Percolation and Minimal Spanning Trees

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Abstract

Let $G = (V, E, w)$ be a weighted connected graph with $|V| < \infty$, $|E| < \infty$. A minimal spanning tree $\mathcal{T}$ is a spanning tree through all the vertices of V such that its total edge length attains the minimum. The geometry of minimal spanning tree (MST) has been a central subject in the theory and applications of combinatorial optimization. The classic greedy algorithm is a very efficient way of constructing a MST.

Our graph is random in the sense that the vertex set is randomly distributed in R^d, $d \geq 2$. The probability distribution of $\mathcal{T}$ contains information about the typical structure of MSTs. Indeed, the probability theory of random MSTs has been well developed already. In this talk, we shall restrict ourself to the link between random MST and percolation theory, and the focus is upon the application of percolation theory to the limit theorems for functionals of MSTs, including total edge length, the number of vertices of fixed degree, and independence number.

Almost all of results surveyed in this talk owe a debt of one sort or another to the classic theorem of Beardwood, Halton and Hammersley (1959) that lays out the basic behavior of the length of the shortest tour through random sample from a general distribution in R^d. The BHH theorem has led to developments of several different types. It turns out that the essential geometrical features (the subadditivity property) of TSP are present in many of the combinatorial optimization problem. Also, the technique of Poissonization and de-Poissonization in the proof of the BHH theorem has became a powerful tool in the analysis of Euclidean combinatorial optimization problems. However, one of drawbacks is that one is hard to get any information on the exact limit value.

J.M. Steele (1989), J.M. Steele, L.A.Shepp and W. F. Eddy (1987) established the basic almost-sure asymptotic behaviors of MST that parallels the BHH theorem. Meanwhile they presented some problems, which require new ideas and have led to further development.

Aldous and Steele (1992) use a completely different approach to the MST probelm. The new approach is based on the study of an analog of the MST for an infinite set of points, in particular the points of a Poisson process on all R^d. The basic philosophy that guided the analysis has found rather wide applications

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in other combinatorial optimization problems, say, random assignment. This is referred to as Local Weak Convergence and Objective Methods in the recent literatures.

How does one define an analog of the MST for an infinite set of points? What structure does the new MST have? This turn out to be challenging mathematical problem and have a close relation with deeper continuum percolation. Alexander (1996) discuss the structure of minimal spanning forest (MSF) in some detail for stationary random graph. In particular, he proved the conjecture of Aldous and Steele (1992) in $d = 2$, which says the MSF constructed by Aldous and Steele is indeed an one-ended infinite tree, thus nailing down the core reason for the effectiveness of the construction.

Although we have an elegant limit as a consequence of objective method, we dopn't have a simple formula for it. So, it has been a major open problem to give a good estimate. Avrma and Bertsmas (1992), Jaillet (1992), Penrose (1996) obtain some asymptotic behaviors for the limit in high dimensional cases.

The central limit theorem has been what a lot of authors are concerned with. Ramey (1982) argued in his thesis that a certain property of continuum percolation would imply a central limit theorem for total edge lengths of MSTs for $d = 2$. However, he did not prove this property of continuum percolation. Only till 1996 did Alexander prove a central limit theorem by Ramey's approach for Poissonized MSTs in the case $d = 2$.

The complete extension to non-Poissonized MSTs is due to Kesten and Lee (1996). They developed the classic martingale approach to random MSTs, later Lee go further and find the stabilization property, which is much stronger than monotonicity property and has been generalized to a wide of variety of random functionals and graphs by Penrose and Yukich (2001,2003,2004). Loosely speaking, the stabilization property means that the minimal spanning tree structure is locally determined.

Unfortunately, the stabilization property is not valid for the independence number of random MSTs, or at least we do not know to how prove it. So, we are supposed to restrict ourselves to Poissonized MST and 2 dimensional case in order to get a central limit theorem for the independence number. It is still an open to prove the central limit theorem in general case.

Before concluding the talk, we briefly review some other link between percolation and MSTs. This includes free energy and empirical distribution function, the AEU and its quantifications.