle-M_3(i,\nu)\big|\le\frac{24}{q}. \tag{28}$$

When this condition is satisfied, by noting

$$\mathrm{IV}:=\Big|q^3 2\frac{1-(\gamma^3+1)\langle q^3c_4\rangle}{p^3}-\frac{2}{\Gamma^3}\Big(1-(\Gamma^3+1)M_3(i,\nu)\Big)\Big|$$
$$\le 2\Big|\frac{1}{\gamma^3}-\frac{1}{\Gamma^3}\Big|\big|1-\langle q^3c_4\rangle\big|+2\Big(1+\frac{1}{\Gamma^3}\Big)\big|\langle q^3c_4\rangle-M_3(i,\nu)\big|\le\frac{2}{q}+\frac{53}{q}=\frac{55}{q},$$

and by putting

$$M_4(i,\nu):=M_2(i,\nu)+\frac{2}{\Gamma^3}\big(1-(\Gamma^3+1)M_3(i,\nu)\big),$$

we have

$$|q^3D_+X_3(c_4)-M_4(i,\nu)|\le|q^3D_+X_2(c_4)-M_2(i,\nu)|+\mathrm{IV}\le\frac{21}{q}+\frac{55}{q}=\frac{76}{q}.$$

By (9) for $N=3$, we see

$$q^3D_+\sigma^2(c_4)\lesseqgtr q^3Z_{3,\pm}(c_4)\lesseqgtr M_4(i,\nu)\pm\frac{76}{q}\pm\frac{2}{q-1}. \tag{29}$$

By assuming (27) and $1\le j\le 8$, we prove

$$q^3Z_{2,\pm}(c_{20+j})\lesseqgtr M_2(i,\nu)\pm\frac{21}{q}-10\Big(\frac{1}{\Gamma^3}\mp\frac{1}{q}\Big)\Big(j-M_3(i,\nu)\mp\frac{24}{q}\Big)\pm\frac{2q}{q-1}, \tag{30}$$

together with the following implication: the condition (27) and $c_{20+j} < c_{15}$ imply

$$
\begin{aligned}
q^3 Z_{3,\pm}(c_{20+j}) \lesseqgtr M_4(i,\nu) \pm \frac{76}{q} \pm \frac{2}{q-1} \\
+ 2\Big(1 - \Big(\frac{1}{\Gamma^3} \mp \frac{1}{q}\Big)\Big(6j + 1 - 7M_3(i,\nu) \mp \frac{168}{q}\Big)\Big).
\end{aligned} \tag{31}
$$

We here prove $c_{28} < c_8$. For $i = 0$, 1, 2, and $-\frac{\Gamma-1}{2} < t < \Gamma - 2$, we have $\langle q^2 c_4\rangle = \frac{i}{3} + \frac{2}{3}\tau \le \frac{2}{3} + \frac{2}{3}(\Gamma - 2)$. Hence $p^3(c_8 - c_{28}) \ge p(\frac{1}{3} - \frac{2}{3}(\Gamma-2)) - 8 \ge \frac{5-2\Gamma}{3} 20\Gamma_\flat - 8 > 0$.

On $[c_4, c_8)$, by $D_+X_2(a) = -10(a - a_{1,-1}) = -10(a - c_4) + D_+X_2(c_4)$ we see $q^3 D_+X_2(c_{20+j}) = q^3 D_+X_2(c_4) - 10\frac{1}{\gamma^3}\big(j - \langle q^3 c_4\rangle\big)$, which proves (30).

Because of $q^3(c_{20+j} - c_{15}) = \frac{1}{\gamma^3}(j - \langle q^3 c_4\rangle) - (1 - \langle q^3 c_4\rangle)$, we see it is increasing in $j$, $q^3(c_{21} - c_{15}) \in (-1, 0)$, and hence $q^3(c_{20+j} - c_{15}) \in (-1, 0)$ by assumption. From this we see $\langle q^3 c_{20+j}\rangle = \Big\langle q^3 c_4 + \frac{q^3}{p^3}\big(j - \langle q^3 c_4\rangle\big)\Big\rangle = \langle q^3 c_4\rangle + \frac{1}{\gamma^3}\big(j - \langle q^3 c_4\rangle\big)$, since the right hand side equals to $q^3(c_{20+j} - c_{15}) + 1 \in (0, 1)$. By (7), we see

$$
\begin{aligned}
q^3 \frac{2}{p^3 q^3} D_+V\big(\langle p^3 c_{20+j}\rangle, \langle q^3 c_{20+j}\rangle\big) = 2\Big(1 - \langle q^3 c_4\rangle - \frac{1}{\gamma^3}\big(j - \langle q^3 c_4\rangle\big)\Big) \\
= q^3 \frac{2}{p^3 q^3} D_+V\big(\langle p^3 c_4\rangle, \langle q^3 c_4\rangle\big) + 2\Big(1 - \frac{1}{\gamma^3}\big(j + 1 - 2\langle q^3 c_4\rangle\big)\Big).
\end{aligned} \tag{32}
$$

Therefore we have $q^3 D_+X_3(c_{20+j}) = q^3 D_+X_3(c_4) + 2\big(1 - \frac{1}{\gamma^3}\big(6j + 1 - 7\langle q^3 c_4\rangle\big)\big)$, which proves (31).

[*The case* $i = 0$.] Since $p$ is odd, $q$ is even and $p - q = 0 \mod 3$, we can write $p = 6S + 2T + 3$ and $q = 2T$. Then we see $\nu = -24((19T-15)^2 - 6(19S+3)^2)$ and can write $\nu = -24 \cdot 19C$ by noting $15^2 - 6 \cdot 3^2 = 3^2 \cdot 19$. We prove $3 \nmid C$. Actually, if $3 \mid C$, then $3 \mid (19T - 15)^2$, $3 \mid T$, and $3 \mid q$ in turn, which contradicts $3 \mid p - q$ and $(p, q) = 1$.

We solve $(19T - 15)^2 - 6(19S + 3)^2 = 19C$, i.e., find solution of Pell equation $x^2 - 6y^2 = 19C$ with $x = 4$, $y = 3 \mod 19$. Solutions of the Pell equation are given by

$$
x_n + y_n\sqrt{6} = (x_0 + y_0\sqrt{6})(5 + 2\sqrt{6})^{n-1}, \tag{33}
$$

where $x_0 + y_0\sqrt{6}$ is the fundamental solution. We denote the field $\mathbf{Q}(\sqrt{6})$ by $K$, and its integer ring by $O_K$. For $m = 1$, 2, 6, and 12, we can verify that $(5 + 2\sqrt{6})^\ell = 1 \mod 19mO_K$ if and only if $18 \mid \ell$. Hence $x_n + y_n\sqrt{6} \mod 19mO_K$ is a periodic sequence with period 18, and we see that $x_n + y_n\sqrt{6} = 4 + 3\sqrt{6} \mod 19mO_K$ holds periodically with period 18 if there exists such $n$. In this case, there exists an $\ell = 0, \ldots, 17$ such that $x_{\ell+18(k-1)} + y_{\ell+18(k-1)}\sqrt{6}$ $(k \ge 1)$ give our solutions, and we define $S(k)$ and $T(k)$ by $(19T(k) - 15) + (19S(k) + 3)\sqrt{6} = x_{\ell+18(k-1)} + y_{\ell+18(k-1)}\sqrt{6}$ $(k \ge 1)$ and put $p(k) = 6S(k) + 2T(k) + 3$, $q(k) = 2T(k)$ $(k \ge 1)$. We find the fundamental solution $x_0 + y_0\sqrt{6}$ using a recipe in section 8.8 of [1] and then check the first 18 terms of $x_n + y_n\sqrt{6}$ to decide if there is a solution of our condition and to determine $\ell$.

By $-2403 \le -24 \cdot 19C \le 3258$, it is enough to consider $C$ satisfying $-7 \le C \le -1$ or $1 \le C \le 5$. Among these $C$, there exists an integer solution of Pell equation $x^2 - 6y^2 = 19C$ if $C = -6, -5, -2, 1, 3, 4$. Since $C = -6, 3$ cannot be candidates, we will consider the cases $\nu = -456, -1824, 2280, 912$. As we show later, we have $q(k) \ge 262$ and see $\frac{24}{q(k)} \le 0.0916$. Hence $M_3(0, -456) = 0.1405$, $M_3(0, -1824) = 0.4090$, $M_3(0, 2280) = 0.6036$, and $M_3(0, 912) = 0.8720$ satisfy (27) and we have (28)–(32).

We consider that case $C = 1$ or $\nu = -456$. We have $(x_0, y_0, \ell) = (13, 5, 15)$, $q \geq 1141090950502840$ and $M_4(0, -456) = 0.3046$. Hence $q^3 D_+ \sigma^2(c_4) > 0$ and $c_4$ cannot be the maximal point.

When $C = 4$ or $\nu = -1824$, we have $(x_0, y_0, \ell) = (26, 10, 2)$, $q(1) = 262$ and $M_4(0, -1824) = 1.0342$. Hence $q^3 D_+ \sigma^2(c_4) > 0$ and $c_4$ cannot be the maximal point.

For $C = -5$ or $\nu = 2280$, we have $(x_0, y_0, \ell) = (17, 8, 4)$ and $(43, 18, 16)$. Since $q(1) = 18496$ or $q(1) = 28964180210424124$, and $M_2(0, 2280) = -2.2008$, we have $q^3 Z_{2,+}(c_4) < 0$ by (26) and $Z_{2,+}(a) < 0$ on $[\,c_4, c_8)$. Hence $c_4$ is the maximal point.

For $C = -2$ or $\nu = 912$, we have $(x_0, y_0, \ell) = (4, 3, 0)$, $q(1) = 2$, $q(2) \geq 10^{18}$. We prove $c_{22} < c_{15} < c_{23}$. By denoting $M_3(0, 912)$ simply denote by $M_3$, we see

$$q^3(c_{22} - c_{15}) \leq \Big(\frac{1}{\Gamma^3} + \frac{1}{q}\Big)\Big(2 - M_3 + \frac{24}{q}\Big) - \Big(1 - M_3 - \frac{24}{q}\Big) = -0.0254,$$
$$q^3(c_{23} - c_{15}) \geq \Big(\frac{1}{\Gamma^3} - \frac{1}{q}\Big)\Big(3 - M_3 - \frac{24}{q}\Big) - \Big(1 - M_3 + \frac{24}{q}\Big) = 0.0653.$$

Since $D_+ V\big(\langle q^3 a\rangle, \langle p^3 a\rangle\big)$ decreases on $[\,c_4, c_{21})$, $[\,c_{21}, c_{22})$, and $[\,c_{22}, c_{15})$, and $D_+ X_2(a)$ is decreasing, $Z_{3,+}$ also decreases on these three intervals. We have $Z_{3,+}(c_4) < 0$ by $M_4(0, 912) = -2.6064$, and $q^3 Z_{3,+}(c_{22}) \leq q^3 Z_{3,+}(c_{21}) \leq -0.7691$ by (31). Hence $Z_{3,+}(a)$ is negative on $[\,c_4, c_{21})$, $[\,c_{21}, c_{22})$ and $[\,c_{22}, c_{15})$, and thereby $\sigma^2(a)$ is decreasing on each interval. If we prove $Z_{2,+}(c_{15}) < 0$, then we can also see $Z_{2,+}(a)$ is negative on $[\,c_{15}, c_8)$ and $\sigma^2(a)$ is decreasing there. By this argument we can conclude that $\sigma^2(a)$ is decreasing on $[\,c_4, c_8)$, and the proof is complete in this case. The estimate $Z_{2,+}(c_{15}) < 0$ is proved by

$$\begin{aligned} q^3 Z_{2,+}(c_{15}) &= q^3 D_+ X_2(c_{15}) + \frac{2q}{q-1} = q^3 D_+ X_2(c_4) - 10 q^3 (c_{15} - c_4) + \frac{2q}{q-1} \\ &= q^3 D_+ X_2(c_4) - 10(1 - \langle q^3 c_4 \rangle) + \frac{2q}{q-1} \\ &\leq M_2(0, 912) + \frac{21}{q} - 10\Big(1 - M_3(0, 912) - \frac{24}{q}\Big) + \frac{2q}{q-1} = -0.1648 < 0. \end{aligned}$$

[*The case* $i = 1$]

By putting $p - q = 6S + 5$ and $q = 2T$, we have $\nu = (38 \cdot 6S + 143)^2 - 6(38T - 21)^2$. Since $143^3 - 6 \cdot 21^2 = 19 \cdot 937$, the right hand side is divisible by 19. By putting $\nu = 19C$, expanding and dividing by 19, we have $-456T^2 + 504T + 2726S^2 + 3432S = C - 937$. Hence we see $24 \mid C - 937$. By $-2403 \leq 19C \leq 3258$, we have $-126 \leq C \leq 171$. We investigate the solution of $x^2 - 6y^2 = 19C$ satisfying $x = 143 \mod 38 \cdot 6$ and $y = 17 \mod 38$. Because $x_n + y_n\sqrt{6} \mod 19mO_K$ for $m = 2$ and 12 has period 18, we see the orbit of $(x_n \mod 38 \cdot 6, y_n \mod 38)$ has period 18. By investigating the fundamental solution for each $C$, we have the following. We can find the solution of above type only for $C = -95, -71, 97, 121$, and 145. Solutions are given by $38 \cdot 6S(k) + 143 + (38T(k) - 21)\sqrt{6} = x_{\ell+18(k-1)} + y_{\ell+18(k-1)}\sqrt{6}$, $p(k) = 6S(k) + 2T(k) + 5$, and $q(k) = 2T(k)$, where $(x_0, y_0, \ell) = (107, 47, 8)$ if $C = -95$, $(x_0, y_0, \ell) = (55, 27, 1)$ if $C = -71$, $(x_0, y_0, \ell) = (203, 81, 12)$ if $C = 97$, $(x_0, y_0, \ell) = (143, 55, 0)$ if $C = 121$, and $(x_0, y_0, \ell) = (59, 11, 10)$ if $C = 145$.

We have $p(1)/q(1) = 31/14$ for $C = -71$, and $p(1)/q(1) = 9/4$ for $C = 121$. Other than these, all solution satisfies $q(k) \geq 10^8$. Hence $M_3(1, -1805) = 0.0963$, $M_3(1, -1349) = 0.3401$, $M_3(1, 1843) = 0.0470$, $M_3(1, 2299) = 0.2908$, and $M_3(1, 2755) = 0.5347$ satisfy (27).

By applying (29), we see $q^3Z_{3,-}(c_4) \geq 1.7684$ for $C = -95$ and $q^3Z_{3,-}(c_4) \geq 0.7980$ for $C = -71$. Hence $c_4$ is not the maximal point in these cases. We can verify $q^3Z_{2,+}(c_4) \leq -0.5871$ for $C = 145$, and $q^3Z_{2,+}(c_4) \leq -0.1487$ for $C = 121$, and see that $c_4$ is maximal in these cases.

We consider the case $C = 97$. Note that $c_{21} < c_{15}$. By (29), we see $q^3Z_{3,+}(c_4) \leq -1.6313$ and see that $\sigma^2(a)$ is decreasing in $[\,c_4, c_{21})$. By (30) we see $q^3Z_{2,+}(c_{21}) \leq -0.5757$ and see that $\sigma^2(a)$ decreases on $[\,c_{21}, c_8)$. Therefore $c_4$ is the maximal point also in this case.

[*The case* $i = 2$]

Since $p - q = 1 \mod 3$ is odd, we see $p - q = 6S + 1$ and $q = 2T$. Hence $\nu = 4((19 \cdot 6S + 11)^2 - 6(19T - 6)^2)$ which is divisible by 19.

Consider $4((19 \cdot 6S + 11)^2 - 6(19T - 6)^2) = 4 \cdot 19C$. Expanding and dividing by 19, we have $19 \cdot 6^2S^2 + 2 \cdot 6 \cdot 11S - 6 \cdot 19T^2 + 2 \cdot 6^2T = C + 5$. Hence we have $6 \mid C + 5$.

If we investigate the equation $x^2 - 6y^2 = 19C$. By $-2403 \leq 4 \cdot 19C \leq 3258$, we see $-31 \leq C \leq 42$. It has a solution with $x = 11 \mod 19 \cdot 6$ and $y = 13 \mod 19$, if $C = -23$, $-5$, $1$, $19$, $25$. Solutions are given by $19 \cdot 6S(k) + 11 + (19T(k) - 6)\sqrt{6} = x_{\ell+18(k-1)} + y_{\ell+18(k-1)}\sqrt{6}$, $p(k) = 6S(k) + 2T(k) + 1$, and $q(k) = 2T(k)$, where $(x_0, y_0, \ell) = (47, 21, 16)$ if $C = -23$, $(x_0, y_0, \ell) = (17, 8, 8)$ if $C = -5$, $(x_0, y_0, \ell) = (13, 5, 1)$ if $C = 1$, $(x_0, y_0, \ell) = (19, 35, 12)$ if $C = 19$, and $(x_0, y_0, \ell) = (23, 3, 4)$ and $(x_0, y_0, \ell) = (31, 9, 15)$ if $C = 25$. As we see below, we have $q(k) \geq 6262$. Hence $M_3(2, 76) = 0.5689$, $M_3(2, 1444) = 0.3004$ , $M_3(2, 1900) = 0.5443$ satisfy (27).

When $C = -23$ and $-5$, we have $q(k) \geq 10^8$, $M_4(2, -1748) = 0.9129$, and $M_4(2, -380) = 0.1833$. Therefore by (29) we see $Z_{3,-}(c_4) > 0$ and verify that $c_4$ is not the maximal point.

For $C = 1$, $19$, $25$, we see $q(k) \geq 10^{14}$ except for the following two cases. One is $C = 1$ and $p(1)/q(1) = 13/6$ which can be omitted here, and the other is $C = 25$, $(x_0, y_0) = (23, 3)$ and $p(1)/q(1) = 13931/6262$.

The right hand side of

$$q^3(c_{24} - c_{15}) \leq \Big(\frac{1}{\Gamma^3} + \frac{1}{q}\Big)\Big(4 - M_3(2, 19 \cdot 4C) + \frac{24}{q}\Big) - \Big(1 - M_3(2, 19 \cdot 4C) - \frac{24}{q}\Big)$$

equals to $-0.1194$ if $C = 1$, $-0.3635$ if $C = 19$, $-0.1418$ if $C = 25$ and $q \neq 6262$, $-0.1357$ if $C = 25$ and $q = 6262$, and we see $c_{24} < c_{15}$.

By $M_4(2, 76) = -0.7870$, $M_4(2, 1444) = -1.5165$, $M_4(2, 1900) = -2.4869$ we see that $q^3D_+Z_{3,+}(c_4) < 0$ and that $\sigma^2(a)$ is decreasing in $[\,c_4, c_{21})$.

Next, we verify $c_{15} < c_8$. By $t < 0$, we see $\tau < 0$, $\langle q^2c_4\rangle = \frac{2}{3} + \frac{2}{3}\tau < \frac{2}{3}$, and hence $c_8 \geq c_4 + \frac{1}{3q^2} \geq c_4 + \frac{1}{q^3} \geq c_{15}$.

We consider the case $C = 25$ and $q \neq 6262$. We have $q^3Z_{3,+}(c_{21}) \leq -1.0663$ by (31), and $q^3Z_{2,+}(c_{22}) \leq -0.8030$ by (30). Hence $\sigma^2(a)$ is decreasing in $[\,c_{21}, c_{22})$ and $[\,c_{22}, c_8)$, and thereby $c_8$ is the maximal point. In case $C = 25$ and $q = 6262$, we see $q^3Z_{3,+}(c_{21}) \leq -1.0479$ and $q^3Z_{2,+}(c_{22}) \leq -0.7936$ to have the same conclusion. In case $C = 19$, we see $q^3Z_{3,+}(c_{21}) \leq -0.4059$ and $q^3Z_{2,+}(c_{22}) \leq -0.5861$ to have the same conclusion.

We consider the case $C = 1$. By (31), we have $q^3Z_{3,+}(c_{22}) \leq -0.4248$ and $q^3Z_{3,+}(c_{23}) \leq -1.5146$. Hence $\sigma^2(a)$ is decreasing in $[\,c_{22}, c_{23})$ and $[\,c_{23}, c_{24})$. By (30), we have $q^3Z_{2,+}(c_{24}) \leq -0.8433 < 0$, and we see that $\sigma^2(a)$ is decreasing in $[\,c_{24}, c_8)$. Hence we only have to prove $\sigma^2(a) \leq \sigma^2(c_4)$ for $a \in [\,c_{21}, c_{22})$. We first verify

$c_{21} < c_{16} < c_{22}$. By $q^3(c_{20+j} - c_{16}) = \frac{1}{\gamma^3}(j - \langle q^3 c_4\rangle) - \frac{1}{\gamma^3-1}$, we see

$$q^3(c_{21} - c_{16}) \le \Big(\frac{1}{\Gamma^3} + \frac{1}{q}\Big)\Big(1 - M_3(2,76) + \frac{24}{q}\Big) - \frac{1}{\Gamma^3 - 1} + \frac{1}{q} = -0.06074 < 0,$$
$$q^3(c_{22} - c_{16}) \ge \Big(\frac{1}{\Gamma^3} - \frac{1}{q}\Big)\Big(2 - M_3(2,76) - \frac{24}{q}\Big) - \frac{1}{\Gamma^3 - 1} - \frac{1}{q} = 0.003007 > 0.$$

By (31) we have $Z_{3,-}(c_{21}) > 0.6649$ and now we prove $Z_{3,+}(c_{16} - 0) < 0$. Note that $p^3 c_{16} - q^3 c_{16}$ is an integer. By $\langle q^3 c_4\rangle \le M_3(2,76) + \frac{24}{q} = 0.5689$ and $0 < \frac{1}{\gamma^3-1} \le \frac{1}{10}$, we have

$$\langle p^3 c_{16}\rangle = \langle q^3 c_{16}\rangle = \Big\langle q^3 c_4 + \frac{1}{\gamma^3 - 1}\Big\rangle = \langle q^3 c_4\rangle + \frac{1}{\gamma^3 - 1}.$$

Combining $q^3 D_+ X_2(c_{16}) = q^3 D_+ X_2(c_4) - 10q^3(c_{16} - c_4) = q^3 D_+ X_2(c_4) - \frac{10}{\gamma^3-1}$ and

$$q^3 \frac{2}{p^3 q^3} D_+ V\big(\langle p^3(c_{16} - 0)\rangle, \langle q^3(c_{16} - 0)\rangle\big) = \frac{2}{p^3}\big(p^3 - p^3\langle q^3 c_{16}\rangle - q^3\langle q^3 c_{16}\rangle\big)$$
$$= 2 - \frac{2(\gamma^3 + 1)}{\gamma^3}\Big(\langle q^3 c_4\rangle + \frac{1}{\gamma^3 - 1}\Big) = q^3 \frac{2}{p^3 q^3} D_+ V\big(\langle p^3 c_4\rangle, \langle q^3 c_4\rangle\big) + 2 - \frac{4}{\gamma^3 - 1},$$

we have

$$q^3 Z_{3,+}(c_{16} - 0) = q^3 D_+ X_3(c_{16} - 0) + \frac{2}{q-1} = q^3 D_+ X_3(c_4) + 2 - \frac{14}{\gamma^3 - 1} + \frac{2}{q-1}$$
$$\le M_4(2,76) + \frac{76}{q} + 2 - \frac{14}{\Gamma^3 - 1} + \frac{14}{q} + \frac{2}{q-1} = -0.1854 < 0.$$

By $0 > Z_{3,+}(c_{16} - 0) \ge Z_{3,+}(c_{16})$, we see that $\sigma^2(a)$ is decreasing in $[\,c_{16}, c_{22})$. We denote the zero of $D_+ X_3(a)$ in $(c_{21}, c_{16})$ by $c_{17}$. It is the maximal point of $X_3(a)$ in $[\,c_{21}, c_{16})$, i.e., $X_3(a) \le X_3(c_{16})$. By denoting $D_+ X_3(a) = -14a + \eta$ in $[\,c_{21}, c_{16})$, we have $c_{17} = \eta/14$. By $D_+ X_3(c_{21}) = -14c_{21} + \eta$, we see $\eta = D_+ X_3(c_{21}) + 14c_{21}$, $D_+ X_3(a) = -14(a - c_{21}) + D_+ X_3(c_{21})$ and $c_{17} = \frac{1}{14} D_+ X_3(c_{21}) + c_{21}$. By integration, we have $X_3(c_{17}) - X_3(c_{21}) = -7(c_{17} - c_{21})^2 + D_+ X_3(c_{21})(c_{17} - c_{21}) = \frac{1}{28} D_+ X_3(c_{21})^2$. In the same way, we have $X_3(c_{21}) - X_3(c_4) = -7\Big(\frac{1 - \langle q^3 c_4\rangle}{p^3}\Big)^2 + \frac{1 - \langle q^3 c_4\rangle}{p^3} D_+ X_3(c_4)$. Applying $D_+ X_2(c_{21}) = D_+ X_2(c_4) - 10\frac{1 - \langle q^3 c_4\rangle}{p^3}$ and (32), we have $D_+ X_3(c_{21}) = D_+ X_3(c_4) + 2\big(\frac{1}{q^3} - 7\frac{1 - \langle q^3 c_4\rangle}{p^3}\big)$. Combining these and $|X_3(a) - \sigma^2(a)| \le \frac{1}{2(pq)^3(pq-1)} =: q^{-6} R$, and applying $\frac{1}{28}(D + 2 - 14F)^2 - 7F^2 + FD = \frac{D^2}{28} + \frac{1}{7} + \frac{D}{7} - 2F$, for $a \in [\,c_{21}, c_{16})$,

$$q^6(\sigma^2(a) - \sigma^2(c_4)) \le q^6(X_3(a) - X_3(c_4)) + R \le q^6(X_3(c_{17}) - X_3(c_4)) + R$$
$$= \frac{1}{28}(q^3 D_+ X_3(c_4))^2 + \frac{1}{7} + \frac{1}{7} q^3 D_+ X_3(c_4) - 2q^3 \frac{1 - \langle q^3 c_4\rangle}{p^3} + R$$
$$\le \frac{1}{28}\Big(M_4(2,76) + \frac{76}{q}\Big)^2 + \frac{1}{7} + \frac{1}{7}\Big(M_4(2,76) + \frac{76}{q}\Big)$$
$$- 2\Big(\frac{1}{\Gamma^3} - \frac{1}{q}\Big)\Big(1 - M_3(2,76) - \frac{24}{q}\Big) + \frac{1}{\gamma^3(pq-1)} = -0.02574 < 0.$$

Hence we have proved $\sigma^2(a) < \sigma^2(c_4)$ for $a \in [\,c_{21}, c_{16})$. □

**Lemma 15** *For positive integers $p$ and $q$ with $2 \nmid p$, $2 \mid q$, $p-q = 0 \mod 3$, $\gcd(p,q) = 1$, $(p,q) \notin E$, and $0 < p - q\Gamma - t_{3,-}$, we have*

$$p - q\Gamma - t_{3,-} \geq \frac{0.03976}{q}. \tag{34}$$

*For $(p,q) \in E$, we have*

$$0 < p - q\Gamma - t_{3,-} \leq \frac{0.003392}{q}. \tag{35}$$

*Proof* We consider the case $t > t_{3,-}$. We may assume $t < 0$, since (34) is trivial otherwise. We have

$$p - q\Gamma - t_{3,-} = \frac{-6((19^2q+720)^2 - 6(19^2\frac{p-q}{3}+350)^2)}{2 \cdot 19^2(19^2q+720)(\frac{2\cdot 19^2(p-q)+2100}{19^2q+720}+\sqrt{6})},$$

where $\frac{2\cdot 19^2(p-q)+2100}{19^2q+720} > \sqrt{6}$ by $t > t_{3,-}$. By denoting $\nu = (19^2q+720)^2 - 6(19^2\frac{p-q}{3}+350)^2$, we have

$$p - q\Gamma - t_{3,-} \leq \frac{-6\nu}{2\cdot 19^4 \cdot 2\sqrt{6}q}.$$

By $720^2 - 6\cdot 350^2 = -2^3\cdot 3\cdot 5^2\cdot 19^2$, $\nu$ is a multiple of $19^2$. We put $\nu = -19^2C$.

By $p-q = q(\gamma-1) \leq q(\Gamma-1) = q\sqrt{6}/2$ we have

$$\frac{2\cdot 19^2(p-q)+2100}{19^2q+720} \leq \frac{19^2\sqrt{6}q+2100}{19^2q+720} \leq \frac{19^2\sqrt{6}\cdot 20+2100}{19^2\cdot 20+720}.$$

For $q \geq 20$, we have $19^2q+720 \leq (19^2+36)q$ and

$$p - q\Gamma - t_{3,-} \geq \frac{1}{q}\frac{6\cdot 19^2C}{2\cdot 19^2(19^2+36)(\frac{19^2\sqrt{6}\cdot 20+2100}{19^2\cdot 20+720}+\sqrt{6})} \geq \frac{0.03976}{q}$$

if $C \geq 26$.

Because of $2 \mid q$, we see $2 \mid \nu$ and $2 \mid C$. We see $3 \nmid C$, because $3 \mid C$ implies $3 \mid (19^2q+720)^2$ or $3 \mid q$ which contradicts with $(p,q) = 1$. We see $4 \nmid C$, because $4 \mid C$ implies $2 \mid (19^2\frac{p-q}{3}+350)^2$ and $2 \mid p-q$ which contradicts with $(p,q) = 1$.

Hence the possible values of $C$ are 2, 10, 14 and 22. Among these, $C = 2$ is the only case having solution. In this case we have $p - q\Gamma - t_{3,-} \leq \frac{6\cdot 19^2\cdot 2}{2\cdot 19^4\cdot 2\sqrt{6}q} = \frac{0.003392}{q}$. We examine the values of $p/q$ in this case. We have $(19^2q+720)^2 - 6\big(19^2\frac{p-q}{3}+350\big)^2 = -2\cdot 19^2$. We see $2 \mid (19^2q+720)$ and $2 \nmid 19^2\frac{p-q}{3}+350$, and hence $2 \mid q$ and $2 \nmid p$. Expanding, we have $19^2q^2+1440q-6\big(19^2\big(\frac{p-q}{3}\big)^2+700\frac{p-q}{3}\big) = 2\cdot 13\cdot 23$. and we see $3 \nmid q$. Common factors of $p$ and $q$ are those of $q$ and $p-q$. Hence common prime factors except for 3 are those of $q$ and $\frac{p-q}{3}$. By the above formula, common prime factors of $q$ and $\frac{p-q}{3}$ are 3, 13 and 23. Hence we see only 13 and 23 can be common prime factors of $p$ and $q$. By noting $(5+2\sqrt{6})^{19\cdot 18} = 1 \mod 19^2O_K$, we see the orbit of the solution has period $18\cdot 19$ and can be calculated concretely by using the fundamental solution. The solutions of $x^2-6y^2 = -2\cdot 19^2$ are given by $x+y\sqrt{6} = z_0(5+2\sqrt{6})^j$, $j \in \mathbf{Z}$, $z_0 = 2+11\sqrt{6}$, $38+19\sqrt{6}$, $122+51\sqrt{6}$. Among these, positive solutions satisfying $x = 720$, $y = 350 \mod 19^2$ can be expressed as $x+y\sqrt{6} = (2+11\sqrt{6})(5+2\sqrt{6})^{171+19\cdot 18j}$, $(j \geq 0)$. By $x = 19^2q+720$ and $y = 19^2\frac{p-q}{3}+350$, we see the expressions of $p_j$ and $q_j$. By calculating it for $j \leq 11$, we see $\gcd(p,q) = 13$ if $\ell = 6$, $\gcd(p,q) = 23$ if $\ell = 11$, and $\gcd(p,q) = 1$ for $\ell \neq 6, 11$. By $(5 + 2\sqrt{6})^7 = 1 \mod 13O_K$ and $(5 + 2\sqrt{6})^{11} = 1 \mod 23O_K$, we see that $\gcd(p_j, q_j) \mod 13$ has period 7, and $\gcd(p_j, q_j) \mod 23$ has period 11. Hence we can conclude that $7 \nmid (j-6)$ and $11 \nmid j$ imply $\gcd(p_j, q_j) = 1$. □

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