Pseudodifferential Operators

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On this course

Purpose: We learn basics of pseudodiffernetial operators.

References:
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Chapter 1
Oscillatory Integrals

§ 1.1 Introduction

○ **Notation**

In this course we use the notation

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \{0\} \cup \mathbb{N}.$$

We usually let $d \in \mathbb{N}$ be the dimension of the **configuration space**. For any **multi-index** $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we define its **length** and **factorial** as

$$|\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \alpha! = (\alpha_1!) \cdot \cdots \cdot (\alpha_d!),$$

respectively. In addition, for any $\alpha, \beta \in \mathbb{N}_0^d$ we let

$$\alpha \le \beta \quad \overset{\text{def}}{\iff} \quad \alpha_j \le \beta_j \ \text{ for all } j = 1, \ldots, d,$$

and define the **binomial coefficient** as

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} \quad \text{if } 0 \le \beta \le \alpha, \qquad \binom{\alpha}{\beta} = 0 \quad \text{otherwise,}$$

where $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_d - \beta_d)$.

For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}, \quad \partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}.$$

Moreover, we introduce the notation

$$D_j = -\mathrm{i}\partial_j, \quad D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}.$$

Then, in particular, we have

$$D^\alpha = (-\mathrm{i})^{|\alpha|}\partial^\alpha.$$

Thoughout the course for any $x, \xi \in \mathbb{R}^d$ we write simply

$$x\xi = x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d, \quad x^2 = x \cdot x, \quad |x| = \sqrt{x \cdot x},$$

and we adopt the **Fourier transform** and its inverse defined as extensions from

$$\mathcal{F}u(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\xi} u(x)\, \mathrm{d}x \quad \text{for } u \in \mathcal{S}(\mathbb{R}^d),$$
$$\mathcal{F}^* f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\xi} f(\xi)\, \mathrm{d}\xi \quad \text{for } f \in \mathcal{S}(\mathbb{R}^d),$$

respectively. Note, in particular, for any $u, v \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$

$$(u, v)_{L^2} = (\mathcal{F}u, \mathcal{F}v)_{L^2}, \quad \mathcal{F}^*\xi^\alpha\mathcal{F}u = D^\alpha u,$$

where $(\cdot,\cdot)_{L^2}$ denotes the L^2-**inner product**, being linear and conjugate-linear in the first and second entries, respectively.

Problem. 1. (**Binomial theorem**) Show for any $\alpha \in \mathbb{N}_0^d$ and $x, y \in \mathbb{R}^d$

$$(x+y)^\alpha = \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} x^{\alpha-\beta} y^\beta; \quad \text{In particular,} \quad \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} = 2^{|\alpha|}.$$

2. (**Leibniz rule**) Show for any $\alpha \in \mathbb{N}_0^d$ and $f, g \in C^{|\alpha|}(\mathbb{R}^d)$

$$\partial^\alpha(fg) = \sum_{\beta \in \mathbb{N}_0^d} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} f)(\partial^\beta g).$$

∘ Partial differential operators

Consider a partial differential operator (PDO) on $\mathbb{R}^d$:

$$A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \quad a_\alpha \in C^\infty(\mathbb{R}^d).$$

If we let

$$a(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha,$$

then we can write for any $u \in C_c^\infty(\mathbb{R}^d)$

$$Au(x) = a(x, D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} a(x, \xi) u(y)\, \mathrm{d}y\mathrm{d}\xi.$$

The last integral makes sense even if we replace the polynomial $a(x,\xi)$ in ξ by a **symbol** growing at most polynomially in $\xi \in \mathbb{R}^d$. That is a **pseudodifferential operator** (ΨDO, or PsDO). We are going to develop a pseudodifferential calculus for an appropriate symbol class, and discuss its applications.

Remark. The last integral has to be interpreted as an iterated integral; The integrand is not integrable in (y,ξ). However, we can also justify it as an **oscillatory integral**, as discussed in the following section.

§ 1.2 Oscillatory Integrals

For any $x \in \mathbb{R}^d$ we let

$$\langle x \rangle = (1+x^2)^{1/2} \in C^\infty(\mathbb{R}^d).$$

Lemma 1.1. 1. For any $x \in \mathbb{R}^d$

$$\frac{1}{\sqrt{2}}(1+|x|) \le \langle x \rangle \le 1+|x|.$$

2. For any $\alpha \in \mathbb{N}_0^d$ there exists $C_\alpha > 0$ such that for any $x \in \mathbb{R}^d$

$$|\partial^\alpha \langle x \rangle| \le C_\alpha \langle x \rangle^{1-|\alpha|}.$$

3. (**Peetre's inequality**) For any $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$

$$\langle x+y \rangle^s \le 2^{|s|} \langle x \rangle^{|s|} \langle y \rangle^s.$$

Proof. 1, 2. We omit the proofs.

3. By the assertion 1 we can estimate

$$\begin{aligned}\langle x+y \rangle &\le 1+|x+y| \le 1+|x|+|y| \\ &\le (1+|x|)(1+|y|) \le 2\langle x \rangle \langle y \rangle.\end{aligned}$$

This implies the assertion for $s \ge 0$. The same estimate also implies

$$\langle y \rangle^{-1} \le 2\langle x \rangle \langle x+y \rangle^{-1}.$$

If we replace x by $-x$, and then y by $x+y$, it follows that

$$\langle x+y \rangle^{-1} \le 2\langle x \rangle \langle y \rangle^{-1},$$

which implies the assertion for $s \le 0$. Hence we are done. □

◦ Oscillatory Integrals

For any $m, \delta \in \mathbb{R}$ we define the set of **amplitude functions** as

$$A_\delta^m(\mathbb{R}^d) = \Big\{ a \in C^\infty(\mathbb{R}^d);\ \forall \alpha \in \mathbb{N}_0^d \ \sup_{x \in \mathbb{R}^d} \langle x \rangle^{-m-\delta|\alpha|} |\partial^\alpha a(x)| < \infty \Big\}.$$

For any $k \in \mathbb{N}_0$ define a **seminorm** $|\cdot|_k$ on $A_\delta^m(\mathbb{R}^d)$ as

$$|a|_k = |a|_{k, A_\delta^m} = \sup\Big\{ \langle x \rangle^{-m-\delta|\alpha|} |\partial^\alpha a(x)|;\ |\alpha| \le k,\ x \in \mathbb{R}^d \Big\}.$$

Remark. Obviously, $A_\delta^m(\mathbb{R}^d)$ is a **Fréchet space** with respect to the family $\{|\cdot|_k\}_{k \in \mathbb{N}_0}$ of seminorms.

Problem. Let $\chi \in \mathcal{S}(\mathbb{R}^d)$. Show $\chi(\epsilon x) \in A^0_{-1}(\mathbb{R}^d)$ uniformly in $\epsilon \in (0,1)$, i.e., for any $\alpha \in \mathbb{N}_0^d$ there exists $C > 0$ such that for any $\epsilon \in (0,1)$ and $x \in \mathbb{R}^d$

$$\left|\partial^\alpha(\chi(\epsilon x))\right| \le C\langle x\rangle^{-|\alpha|}.$$

Solution. Take any $\alpha \in \mathbb{N}_0^d$. Since χ is rapidly decreasing, we can compute and bound it as

$$\begin{aligned}\left|\partial^\alpha(\chi(\epsilon x))\right| &= \epsilon^{|\alpha|}\left|(\partial^\alpha\chi)(\epsilon x)\right| \le C\epsilon^{|\alpha|}\langle \epsilon x\rangle^{-|\alpha|}\\ &\le C\epsilon^{|\alpha|}\left(\epsilon^2+\epsilon^2x^2\right)^{-|\alpha|/2} = C\langle x\rangle^{-|\alpha|}.\end{aligned}$$

Hence we are done. □

Remark. Of course, for any fixed $\epsilon \in (0,1)$ we have $\chi(\epsilon x) \in A^m_\delta(\mathbb{R}^d)$ for all $m, \delta \in \mathbb{R}$.

Theorem 1.2. Let Q be a non-degenerate real symmetric matrix of order d, and let $m \in \mathbb{R}$ and $\delta < 1$. Then for any $a \in A^m_\delta(\mathbb{R}^d)$ and $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$ there exists the limit

$$I_Q(a) := \lim_{\epsilon\to+0}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\chi(\epsilon x)a(x)\,\mathrm{d}x, \qquad (\spadesuit)$$

and it is independent of choice of $\chi \in \mathcal{S}(\mathbb{R}^d)$. Moreover, there exist $k \in \mathbb{N}_0$ and $C > 0$ such that for any $a \in A^m_\delta(\mathbb{R}^d)$

$$|I_Q(a)| \le C|a|_{k,A^m_\delta}.$$

Remark. The last bound implies $I_Q\colon A^m_\delta(\mathbb{R}^d) \to \mathbb{C}$ is continuous.

Proof. Noting that for any $x, y \in \mathbb{R}^d$

$$y\partial\Bigl(\frac{xQx}{2}\Bigr) = \frac{1}{2}\sum_{j=1}^d y_j(e_jQx + xQe_j) = yQx,$$

we can deduce

$$\mathrm{e}^{\mathrm{i}xQx/2} = {}^tL\mathrm{e}^{\mathrm{i}xQx/2};\quad {}^tL = \langle x\rangle^{-2}\bigl(1 + xQ^{-1}D\bigr).$$

Substitute the above identity into the integrand of ($\spadesuit$), and integrate it by parts. Repeat this precedure, and we obtain

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\chi(\epsilon x)a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}L^k\bigl(\chi(\epsilon x)a(x)\bigr)\,\mathrm{d}x$$

for any $k \in \mathbb{N}_0$. Since L is of the form

$$L = c_0 + \sum_{j=1}^d c_j\partial_j;\quad c_0 \in A^{-2}_{-1}(\mathbb{R}^d),\quad c_j \in A^{-1}_{-1}(\mathbb{R}^d),$$

there exists $C > 0$ such that for any $\epsilon \in (0,1)$ and $a \in A^m_\delta(\mathbb{R}^d)$

$$\left|L^k\bigl(\chi(\epsilon x)a(x)\bigr)\right| \le C|a|_{k,A^m_\delta}\langle x\rangle^{m-k\min\{2,1-\delta\}}. \qquad (\heartsuit)$$

We also note there exists a pointwise limit

$$\lim_{\epsilon\to+0} L^k\bigl(\chi(\epsilon x)a(x)\bigr) = L^ka(x).$$

Then, if we choose $k \in \mathbb{N}_0$ such that $m - k\min\{2, 1-\delta\} < -d$, it follows by the Lebesgue convergence theorem that

$$I_Q(a) = \lim_{\epsilon\to+0}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\chi(\epsilon x)a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}L^ka(x)\,\mathrm{d}x.$$

Certainly the last expression is independent of χ. Combined with ($\heartsuit$), it also implies the asserted bound. We are done. □

Remarks. 1. The limit (♠) from Theorem 1.2 is called an **oscillatory integral**, and is denoted simply by

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\chi(\epsilon x) a(x)\,\mathrm{d}x.$$

The notation is compatible with the case $a \in L^1(\mathbb{R}^d)$.

2. We can also define the oscillatory integral as

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} L^k a(x)\,\mathrm{d}x,$$

where L^k is from the proof of Theorem 1.2. Practically, in order to compute an oscillatory integral we may implement *any* formal integrations by parts until the integrand gets integrable, see also Lemma 1.3.3.

Lemma 1.3. Let Q be a non-degenerate real symmetric matrix of order d, and let $a \in A^m_\delta(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$.

1. For any $c \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \mathrm{e}^{\mathrm{i}cQc/2} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}yQy/2}\big(\mathrm{e}^{\mathrm{i}cQy} a(y+c)\big)\,\mathrm{d}y.$$

2. For any real invertible matrix P of order d

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2} a(Py)|\det P|\,\mathrm{d}y.$$

3. For any $\alpha \in \mathbb{N}_0^d$

$$\int_{\mathbb{R}^d} \big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\big) a(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} \partial^\alpha a(x)\,\mathrm{d}x.$$

Proof. 1 and 2. We can prove 1 and 2 very similarly, and here we disucss only 2. Let $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$, and then by definition of the oscillatory integral

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2} a(x)\,\mathrm{d}x &= \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\chi(\epsilon x) a(x)\,\mathrm{d}x \\ &= \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2}\chi(\epsilon Py) a(Py)|\det P|\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y({}^tPQP)y/2} a(Py)|\det P|\,\mathrm{d}y.\end{aligned}$$

This implies the assertion.

3. Similarly to the above, let $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$. Then

$$\begin{aligned}&\int_{\mathbb{R}^d} \big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\big) a(x)\,\mathrm{d}x \\ &= \lim_{\epsilon\to+0} \int_{\mathbb{R}^d} \big(\partial^\alpha \mathrm{e}^{\mathrm{i}xQx/2}\big)\chi(\epsilon x) a(x)\,\mathrm{d}x \\ &= \lim_{\epsilon\to+0} (-1)^{|\alpha|} \Bigg[\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\chi(\epsilon x)\partial^\alpha a(x)\,\mathrm{d}x \\ &\qquad + \sum_{|\beta|\ge 1} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xQx/2}\big(\partial^\beta \chi(\epsilon x)\big)\big(\partial^{\alpha-\beta} a(x)\big)\,\mathrm{d}x\Bigg].\end{aligned}$$

For the second integral in the above square brackets we can further implement integrations by parts, e.g., by using L from the proof of Theorem 1.2, and then we can verify that it converges to 0 as $\epsilon \to +0$. Thus we obtain the assertion. □

§ 1.3 Expansion Formula

Definition. Let Q be a non-degenerate real symmetric matrix of order d, and let $u \in \mathcal{S}'(\mathbb{R}^d)$. We define

$$\mathrm{e}^{\mathrm{i}DQD/2}u = \mathcal{F}^*\mathrm{e}^{\mathrm{i}\xi Q\xi/2}\mathcal{F}u \in \mathcal{S}'(\mathbb{R}^d).$$

Theorem 1.4. Let Q be a non-degenerate real symmetric matrix of order d, and let $a \in A^m_\delta(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$. Then

$$\mathrm{e}^{\mathrm{i}DQD/2}a(x) = \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{(2\pi)^{d/2}|\det Q|^{1/2}}\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}yQ^{-1}y/2}a(x+y)\,\mathrm{d}y.$$

Remark. As for $a \in A^m_\delta(\mathbb{R}^d)$ we can compute pointwise values of $\mathrm{e}^{\mathrm{i}DQD/2}a$ as an oscillatory integral.

Theorem 1.5. There exists $C > 0$ dependent only on the dimension d such that for any non-degenerate real symmetric matrix Q of order d, $a \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$

$$\mathrm{e}^{\mathrm{i}DQD/2}a(x) = \sum_{k=0}^{N-1}\frac{\mathrm{i}^k}{2^k k!}(DQD)^k a(x) + R_N(a)$$

with

$$\big|R_N(a)\big| \le \frac{C}{2^N N!}\sum_{|\alpha|\le d+1}\big\|\partial^\alpha(DQD)^N a\big\|_{L^1}.$$

Lemma 1.6. Let Q be a non-degenerate real symmetric matrix of order d. Then

$$\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\Big)(\xi) = \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{|\det Q|^{1/2}}\mathrm{e}^{-\mathrm{i}\xi Q^{-1}\xi/2}.$$

Proof. Step 1. We first let $d = 1$. Since $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is continuous, we can proceed as

$$\begin{aligned}\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}Qx^2/2}\Big)(\xi) &= \lim_{\epsilon\to+0}\Big(\mathcal{F}\mathrm{e}^{-(\epsilon-\mathrm{i}Q)x^2/2}\Big)(\xi)\\ &= \lim_{\epsilon\to+0}\Big(\epsilon-\mathrm{i}Q\Big)^{-1/2}\mathrm{e}^{-(\epsilon-\mathrm{i}Q)^{-1}\xi^2/2}\\ &= \frac{\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}}{|Q|^{1/2}}\mathrm{e}^{-\mathrm{i}Q^{-1}\xi^2/2}.\end{aligned}$$

Thus the assertion for $d = 1$ is verified.

Step 2. There exists an invertible real matrix P such that

$${}^tPQP = \operatorname{diag}(I_p, -I_q),$$

where I_p, I_q are the identity matrices of order $p, q \in \mathbb{N}_0$ with $p + q = d$, respectively. Changing variables as $x = Py$ and spliting $y = (y', y'') \in \mathbb{R}^p \times \mathbb{R}^q$, we can compute

$$\begin{aligned}&\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\Big)(P^{-1}\eta)\\ &= \lim_{\epsilon\to+0}\Big(\mathcal{F}\mathrm{e}^{\mathrm{i}xQx/2}\mathrm{e}^{-\epsilon x({}^tP^{-1}P^{-1})x}\Big)(P^{-1}\eta)\\ &= \lim_{\epsilon\to+0}(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}y\eta}\mathrm{e}^{\mathrm{i}(y'^2-y''^2)/2}\mathrm{e}^{-\epsilon y^2}|\det P|\,\mathrm{d}y\\ &= |\det P|\mathrm{e}^{\mathrm{i}\pi(\operatorname{sgn} Q)/4}\mathrm{e}^{-\mathrm{i}(\eta'^2-\eta''^2)/2},\end{aligned}$$

where in the last equality we use the result from Step 1. Finally let $\eta = P\xi$, and we obtain the assertion. □

Proof of Theorem 1.4. Let $a \in C_c^\infty(\mathbb{R}^d)$. Then it follows by change of variables, the Plancherel theorem and Lemma 1.6

$$\begin{aligned} e^{iDQD/2}a(x) &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\xi Q\xi/2}\Big(\int_{\mathbb{R}^d} e^{-iy\xi}a(x+y)\,dy\Big)\,d\xi \\ &= \frac{e^{i\pi(\operatorname{sgn} Q)/4}}{(2\pi)^{d/2}|\det Q|^{1/2}}\int_{\mathbb{R}^d} e^{-iyQ^{-1}y/2}a(x+y)\,dy. \end{aligned}$$

Then, since the right-hand side of the asserted identity is continuous on $A_\delta^m(\mathbb{R}^d)$ byTheorem 1.2, we obtain the assertion. □

Proof of Theorem 1.5. Recall by Taylor's theorem for any $N \in \mathbb{N}$ and $t \in \mathbb{R}$

$$e^{it} = \sum_{k=0}^{N-1}\frac{(it)^k}{k!} + \frac{i^N}{(N-1)!}\int_0^t e^{is}(t-s)^{N-1}\,ds,$$

so that we can write

$$e^{i\xi Q\xi/2} = \sum_{k=0}^{N-1}\frac{(i\xi Q\xi)^k}{2^k k!} + r_N(\xi); \quad |r_N(\xi)| \le \frac{|\xi Q\xi|^N}{2^N N!}.$$

Substitute the above expansion into the definition of $e^{iDQD/2}a$ and implement the Fourier inversion formula, and then

$$e^{iDQD/2}a(x) = \sum_{k=0}^{N-1}\frac{i^k}{2^k k!}(DQD)^k u(x) + R_N(a)$$

with

$$|R_N(a)| \le \frac{1}{(2\pi)^{d/2}2^N N!}\int_{\mathbb{R}^d}\Big|\Big(\mathcal{F}(DQD)^N a\Big)(\xi)\Big|\,d\xi.$$

Finally it suffices to show that for any $v \in C_c^\infty(\mathbb{R}^d)$

$$\|\mathcal{F}v\|_{L^1} \le C\sum_{|\alpha|\le d+1}\|\partial^\alpha v\|_{L^1}.$$

However, it is clear since

$$\mathcal{F}v(\xi) = (2\pi)^{-d/2}\langle\xi\rangle^{-2(d+1)}\int_{\mathbb{R}^d} e^{-ix\xi}(1+\xi D)^{d+1}v(x)\,dx.$$

Thus we are done. □

Corollary 1.7 (**Stationary phase theorem**)**.** There exists $C > 0$ dependent only on the dimension d such that for any non-degenerate real symmetric matrix Q of order d, $a \in C_c^\infty(\mathbb{R}^d)$, $N \in \mathbb{N}$ and $h > 0$

$$\begin{aligned} &\int_{\mathbb{R}^d} e^{ixQx/(2h)}a(x)\,dx \\ &= \sum_{k=0}^{N-1}\frac{(2\pi)^{d/2}h^{k+d/2}e^{i\pi(\operatorname{sgn} Q)/4}}{|\det Q|^{1/2}(2i)^k k!}\Big((DQ^{-1}D)^k a\Big)(0) + R_N(a,h) \end{aligned}$$

with

$$\Big|R_N(a,h)\Big| \le \frac{Ch^{N+d/2}}{|\det Q|^{1/2}2^N N!}\sum_{|\alpha|\le d+1}\Big\|\partial^\alpha(DQ^{-1}D)^N a\Big\|_{L^1}.$$

Proof. The assertion is clear by Theorems 1.4 and 1.5. □

Remarks. 1. As $h \to +0$, the rapid oscillatory factor $\mathrm{e}^{\mathrm{i}xQx/(2h)}$ cancels contributions from the amplitude a. However, the oscillation is slightly milder at the stationary point $x = 0$ of the phase function. This is why the behavior of a at around $x = 0$ dominates the asymptotics.

2. The **semiclassical parameter** $h > 0$, rooted in the **Planck constant**, plays a fundamental role in the **semiclassical analysis**. However, in this course we do not discuss it.

Problem. Show the following extended version of the "pointwise Fourier inversion formula": For any $a \in A^m_\delta(\mathbb{R}^d)$ with $m \in \mathbb{R}$ and $\delta < 1$ and for any $\alpha \in \mathbb{N}_0^d$ and $x' \in \mathbb{R}^d$

$$(2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} \xi^\alpha a(x)\,\mathrm{d}x\mathrm{d}\xi = (D^\alpha a)(x').$$

Remark. This is an oscillatory integral on $\mathbb{R}^{2d} = \mathbb{R}^d_x \times \mathbb{R}^d_\xi$, not on $\mathbb{R}^d$, with a phase function

$$-x\xi = 4^{-1}\big((x-\xi)^2 - (x+\xi)^2\big)$$

and an amplitude $\mathrm{e}^{\mathrm{i}x'\xi}\xi^\alpha a(x) \in A^{|\alpha|+\max\{m,0\}}_{\max\{\delta,0\}}(\mathbb{R}^{2d})$.

Solution. By Lemma 1.3 it suffices to prove the assertion for $\alpha = 0$. By definition of oscillatory integrals, take any $\chi \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(0) = 1$, and then we can compute

$$\begin{aligned}
&(2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} a(x)\,\mathrm{d}x\mathrm{d}\xi \\
&= \lim_{\epsilon\to+0} (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x'-x)\xi} \chi(\epsilon x)\chi(\epsilon\xi) a(x)\,\mathrm{d}x\mathrm{d}\xi \\
&= \lim_{\epsilon\to+0} (2\pi\epsilon)^{-d/2} \int_{\mathbb{R}^d} (\mathcal{F}\chi)((x-x')/\epsilon)\chi(\epsilon x) a(x)\,\mathrm{d}x \\
&= \lim_{\epsilon\to+0} (2\pi)^{-d/2} \int_{\mathbb{R}^d} (\mathcal{F}\chi)(\eta)\chi(\epsilon(x'+\epsilon\eta)) a(x'+\epsilon\eta)\,\mathrm{d}\eta \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} a(x')(\mathcal{F}\chi)(\eta)\,\mathrm{d}\eta \\
&= a(x').
\end{aligned}$$

Hence we are done. □

Chapter 2
Pseudodifferential Calculus

§ 2.1 Pseudodifferential Operators

Definition. Let $m, \rho, \delta \in \mathbb{R}$. We denote by $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ the set of all the functions $a \in C^\infty(\mathbb{R}^{2d})$ satisfying that for any $\alpha, \beta \in \mathbb{N}_0^d$ there exists $C > 0$ such that for any $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|}.$$

We call $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ the **Kohn–Nirenberg** (or **Hörmander**) **symbol class**, and its element a **symbol of order** m. In addition, we set

$$S^\infty_{\rho,\delta}(\mathbb{R}^{2d}) = \bigcup_{m\in\mathbb{R}} S^m_{\rho,\delta}(\mathbb{R}^{2d}), \quad S^{-\infty}(\mathbb{R}^{2d}) = \bigcap_{m\in\mathbb{R}} S^m_{\rho,\delta}(\mathbb{R}^{2d}).$$

We often write $S^m(\mathbb{R}^{2d}) = S^m_{1,0}(\mathbb{R}^{2d})$ for short.

Remarks. 1. In order to have an appropriate pseudodifferential calculus available it is typically assumed that

$$0 \le \delta < \rho \le 1, \quad \text{or} \quad 1-\rho \le \delta < \rho \le 1.$$

2. Some authors define $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ as the set of all the functions $a \in C^\infty(\mathbb{R}^{2d})$ satisfying that for any $\alpha, \beta \in \mathbb{N}_0^d$ and $K \Subset \mathbb{R}^d$ there exists $C > 0$ such that for any $(x,\xi) \in K \times \mathbb{R}^d$

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|}.$$

3. There are many other variations of symbol classes, including semiclassical ones.

4. The symbol class $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ is a Fréchet space with respect to a family of seminorms given by

$$|a|_j = |a|_{j,S^m_{\rho,\delta}} = \sup\Big\{\langle\xi\rangle^{-m-\delta|\alpha|+\rho|\beta|}\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right|; \\ |\alpha| + |\beta| \le j,\ (x,\xi) \in \mathbb{R}^{2d}\Big\}.$$

Problem. 1. Show that, if $l \le m$, $\sigma \ge \rho$ and $\epsilon \le \delta$, then

$$S^l_{\sigma,\epsilon}(\mathbb{R}^{2d}) \subset S^m_{\rho,\delta}(\mathbb{R}^{2d}).$$

2. Show that for any $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$, $b \in S^l_{\rho,\delta}(\mathbb{R}^{2d})$ and $\alpha, \beta \in \mathbb{N}_0^d$

$$\partial_x^\alpha \partial_\xi^\beta a \in S^{m+\delta|\alpha|-\rho|\beta|}_{\rho,\delta}(\mathbb{R}^{2d}), \quad ab \in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d}).$$

Solution. We omit it. □

Examples. 1. Consider

$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha; \quad a_\alpha \in C^\infty(\mathbb{R}^d).$$

If a_α for all $|\alpha| \le m$ satisfy that for any $\beta \in \mathbb{N}_0^d$

$$\sup_{x\in\mathbb{R}^d} |\partial^\beta a_\alpha(x)| < \infty, \qquad (\heartsuit)$$

then obviously $a \in S^m(\mathbb{R}^{2d})$. Even if a_α dissatisfy $(\heartsuit)$, take any $\chi \in C_c^\infty(\mathbb{R}^d)$, and then

$$\chi(x)a(x,\xi) \in S^m(\mathbb{R}^{2d}).$$

We can still discuss local properties of a PDO by letting $\chi(x) = 1$ in a neighborhood of a point of our interest.

2. For any $m \in \mathbb{R}$ we have $\langle\xi\rangle^m \in S^m(\mathbb{R}^{2d})$.

3. Assume $a \in C^\infty(\mathbb{R}^{2d})$ is **positively homogeneous of degree** $m \in \mathbb{R}$ in $|\xi| \geq 1$, i.e., for any $x \in \mathbb{R}^d$, $|\xi| \geq 1$ and $t \geq 1$

$$a(x, t\xi) = t^m a(x, \xi).$$

In addition, assume for simplicity

$$\pi_1(\operatorname{supp} a) \Subset \mathbb{R}^d,$$

where $\pi_1 \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is the first projection. Then we have $a \in S^m(\mathbb{R}^{2d})$.

Definition. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $\rho > -1$ and $\delta < 1$. Define the **pseudodifferential operator** $a(x, D)$ **of order** m as, for any $u \in \mathcal{S}(\mathbb{R}^d)$,

$$a(x, D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} a(x, \xi) u(y)\, \mathrm{d}y\mathrm{d}\xi.$$

We denote

$$\Psi^m_{\rho,\delta}(\mathbb{R}^d) = \left\{a(x, D);\ a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})\right\},$$

and similarly for $\Psi^\infty_{\rho,\delta}(\mathbb{R}^d)$, $\Psi^{-\infty}(\mathbb{R}^d)$ and $\Psi^m(\mathbb{R}^d)$. In particular, an element of $\Psi^{-\infty}(\mathbb{R}^d)$ is called a **smoothing operator**.

Remarks. 1. Such a systematic procedure to assign operators to symbols is called a **quantization**, as in the quantum mechanics. There are various quantizations.

2. It is also common to use the notation $\mathrm{Op}(a)$ for $a(x, D)$.

3. The **semiclassical pseudodifferential operator** is defined as

$$\mathrm{Op}_h(a) = a(x, hD).$$

Here $h > 0$ is the semiclassical parameter.

4. The operator $\mathrm{e}^{\mathrm{i}DQD/2}$ from the previous chapter may be considered as a pseudodifferential operator, but the associated symbol $\mathrm{e}^{\mathrm{i}\xi Q\xi/2}$ is in a much worse class.

Theorem 2.1. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $\rho > -1$ and $\delta < 1$. Then $a(x, D)$ is a continuous operator on $\mathcal{S}(\mathbb{R}^d)$.

Proof. For any $N \in \mathbb{N}_0$ we can write

$$a(x, D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} \langle\xi\rangle^{-2N} a(x, \xi) \langle D_y\rangle^{2N} u(y)\, \mathrm{d}y\mathrm{d}\xi.$$

Here the integrand is estimated as, for any $\beta \in \mathbb{N}_0^d$,

$$\begin{aligned} &\left|\partial_x^\beta \mathrm{e}^{\mathrm{i}(x-y)\xi} \langle\xi\rangle^{-2N} a(x, \xi) \langle D_y\rangle^{2N} u(y)\right| \\ &\qquad \leq C_\alpha \langle\xi\rangle^{m+|\beta|-2N} \left|\langle D_y\rangle^{2N} u(y)\right|, \end{aligned}$$

and hence we can differentiate $a(x, D)u(x)$ as much as we want by retaking N be larger beforehand. Thus for any $\beta \in \mathbb{N}_0^d$

$$\partial^\beta a(x,D)u(x) = (2\pi)^{-d} \sum_{\tau \in \mathbb{N}_0^d} \binom{\beta}{\tau} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} \cdot (\mathrm{i}\xi)^{\beta-\tau} \langle \xi \rangle^{-2N} \partial_x^\tau a(x,\xi) \langle D_y \rangle^{2N} u(y)\, \mathrm{d}y\mathrm{d}\xi.$$

Futhermore, by Lemma 1.3 for any $\alpha \in \mathbb{N}_0^d$

$$x^\alpha \partial^\beta a(x,D)u(x) = (2\pi)^{-d} \sum_{\tau,\sigma \in \mathbb{N}_0^d} \binom{\alpha}{\sigma}\binom{\beta}{\tau} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} y^{\alpha-\sigma} \cdot \Big((-D_\xi)^\sigma (\mathrm{i}\xi)^{\beta-\tau} \langle \xi \rangle^{-2N} \partial_x^\tau a(x,\xi) \Big) \langle D_y \rangle^{2N} u(y)\, \mathrm{d}y\mathrm{d}\xi.$$

Therefore for any $k \in \mathbb{N}_0$ by letting N be sufficiently large we can find $C > 0$ and $l \in \mathbb{N}_0$ such that for any $u \in \mathcal{S}(\mathbb{R}^d)$

$$|a(x,D)u|_{k,\mathcal{S}} \le C|u|_{l,\mathcal{S}}.$$

This implies the assertion. □

§ 2.2 Asymptotic Summation

Theorem 2.2. For each $j \in \mathbb{N}_0$ given $a_j \in S^{m_j}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$m := m_0 > m_1 > m_2 > \cdots > m_j \to -\infty \quad \text{as } j \to \infty,$$

and $\rho \le 1$ and $\delta \in \mathbb{R}$. Then there exists $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ such that for any $k \in \mathbb{N}_0$

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{\rho,\delta}(\mathbb{R}^{2d}). \tag{♠}$$

Such a is unique up to $S^{-\infty}(\mathbb{R}^{2d})$. Moreover, one can choose $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$\operatorname{supp} a \subset \overline{\Big(\bigcup_{j=0}^{\infty} \operatorname{supp} a_j \Big)}. \tag{♡}$$

Definition. Under the setting of Theorem 2.2 we write

$$a \sim \sum_{j=0}^{\infty} a_j,$$

and call it the **asymptotic sum** or **asymptotic expansion**. In addition, when $a_0 \not\equiv 0$, we call a_0 the **principal symbol** of a, or of $A := a(x,D)$, and often write it as

$$\sigma(A) = a_0.$$

Note the principal symbol is not unique by definition, and the above identity has to be understood up to lower order errors.

Proof. Step 1. Fix $\chi \in C^\infty(\mathbb{R}^d)$ satisfying

$$\chi(\xi) = \begin{cases} 0 & \text{for } |\xi| \le 1, \\ 1 & \text{for } |\xi| \ge 2, \end{cases}$$

and we construct $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ of the form

$$a(x,\xi) = \sum_{j=0}^{\infty} \chi(\epsilon_j \xi) a_j(x,\xi)$$

with

$$1 > \epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \to +0.$$

Note the above sum is locally finite, and hence is locally bounded and smooth. Note also, then, (♡) is automatically satisfied.

Step 2. Here we are going to choose

$$1 > \epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \to +0$$

such that for any $j \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^d$ with $|\alpha| + |\beta| \le j$

$$\left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon_j \xi) a_j(x,\xi))\right| \le 2^{-j} \langle \xi \rangle^{m_j + 1 + \delta|\alpha| - \rho|\beta|} \qquad (\clubsuit)$$

For that we note for any $j \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^d$ there exists $C_{j\alpha\beta} > 0$ such that uniformly in $\epsilon \in (0,1)$

$$\left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon \xi) a_j(x,\xi))\right| \le C_{j\alpha\beta} \langle \xi \rangle^{m_j + \delta|\alpha| - \rho|\beta|}, \qquad (\diamondsuit)$$

since

$$\epsilon \le 2|\xi|^{-1} \le 4(1+|\xi|)^{-1} \quad \text{on } \operatorname{supp}\left(\partial_\xi^\gamma(\chi(\epsilon\xi))\right) \text{ with } |\gamma| \ge 1.$$

However, since

$$1 \le \epsilon|\xi| \le \epsilon\langle\xi\rangle \quad \text{on } \operatorname{supp} \chi(\epsilon\xi),$$

we can further deduce uniformly in $\epsilon \in (0,1)$

$$\left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon \xi) a_j(x,\xi))\right| \le C_{j\alpha\beta}\, \epsilon \langle \xi \rangle^{m_j + 1 + \delta|\alpha| - \rho|\beta|}.$$

Now we first choose

$$\epsilon_0 < \min\left\{1, (C_{000})^{-1}\right\},$$

and then $(\clubsuit)$ is satisfied for $j = 0$. Next, suppose we have found $\epsilon_0, \ldots, \epsilon_{j-1}$ as claimed, and then it suffices to choose

$$\epsilon_j < \min\left\{j^{-1}, \epsilon_{j-1}, 2^{-j}(C_{j\alpha\beta})^{-1};\ |\alpha| + |\beta| \le j\right\}.$$

Thus by induction we obtain $\epsilon_0, \epsilon_1, \ldots$ as claimed.

Step 3. Here we prove a from Steps 1 and 2 belongs to $S^m_{\rho,\delta}(\mathbb{R}^{2d})$. In fact, for any $\alpha, \beta \in \mathbb{N}_0^d$, if we choose $k \in \mathbb{N}_0$ such that

$$k \ge |\alpha| + |\beta| \quad \text{and} \quad m_k + 1 \le m,$$

then by $(\diamondsuit)$ and $(\clubsuit)$

$$\begin{aligned}
\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| &\le \sum_{j=0}^{k-1} \left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon_j \xi) a_j(x,\xi))\right| \\
&\quad + \sum_{j=k}^{\infty} \left|\partial_x^\alpha \partial_\xi^\beta (\chi(\epsilon_j \xi) a_j(x,\xi))\right| \\
&\le \sum_{j=0}^{k-1} C_{j\alpha\beta} \langle\xi\rangle^{m_j + \delta|\alpha| - \rho|\beta|} + \sum_{j=k}^{\infty} 2^{-j} \langle\xi\rangle^{m_j + 1 + \delta|\alpha| - \rho|\beta|} \\
&\le C'_{\alpha\beta} \langle\xi\rangle^{m + \delta|\alpha| - \rho|\beta|}.
\end{aligned}$$

This implies the claim.

Step 4. Let us verify $(\spadesuit)$. For any $k \in \mathbb{N}_0$ decompose

$$a - \sum_{j=0}^{k-1} a_j = \sum_{j=0}^{k-1} \left(\chi(\epsilon_j \xi) - 1\right) a_j(x,\xi) + \sum_{j=k}^{\infty} \chi(\epsilon_j \xi) a_j(x,\xi).$$

Then the first sum on the right-hand side belongs to $S^{-\infty}(\mathbb{R}^{2d})$ since it vanishes for $|\xi| \ge 2/\epsilon_k$, while the second to $S^{m_k}_{\rho,\delta}(\mathbb{R}^{2d})$ similarly to Step 3. Thus the claim follows.

Step 5. Finally we discuss the uniqueness up to $S^{-\infty}(\mathbb{R}^{2d})$. If both of $a, b \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ satisfy $(\spadesuit)$, then for any $k \in \mathbb{N}_0$

$$a - b = \left(a - \sum_{j=0}^{k-1} a_j\right) - \left(b - \sum_{j=0}^{k-1} a_j\right) \in S^{m_k}_{\rho,\delta}(\mathbb{R}^{2d}),$$

so that $a - b \in S^{-\infty}(\mathbb{R}^{2d})$. Thus we are done. $\square$

Definition. Let $m \in \mathbb{R}$. $a \in S^m(\mathbb{R}^{2d})$, or $a(x,D) \in \Psi^m(\mathbb{R}^d)$, is **classical** (or **polyhomogeneous**) if a has an expansion

$$a \sim \sum_{j=0}^{\infty} a_j$$

such that, for each $j \in \mathbb{N}_0$, $a_j \in S^{m-j}(\mathbb{R}^{2d})$ is positively homogeneous of degree $m-j$ in $\xi \neq 0$. Although we actually need modifications around $\xi = 0$, we often abuse notation as above. We denote

$$S^m_{\mathrm{cl}}(\mathbb{R}^{2d}) = \Big\{a \in S^m(\mathbb{R}^{2d});\ a \text{ is classical}\Big\},$$
$$\Psi^m_{\mathrm{cl}}(\mathbb{R}^d) = \Big\{a(x,D);\ a \in S^m_{\mathrm{cl}}(\mathbb{R}^{2d})\Big\}.$$

Remark. Under homogeneity the principal symbol is unique.

Examples. 1. Any partial differential operator of order $m \in \mathbb{N}_0$:

$$A = a(x,D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

where $a_\alpha \in C^\infty(\mathbb{R}^d)$ has bounded derivatives, is classical. The principal symbol is given by

$$\sigma(A)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

2. For any $m \in \mathbb{R}$ the operator $\langle D\rangle^m \in \Psi^m(\mathbb{R}^{2d})$ is classical. In fact, by the Taylor expansion for any $|\xi| > 1$

$$\langle\xi\rangle^m = |\xi|^m\Big(1 + |\xi|^{-2}\Big)^{m/2} = \sum_{j=0}^{\infty} \frac{(m/2)(m/2-1)\cdots(m/2-j+1)}{j!}|\xi|^{m-2j}.$$

Problem (**Borel's theorem**)**.** Show that, given $c_\alpha \in \mathbb{R}$ for all $\alpha \in \mathbb{N}_0^d$, there exists $f \in C^\infty(\mathbb{R}^d)$ such that for any $\alpha \in \mathbb{N}_0^d$

$$(\partial^\alpha f)(0) = c_\alpha.$$

Solution. Step 1. Fix $\chi \in C^\infty(\mathbb{R}^d)$ satisfying

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| \ge 2, \end{cases}$$

and we construct $f \in C^\infty(\mathbb{R}^d)$ of the form

$$f(x) = \sum_{j=0}^{\infty} \chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha; \quad 1 < R_0 < R_1 < \cdots < R_j \to \infty.$$

Note the above sum is locally finite on $\mathbb{R}^d \setminus \{0\}$, hence locally bounded there. In addition, it is obviously finite at $x = 0$.

Step 2. Here we are going to choose

$$1 < R_0 < R_1 < \cdots < R_j \to \infty$$

such that any $j \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^d$ with $|\beta| \le j$

$$\left|\partial^\beta\left(\chi(R_j x)\sum_{|\alpha|=j}\frac{c_\alpha}{\alpha!}x^\alpha\right)\right| \le 2^{-j}|x|^{j-1-|\beta|}$$

Note that, thanks to supporting property of $\chi(Rx)$, for any $j \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^d$ there exists $C_{j\beta} > 0$ such that uniformly in $R \ge 1$

$$\left|\partial^\beta\left(\chi(Rx)\sum_{|\alpha|=j}\frac{c_\alpha}{\alpha!}x^\alpha\right)\right| \le C_{j\beta}R^{-1}|x|^{j-1-|\beta|}.$$

Then we can discuss similarly to the proof of Theorem 2.2. We omit the details.

Step 3. Now let $\beta \in \mathbb{N}_0^d$, and consider the following series:

$$\sum_{j=0}^{\infty} \partial^\beta \left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha \right) = \sum_{j=0}^{|\beta|} \partial^\beta \left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha \right) + \sum_{j=|\beta|+1}^{\infty} \partial^\beta \left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha \right).$$

The sum is pointwise finite on $\mathbb{R}^d$ similarly to Step 1. Moreover, it is uniformly and absolutely convergent due to the result from Step 2. Since $\beta \in \mathbb{N}_0^d$ is arbitrary, we can conclude $f \in C^\infty(\mathbb{R}^d)$ by induction, and differetiate it under the summation. Thus

$$(\partial^\beta f)(0) = \sum_{j=0}^{\infty} \partial^\beta \left(\chi(R_j x) \sum_{|\alpha|=j} \frac{c_\alpha}{\alpha!} x^\alpha \right) \Bigg|_{x=0} = c_\beta.$$

We are done. □

§ 2.3 Formal Adjoint

Theorem 2.3. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\delta \neq 1$, and define

$$a^*(x,\xi) = \mathrm{e}^{\mathrm{i}D_x D_\xi} \overline{a}(x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}y\eta} \overline{a}(x+y, \xi+\eta)\, \mathrm{d}y\mathrm{d}\eta.$$

Then $a^* \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$, and

$$a(x,D)^* = a^*(x,D).$$

Moreover, if $\delta < \rho$, then

$$a^* \sim \sum_{\alpha \in \mathbb{N}_0^d} \frac{1}{\mathrm{i}^{|\alpha|} \alpha!} \partial_x^\alpha \partial_\xi^\alpha \overline{a}.$$

Remarks. 1. The **formal adjoint** of an operator A on $\mathcal{S}(\mathbb{R}^d)$ is an operator A^* on $\mathcal{S}(\mathbb{R}^d)$ such that for any $u, v \in \mathcal{S}(\mathbb{R}^d)$

$$(Au, v) = (u, A^* v).$$

2. By Proposition 2.5 below we can also see uniqueness of the "adjoint symbol" $a^* \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$.

Proof. Step 1. We first show $a^* \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$. For that we are going to prove for any $\alpha, \beta \in \mathbb{N}_0^d$

$$\left| \partial_x^\alpha \partial_\xi^\beta a^*(x,\xi) \right| \le C_{\alpha\beta} \langle \xi \rangle^{m+\delta|\alpha|-\rho|\beta|}. \qquad (\diamondsuit)$$

However, since, as we can easily see,

$$\partial_x^\alpha \partial_\xi^\beta a^*(x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}y\eta} (\partial_x^\alpha \partial_\xi^\beta \overline{a})(x+y, \xi+\eta)\, \mathrm{d}y\mathrm{d}\eta,$$

it suffices to prove $(\diamondsuit)$ only for $\alpha = \beta = 0$.

Fix any $\chi \in C^\infty(\mathbb{R}^d)$ satisfying

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| \ge 2, \end{cases}$$

and we set

$$\begin{aligned} \chi_1(\xi,y,\eta) &= \chi\big(\langle\xi\rangle^\delta y\big)\chi\big(2\epsilon^{-1}\langle\xi\rangle^{-\rho}\eta\big), \\ \chi_2(\xi,y,\eta) &= \big[1 - \chi\big(\langle\xi\rangle^\delta y\big)\big]\chi\big(2\epsilon^{-1}\langle\xi\rangle^{-\rho}\eta\big), \\ \chi_3(\xi,y,\eta) &= \chi\big(\epsilon^{-1}\langle\xi\rangle^{-1}\eta\big) - \chi\big(2\epsilon^{-1}\langle\xi\rangle^{-\rho}\eta\big), \\ \chi_4(\xi,y,\eta) &= 1 - \chi\big(\epsilon^{-1}\langle\xi\rangle^{-1}\eta\big), \end{aligned}$$

where $\epsilon > 0$ is a fixed small constant such that

$$\begin{aligned} &c\langle\xi\rangle \le \langle\xi+\eta\rangle \le C\langle\xi\rangle \quad \text{on } \operatorname{supp}\chi_1 \cup \operatorname{supp}\chi_2 \cup \operatorname{supp}\chi_3, \\ &\langle\xi\rangle \le C\langle\eta\rangle,\ \langle\xi+\eta\rangle \le C\langle\eta\rangle \quad \text{on } \operatorname{supp}\chi_4. \end{aligned} \qquad (\spadesuit)$$

Using these cut-off functions, we split a^* into four parts as

$$a^* = I_1 + I_2 + I_3 + I_4$$

with

$$I_j(x,\xi) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}y\eta} \chi_j(\xi,y,\eta)\bar{a}(x+y,\xi+\eta)\,\mathrm{d}y\mathrm{d}\eta.$$

The terms I_2, I_3 and I_4 are estimated by integrations by parts. In fact, to estimate I_2, let

$${}^tL_1 = \left\langle \langle\xi\rangle^{-\rho}\eta \right\rangle^{-2}\left(1 - \langle\xi\rangle^{-2\rho}\eta D_y\right), \quad {}^tL_2 = -|y|^{-2} y D_\eta.$$

Then, noting (♠), we have for any $N \geq d+1$

$$\begin{aligned} |I_2(x,\xi)| &\leq C_1 \int_{\mathbb{R}^{2d}} \left| L_2^N L_1^N \chi_2(\xi,y,\eta)\bar{a}(x+y,\xi+\eta)\right| \mathrm{d}y\mathrm{d}\eta \\ &\leq C_2 \int_{\mathbb{R}^{2d}} \left\langle \langle\xi\rangle^{\delta} y\right\rangle^{-N} \left\langle \langle\xi\rangle^{-\rho}\eta\right\rangle^{-N} \langle\xi\rangle^{m-(\rho-\delta)N}\,\mathrm{d}y\mathrm{d}\eta \\ &\leq C_3 \langle\xi\rangle^{m-(\rho-\delta)(N-d)}. \end{aligned}$$

Thus I_2 satisfies (◇) for $\alpha=\beta=0$. Similarly, as for I_3, let

$${}^tL_3 = -|\eta|^{-2}\eta D_y, \quad {}^tL_4 = \left\langle \langle\xi\rangle^{\delta} y\right\rangle^{-2}\left(1 - \langle\xi\rangle^{2\delta} y D_\eta\right).$$

Then, noting (♠), we have for any $N \geq d+1$

$$\begin{aligned} |I_3(x,\xi)| &\leq C_4 \int_{\mathbb{R}^{2d}} \left| L_3^N L_4^N \chi_3(\xi,y,\eta)\bar{a}(x+y,\xi+\eta)\right| \mathrm{d}y\mathrm{d}\eta \\ &\leq C_5 \int_{\mathbb{R}^{2d}} \left(\eta + \langle\xi\rangle^{\rho}\right)^{-N} \left\langle \langle\xi\rangle^{\delta} y\right\rangle^{-N} \langle\xi\rangle^{m+\delta N}\,\mathrm{d}y\mathrm{d}\eta \\ &\leq C_6 \langle\xi\rangle^{m-(\rho-\delta)(N-d)}. \end{aligned}$$

Thus I_3 also satisfies (◇) for $\alpha=\beta=0$. As for I_4, let

$${}^tL_{y,\eta} = \langle (y,\eta)\rangle^{-2}(1 - \eta D_y - y D_\eta),$$

and fix $N_0 \in \mathbb{N}$ such that

$$-N_0 + |m| + \delta N_0 < -2d.$$

Then, noting (♠), we have for any $N \geq N_0$

$$\begin{aligned} |I_4| &\leq C_4 \int_{\mathbb{R}^{2d}} \left| L_{y,\eta}^N \chi_4(\xi,y,\eta)\bar{a}(x+y,\xi+\eta)\right| \mathrm{d}y\mathrm{d}\eta \\ &\leq C_5 \int_{\eta \geq \epsilon\langle\xi\rangle} \langle (y,\eta)\rangle^{-N} \langle\eta\rangle^{|m|+\delta N}\,\mathrm{d}y\mathrm{d}\eta \\ &\leq C_6 \langle\xi\rangle^{-(1-\delta)(N-N_0)}. \end{aligned}$$

Thus by letting N be large I_3 satisfies (◇) for $\alpha=\beta=0$.

Finally consider I_1. We change variables and use Theorem 1.4, so that

$$\begin{aligned} I_1 &= (2\pi)^{-d}\langle\xi\rangle^{d(\rho-\delta)} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}\langle\xi\rangle^{\rho-\delta} y\eta} \chi(y)\chi(\eta/\epsilon) \\ &\qquad \cdot \bar{a}\left(x + \langle\xi\rangle^{-\delta} y, \xi + \langle\xi\rangle^{\rho}\eta\right) \mathrm{d}y\mathrm{d}\eta \\ &= \mathrm{e}^{\mathrm{i}\langle\xi\rangle^{\delta-\rho} D_y D_\eta} \chi(y)\chi(\eta/\epsilon)\bar{a}\left(x + \langle\xi\rangle^{-\delta} y, \xi + \langle\xi\rangle^{\rho}\eta\right)\Big|_{(y,\eta)=(0,0)}. \end{aligned}$$

Apply Theorem 1.5, and then we obtain for any $N \in \mathbb{N}$

$$I_1 = \sum_{k=0}^{N-1} \frac{\mathrm{i}^k}{k!}(D_x D_\xi)^k \bar{a}(x,\xi) + R_N(x,\xi)$$

with

$$\begin{aligned} |R_N(x,\xi)| &\leq \frac{C_7}{N!}\langle\xi\rangle^{-(\rho-\delta)N} \sum_{|\alpha|\leq 2d+1} \Big\| \partial^\alpha (D_y D_\eta)^N \chi(y)\chi(\eta/\epsilon) \\ &\qquad \cdot \bar{a}\left(x + \langle\xi\rangle^{-\delta} y, \xi + \langle\xi\rangle^{\rho}\eta\right)\Big\|_{L^1_{y,\eta}} \\ &\leq C_8 \langle\xi\rangle^{m-(\rho-\delta)N}. \end{aligned}$$

Thus we can estimate I_1 as desired, and the claim is verified.

Step 2. The asserted asymptotic expansion is essentially done in Step 1. We omit the details.

Step 3. Finally we prove $a^*(x,D)$ is the formal adjoint of $a(x,D)$. For any $u,v\in\mathcal{S}(\mathbb{R}^d)$ we rewrite

$$\begin{aligned}&(2\pi)^{3d/2}(a(x,D)u,v)\\&=(2\pi)^{d/2}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^{2d}}\mathrm{e}^{\mathrm{i}(x-y)\xi}a(x,\xi)u(y)\,\mathrm{d}y\mathrm{d}\xi\Big)\overline{v}(x)\,\mathrm{d}x\\&=\int_{\mathbb{R}^{2d}}\mathrm{e}^{-\mathrm{i}x\eta}\Big(\int_{\mathbb{R}^{2d}}\mathrm{e}^{-\mathrm{i}y\xi}a(x,\xi)u(x+y)(\mathcal{F}^*\overline{v})(\eta)\,\mathrm{d}y\mathrm{d}\xi\Big)\,\mathrm{d}x\mathrm{d}\eta.\end{aligned}$$

Implement integrations by parts in (y,ξ), so that the integrand gets integrable in (y,ξ,x,η). Then by Fubini's theorem and Lemma 1.3 we can rewrite it as an oscillatory integral in (y,ξ,x,η)

as

$$\begin{aligned}&(2\pi)^{3d/2}(a(x,D)u,v)\\&=\int_{\mathbb{R}^{4d}}\mathrm{e}^{-\mathrm{i}x\eta-\mathrm{i}y\xi}a(x,\xi)u(x+y)(\mathcal{F}^*\overline{v})(\eta)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}x\mathrm{d}\eta\\&=\int_{\mathbb{R}^{4d}}\mathrm{e}^{-\mathrm{i}y\eta+\mathrm{i}x\xi}a(x+y,\xi+\eta)u(y)(\mathcal{F}^*\overline{v})(\eta)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}x\mathrm{d}\eta.\end{aligned}$$

Next, again, implement integrations by parts in (x,ξ) to have an integrable integrand, and apply Fubini's theorem. Then the definition of a^* appears, and we obtain

$$\begin{aligned}&(2\pi)^{3d/2}(a(x,D)u,v)\\&=(2\pi)^d\int_{\mathbb{R}^{2d}}\mathrm{e}^{-\mathrm{i}y\eta}\overline{a^*(y,\eta)}u(y)(\mathcal{F}^*\overline{v})(\eta)\,\mathrm{d}y\mathrm{d}\eta\\&=(2\pi)^{3d/2}(u,a^*(x,D)v).\end{aligned}$$

Hence we are done. □

Example. Let

$$A=a(x,D)=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha,\qquad a_\alpha\in C^\infty(\mathbb{R}^d).$$

Then the formal adjoint of A on $C_\mathrm{c}^\infty(\mathbb{R}^d)$ is computed by the Leibniz rule as

$$A^*=\sum_{|\alpha|\le m}D^\alpha\overline{a}_\alpha(x)=\sum_{\beta\in\mathbb{N}_0^d}\sum_{|\alpha|\le m}\binom{\alpha}{\beta}(D^\beta\overline{a}_\alpha)(x)D^{\alpha-\beta}.$$

Hence the adjoint symbol a^* is given by

$$a^*(x,\xi)=\sum_{\beta\in\mathbb{N}_0^d}\sum_{|\alpha|\le m}\binom{\alpha}{\beta}(D^\beta\overline{a}_\alpha)(x)\xi^{\alpha-\beta}=\sum_{\beta\in\mathbb{N}_0^d}\frac{1}{\mathrm{i}^{|\beta|}\beta!}\partial_x^\beta\partial_\xi^\beta\overline{a}(x,\xi),$$

which coincides with the asymptotic expansion.

◦ **Extension to tempered disributions**

Corollary 2.4. Let $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m\in\mathbb{R}$, $0\le\delta\le\rho\le 1$ and $\delta\neq 1$. Then $a(x,D)$ extends as a continuous operator on $\mathcal{S}'(\mathbb{R}^d)$.

Proof. For any $u\in\mathcal{S}'(\mathbb{R}^d)$ define $a(x,D)u\in\mathcal{S}'(\mathbb{R}^d)$ as, for any $\phi\in\mathcal{S}(\mathbb{R}^d)$,

$$(a(x,D)u,\phi)=(u,a^*(x,D)\phi).$$

Obviously this provides a continuous extension of $a(x,D)$ from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. We are done. □

Proposition 2.5. Let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\delta \ne 1$. Then for any $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$

$$\mathrm{e}^{-\mathrm{i}x\xi} a(x,D)\mathrm{e}^{\mathrm{i}x\xi} = a(x,\xi).$$

In particular, the quantization

$$S^m_{\rho,\delta}(\mathbb{R}^{2d}) \to \Psi^m_{\rho,\delta}(\mathbb{R}^d), \quad a(x,\xi) \mapsto a(x,D)$$

is bijective.

Proof. For any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we can compute

$$\begin{aligned}
&(2\pi)^{3d/2}\Big(\mathrm{e}^{-\mathrm{i}x\xi}a(x,D)\mathrm{e}^{\mathrm{i}x\xi},\phi\Big) = (2\pi)^{3d/2}\Big(\mathrm{e}^{\mathrm{i}x\xi}, a^*(x,D)\mathrm{e}^{\mathrm{i}x\xi}\phi\Big)\\
&= \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\xi}\Big(\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}(x-y)\eta - \mathrm{i}y\xi}\overline{a^*(x,\eta)}\Big(\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}y\zeta}\overline{\mathcal{F}\phi(\zeta)}\,\mathrm{d}\zeta\Big)\,\mathrm{d}y\mathrm{d}\eta\Big)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}y\eta}\overline{a^*(x,\xi+\eta)}\Big(\int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}(x+y)\zeta}\overline{\mathcal{F}\phi(\zeta)}\,\mathrm{d}\zeta\Big)\,\mathrm{d}y\mathrm{d}\eta\Big)\,\mathrm{d}x.
\end{aligned}$$

We integrate by parts in (y,η) to make the integrand integrable in (ζ,y,η). Then apply the Fubini's theorem, and we can proceed

$$\begin{aligned}
&(2\pi)^{3d/2}\Big(\mathrm{e}^{-\mathrm{i}x\xi}a(x,D)\mathrm{e}^{\mathrm{i}x\xi},\phi\Big)\\
&= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y\eta - \mathrm{i}(x+y)\zeta}\overline{a^*(x,\xi+\eta)}\,\overline{\mathcal{F}\phi(\zeta)}\,\mathrm{d}y\mathrm{d}\eta\Big)\,\mathrm{d}\zeta\Big)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^{2d}}\Big(\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}y\eta - \mathrm{i}x\zeta}\overline{a^*(x,\xi+\eta+\zeta)}\,\overline{\mathcal{F}\phi(\zeta)}\,\mathrm{d}y\mathrm{d}\eta\Big)\,\mathrm{d}\zeta\Big)\,\mathrm{d}x.
\end{aligned}$$

We integrate by parts in (y,η) and in (x,ζ), and then we can verify

$$\begin{aligned}
&(2\pi)^{3d/2}\Big(\mathrm{e}^{-\mathrm{i}x\xi}a(x,D)\mathrm{e}^{\mathrm{i}x\xi},\phi\Big)\\
&= \int_{\mathbb{R}^{4d}} \mathrm{e}^{\mathrm{i}y\eta - \mathrm{i}x\zeta}\overline{a^*(x-y,\xi+\eta)}\,\overline{\mathcal{F}\phi(\zeta)}\,\mathrm{d}y\mathrm{d}\eta\mathrm{d}\zeta\mathrm{d}x\\
&= (2\pi)^{d/2}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}y\eta}\overline{a^*(x-y,\xi+\eta)}\,\mathrm{d}y\mathrm{d}\eta\Big)\overline{\phi(x)}\,\mathrm{d}x\\
&= (2\pi)^{3d/2}\int_{\mathbb{R}^d}(a^*)^*(x,\xi)\overline{\phi(x)}\,\mathrm{d}x.
\end{aligned}$$

Since (passing through the Fourier space expression)

$$(a^*)^* = \mathrm{e}^{\mathrm{i}D_xD_\xi}\overline{\Big(\mathrm{e}^{\mathrm{i}D_xD_\xi}\overline{a}\Big)} = a,$$

we obtain the assertion. □

§ 2.4 Composition

Theorem 2.6. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ and $b \in S^l_{\rho,\delta}(\mathbb{R}^{2d})$ with $m,l \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\delta \ne 1$. Then there uniquely exists $a\#b \in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a(x,D)\circ b(x,D) = (a\#b)(x,D).$$

Moreover, $a\#b$ is expressed as

$$\begin{aligned}
(a\#b)(x,\xi) &= \mathrm{e}^{\mathrm{i}D_yD_\eta}a(x,\eta)b(y,\xi)\Big|_{(y,\eta)=(x,\xi)}\\
&= (2\pi)^{-d}\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}y\eta}a(x,\xi+\eta)b(x+y,\xi)\,\mathrm{d}y\mathrm{d}\eta,
\end{aligned} \tag{$\heartsuit$}$$

and, if $\delta < \rho$, then

$$a\#b \sim \sum_{\alpha\in\mathbb{N}_0^d} \frac{1}{\mathrm{i}^{|\alpha|}\alpha!}(\partial_\xi^\alpha a)(\partial_x^\alpha b).$$

Proof. Let $a\#b$ be given by (♡). Then we can verify $a\#b \in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d})$ and the asserted asymptotic expansion similarly to Steps 1 and 2 of the proof of Theorem 2.3. We omit the details. The uniqueness of the "composite symbol" is clear by Proposition 2.5 as long as it exists. Hence it remainds to show

$$a(x,D) \circ b(x,D) = (a\#b)(x,D),$$

where $a\#b$ is given by (♡). For any $u \in \mathcal{S}(\mathbb{R}^d)$ we can rewrite by change of variables

$$\begin{aligned}&(2\pi)^{2d} a(x,D) \circ b(x,D)u(x)\\ &= \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}y\xi} a(x,\xi)\Big(\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}z\eta} b(x+y,\eta)u(x+y+z)\,\mathrm{d}z\mathrm{d}\eta\Big)\,\mathrm{d}y\mathrm{d}\xi.\end{aligned}$$

Integrate it by parts in (z,η) sufficiently many times, and then in (y,ξ), so that the resulting integrand gets integrable in (z,η,y,ξ).

Then by Fubini's theorem and Lemma 1.3 we can rewrite it as

$$\begin{aligned}&(2\pi)^{2d} a(x,D) \circ b(x,D)u(x)\\ &= \int_{\mathbb{R}^{4d}} \mathrm{e}^{-\mathrm{i}y\xi-\mathrm{i}z\eta} a(x,\xi)b(x+y,\eta)u(x+y+z)\,\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi\\ &= \int_{\mathbb{R}^{4d}} \mathrm{e}^{-\mathrm{i}y\xi-\mathrm{i}z\eta} a(x,\xi+\eta)b(x+y,\eta)u(x+z)\,\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi.\end{aligned}$$

Again, integrate it by parts first in (y,ξ), and then in (z,η), and apply Fubini's theorem. (Note integrations by parts in (z,η) do not make anything worse.) Then we obtain

$$\begin{aligned}&(2\pi)^{2d} a(x,D) \circ b(x,D)u(x)\\ &= (2\pi)^d \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}z\eta}(a\#b)(x,\eta)u(x+z)\,\mathrm{d}z\mathrm{d}\eta\\ &= (2\pi)^{2d}(a\#b)(x,D)u(x).\end{aligned}$$

Hence we are done. □

Example. Let

$$A = a(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha, \quad B = b(x,D) = \sum_{|\beta|\le l} b_\beta(x)D^\beta$$

with $a_\alpha, b_\beta \in C^\infty(\mathbb{R}^d)$. Then by the Leibniz rule

$$AB = \sum_{\gamma\in\mathbb{N}_0^d}\sum_{|\alpha|\le m}\sum_{|\beta|\le l}\binom{\alpha}{\gamma} a_\alpha(x)(D^\gamma b_\beta)(x)D^{\alpha+\beta-\gamma}.$$

Hence the composite symbol $a\#b$ is given by

$$\begin{aligned}(a\#b)(x,\xi) &= \sum_{\gamma\in\mathbb{N}_0^d}\Big(\sum_{|\alpha|\le m}\binom{\alpha}{\gamma} a_\alpha(x)\xi^{\alpha-\gamma}\Big)\Big(\sum_{|\beta|\le l}(D^\gamma b_\beta)(x)\xi^\beta\Big)\\ &= \sum_{\gamma\in\mathbb{N}_0^d}\frac{1}{\mathrm{i}^{|\gamma|}\gamma!}(\partial_\xi^\gamma a(x,\xi))(\partial_x^\gamma b(x,\xi)),\end{aligned}$$

being compatible with the asymptotic expansion.

○ **Commutator and Possion bracket**

Definition. 1. Define the **commutator** of operators A, B on $\mathcal{S}(\mathbb{R}^d)$ as

$$[A,B] = AB - BA.$$

2. Define the **Poisson bracket** of $a, b \in C^1(\mathbb{R}^{2d})$ as

$$\{a,b\} = \frac{\partial a}{\partial \xi}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial b}{\partial \xi} \in C(\mathbb{R}^{2d}).$$

Corollary 2.7. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ and $b \in S^l_{\rho,\delta}(\mathbb{R}^{2d})$ with $m, l \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$.

1. If $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$, then
$$a \# b \in S^{-\infty}(\mathbb{R}^{2d}).$$

2. One has
$$[a(x,D), b(x,D)] \in \Psi^{m+l-(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^d),$$
and the associated symbol satisfies
$$(a\#b - b\#a) + \mathrm{i}\{a, b\} \in S^{m+l-2(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^{2d}).$$

Proof. The assertions are clear by Theorem 2.6. □

Remark. According to Theorems 2.3 and 2.6, a multiplication operator by
$$a(x,\xi) \quad \text{on the phase space } \mathbb{R}^{2d}$$
may be "comparable" to a pseudodifferential operator
$$a(x,D) \quad \text{on the configuration space } \mathbb{R}^{d}$$
up to errors of lower order. Such a comparison gets more accurate in the high energy (frequency) limit $|\xi| \to \infty$.

§ 2.5 Parametrix

Definition. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$.

1. We say $a(x,\xi)$, or $a(x,D)$, is **elliptic** if there exists $\epsilon, R > 0$ such that for any $(x,\xi) \in \mathbb{R}^{2d}$ with $|\xi| \geq R$
$$|a(x,\xi)| \geq \epsilon |\xi|^m.$$

2. We call $b(x,D) \in \Psi^{-m}_{\rho,\delta}(\mathbb{R}^d)$ a **parametrix** for $a(x,D)$ if
$$a(x,D) \circ b(x,D) - 1 \in \Psi^{-\infty}(\mathbb{R}^d),$$
$$b(x,D) \circ a(x,D) - 1 \in \Psi^{-\infty}(\mathbb{R}^d).$$

Problem. Show a parametrix is unique up to $\Psi^{-\infty}(\mathbb{R}^d)$ if it exists.

Theorem 2.8. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$. The following conditions are equivalent to each other:

1. a is elliptic;

2. There exists $b_0 \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$ such that
$$a(x,D) \circ b_0(x,D) - 1 \in \Psi^{-(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^d) \qquad (\spadesuit)$$
or
$$b_0(x,D) \circ a(x,D) - 1 \in \Psi^{-(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^d); \qquad (\heartsuit)$$

3. $a(x,D)$ has a parametrix $b(x,D) \in \Psi^{-m}_{\rho,\delta}(\mathbb{R}^d)$.

Proof. 1 ⇒ 2. Take $\chi \in C^\infty(\mathbb{R}^d)$ such that

$$\chi(\xi) = \begin{cases} 0 & \text{for } |\xi| \le 1, \\ 1 & \text{for } |\xi| \ge 2, \end{cases}$$

and set for large $R > 0$

$$b_0(x,\xi) = \chi(\xi/R)a(x,\xi)^{-1}.$$

Then we can easily verify $b_0 \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$. Moreover, by Theorem 2.6 it clearly satisfies both (♠) and (♡).

2 ⇒ 3. We first note that by Corollary 2.7, if either (♠) or (♡) holds, then both of them hold. Let $b_0 \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$ be as in the condition 2, and we set

$$r = a\#b_0 - 1 \in S^{-(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^{2d}).$$

Then, since

$$b_0\#(-r)^{\#j} \in S^{-m-j(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^{2d}),$$

we can take their asymptotic sum: For some $b \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$

$$b \sim \sum_{j=0}^{\infty} b_0\#(-r)^{\#j}.$$

Now we have $a\#b - 1 \in S^{-\infty}(\mathbb{R}^d)$. In fact, noting

$$a\#b_0\#(-r)^{\#j} = (-r)^{\#j} - (-r)^{\#(j+1)},$$

we have for any $k \in \mathbb{N}$

$$a\#b - 1 = a\#\left(b - \sum_{j=0}^{k-1} b_0\#(-r)^{\#j}\right) - (-r)^{\#k} \in S^{-k(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^{2d}).$$

Similarly, we can construct $c \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$c\#a - 1 \in S^{-\infty}(\mathbb{R}^{2d}).$$

However, then

$$\begin{aligned} b &= c\#a\#b + (1 - c\#a)\#b \\ &= c + c\#(a\#b - 1) + (1 - c\#a)\#b, \end{aligned}$$

so that

$$b - c \in S^{-\infty}(\mathbb{R}^{2d}).$$

Thus $b(x,D)$ gives a parametrix for $a(x,D)$ as desired.

3 ⇒ 1. By the assumption and Theorem 2.6 there exists $C_1 > 0$ such that

$$|a(x,\xi)b(x,\xi) - 1| \le C_1\langle\xi\rangle^{-(\rho-\delta)}$$

On the ohter hand, since $b \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$, there exists $C_2 > 0$ such that

$$|a(x,\xi)b(x,\xi)| \le C_2|a(x,\xi)|\langle\xi\rangle^{-m}.$$

Hence, combining these estimates, we obtain

$$\begin{aligned} |a(x,\xi)| &\ge C_2^{-1}|a(x,\xi)b(x,\xi)|\langle\xi\rangle^m \\ &\ge C_2^{-1}\big(1 - |a(x,\xi)b(x,\xi) - 1|\big)\langle\xi\rangle^m \\ &\ge C_2^{-1}\big(1 - C_1\langle\xi\rangle^{-(\rho-\delta)}\big)\langle\xi\rangle^m, \end{aligned}$$

implying the ellipticity of a. □

§ 2.6 Weyl Quantization

Definition. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $\rho > -1$ and $\delta < 1$, and let $t \in [0,1]$. Define the t-**quantization** of a as, for any $u \in \mathcal{S}(\mathbb{R}^d)$,

$$a^t(x,D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} a\Big((1-t)x + ty, \xi\Big) u(y)\, \mathrm{d}y \mathrm{d}\xi.$$

In particular, we call:

1. $a(x,D) = a^0(x,D)$ the **standard** (or **left**) **quantization**;
2. $a^1(x,D)$ the **right quantization**;
3. $a^{\mathrm{W}}(x,D) := a^{1/2}(x,D)$ the **Weyl quantization**.

○ **Continuity**

Proposition 2.9. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $\rho > -1$ and $\delta < 1$, and let $t \in [0,1]$. Then $a^t(x,D)$ is a continuous operator on $\mathcal{S}(\mathbb{R}^d)$.

Proof. We can prove it similarly to Theorem 2.1. The details are omitted. □

Problem. Fill out the details of the above proof.

Proposition 2.10. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $\rho > -1$ and $\delta < 1$, and let $t \in [0,1]$. Then

$$a^t(x,D)^* = (\bar{a})^{1-t}(x,D).$$

In particular, the following holds.

1. $a^t(x,D)$ extends as a continuous operator on $\mathcal{S}'(\mathbb{R}^d)$.
2. If a is real-valued, $a^{\mathrm{W}}(x,D)$ is **formally self-adjoint**, i.e.,
$$a^{\mathrm{W}}(x,D)^* = a^{\mathrm{W}}(x,D).$$

Proof. We prove only the former assertion since the latter ones are obvious. We implement integrations by parts to change the order of integrations as follows. Take large $N \in \mathbb{N}_0$ such that

$$m - 2(1-\delta)N < -d,$$

and then we can compute

$$\begin{aligned}
&(2\pi)^d (a^t(x,D)u, v) \\
&= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} a\Big((1-t)x+ty,\xi\Big) u(y)\, \mathrm{d}y\mathrm{d}\xi \Big) \bar{v}(x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^{3d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} \langle \xi \rangle^{-2N} \langle D_y \rangle^{2N} a\Big((1-t)x+ty,\xi\Big) u(y)\bar{v}(x)\, \mathrm{d}x\mathrm{d}\xi\mathrm{d}y \\
&= \int_{\mathbb{R}^{3d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} \langle \xi \rangle^{-4N} \\
&\qquad \cdot \langle D_x \rangle^{2N} \langle D_y \rangle^{2N} a\Big((1-t)x+ty,\xi\Big) u(y)\bar{v}(x)\, \mathrm{d}x\mathrm{d}\xi\mathrm{d}y \\
&= \int_{\mathbb{R}^{3d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} \langle \xi \rangle^{-2N} \langle D_x \rangle^{2N} a\Big((1-t)x+ty,\xi\Big) \bar{v}(x) u(y)\, \mathrm{d}x\mathrm{d}\xi\mathrm{d}y \\
&= \int_{\mathbb{R}^d} u(y) \Big(\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} a\Big((1-t)x+ty,\xi\Big) \bar{v}(x)\, \mathrm{d}x\mathrm{d}\xi \Big)\, \mathrm{d}y \\
&= (2\pi)^d (u, (\bar{a})^{1-t}(x,D)v).
\end{aligned}$$

Hence we obtain the former assertion. We are done. □

∘ Change of quantization

Theorem 2.11. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\delta \neq 1$, and let $t, s \in [0, 1]$ with $t \neq s$. There uniquely exists $b \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a^t(x, D) = b^s(x, D). \tag{$\diamondsuit$}$$

Moreover, b is expressed as

$$\begin{aligned} b(x, \xi) &= \mathrm{e}^{\mathrm{i}(t-s)D_xD_\xi} a(x, \xi) \\ &= (2\pi)^{-d}|t - s|^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}y\eta/(t-s)} a(x + y, \xi + \eta)\,\mathrm{d}y\mathrm{d}\eta, \end{aligned} \tag{$\clubsuit$}$$

and, if $\delta < \rho$, then

$$b \sim \sum_{\alpha \in \mathbb{N}_0^d} \frac{(t - s)^{|\alpha|}}{\mathrm{i}^{|\alpha|}\alpha!} \partial_x^\alpha \partial_\xi^\alpha a.$$

Proof. Step 1. We first let b be given by ($\clubsuit$). Then we can verify $b \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ and the asserted asymptotic expansion in exactly the same way as in the proof of Theorem 2.3. We omit the details.

Step 2. Next we prove ($\diamondsuit$) for b given by ($\clubsuit$), but only present the outline. By ($\clubsuit$) we can write

$$\begin{aligned} &(2\pi)^{2d} b^s(x, D)u(x) \\ &= |t - s|^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-z)\xi} \Big(\int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}y\eta/(t-s)} \\ &\qquad \cdot a((1 - s)x + sz + y, \xi + \eta)\,\mathrm{d}y\mathrm{d}\eta \Big) u(z)\,\mathrm{d}z\mathrm{d}\xi. \end{aligned}$$

We change variables, integrate it by parts and change the order of integrations, so that

$$\begin{aligned} &(2\pi)^{2d} b^s(x, D)u(x) \\ &= \int_{\mathbb{R}^{4d}} \mathrm{e}^{-\mathrm{i}z\xi-\mathrm{i}y\eta} a(x + sz + (t - s)y, \xi + \eta)u(x + z)\,\mathrm{d}y\mathrm{d}\eta\mathrm{d}z\mathrm{d}\xi. \end{aligned}$$

We further change variables, and apply the Fourier inversion formula:

$$\begin{aligned} &(2\pi)^{2d} b^s(x, D)u(x) \\ &= \int_{\mathbb{R}^{4d}} \mathrm{e}^{-\mathrm{i}z\xi-\mathrm{i}y\eta} a(x + sz + ty, \eta)u(x + y + z)\,\mathrm{d}y\mathrm{d}\eta\mathrm{d}z\mathrm{d}\xi \\ &= (2\pi)^d \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}y\eta} a(x + ty, \eta)u(x + y)\,\mathrm{d}y\mathrm{d}\eta \\ &= (2\pi)^{2d} a^t(x, D)u(x). \end{aligned}$$

Hence ($\diamondsuit$) is verified for b given by ($\clubsuit$).

Step 3. We finally discuss the uniqueness. Suppose that both $b, c \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ satisfy ($\diamondsuit$). If we let

$$\tilde{b} = \mathrm{e}^{\mathrm{i}sD_xD_\xi} b, \quad \tilde{c} = \mathrm{e}^{\mathrm{i}sD_xD_\xi} c,$$

then we have $\tilde{b}(x, D) = \tilde{c}(x, D)$, so that by Proposition 2.5

$$\tilde{b} = \tilde{c}.$$

Now we note that $\mathrm{e}^{\mathrm{i}sD_xD_\xi}$ is bijective from $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ to itself, since $\mathrm{e}^{\pm\mathrm{i}sD_xD_\xi}$ map it into itself, being the inverses to each other on $\mathcal{S}'(\mathbb{R}^{2d})$. Hence we can conclude $b = c$. We are done. □

∘ **Composition**

Theorem 2.12. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ and $b \in S^l_{\rho,\delta}(\mathbb{R}^{2d})$ with $m, l \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\delta \ne 1$, and let $t \in [0,1]$. Then there uniquely exists $a\#^t b \in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a^t(x,D) \circ b^t(x,D) = (a\#^t b)^t(x,D).$$

Moreover, $a\#^t b$ is given by

$$\begin{aligned}(a\#^t b)(x,\xi) &\\ &= \mathrm{e}^{\mathrm{i}(D_yD_\eta - D_zD_\zeta)} a((1-t)x+tz,\eta) b((1-t)y+tx,\zeta)\Big|_{\substack{y=z=x,\\ \eta=\zeta=\xi}} \qquad (\spadesuit)\\ &= (2\pi)^{-2d}\int_{\mathbb{R}^{4d}} \mathrm{e}^{-\mathrm{i}(y\eta - z\zeta)} a(x+tz,\xi+\eta)\\ &\qquad \cdot b((1-t)y+x,\xi+\zeta)\,\mathrm{d}y\mathrm{d}\eta\mathrm{d}z\mathrm{d}\zeta,\end{aligned}$$

and, if $\delta < \rho$, then

$$\begin{aligned}a\#^t b \sim \sum_{k\in\mathbb{N}_0} \frac{1}{\mathrm{i}^k k!}(\partial_y\partial_\eta - \partial_z\partial_\zeta)^k &\\ a((1-t)x+tz,\eta)b((1-t)y+tx,\zeta)\Big|_{\substack{y=z=x,\\ \eta=\zeta=\xi}}.\end{aligned}$$

Proof. Step 1. Here we prove $a\#^t b$ given by ($\spadesuit$) belongs to $S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d})$. However, we only present the strategy since the proof is similar to that of Theorem 2.3. It suffices to show

$$\left|(a\#^t b)(x,\xi)\right| \le C\langle\xi\rangle^{m+l}.$$

Fix any $\chi \in C^\infty(\mathbb{R}^{2d})$ satisfying

$$\chi(x,y) = \begin{cases} 1 & \text{for } |(x,y)| \le 1,\\ 0 & \text{for } |(x,y)| \ge 2,\end{cases}$$

and we set

$$\begin{aligned}\chi_1(\xi,y,\eta) &= \chi\Big(\langle\xi\rangle^\delta y, \langle\xi\rangle^\delta z\Big)\chi\Big(2\epsilon^{-1}\langle\xi\rangle^{-\rho}\eta, 2\epsilon^{-1}\langle\xi\rangle^{-\rho}\zeta\Big),\\ \chi_2(\xi,y,\eta) &= \Big[1-\chi\Big(\langle\xi\rangle^\delta y, \langle\xi\rangle^\delta z\Big)\Big]\chi\Big(2\epsilon^{-1}\langle\xi\rangle^{-\rho}\eta, 2\epsilon^{-1}\langle\xi\rangle^{-\rho}\zeta\Big),\\ \chi_3(\xi,y,\eta) &= \chi\Big(\epsilon^{-1}\langle\xi\rangle^{-1}\eta, \epsilon^{-1}\langle\xi\rangle^{-1}\zeta\Big)\\ &\qquad - \chi\Big(2\epsilon^{-1}\langle\xi\rangle^{-\rho}\eta, 2\epsilon^{-1}\langle\xi\rangle^{-\rho}\zeta\Big),\\ \chi_4(\xi,y,\eta) &= 1-\chi\Big(\epsilon^{-1}\langle\xi\rangle^{-1}\eta, \epsilon^{-1}\langle\xi\rangle^{-1}\zeta\Big),\end{aligned}$$

where $\epsilon > 0$ is a sufficiently small constant. The we split $a\#^t b$, using these cut-off functions, and estimate them similarly to Theorem 2.3. We omit the rest of the arguments.

Step 2. The asserted asymptotic expansion is obtained similarly to Theorem 2.3. We omit the details.

Step 3. Now, let $a\#^t b$ be given in the assertion, and we prove

$$a^t(x,D) \circ b^t(x,D) = (a\#^t b)^t(x,D).$$

For that we first construct $c \in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a^t(x,D) \circ b^t(x,D) = c(x,D),$$

and then verify

$$\mathrm{e}^{-\mathrm{i}tD_xD_\xi}c = a\#^t b.$$

The following computations can be verified by integrations by parts, change of variables and change of order of integrations, though the details are omitted for simplicity. For any $u \in \mathcal{S}(\mathbb{R}^d)$

we compute

$$
\begin{aligned}
&(2\pi)^{3d} a^t(x,D) \circ b^t(x,D) u(x) \\
&= (2\pi)^{3d} a^t(x,D) \circ b^t(x,D) (\mathcal{F}^* \mathcal{F} u)(x) \\
&= \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} a((1-t)x + ty, \xi) \Big[\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(y-z)\eta} b((1-t)y + tz, \eta) \\
&\qquad\qquad \cdot \Big(\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(z-w)\zeta} u(w)\, \mathrm{d}w \mathrm{d}\zeta \Big) \mathrm{d}z \mathrm{d}\eta \Big] \,\mathrm{d}y \mathrm{d}\xi \\
&= \int_{\mathbb{R}^{6d}} \mathrm{e}^{-\mathrm{i}y\xi - \mathrm{i}z\eta - \mathrm{i}w\zeta} a(x + ty, \xi) b(x + y + tz, \eta) \\
&\qquad\qquad \cdot u(x + y + z + w)\, \mathrm{d}w \mathrm{d}\zeta \mathrm{d}z \mathrm{d}\eta \mathrm{d}y \mathrm{d}\xi \\
&= \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\mathrm{i}w\zeta} \Big(\int_{\mathbb{R}^{4d}} \mathrm{e}^{-\mathrm{i}y\xi - \mathrm{i}z\eta} a(x + ty, \zeta + \xi) \\
&\qquad\qquad \cdot b(x + y + tz, \zeta + \eta) \mathrm{d}z \mathrm{d}\eta \mathrm{d}y \mathrm{d}\xi \Big) u(x + w)\, \mathrm{d}w \mathrm{d}\zeta.
\end{aligned}
$$

Hence we should set

$$
\begin{aligned}
c(x,\zeta) = (2\pi)^{-2d} \int_{\mathbb{R}^{4d}} &\mathrm{e}^{-\mathrm{i}y\xi - \mathrm{i}z\eta} a(x + ty, \zeta + \xi) \\
&\cdot b(x + y + tz, \zeta + \eta)\, \mathrm{d}z \mathrm{d}\eta \mathrm{d}y \mathrm{d}\xi.
\end{aligned}
$$

Similarly to Theorem 2.12, we can show $c \in S^{m+l}_{\rho,\delta}(\mathbb{R}^{2d})$. Then we further proceed along with the Fourier inversion formula

$$
\begin{aligned}
&(2\pi)^{3d} \mathrm{e}^{-\mathrm{i}t D_x D_\zeta} c(x,\zeta) \\
&= |t|^{-d} \int_{\mathbb{R}^{6d}} \mathrm{e}^{\mathrm{i}w\theta/t - \mathrm{i}y\xi - \mathrm{i}z\eta} a(x + w + ty, \zeta + \theta + \xi) \\
&\qquad\qquad \cdot b(x + w + y + tz, \zeta + \theta + \eta)\, \mathrm{d}z \mathrm{d}\eta \mathrm{d}y \mathrm{d}\xi \mathrm{d}w \mathrm{d}\theta \\
&= \int_{\mathbb{R}^{6d}} \mathrm{e}^{-\mathrm{i}y\xi + \mathrm{i}w\eta + \mathrm{i}z\theta} a(x + tw, \zeta + \xi) \\
&\qquad\qquad \cdot b(x + (1-t)y + tz, \zeta + \eta)\, \mathrm{d}z \mathrm{d}\eta \mathrm{d}y \mathrm{d}\xi \mathrm{d}w \mathrm{d}\theta.
\end{aligned}
$$

Hence with the Fourier inversion formula

$$
\begin{aligned}
(2\pi)^{3d} \mathrm{e}^{-\mathrm{i}t D_x D_\zeta} c(x,\zeta) &= (2\pi)^{d} \int_{\mathbb{R}^{4d}} \mathrm{e}^{-\mathrm{i}y\xi + \mathrm{i}w\eta} a(x + tw, \zeta + \xi) \\
&\qquad \cdot b(x + (1-t)y, \zeta + \eta)\, \mathrm{d}\eta \mathrm{d}y \mathrm{d}\xi \mathrm{d}w \\
&= (2\pi)^{3d} (a \#^t b)(x,\zeta).
\end{aligned}
$$

Thus we obtain the claim.

Step 4. Finally it remains to discuss the uniqueness. The uniqueness of the "t-symbol" can be shown as in Step 3 of the proof of Theorem 2.11, and we omit it. Thus we are done. □

Corollary 2.13. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ and $b \in S^l_{\rho,\delta}(\mathbb{R}^{2d})$ with $m, l \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$. Then

$$
a \#^{\mathrm{W}} b := a \#^{1/2} b \sim \sum_{\alpha,\beta \in \mathbb{N}_0^d} \frac{(-1)^{|\alpha|}}{(2\mathrm{i})^{|\alpha|+|\beta|} \alpha! \beta!} (\partial_x^\alpha \partial_\xi^\beta a)(\partial_\xi^\alpha \partial_x^\beta b).
$$

Moreover,

$$
a \#^{\mathrm{W}} b - b \#^{\mathrm{W}} a + \mathrm{i}\{a, b\} \in S^{m+l-3(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^{2d}).
$$

Proof. The expansion is verified by Theorem 2.12 and the multinomial theorem. Under interchange of the indices α and β a partial sum over $|\alpha| + |\beta| = k \in \mathbb{N}_0$ is even or odd according to k even or odd, respectively. Thus the latter assertion follows. □

Problem. Let $a \in S^0_{0,0}(\mathbb{R}^{2d})$.

1. Verify
$$\mathcal{F}a^{\mathsf{W}}(x, D_x)\mathcal{F}^* = a^{\mathsf{W}}(-D_\xi, \xi) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d). \qquad (\heartsuit)$$

2. For any $t \in \mathbb{R}$ define the **free Schrödinger propagator** as
$$\mathrm{e}^{\mathrm{i}t\Delta/2} = \mathcal{F}^*\mathrm{e}^{-\mathrm{i}t\xi^2/2}\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d).$$
Then verify
$$\mathrm{e}^{-\mathrm{i}t\Delta/2}a^{\mathsf{W}}(x, D)\mathrm{e}^{\mathrm{i}t\Delta/2} = a^{\mathsf{W}}(x + tD, D).$$

Remarks. 1. These identities support the idea that $a^{\mathsf{W}}(x, D)$ is merely a multiplication operator by $a(x, \xi)$ on $\mathbb{R}^{2d}$, with $\mathcal{F}$ and $\mathrm{e}^{\mathrm{i}t\Delta}$ being symplectic transforms
$$(x, \xi) \mapsto (-\xi, x), \quad (x, \xi) \mapsto (x + t\xi, \xi),$$
respectively.

2. Due to the symmetry $(\heartsuit)$ in x and ξ, it is also possible to develop the theory of ΨDOs for symbols satisfying
$$\left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \le C_{\alpha\beta}\langle x\rangle^{m-\rho|\alpha|+\delta|\beta|}.$$
Such a class is useful, for example, in the quantum scattering theory. This is just an example of various symbol classes.

Chapter 3
Pseudodifferential Estimates

§ 3.1 L^2-boundedness

Theorem 3.1. Let $0 \le \delta < \rho \le 1$. Then there exist $C > 0$ and $j \in \mathbb{N}_0$ such that for any $a \in S^0_{\rho,\delta}(\mathbb{R}^{2d})$ and $u \in \mathcal{S}(\mathbb{R}^d)$
$$\|a(x, D)u\|_{L^2} \le C|a|_{j,S^0_{\rho,\delta}}\|u\|_{L^2}.$$
In particular, $a(x, D)$ is bounded on $L^2(\mathbb{R}^d)$.

Remark. Recall the seminorm $|\cdot|_{j,S^m_{\rho,\delta}}$ on $S^m_{\rho,\delta}(\mathbb{R}^{2d})$ is defined as
$$|a|_j = |a|_{j,S^m_{\rho,\delta}} = \sup\Big\{\langle\xi\rangle^{-m-\delta|\alpha|+\rho|\beta|}\left|\partial_x^\alpha\partial_\xi^\beta a(x, \xi)\right|; \ |\alpha| + |\beta| \le j,\ (x, \xi) \in \mathbb{R}^{2d}\Big\}.$$

Proposition 3.2 (Schur's lemma). Let $K\colon \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{C}$ be measurable, and assume there exist $\alpha,\beta \ge 0$ such that

$$\int_{\mathbb{R}^d} |K(x,y)|\,\mathrm{d}y \le \alpha \quad \text{for a.e. } x\in\mathbb{R}^d,$$
$$\int_{\mathbb{R}^d} |K(x,y)|\,\mathrm{d}x \le \beta \quad \text{for a.e. } y\in\mathbb{R}^d.$$

Then, for any $u \in L^2(\mathbb{R}^d)$ and for a.e. $x\in\mathbb{R}^d$, $K(x,\cdot)u$ is integrable, and

$$\Big\| \int_{\mathbb{R}^d} K(\cdot,y)u(y)\,\mathrm{d}y \Big\|_{L^2} \le (\alpha\beta)^{1/2}\|u\|_{L^2}.$$

Proof. Let $u \in L^2(\mathbb{R}^d)$. Then by Fubini's theorem and the Cauchy–Schwarz inequality

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|K(x,y)u(y)|\,\mathrm{d}y\right)^2\mathrm{d}x\\
&\le \int_{\mathbb{R}^{3d}}|K(x,y)||K(x,z)||u(y)||u(z)|\,\mathrm{d}y\mathrm{d}z\mathrm{d}x\\
&\le \frac12\int_{\mathbb{R}^{3d}}|K(x,y)||K(x,z)||u(y)|^2\,\mathrm{d}y\mathrm{d}z\mathrm{d}x\\
&\quad+\frac12\int_{\mathbb{R}^{3d}}|K(x,y)||K(x,z)||u(z)|^2\,\mathrm{d}y\mathrm{d}z\mathrm{d}x\\
&\le \int_{\mathbb{R}^d}|u(y)|^2\left(\int_{\mathbb{R}^d}|K(x,y)|\left(\int_{\mathbb{R}^d}|K(x,z)|\,\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y\\
&\le \alpha\beta\|u\|_{L^2}^2.
\end{aligned}$$

Hence by Fubini's theorem again the assertion is verified. □

Proof of Theorem 3.1. For simplicity we shall not keep track of dependence of constants on seminorms, but it is not difficult.

Step 1. We first prove the assertion for $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m<-d$. Let $u\in\mathcal{S}(\mathbb{R}^d)$. By the assumption and Fubini's theorem

$$a(x,D)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}(x-y)\xi}a(x,\xi)\,\mathrm{d}\xi\right)u(y)\,\mathrm{d}y,$$

so that $a(x,D)$ has the Schwartz kernel

$$K(x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}(x-y)\xi}a(x,\xi)\,\mathrm{d}\xi.$$

By integrations by parts we can verify that for any $N\in\mathbb{N}_0$

$$|K(x,y)| \le C_1\langle x-y\rangle^{-2N}.$$

Schur's lemma applies for large N, hence $a(x,D)\in\mathcal{B}(L^2(\mathbb{R}^d))$.

Step 2. Next we prove the assertion for $a\in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m<0$. By Step 1 and induction it suffices to show, if for some $l<0$

$$\Psi^l_{\rho,\delta}(\mathbb{R}^d)\subset\mathcal{B}(L^2(\mathbb{R}^d)), \qquad (\clubsuit)$$

then

$$\Psi^{l/2}_{\rho,\delta}(\mathbb{R}^d)\subset\mathcal{B}(L^2(\mathbb{R}^d)).$$

Suppose ($\clubsuit$), and take any $a\in S^{l/2}_{\rho,\delta}(\mathbb{R}^{2d})$. Then for any $u\in\mathcal{S}(\mathbb{R}^d)$ by the Cauchy–Schwarz inequality

$$\|a(x,D)u\|_{L^2}^2 \le \|a^*(x,D)a(x,D)u\|_{L^2}\|u\|_{L^2}.$$

However, by $a^*(x,D)a(x,D)\in\Psi^l_{\rho,\delta}(\mathbb{R}^d)$ and ($\clubsuit$) it follows that

$$\|a(x,D)\|_{\mathcal{B}(L^2)} \le \|a^*(x,D)a(x,D)\|_{\mathcal{B}(L^2)}^{1/2} < \infty.$$

Thus the claim is verified.

Step 3. Finally let $a \in S^0_{\rho,\delta}(\mathbb{R}^{2d})$. We set

$$b(x,\xi) = \sqrt{2|a|_0^2 - |a(x,\xi)|^2} \in S^0_{\rho,\delta}(\mathbb{R}^{2d}).$$

Then there exists $c \in S^{-(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$a^*\#a + b^*\#b = 2|a|_0^2 + c.$$

Now for any $u \in \mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned}\|a(x,D)u\|_{L^2}^2 &\le \|a(x,D)u\|_{L^2}^2 + \|b(x,D)u\|_{L^2}^2 \\ &= 2|a|_0^2\|u\|_{L^2}^2 + (c(x,D)u,u)_{L^2} \\ &\le \Big(2|a|_0^2 + \|c(x,D)\|_{\mathcal{B}(L^2)}\Big)\|u\|_{L^2}^2,\end{aligned}$$

and hence we obtain the assertion. □

◦ **Calderón–Vaillancourt theorem**

Theorem 3.3 (Calderón–Vaillancourt). There exist $C > 0$ and $j \in \mathbb{N}_0$ such that for any $a \in S^0_{0,0}(\mathbb{R}^{2d})$ and $u \in \mathcal{S}(\mathbb{R}^d)$

$$\|a(x,D)u\|_{L^2} \le C|a|_{j,S^0_{0,0}}\|u\|_{L^2}.$$

In particular, $a(x,D)$ is bounded on $L^2(\mathbb{R}^d)$.

Lemma 3.4 (Cotlar–Stein lemma). Let $\mathcal{H}$ be a Hilbert space, and assume a family $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H})$ satisfies for some $M \ge 0$

$$\sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \|A_jA_k^*\|_{\mathcal{B}(\mathcal{H})}^{1/2} \le M, \quad \sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \|A_j^*A_k\|_{\mathcal{B}(\mathcal{H})}^{1/2} \le M.$$

Then the series

$$S := \sum_{j\in\mathbb{N}} A_j$$

converges strongly in $\mathcal{B}(\mathcal{H})$, and

$$\|S\|_{\mathcal{B}(\mathcal{H})} \le M.$$

Proof. Step 1. Here we prove that for any $n \in \mathbb{N}$

$$\|S_n\| \le M; \quad S_n := \sum_{j=1}^n A_j \in \mathcal{B}(\mathcal{H}).$$

For that we shall compute and bound $\|S_n\|^{2m}$ for $m \in \mathbb{N}$. Since $S_n^*S_n$ is bounded on $\mathcal{H}$, we have

$$\|S_n\|^2 = \sup_{\|u\|_{\mathcal{H}}=1}\|S_nu\|^2 = \sup_{\|u\|_{\mathcal{H}}=1}(S_n^*S_nu,u) = \|S_n^*S_n\|.$$

Then, since $S_n^*S_n$ is self-adjoint,

$$\|S_n\|^{2m} = \|S_n^*S_n\|^m = \|(S_n^*S_n)^m\|.$$

Hence we are lead to compute and bound

$$(S^*S)^m = \sum_{j_1,\dots,j_{2m}=1}^n A_{j_1}^*A_{j_2}\cdots A_{j_{2m-1}}^*A_{j_{2m}}.$$

Denote the above summand by $A_{j_1\cdots j_{2m}}$. Then we have

$$\|A_{j_1\cdots j_{2m}}\| \le \|A^*_{j_1}A_{j_2}\|\cdots\|A^*_{j_{2m-1}}A_{j_{2m}}\|,$$

and

$$\|A_{j_1\cdots j_{2m}}\| \le \|A^*_{j_1}\|\|A_{j_2}A^*_{j_3}\|\cdots\|A_{j_{2m-2}}A^*_{j_{2m-1}}\|\|A_{j_{2m}}\|.$$

Noting

$$\|A_j\| = \|A^*_j\| = \|A^*_jA_j\|^{1/2} \le M,$$

we can deduce

$$\|A_{j_1\cdots j_{2m}}\| \le M\Big(\|A^*_{j_1}A_{j_2}\|\|A_{j_2}A^*_{j_3}\|\cdots\|A^*_{j_{2m-1}}A_{j_{2m}}\|\Big)^{1/2}.$$

Therefore by the assumption

$$\|S_n\|^{2m} \le nM^{2m}, \quad \text{or} \quad \|S_n\| \le n^{1/(2m)}M.$$

Now by letting $m\to\infty$ we obtain the claim.

Step 2. To prove S_n is strongly convergent as $n\to\infty$ we split

$$\mathcal{H} = \mathcal{G}\oplus\mathcal{G}^\perp; \quad \mathcal{G} = \overline{\mathrm{span}\Big(\bigcup_{k\in\mathbb{N}}\mathrm{Ran}\,A^*_k\Big)}.$$

Note $S_n\equiv 0$ on $\mathcal{G}^\perp$ since for any $u\in\mathcal{G}^\perp$ and $v\in\mathcal{H}$

$$(S_nu,v) = \sum_{j=1}^n (u, A^*_jv) = 0.$$

Thus it suffices to discuss the limit of S_nu for $u\in\mathcal{G}$, however, due to Step 1 and the density argument it further reduces to the case $u\in\mathrm{Ran}\,A^*_k$. Let $u=A^*_kv$ for some $v\in\mathcal{H}$, and then

$$\sum_{j=1}^n\|A_ju\| \le \sum_{j=1}^n\|A_jA^*_k\|^{1/2}\|A_jA^*_k\|^{1/2}\|v\| \le M^2\|v\|.$$

This implies S_nu is absolutely convergent for $u\in\mathrm{Ran}\,A^*_k$.

Step 3. Finally we estimate $\|S\|$. However, it is straightforward. For any $u\in\mathcal{H}$

$$\|Su\| = \lim_{n\to\infty}\|S_nu\| \le \lim_{n\to\infty}\|S_n\|\|u\| \le M\|u\|.$$

Hence we are done. □

Proof of Theorem 3.3. Step 1. By Theorem 2.11 it suffices to show $a^{\mathsf{W}}(x,D)$ is bounded on $L^2(\mathbb{R}^d)$. Let $\chi\in C^\infty_{\mathrm{c}}(\mathbb{R}^{2d})$ be such that

$$\sum_{\mu\in\mathbb{Z}^{2d}}\chi_\mu = 1; \quad \chi_\mu(\cdot) = \chi(\cdot-\mu)$$

(construction of such χ is left to the reader as a **Problem**), and we **microlocally** cut off and set

$$a_\mu = \chi_\mu a, \quad A_\mu = a^{\mathsf{W}}_\mu(x,D).$$

Step 2. Here we let $u\in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$, and prove pointwise convergence

$$a^{\mathsf{W}}(x,D)u(x) = \sum_{\mu\in\mathbb{Z}^{2d}}A_\mu u(x). \qquad (\spadesuit)$$

We introduce

$${}^tL_1 = \langle\xi\rangle^{-2}(1-\xi D_y),$$

and rewrite a partial sum of the right-hand side of ($\spadesuit$) as

$$\sum_{|\mu|\le n}A_\mu u(x) = (2\pi)^{-d}\int_{\mathbb{R}^{2d}}\mathrm{e}^{\mathrm{i}(x-y)\xi}\sum_{|\mu|\le n}L_1^Na_\mu\Big(\frac{x+y}{2},\xi\Big)u(y)\,\mathrm{d}y\mathrm{d}\xi.$$

Since the partition $\{\chi_\mu\}_{\mu\in\mathbb{Z}^{2d}}$ of unity is uniformly locally finite, we have for any $(x,y,\xi)\in\mathbb{R}^{3d}$ and $n\in\mathbb{N}_0$

$$\Big|\sum_{|\mu|\le n}L_1^Na_\mu\Big(\frac{x+y}{2},\xi\Big)u(y)\Big| \le C_{1,N}|a|_N\langle y\rangle^{-N}\langle\xi\rangle^{-N}.$$

Hence by the Lebesgue convergence theorem

$$\sum_{\mu\in\mathbb{Z}^{2d}} A_\mu u(x) = (2\pi)^{-d}\int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} L_1^N a\Big(\frac{x+y}{2},\xi\Big)u(y)\,\mathrm{d}y\mathrm{d}\xi,$$

and we obtain (♠).

Step 3. Now it suffices to verify the assumptions of the Cotlar–Stein lemma for $\mathcal{H}=L^2(\mathbb{R}^d)$ and $\{A_\mu\}_{\mu\in\mathbb{Z}^{2d}}$. Let us write

$$A_\mu A_\nu^* u(x) = \int_{\mathbb{R}^d} K_{\mu\nu}(x,y)u(y)\,\mathrm{d}y$$

with

$$K_{\mu\nu}(x,y) = (2\pi)^{-2d}\int_{\mathbb{R}^{3d}} \mathrm{e}^{\mathrm{i}(x\xi-z\xi+z\eta-y\eta)} \cdot a_\mu\Big(\frac{x+z}{2},\xi\Big)\bar{a}_\nu\Big(\frac{y+z}{2},\eta\Big)\,\mathrm{d}\eta\mathrm{d}z\mathrm{d}\xi.$$

We are going to apply Schur's lemma. Note $K_{\mu\nu}\in C^\infty(\mathbb{R}^{2d})$. Set

$${}^tL_2 = \Big\langle(x-y,\xi-\eta)\Big\rangle^{-2}\Big(1+(x-y)(D_\xi+D_\eta)-(\xi-\eta)D_z\Big),$$

and we rewrite

$$K_{\mu\nu}(x,y) = (2\pi)^{-2d}\int_{\mathbb{R}^{3d}} \mathrm{e}^{\mathrm{i}(x\xi-z\xi+z\eta-y\eta)} \cdot L_2^N a_\mu\Big(\frac{x+z}{2},\xi\Big)\bar{a}_\nu\Big(\frac{y+z}{2},\eta\Big)\,\mathrm{d}\eta\mathrm{d}z\mathrm{d}\xi.$$

Note on the support of the integrand we have for $N\ge 2d+2$

$$\begin{aligned}&\Big|L_2^N a_\mu\Big(\frac{x+z}{2},\xi\Big)\bar{a}_\nu\Big(\frac{y+z}{2},\eta\Big)\Big|\\ &\le C_{2,N}|a|_N^2\Big\langle(x-y,\xi-\eta)\Big\rangle^{-N}\\ &\le C_{3,N}|a|_N^2\langle\mu-\nu\rangle^{d+1-N}\langle x-y\rangle^{-d-1},\end{aligned}$$

so that

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K_{\mu\nu}(x,y)|\,\mathrm{d}y \le C_{4,N}|a|_N^2\langle\mu-\nu\rangle^{2d+2-N},$$

and

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}|K_{\mu\nu}(x,y)|\,\mathrm{d}x \le C_{4,N}|a|_N^2\langle\mu-\nu\rangle^{2d+2-N}.$$

Hence by Schur's lemma it follows that

$$\|A_\mu A_\nu^*\| \le C_{4,N}|a|_N^2\langle\mu-\nu\rangle^{2d+2-N}.$$

Similarly we obtain

$$\|A_\mu^* A_\nu\| \le C_{5,N}|a|_N^2\langle\mu-\nu\rangle^{2d+2-N}.$$

Now the Cotlar–Stein lemma applies for sufficiently large N, and along with Step 2 we obtain the assertion. □

§ 3.2 Sobolev Spaces

Definition. 1. Define the **weighted L^2-space** of order $s\in\mathbb{R}$ as

$$L^2_s(\mathbb{R}^d) = \Big\{u\in\mathcal{S}'(\mathbb{R}^d);\ \langle x\rangle^s u\in L^2(\mathbb{R}^d)\Big\},$$

which is a Hilbert space with respect to the inner product

$$(u,v)_{L^2_s} = \int_{\mathbb{R}^d}\langle x\rangle^{2s}u(x)\overline{v(x)}\,\mathrm{d}x.$$

2. Define the **Sobolev space** of order $s\in\mathbb{R}$ as

$$H^s(\mathbb{R}^d) = \Big\{u\in\mathcal{S}'(\mathbb{R}^d);\ \mathcal{F}u\in L^2_s(\mathbb{R}^d)\Big\},$$

which is a Hilbert space with respect to the inner product

$$(u,v)_{H^s} = \int_{\mathbb{R}^d}\langle\xi\rangle^{2s}(\mathcal{F}u)(\xi)\overline{(\mathcal{F}v)(\xi)}\,\mathrm{d}\xi.$$

We further set

$$H^\infty(\mathbb{R}^d) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d), \quad H^{-\infty}(\mathbb{R}^d) = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^d).$$

Note that for any $s < t$

$$\mathcal{S}(\mathbb{R}^d) \subset H^\infty(\mathbb{R}^d) \subset H^t(\mathbb{R}^d) \subset H^s(\mathbb{R}^d) \subset H^{-\infty}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d).$$

Proposition 3.5. Let $s \in \mathbb{R}$. Then $\mathcal{S}(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$.

Proof. It is straightforward if we discuss it in the Fourier space. We omit the details. □

Theorem 3.6 (Sobolev embedding theorem). Let $s \in \mathbb{R}$ and $k \in \mathbb{N}_0$ with $s > k + d/2$. Then

$$H^s(\mathbb{R}^d) \subset C^k_{\mathrm{b}}(\mathbb{R}^d).$$

Moreover, there exists $C > 0$ such that for any $u \in H^s(\mathbb{R}^d)$

$$\|u\|_{C^k_{\mathrm{b}}} = \sup\bigl\{|\partial^\alpha u(x)|;\ |\alpha| \le k,\ x \in \mathbb{R}^d\bigr\} \le C\|u\|_{H^s}.$$

Therefore the embedding $H^s(\mathbb{R}^d) \hookrightarrow C^k_{\mathrm{b}}(\mathbb{R}^d)$ is continuous.

Proof. Let $s > k + d/2$. We first note that for any $u \in \mathcal{S}(\mathbb{R}^d)$, $|\alpha| \le k$ and $x \in \mathbb{R}^d$

$$\begin{aligned} |D^\alpha u(x)| &= (2\pi)^{-d/2}\left|\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\xi}\xi^\alpha(\mathcal{F}u)(\xi)\,\mathrm{d}\xi\right| \\ &\le (2\pi)^{-d/2}\left(\int_{\mathbb{R}^d} |\xi|^{2|\alpha|}\langle\xi\rangle^{-2s}\,\mathrm{d}\xi\right)^{1/2}\|u\|_{H^s} = C\|u\|_{H^s}. \end{aligned}$$

Let $v \in H^s(\mathbb{R}^d)$. Take a sequence $(v_n)_{n\in\mathbb{N}}$ on $\mathcal{S}(\mathbb{R}^d)$ such that

$$v_n \to v \quad \text{in } H^s(\mathbb{R}^d).$$

Due to the above bound $(v_n)_{n\in\mathbb{N}}$ is also a Cauchy sequance on $C^k_{\mathrm{b}}(\mathbb{R}^d)$, and thus there exists $w \in C^k_{\mathrm{b}}(\mathbb{R}^d)$ such that

$$v_n \to w \quad \text{in } C^k_{\mathrm{b}}(\mathbb{R}^d).$$

By uniquness of limit in $\mathcal{S}'(\mathbb{R}^d)$ it follows that $v = w \in C^k_{\mathrm{b}}(\mathbb{R}^d)$. The asserted bound also follows from the above one. □

Proposition 3.7. Let $s, t \in \mathbb{R}$. The operator $\langle D\rangle^s$ is unitary as

$$H^{t+s}(\mathbb{R}^d) \to H^t(\mathbb{R}^d).$$

Moreover, it also gives linear isomorphisms

$$H^\infty(\mathbb{R}^d) \to H^\infty(\mathbb{R}^d), \quad H^{-\infty}(\mathbb{R}^d) \to H^{-\infty}(\mathbb{R}^d).$$

Proof. By the Fourier transform we may reduce the assertion to that for the corresponding weighted L^2-spaces. Then the proof is straightforward. We omit the details. □

Theorem 3.8. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ or $a \in S^m_{0,0}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$, and let $s \in \mathbb{R}$. Then $a(x,D)$ is bounded as $H^s(\mathbb{R}^d) \to H^{s-m}(\mathbb{R}^d)$.

Proof. Set

$$b(x,\xi) = \langle\xi\rangle^{s-m} \# a(x,\xi) \# \langle\xi\rangle^{-s} \in S^0_{\rho,\delta}(\mathbb{R}^{2d}) \text{ or } S^0_{0,0}(\mathbb{R}^{2d}).$$

By Theorems 3.1 or 3.3 there exists $C > 0$ such that for any $u \in L^2(\mathbb{R}^d)$

$$\|b(x,D)u\|_{L^2} \le C\|u\|_{L^2}.$$

Now we let $u = \langle D\rangle^s v$ with $v \in \mathcal{S}(\mathbb{R}^d)$, and then it follows that

$$\|a(x,D)v\|_{H^{s-m}} \le C\|v\|_{H^s}.$$

Since $\mathcal{S}(\mathbb{R}^d) \subset H^s(\mathbb{R}^d)$ is dense, the assertion is verified. □

◦ **Smoothing operators**

Proposition 3.9. Let $a \in S^{-\infty}(\mathbb{R}^{2d})$.

1. For any $u \in \mathcal{S}'(\mathbb{R}^d)$ there exists $N \in \mathbb{N}_0$ such that
$$a(x,D)u \in \langle x\rangle^N H^\infty(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d).$$

2. $a(x,D)$ has the Schwartz kernel $K(x, x-y)$ with $K \in C^\infty(\mathbb{R}^{2d})$ satisfying for any $\alpha, \beta, \gamma \in \mathbb{N}_0^d$
$$\sup_{(x,z)\in\mathbb{R}^{2d}} \left| z^\alpha \partial_x^\beta \partial_z^\gamma K(x,z) \right| < \infty.$$

3. Conversely, any operator with the Schwartz kernel $K(x, x-y)$ satisfying the above properties belongs to $\Psi^{-\infty}(\mathbb{R}^d)$.

Proof. 1. Due to the **structure of** $\mathcal{S}'(\mathbb{R}^d)$ for any $u \in \mathcal{S}'(\mathbb{R}^d)$ there exists $N \in \mathbb{N}_0$ and $s \in \mathbb{R}$ such that

$$v := \langle x\rangle^{-2N} u \in H^s(\mathbb{R}^d).$$

Then we can write for some $b_\alpha \in S^{-\infty}(\mathbb{R}^{2d})$

$$\begin{aligned} a(x,D)u(x) &= (2\pi)^{-d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}x\xi}\Big(\langle D_\xi\rangle^{2N} \mathrm{e}^{-\mathrm{i}y\xi}\Big) a(x,\xi) v(y)\,\mathrm{d}y\mathrm{d}\xi \\ &= (2\pi)^{-d} \sum_{|\alpha|\le 2N} x^\alpha \int_{\mathbb{R}^{2d}} \mathrm{e}^{\mathrm{i}(x-y)\xi} b_\alpha(x,\xi) v(y)\,\mathrm{d}y\mathrm{d}\xi, \end{aligned}$$

so that by Theorem 3.8

$$\langle x\rangle^{-2N} a(x,D)u(x) = \sum_{|\alpha|\le 2N} x^\alpha \langle x\rangle^{-2N} b_\alpha(x,D)v(x) \in H^\infty(\mathbb{R}^d).$$

The inclusion $H^\infty(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$ is obvious by Theorem 3.6.

2. For any $u \in \mathcal{S}(\mathbb{R}^d)$ we can write by Fubini's theorem

$$a(x,D)u(x) = \int_{\mathbb{R}^d} K(x, x-y) u(y)\,\mathrm{d}y$$

with

$$K(x,z) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}z\xi} a(x,\xi)\,\mathrm{d}\xi.$$

The asserted properties of K follows by integrations by parts.

3. We can construct the associated symbol as

$$a(x,\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}z\xi} K(x,z)\,\mathrm{d}z.$$

It is easy to see $a \in S^{-\infty}(\mathbb{R}^{2d})$, and that $a(x,D)$ in fact has the Schwartz kernel $K(x, x-y)$. We omit the details. □

◦ **Compactness criterion**

Theorem 3.10. Let $a \in S^0_{\rho,\delta}(\mathbb{R})$ with $0 \le \delta < \rho \le 1$ or $\rho = \delta = 0$, and assume for any $\alpha, \beta \in \mathbb{N}_0^d$ there exists $m \in L^\infty(\mathbb{R}^{2d})$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le m(x,\xi)\langle\xi\rangle^{\delta|\alpha|-\rho|\beta|}$$

and

$$\lim_{|(x,\xi)|\to\infty} m(x,\xi) = 0.$$

Then $a(x,D)$ is a compact operator on $L^2(\mathbb{R}^d)$.

Proof. We first reduce the proof to the case $a \in C_c^\infty(\mathbb{R}^{2d})$. In fact, take any $\chi \in C_c^\infty(\mathbb{R}^{2d})$ such that

$$\chi(x,\xi) = \begin{cases} 1 & \text{for } |(x,\xi)| \le 1, \\ 0 & \text{for } |(x,\xi)| \ge 2, \end{cases}$$

and set for $\epsilon > 0$

$$a_\epsilon(x,\xi) = \chi(\epsilon x, \epsilon\xi)a(x,\xi).$$

Then by the assumption we can see for any $j \in \mathbb{N}_0$

$$|a - a_\epsilon|_{S^0_{\rho,\delta}} \to 0 \quad \text{as } \epsilon \to +0.$$

This implies by Theorems 3.1 or 3.3

$$\lim_{\epsilon\to+0} a_\epsilon(x,D) = a(x,D) \quad \text{in } \mathcal{B}(L^2(\mathbb{R}^d)),$$

and thus the reduction is valid.

Now suppose $a \in C_c^\infty(\mathbb{R}^{2d})$, and let $(u_j)_{j\in\mathbb{N}}$ be a bounded sequence on $L^2(\mathbb{R}^d)$. By the assumption there exists a compact subset $K \subset \mathbb{R}^d$ such that for any $j \in \mathbb{N}$

$$\operatorname{supp} a(x,D)u_j \subset K.$$

In addition, since $a(x,D) \in \Psi^{-\infty}(\mathbb{R}^d)$, by Theorems 3.6 and 3.8 there exists $C > 0$ such that for any $j \in \mathbb{N}$, $|\alpha| \le 1$ and $x \in \mathbb{R}^d$

$$\left|\partial^\alpha a(x,D)u_j(x)\right| \le C.$$

Then by the Ascoli–Arzelà theorem we can choose a uniformly convergent subsequence of $(a(x,D)u_j)_{j\in\mathbb{N}}$, and it also converges in $L^2(\mathbb{R}^d)$. Hence we are done. □

Remark. Let us present a heuristic. Let a be as in Theorem 3.10, and take any bounded sequence $(u_j)_{j\in\mathbb{N}}$ on $L^2(\mathbb{R}^d)$. Suppose we could regard $u_j(x)$ as a function $u_j(x,\xi)$ on $\mathbb{R}^{2d}$, and look at

$$a(x,\xi)u_j(x,\xi) \quad \text{instead of } a(x,D)u_j(x).$$

By the assumption and the **uncertainty principle** $u_j(x,\xi)$ should be "uniformly bounded" on $\mathbb{R}^{2d}$. Thus we would have

$$|a(x,\xi)u_j(x,\xi)| \le Cm(x,\xi)$$

uniformly in $j \in \mathbb{N}$. Then by the diagonal argument we would be able to extract a subsequence of $(a(x,\xi)u_j(x,\xi))_{j\in\mathbb{N}}$ that converges on any compact subsets of $\mathbb{R}^{2d}$.

§ 3.3 Gårding-Type Inequalities

Theorem 3.11 (**Elliptic a priori estimate**). Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ be elliptic with $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$, and let $s, t \in \mathbb{R}$. Then there exists $C > 0$ such that for any $u \in \mathcal{S}(\mathbb{R}^d)$

$$\|u\|_{H^{s+m}} \le C\Big(\|a(x,D)u\|_{H^s} + \|u\|_{H^t}\Big)$$

Proof. By the assumption and Theorem 2.8 there exist $b \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$ and $r \in S^{-\infty}(\mathbb{R}^{2d})$ such that

$$1 = b(x,D)a(x,D) + r(x,D),$$

so that for any $u \in \mathcal{S}(\mathbb{R}^d)$

$$\langle D\rangle^{s+m}u = \langle D\rangle^{s+m}b(x,D)a(x,D)u + \langle D\rangle^{s+m}r(x,D)u. \qquad (\spadesuit)$$

Then the assertion follows by Proposition 3.8. □

Example. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ be elliptic with $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$. Given $f \in H^s(\mathbb{R}^d)$ with $s \in \mathbb{R}$, we consider an inhomogeneous elliptic equation

$$a(x,D)u = f.$$

Suppose we find a solution u in a *wide* Sobolev space $H^{-N}(\mathbb{R}^d)$ with $N \gg 1$. However, then it automatically follows by the a priori estimate, or more presicely by ($\spadesuit$), that

$$u \in H^{s+m}(\mathbb{R}^d).$$

We can always recover the regularity of a solution u. Such a property is called the **elliptic regularity**. See also Theorem 4.1.

Theorem 3.12 (**Gårding inequality**). Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$. Assume there exist $\epsilon_0 > 0$ and $R \ge 0$ such that for any $x \in \mathbb{R}^d$ and $|\xi| \ge R$

$$\mathrm{Re}\, a(x,\xi) \ge \epsilon_0 \langle\xi\rangle^m.$$

Then for any $\epsilon \in (0, \epsilon_0)$ and $l < m$ there exists $C > 0$ such that, as **quadratic forms** on $H^{m/2}(\mathbb{R}^d)$,

$$\mathrm{Re}(a(x,D)) \ge \epsilon\langle D\rangle^m - C\langle D\rangle^l,$$

i.e., for any $u \in H^{m/2}(\mathbb{R}^d)$

$$\mathrm{Re}\Big(a(x,D)u, u\Big)_{L^2} \ge \epsilon\|u\|^2_{H^{m/2}} - C\|u\|^2_{H^{l/2}}.$$

Remarks. 1. In general, for an operator A we define

$$\mathrm{Re}\, A = \frac{1}{2}(A + A^*), \quad \mathrm{Im}\, A = \frac{1}{2\mathrm{i}}(A - A^*).$$

These conform with the associated quadratic forms as

$$(\mathrm{Re}\, Au, u) = \mathrm{Re}(Au, u), \quad (\mathrm{Im}\, Au, u) = \mathrm{Im}(Au, u).$$

2. We can say symbol estimates are translated into the associated operators up to lower order errors.
3. Inner product is more informative than norm.

Problem. Deduce the elliptic a priori estimate from the Gårding inequality.

Proof. Take sufficiently large $C_1 > 0$, so that for any $(x, \xi) \in \mathbb{R}^{2d}$

$$\operatorname{Re} a(x,\xi) \geq \epsilon_0 \langle \xi \rangle^m - C_1 \langle \xi \rangle^{m-\rho+\delta}.$$

Set for any $\epsilon' \in (\epsilon, \epsilon_0)$

$$b(x,\xi) = \left(\operatorname{Re} a(x,\xi) - \epsilon' \langle \xi \rangle^m + C_1 \langle \xi \rangle^{m-\rho+\delta}\right)^{1/2} \in S^{m/2}_{\rho,\delta}(\mathbb{R}^{2d}).$$

Then there exists $c \in S^{m-\rho+\delta}_{\rho,\delta}(\mathbb{R}^{2d})$ such that

$$\frac{1}{2}\left(a(x,\xi) + a^*(x,\xi)\right) = (b^* \# b)(x,\xi) + \epsilon' \langle \xi \rangle^m - c(x,\xi).$$

Hence we obtain for sufficiently large $C_2 > 0$

$$\begin{aligned}\operatorname{Re} a(x,D) &= b^*(x,D) b(x,D) + \epsilon' \langle D \rangle^m - c(x,D) \\ &\geq \epsilon' \langle D \rangle^m - C_2 \langle D \rangle^{m-\rho+\delta}.\end{aligned}$$

Finally for any $l < m$ we can find $C_3 > 0$ such that

$$-C_2 \langle D \rangle^{m-\rho+\delta} \geq -(\epsilon' - \epsilon) \langle D \rangle^m - C_3 \langle D \rangle^l.$$

Thus we obtain the assertion. □

Theorem 3.13 (Sharp Gårding inequality). Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$. Assume there exists $R \geq 0$ such that for any $x \in \mathbb{R}^d$ and $|\xi| \geq R$

$$\operatorname{Re} a(x,\xi) \geq 0.$$

There exists $C > 0$ such that, as quadratic forms on $H^{m/2}(\mathbb{R}^d)$,

$$\operatorname{Re}(a(x,D)) \geq -C \langle D \rangle^{m-\rho+\delta}.$$

Remark. The **Fefferman–Phong inequality** further improves the right-hand side of the sharp Gårding inequality.

Proof. We omit the proof. □

Problem. Deduce the Gårding inequality from the sharp Gårding inequality.

Chapter 4

Application I: Analysis of Singularities

§ 4.1 Pseudolocality

Definition. Define the **support** and **singular support** of $u \in \mathcal{S}'(\mathbb{R}^d)$ as

$$\operatorname{supp} u = \Big(\bigcup\big\{U \subset \mathbb{R}^d;\ U \text{ is open, and } u|_U \equiv 0\big\}\Big)^c,$$
$$\operatorname{sing\,supp} u = \Big(\bigcup\big\{U \subset \mathbb{R}^d;\ U \text{ is open, and } u|_U \in C^\infty(U)\big\}\Big)^c,$$

respectively.

Remark. By definition $u|_U \equiv 0$ iff

$$\langle u, \phi\rangle = 0 \quad \text{for any } \phi \in C_c^\infty(U).$$

Similarly, $u|_U \in C^\infty(U)$ iff there exists $v \in C^\infty(U)$ such that

$$\langle u, \phi\rangle = \int_U v(x)\phi(x)\,\mathrm{d}x \quad \text{for any } \phi \in C_c^\infty(U).$$

Theorem 4.1. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$.

1. $a(x,D)$ is **pseudolocal**, i.e., for any $u \in \mathcal{S}'(\mathbb{R}^d)$

$$\operatorname{sing\,supp} a(x,D)u \subset \operatorname{sing\,supp} u.$$

2. If a is elliptic, $a(x,D)$ is **hypoelliptic**, i.e., for any $u \in \mathcal{S}'(\mathbb{R}^d)$

$$\operatorname{sing\,supp} a(x,D)u = \operatorname{sing\,supp} u.$$

Remark. An operator A on $\mathcal{S}'(\mathbb{R}^d)$ is said to be **local** if it satisfies for any $u \in \mathcal{S}'(\mathbb{R}^d)$

$$\operatorname{supp} Au \subset \operatorname{supp} u.$$

See also Proposition 4.2 below.

Proof. 1. Let $u \in \mathcal{S}'(\mathbb{R}^d)$. Let $U \subset \mathbb{R}^d$ be an open subset such that

$$u|_U \in C^\infty(U).$$

Take any $\chi_1 \in C_c^\infty(U)$, and choose $\chi_2 \in C_c^\infty(U)$ such that

$$\chi_2 = 1 \quad \text{on a neighborhood of } \operatorname{supp}\chi_1.$$

We decompose

$$\chi_1 a(x,D)u = \chi_1 a(x,D)\chi_2 u + \chi_1 a(x,D)(1-\chi_2)u.$$

Then, since $\chi_2 u \in \mathcal{S}(\mathbb{R}^d)$,

$$\chi_1 a(x,D)\chi_2 u \in \mathcal{S}(\mathbb{R}^d).$$

On the other hand, since $\chi_1 a(x,D)(1-\chi_2) \in \Psi^{-\infty}(\mathbb{R}^d)$,

$$\chi_1 a(x,D)(1-\chi_2)u \in \mathcal{S}(\mathbb{R}^d).$$

Thus we obtain $\chi_1 a(x,D)u \in \mathcal{S}(\mathbb{R}^d)$, and hence

$$(a(x,D)u)|_U \in C^\infty(U).$$

This implies the assertion.

2. If a is elliptic, then by Theorem 2.8 there exist $b \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$ and $r \in S^{-\infty}(\mathbb{R}^{2d})$ such that for any $u \in \mathcal{S}'(\mathbb{R}^d)$

$$u = b(x,D)a(x,D)u + r(x,D)u.$$

Then by Proposition 3.9 and the assertion 1

$$\operatorname{sing\,supp} u \subset \operatorname{sing\,supp} b(x,D)a(x,D)u \subset \operatorname{sing\,supp} a(x,D)u.$$

Thus the assertion follows. □

◦ **Topic: Local ΨDOs**

Proposition 4.2. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, $\delta \ne 1$ and $\rho \ne 0$. $a(x,D)$ is local if and only if it is a PDO.

Proof. Step 1. First, assume $m < -d$, and we show $a \equiv 0$. In this case we can write for any $u \in \mathcal{S}(\mathbb{R}^d)$

$$a(x,D)u(x) = \int_{\mathbb{R}^d} K(x,y)u(y)\,\mathrm{d}y$$

with

$$K(x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(x-y)\xi}a(x,\xi)\,\mathrm{d}\xi.$$

By the locality we obtain $K(x,y) = 0$ for $x \ne y$, hence the claim.

Step 2. Next, let $\alpha \in \mathbb{N}_0^d$, and we prove $(\partial_\xi^\alpha a)(x,D)$ is also local. However, it is straightforward since by integration by parts we can write for any $u \in \mathcal{S}(\mathbb{R}^d)$

$$(\partial_\xi^\alpha a)(x,D)u(x) = (-\mathrm{i})^{|\alpha|}\sum_{\beta\in\mathbb{N}_0^d}(-1)^{|\beta|}\binom{\alpha}{\beta}x^{\alpha-\beta}a(x,D)x^\beta u(x).$$

Step 3. Here we prove the assertion. By Taylor's theorm and Steps 2 and 1 it follows that for any $N \in \mathbb{N}_0$ with $m - \rho N < -d$

$$a(x,\xi) = \sum_{|\alpha|\le N}\frac{1}{\alpha!}(\partial_\xi^\alpha a)(x,0)\xi^\alpha.$$

This implies $a(x,D)$ is a PDO. □

§ 4.2 Wave Front Set

Definition. We say $\Gamma \subset \mathbb{R}^d$ is **conic** if it satisfies

$$\xi \in \Gamma,\ t > 0 \ \Rightarrow\ t\xi \in \Gamma.$$

We also say $\Gamma' \subset \mathbb{R}^{2d}$ is **conic** if it satisfies

$$(x,\xi) \in \Gamma',\ t > 0 \ \Rightarrow\ (x,t\xi) \in \Gamma'.$$

In the following we shall write

$$\mathbb{R}^{2d}\setminus 0 = \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$$

for short.

Definition. The **wave front set** of $u \in \mathcal{S}'(\mathbb{R}^d)$:

$$\mathrm{WF}(u) \subset \mathbb{R}^{2d}\setminus 0$$

is defined such that $(x_0,\xi_0) \notin \mathrm{WF}(u)$ if and only if there exist $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi(x_0) \ne 0$ and a conic neighborhood $\Gamma \subset \mathbb{R}^d\setminus\{0\}$ of ξ_0 such that for any $N \ge 0$ there exists $C_N \ge 0$ such that

$$\bigl|(\mathcal{F}\chi u)(\xi)\bigr| \le C_N\langle\xi\rangle^{-N} \quad \text{for } \xi \in \Gamma.$$

Remark. By definition $\mathrm{WF}(u) \subset \mathbb{R}^{2d}\setminus 0$ is closed and conic.

Theorem 4.3. Let $u \in \mathcal{S}'(\mathbb{R}^d)$. Then

$$\pi(\mathrm{WF}(u)) = \operatorname{sing\,supp} u,$$

where

$$\pi \colon \mathbb{R}^{2d} \setminus 0 \to \mathbb{R}^d,\ (x, \xi) \mapsto x$$

is the first projection.

Remark. $\mathrm{WF}(u)$ represents "direction-wise singularities" at each point.

Proof. Step 1. Let $x_0 \notin \pi(\mathrm{WF}(u))$. For each $\xi \in \mathbb{S}^{d-1}$ we have

$$(x_0, \xi) \notin \mathrm{WF}(u),$$

so that we can find $\chi \in C_c^\infty(\mathbb{R}^d)$ and $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ as in the definition of the wave front set. Since $\mathbb{S}^{d-1}$ is compact, we can choose $\xi_j \in \mathbb{S}^{d-1}$, $j = 1, \dots, k$, and the corresponding χ_j and Γ_j such that

$$\bigcup_{j=1}^{k} \Gamma_j = \mathbb{R}^d \setminus \{0\}.$$

Now we set

$$\chi = \chi_1 \cdots \chi_k \in C_c^\infty(\mathbb{R}^d).$$

Then obviously $\chi(x_0) \neq 0$, and moreover we can verify that for any $N \geq 0$ there exists $C_N > 0$ such that

$$|(\mathcal{F}\chi u)(\xi)| \leq C_N \langle \xi \rangle^{-N} \quad \text{for } \xi \in \mathbb{R}^d.$$

(The verification is left to the reader as a **Problem**.) Thus

$$\chi u = \mathcal{F}^* \mathcal{F} \chi u \in C^\infty(\mathbb{R}^d),$$

and this implies $x_0 \notin \operatorname{sing\,supp} u$.

Step 2. Conversely, let $x_0 \notin \operatorname{sing\,supp} u$. Then there exists $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi(x_0) \neq 0, \quad \chi u \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d).$$

Since $\mathcal{F}\chi u \in \mathcal{S}(\mathbb{R}^d)$, for any $N \geq 0$ there exists $C_N > 0$ such that

$$|(\mathcal{F}\chi u)(\xi)| \leq C_N \langle \xi \rangle^{-N} \quad \text{for } \xi \in \mathbb{R}^d.$$

Thus for any $\xi \in \mathbb{R}^d \setminus \{0\}$ we obtain $(x_0, \xi) \notin \mathrm{WF}(u)$. □

Problem. Compute the wave front sets of the following distributions.

1. The Dirac delta funcion δ on $\mathbb{R}^d$;
2. $\delta(x') \otimes 1(x'')$ for $(x', x'') \in \mathbb{R}^p \times \mathbb{R}^q$;
3. $\delta_{\mathbb{S}^{d-1}}$ on $\mathbb{R}^d$;
4. $(x + \mathrm{i}0)^{-1}$ on $\mathbb{R}$;
5. The characteristic function χ_Γ of an angular domain $\Gamma \subset \mathbb{R}^2$.

§ 4.3 Microlocal Ellipticity

Definition. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$.

1. We say $a(x,\xi)$, or $a(x,D)$, is **elliptic at** $x_0 \in \mathbb{R}^d$ if there exists $\epsilon, R > 0$ and a neighborhood $\Omega \subset \mathbb{R}^d$ of x_0 such that for any $x \in \Omega$ and $|\xi| \ge R$
$$|a(x,\xi)| \ge \epsilon |\xi|^m.$$

2. We say $a(x,\xi)$, or $a(x,D)$, is **elliptic at** $(x_0,\xi_0) \in \mathbb{R}^{2d} \setminus 0$ if there exist $\epsilon, R > 0$ and a conic neighborhood $\Gamma \subset \mathbb{R}^{2d}$ of (x_0,ξ_0) such that for any $(x,\xi) \in \Gamma$ with $|\xi| \ge R$
$$|a(x,\xi)| \ge \epsilon |\xi|^m.$$

3. Define the **characteristic set** of $a(x,\xi)$, or $a(x,D)$, as
$$\begin{aligned}\operatorname{char} a &= \operatorname{char}(a(x,D)) \\ &= \big\{(x,\xi) \in \mathbb{R}^{2d} \setminus 0;\ a \text{ is not elliptic at } (x,\xi)\big\}.\end{aligned}$$

Remark. By definition $\operatorname{char} a \subset \mathbb{R}^{2d} \setminus 0$ is closed and conic. Note, if a is elliptic, it is elliptic at any $(x,\xi) \in \mathbb{R}^{2d} \setminus 0$ and $\operatorname{char} a = \emptyset$.

Theorem 4.4. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $(x_0,\xi_0) \in \mathbb{R}^{2d} \setminus 0$. Then $(x_0,\xi_0) \notin \mathrm{WF}(u)$ if and only if there exists $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$ such that it is elliptic at (x_0,ξ_0) and
$$a(x,D)u \in C^\infty(\mathbb{R}^d).$$

Proof. Necessity. First let $(x_0,\xi_0) \notin \mathrm{WF}(u)$. Take $\chi \in C^\infty_c(\mathbb{R}^d)$ and $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ as in the definition of the wave front set. Let $\eta \in C^\infty(\mathbb{R}^d)$ be such that
$$\eta(\xi_0) \neq 0, \quad \operatorname{supp} \eta \subset \Gamma, \quad \eta(t\xi) = \eta(\xi) \text{ for } t \ge 1 \text{ and } |\xi| \ge 1.$$
Then for any $N \ge 0$ there exists $C_N > 0$ such that
$$|\eta(\xi)(\mathcal{F}\chi u)(\xi)| \le C_N \langle \xi \rangle^{-N} \quad \text{for all } \xi \in \mathbb{R}^d,$$

which implies
$$(\bar\chi(x)\bar\eta(D))^* u = \mathcal{F}^* \eta \mathcal{F} \chi u \in C^\infty(\mathbb{R}^d).$$
Thus it suffices to take $a(x,\xi) = (\bar\chi(x)\bar\eta(\xi))^* \in S^0(\mathbb{R}^{2d})$.

Sufficiency. Conversely, assume we can find $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ as in the assertion. Note we may assume
$$\operatorname{supp} u \Subset \mathbb{R}^d, \quad \operatorname{supp} a(x,D)u \Subset \mathbb{R}^d.$$
In fact, take $\phi, \psi \in C^\infty_c(\mathbb{R}^d)$ such that
$$\phi(x_0) \neq 0, \quad \psi = 1 \text{ on } \operatorname{supp} \phi,$$
and decompose
$$\phi(x)a(x,D)u = \phi(x)a(x,D)\psi(x)u + \phi(x)a(x,D)(1-\psi(x))u.$$
Then it suffices to prove the assertion for ψu and ϕa instead of u and a, respectively.

Next, by the assumption there exist $\epsilon, R > 0$ and a conic neighborhood $\Gamma \subset \mathbb{R}^{2d}$ of (x_0, ξ_0) such that

$$|a(x,\xi)| \ge \epsilon|\xi|^m \quad \text{for } (x,\xi) \in \Gamma \text{ with } |\xi| \ge R.$$

Then we can construct $b \in S^{-m}_{\rho,\delta}(\mathbb{R}^{2d})$ and $r \in S^{-\infty}(\mathbb{R}^{2d})$ such that

$$b(x,D)a(x,D) = \eta(D)\chi(x) + r(x,D),$$

where $\chi, \eta \in C^\infty(\mathbb{R}^d)$ satisfy

$$\chi(x_0)\eta(R\xi_0/|\xi_0|) \ne 0, \quad \operatorname{supp}\chi\eta \subset \Gamma,$$
$$\eta(t\xi) = \eta(\xi) \text{ for } |\xi| \ge R \text{ and } t \ge 1.$$

In fact, let $b_0 = \chi\eta a^{-1}$, and then there exist $c_1 \in S^{-\rho+\delta}_{\rho,\delta}(\mathbb{R}^{2d})$ and $r_1 \in S^{-\infty}(\mathbb{R}^{2d})$ such that

$$b_0 \# a = \eta \# \chi + c_1 + r_1, \quad \operatorname{supp} c_1 \subset \operatorname{supp}\chi\eta.$$

Then, let $b_1 = -c_1 a^{-1}$, and there exist $c_2 \in S^{-2(\rho-\delta)}_{\rho,\delta}(\mathbb{R}^{2d})$ and $r_2 \in S^{-\infty}(\mathbb{R}^{2d})$ such that

$$b_1 \# a = -c_1 + c_2 + r_2, \quad \operatorname{supp} c_2 \subset \operatorname{supp}\chi\eta.$$

Repeat this procedure, and we take the asymptotic sum

$$b \sim \sum_{j=0}^{\infty} b_j,$$

which satisfies the claimed identity.

Now we obtain, noting the support of u and $a(x,D)u$,

$$\eta(D)\chi u = b(x,D)a(x,D)u - r(x,D)u \in \mathcal{S}(\mathbb{R}^d),$$

cf. Proposition 3.9. Therefore $(x_0, \xi_0) \notin \mathrm{WF}(u)$. □

Theorem 4.5. Let $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ with $m \in \mathbb{R}$ and $0 \le \delta < \rho \le 1$. Then for any $u \in \mathcal{S}'(\mathbb{R}^d)$

$$\mathrm{WF}(a(x,D)u) \subset \mathrm{WF}(u) \subset \mathrm{WF}(a(x,D)u) \cup \operatorname{char} a.$$

In particular, if a is elliptic, then for any $u \in \mathcal{S}'(\mathbb{R}^d)$

$$\mathrm{WF}(a(x,D)u) = \mathrm{WF}(u).$$

Remarks. 1. These are microlocal refinements of pseudolocality and hypoellipticity, see Theorem 4.1.

2. If $a(x,D)$ is elliptic, the wave front set of a solution u to

$$a(x,D)u = f$$

is completely determined by that of f: $\mathrm{WF}(u) = \mathrm{WF}(f)$.

Proof. Step 1. Assume $(x_0, \xi_0) \notin \mathrm{WF}(a(x,D)u) \cup \operatorname{char} a$. Then, since $(x_0, \xi_0) \notin \mathrm{WF}(a(x,D)u)$, by Theorem 4.4 there exists $b \in S^l_{\sigma,\epsilon}(\mathbb{R}^{2d})$ with $l \in \mathbb{R}$ and $0 \le \epsilon < \sigma \le 1$ such that it is elliptic at (x_0, ξ_0) and

$$b(x,D)a(x,D)u \in C^\infty(\mathbb{R}^d).$$

On the other hand, since $(x_0, \xi_0) \notin \operatorname{char} a$, $b \# a$ is also elliptic at (x_0, ξ_0). Hence by Theorem 4.4 we obtain $(x_0, \xi_0) \notin \mathrm{WF}(u)$.

Step 2. Next, let $(x_0, \xi_0) \notin \mathrm{WF}(u)$. Take $\chi, \tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$ and $\eta, \tilde{\eta} \in C^\infty(\mathbb{R}^d)$ such that

$$\begin{aligned}&\chi(x_0)\eta(\xi_0) \neq 0, \quad \tilde{\eta}(D)\tilde{\chi}(x)u \in H^\infty(\mathbb{R}^d)\\ &\eta(t\xi) = \eta(\xi),\ \tilde{\eta}(t\xi) = \tilde{\eta}(\xi) \quad \text{for } t \geq 1 \text{ and } |\xi| \geq |\xi_0|\\ &\tilde{\chi}(x)\tilde{\eta}(\xi) = 1 \quad \text{on a neighborhood of } \operatorname{supp}\chi(x)\eta(\xi).\end{aligned}$$

We decompose

$$\begin{aligned}\eta(D)\chi(x)a(x,D)u = {}&\eta(D)\chi(x)a(x,D)\tilde{\eta}(D)\tilde{\chi}(x)u\\ &+ \eta(D)\chi(x)a(x,D)\big(1 - \tilde{\eta}(D)\tilde{\chi}(x)\big)u.\end{aligned}$$

Then the first term on the right-hand side belongs to $H^\infty(\mathbb{R}^d)$. In addition, since

$$\eta(D)\chi(x)a(x,D)\big(1 - \tilde{\eta}(D)\tilde{\chi}(x)\big) \in \Psi^{-\infty}(\mathbb{R}^d),$$

the second term belongs to $C^\infty(\mathbb{R}^d)$. Thus we obtain $(x_0, \xi_0) \notin \mathrm{WF}(a(x,D)u)$. We are done. □

§ 4.4 Propagation of Wave Front Set

∘ Hamilton flow

Definition. Let $\Gamma \subset \mathbb{R}^{2d}$ be open. Define the **Hamilton vector field** associated with a **Hamiltonian** $p \in C^\infty(\Gamma; \mathbb{R})$ as

$$H_p = \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi} = \sum_{j=1}^d \left(\frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}\right) \in \mathfrak{X}(\Gamma).$$

In addition, a solution to the **Hamilton equations**

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{\partial p}{\partial \xi_j}(x,\xi), \quad \frac{\mathrm{d}\xi_j}{\mathrm{d}t} = -\frac{\partial p}{\partial x_j}(x,\xi), \quad j = 1, \ldots, d,$$

is called a **bicharacteristic** of p.

Proposition 4.6. Let $p \in C^\infty(\Gamma; \mathbb{R})$ with $\Gamma \subset \mathbb{R}^{2d}$ open. For any bicharacteristic $\gamma\colon I \to \Gamma$, $I \subset \mathbb{R}$, of p, $p \circ \gamma$ is constant on I.

Proof. Let us write simply $\gamma = (x, \xi)$. Then by definition

$$\frac{\mathrm{d}}{\mathrm{d}t}p(x,\xi) = \sum_{j=1}^d \left(\frac{\mathrm{d}x_j}{\mathrm{d}t}\frac{\partial p}{\partial x_j}(x,\xi) + \frac{\mathrm{d}\xi_j}{\mathrm{d}t}\frac{\partial p}{\partial \xi_j}(x,\xi)\right) = 0.$$

Hence the assertion follows. □

Definition. A bicharacteristic γ of p is called a **null bicharacteristic** if $p \circ \gamma \equiv 0$.

Proposition 4.7. Let $\Gamma \subset \mathbb{R}^{2d} \setminus 0$ be open and conic, and let $p \in C^\infty(\Gamma; \mathbb{R})$ be positively homogeneous of degree $m \in \mathbb{R}$ in $\xi \neq 0$. If

$$\gamma(t; y, \eta) = (x(t; y, \eta), \xi(t; y, \eta)), \quad \gamma(0; y, \eta) = (y, \eta),$$

is a bicharacteristic of p, then for any $\lambda > 0$

$$\gamma_{\pm,\lambda}(t; y, \eta) := \Big(x\big(\pm\lambda^{m-1}t; y, \eta\big), \lambda\xi\big(\pm\lambda^{m-1}t; y, \eta\big)\Big)$$

are bicharacteristics of $\pm p$, respectively.

Proof. It is straightforward due to direct computations. □

∘ **Propagation theorem**

Theorem 4.8. Let $a \in S^m_{\mathrm{cl}}(\mathbb{R}^{2d})$ with principal symbol p, and let $u, f \in \mathcal{S}'(\mathbb{R}^d)$ satisfy

$$a(x, D)u = f.$$

Let $\gamma\colon [0, T] \to \mathbb{R}^{2d} \setminus 0$ be a null bicharacteristic of $\mathrm{Re}\, p$, and suppose for some conic neighborhood $\Gamma \subset \mathbb{R}^{2d} \setminus 0$ of $\gamma([0, T])$

$$\mathrm{Im}\, p \geq 0 \quad \text{in } \Gamma.$$

If

$$\gamma(0) \in \mathrm{WF}(u) \quad \text{and} \quad \gamma([0, T]) \cap \mathrm{WF}(f) = \emptyset,$$

then $\gamma(T) \in \mathrm{WF}(u)$.

Remarks. 1. $\mathrm{WF}(u)$ propagates forward/backward along the null bicharacteristics of $\mathrm{Re}\, p$ where $\pm \mathrm{Im}\, p \geq 0$, respectively, until they hit $\mathrm{WF}(f)$. As for the backward propagation for $\mathrm{Im}\, p \leq 0$, it suffices to apply the assertion to

$$-a(x, D)u = -f$$

along with Proposition 4.7. Note, if $\mathrm{Im}\, p \equiv 0$, then $\mathrm{WF}(u)$ propagates both forward and backward, see Corollary 4.9 below.

2. In other words, along null bicharacteristics, singularities may only be amplified/damped according to $\pm \mathrm{Im}\, p \geq 0$, respectively. We avoid $\mathrm{WF}(f)$ since the **external force** f could create or annihilate singularities there.

3. The conclusion is equivalent to the *converse* propagation of regularities: "If

$$\gamma(T) \notin \mathrm{WF}(u) \quad \text{and} \quad \gamma([0, T]) \cap \mathrm{WF}(f) = \emptyset,$$

then $\gamma(0) \notin \mathrm{WF}(u)$." In fact, the proof keeps track of propagation of regularities.

4. Recall Theorem 4.5 implies

$$\mathrm{WF}(u) \cap (\mathrm{char}\, p)^c = \mathrm{WF}(f) \cap (\mathrm{char}\, p)^c.$$

This is why we consider only the *null* bicharacteristics. (However, note also

$$\mathrm{char}\, p = \{\mathrm{Re}\, p = 0\} \cap \{\mathrm{Im}\, p = 0\},$$

see Corollary 4.10 below.)

Corollary 4.9. Let $a \in S^m_{\mathrm{cl}}(\mathbb{R}^{2d})$ have a real principal symbol p, and let $u, f \in \mathcal{S}'(\mathbb{R}^d)$ satisfy

$$a(x, D)u = f.$$

If $\gamma\colon [0, T] \to \mathbb{R}^{2d} \setminus 0$ is a null bicharacteristic of p such that $\gamma([0, T]) \cap \mathrm{WF}(f) = \emptyset$, then either

$$\gamma([0, T]) \subset \mathrm{WF}(u) \quad \text{or} \quad \gamma([0, T]) \subset (\mathrm{WF}(u))^c$$

holds.

Proof. The assertion is obvious by Theorem 4.8 and the subsequent remarks. □

Corollary 4.10. Let $a \in S^m_{\mathrm{cl}}(\mathbb{R}^{2d})$ have a principal symbol p with $\mathrm{Im}\, p \geq 0$, and let $u \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in C^\infty(\mathbb{R}^d)$ satisfy

$$a(x, D)u = f.$$

If $\gamma\colon [0, T] \to \mathbb{R}^{2d} \setminus 0$ is a null bicharacteristic of $\mathrm{Re}\, p$ such that $\mathrm{Im}\, p(\gamma(T)) > 0$, then

$$\gamma([0, T]) \subset (\mathrm{WF}(u))^c$$

holds.

Proof. The assertion is obvious by Theorems 4.5 and 4.8, and the remarks subsequent to Theorems 4.8. □

Example. Consider the 1D wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(t, x) = 0 \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}.$$

We can apply Theorems 4.5 and 4.8, or Corollary 4.9, with

$$a(t, x, \tau, \xi) = p(t, x, \tau, \xi) = -\tau^2 + \xi^2, \quad f = 0,$$

and conclude that $\mathrm{WF}(u)$ is a subset of the **light cone**

$$\left\{(t, x, \tau, \xi) \in \mathbb{R}^4 \setminus 0;\ -\tau^2 + \xi^2 = 0\right\}$$

and that $\mathrm{WF}(u)$ is invariant under the Hamilton flow of p. Note all the null bicharacteristics of p are given by

$$(t, x, \tau, \xi) = (t_0 - 2s\tau_0, x_0 + 2s\xi_0, \tau_0, \xi_0) \quad \text{with } -\tau_0^2 + \xi_0^2 = 0.$$

Outline of proof of Theorem 4.8. Step 1. We microlocalize in a conic neighborhood of $\gamma([0, T])$ with factor $|D|^{1-m}$, so that we may let

$$m = 1, \quad \mathrm{Im}\, p \geq 0, \quad f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d), \quad u \in H^s(\mathbb{R}^d) \text{ for some } s \in \mathbb{R}.$$

In fact, choose $\chi \in S^{1-m}_{\mathrm{cl}}(\mathbb{R}^{2d})$ and $\tilde{\chi} \in S^0_{\mathrm{cl}}(\mathbb{R}^{2d})$ both supported in a small conic neighborhood of $\gamma([0, T])$ such that

$$\chi(x, \xi) = |\xi|^{1-m} \text{ in a conic neighborhood of } \gamma([0, T]),$$
$$\tilde{\chi}(x, \xi) = 1 \text{ in a conic neighborhood of } \mathrm{supp}\, \chi.$$

Then the claim follows by the decomposition

$$\begin{aligned}&\chi(x, D)a(x, D)\tilde{\chi}(x, D)u \\ &= \chi(x, D)f - \chi(x, D)a(x, D)(1 - \tilde{\chi}(x, D))u,\end{aligned}$$

and the structure of compactly supported distributions. Note $\gamma([0, T])$ remains the same up to scaling of time parameter.

Step 2. Let $(y, \eta) \in \mathbb{R}^{2d} \setminus 0$, and take $\psi \in S^s_{\mathrm{cl}}(\mathbb{R}^{2d})$ supported in a small conic neighborhood of (y, η) with

$$\psi(x, \xi) = \langle \epsilon\xi \rangle^{-1/2} \langle \xi \rangle^{s+1/2} \text{ in a conic neighborhood of } (y, \eta).$$

Here $\epsilon \in (0, 1]$ is a parameter to be let $\epsilon \to 0$, cf. **Yosida approximation**. Now we solve a **transport equation**

$$\frac{\partial}{\partial t} b - \{\mathrm{Re}\, p, b\} = 0, \quad b(0, x, \xi) = \psi(x, \xi).$$

In fact, if $\gamma(t; x, \xi)$ is a bicharacteristic with initial data (x, ξ),

$$\frac{\partial}{\partial t} b(t, \gamma(t; x, \xi)) = 0, \quad \text{and hence} \quad b(t, x, \xi) = \psi(\gamma(-t, x, \xi)).$$

Note b are bounded in $S^{s+1/2}_{\mathrm{cl}}(\mathbb{R}^{2d})$ for $t \in [0, T]$ and $\epsilon \in (0, 1]$.

Step 3. In the following let us write for short

$$A = a(x,D), \quad P_r = (\mathrm{Re}\,p)^{\mathsf{W}}(x,D), \quad P_i = (\mathrm{Im}\,p)^{\mathsf{W}}(x,D),$$
$$B = b^{\mathsf{W}}(t,x,D), \quad R = r^{\mathsf{W}}(t,x,D), \quad \ldots.$$

Here we are going to show there exist $\mu > 0$ and $r \in S^{2s}_{\mathrm{cl}}(\mathbb{R}^{2d})$, bounded uniformly in $t \in [0,T]$ and $\epsilon \in (0,1]$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mu t}B^2) - 2\mathrm{e}^{\mu t}\,\mathrm{Im}(A^*B^2) \geq R,$$

as quadratic forms, e.g., on $\mathcal{S}(\mathbb{R}^d)$. In fact, we can compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mu t}B^2) &= \mu\mathrm{e}^{\mu t}B^2 + \mathrm{i}\mathrm{e}^{\mu t}[P_r,B]B + \mathrm{i}\mathrm{e}^{\mu t}B[P_r,B] + R_1 \\ &= \mu\mathrm{e}^{\mu t}B^2 + 2\mathrm{e}^{\mu t}\,\mathrm{Im}(P_rB^2) + R_1 \\ &= \mu\mathrm{e}^{\mu t}B^2 + 2\mathrm{e}^{\mu t}\,\mathrm{Im}(A^*B^2) + 2\mathrm{e}^{\mu t}\,\mathrm{Re}(P_iB^2) \\ &\quad + 2\mathrm{e}^{\mu t}\,\mathrm{Im}\Big((P_r - \mathrm{i}P_i - A^*)B^2\Big) + R_1,\end{aligned}$$

where $R_1 \in \Psi^{2s}_{\mathrm{cl}}(\mathbb{R}^d)$. We continue by using the L^2-boundedness theorem and the sharp Gårding inequality as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mu t}B^2) &= \mu\mathrm{e}^{\mu t}B^2 + 2\mathrm{e}^{\mu t}\,\mathrm{Im}(A^*B^2) \\ &\quad + 2\mathrm{e}^{\mu t}BP_iB + \mathrm{e}^{\mu t}[[P_i,B],B] \\ &\quad + 2\mathrm{e}^{\mu t}B\Big(\mathrm{Im}(P_r - \mathrm{i}P_i - A^*)\Big)B \\ &\quad + 2\mathrm{e}^{\mu t}\,\mathrm{Im}\Big([P_r - \mathrm{i}P_i - A^*, B]B\Big) + R_1 \\ &= (\mu - C_1)\mathrm{e}^{\mu t}B^2 + 2\mathrm{e}^{\mu t}\,\mathrm{Im}(A^*B^2) + R_2\end{aligned}$$

with $R_2 \in \Psi^{2s}_{\mathrm{cl}}(\mathbb{R}^d)$. Therefore the claim follows for large $\mu > 0$.

Step 4. Now let $\gamma(T;y,\eta) \notin \mathrm{WF}(u)$. By Step 3 and the fundamental theorem of calculus

$$\|B(0)u\|^2_{L^2} \leq \mathrm{e}^{\mu T}\|B(T)u\|^2_{L^2} + C\Big(\|u\|^2_{H^s} + \|f\|^2_{H^{s+1}}\Big)$$

uniformly in $\epsilon \in (0,1]$. If we choose $\mathrm{supp}\,\psi$ small enough, and let and let $\epsilon \to +0$, then by the monotone convergence theorem

$$u \text{ is } H^{s+1/2} \text{ in a (microlocal) neighborhood of } (y,\eta).$$

Hence u is $H^{s+1/2}$ in a neighborhood of $\gamma([0,T])$. We repeat the above arguments, and obtain at last u is C^∞ in a neighborhood of $\gamma([0,T])$. (We have to be careful that these neighborhoods should not shrink to $\gamma([0,T])$.) Thus we are done. □

Chapter 5

Application II: Local Solvability of PDOs

§ 5.1 Local Solvability

◦ Definition and reduction

Throughout the chapter we study a PDO

$$a(x,D)=\sum_{|\alpha|\le m} a_\alpha(x)D^\alpha;\quad a_\alpha\in C^\infty(\mathbb{R}^d).$$

Definition. $a(x,D)$ is **locally solvable** at $x_0\in\mathbb{R}^d$ if there exists a neighborhood $U\subset\mathbb{R}^d$ of x_0 such that for any $f\in C^\infty(\mathbb{R}^d)$ there exists $u\in\mathcal{S}'(\mathbb{R}^d)$ satisfying

$$a(x,D)u=f\quad\text{on } U.$$

Theorem 5.1. 1. If $a(x,D)$ is locally solvable at $x_0\in\mathbb{R}^d$, then there exist a neighborhood $U\subset\mathbb{R}^d$ of x_0, $s,t\in\mathbb{R}$ and $c>0$ such that for any $v\in C_c^\infty(U)$

$$\|a^*(x,D)v\|_{H^{-s}}\ge c\|v\|_{H^{-t}}.$$

2. Conversely, if there exist $U\subset\mathbb{R}^d$, $s,t\in\mathbb{R}$ and $c>0$ as above, then for any $f\in H^t(\mathbb{R}^d)$ there exists $u\in H^s(\mathbb{R}^d)$ such that

$$a(x,D)u=f\quad\text{on } U.$$

In partiucular, $a(x,D)$ is locally solvable at x_0.

Remark. We may say, very roughly, $a(x,D)\colon H^s\to H^t$ is surjective if and only if $a^*(x,D)\colon H^{-t}\to H^{-s}$ is injective.

Proof. 1. Step 1. Assume $a(x,D)$ is locally solvable at x_0, and take a neighborhood $U\subset\mathbb{R}^d$ of x_0 as in the definition. We may let U be bounded. For each $v\in C_c^\infty(U)$ we define

$$\phi_v\colon X:=H^\infty(\mathbb{R}^d)\to\mathbb{C},\quad f\mapsto(f,v)_{L^2},$$

and set for each $n,k\in\mathbb{N}_0$

$$X_{n,k}=\Big\{f\in X;\ \forall v\in C_c^\infty(U)\ |\phi_v(f)|\le n\|a^*(x,D)v\|_{H^k}\Big\}.$$

We are going to apply the Baire category theorem for X and $X_{n,k}$. Note X is a complete metric space with respect to a distance given by

$$d(f,g)=\sum_{k\in\mathbb{N}_0}\frac{1}{2^k}\frac{\|f-g\|_{H^k}}{1+\|f-g\|_{H^k}}.$$

Step 2. We verify the assumptions the Baire category theorem. To see $X_{n,k}\subset X$ is closed let us rewrite

$$X_{n,k}=\bigcap_{v\in C_c^\infty(U)}\Big\{f\in X;\quad |\phi_v(f)|\le n\|a^*(x,D)v\|_{H^k}\Big\}.$$

Thus it suffices to show ϕ_v is continuous, however it is staightforward since

$$|\phi_v(f)|=|(f,v)_{L^2}|\le\|f\|_{H^0}\|v\|_{H^0}.$$

Next we prove $X_{n,k}$ with $n,k\in\mathbb{N}_0$ exhaust X. Take any $f\in X\subset C^\infty(\mathbb{R}^d)$, and then by the assumption there exists $u\in\mathcal{S}'(\mathbb{R}^d)$ such that

$$a(x,D)u=f\quad\text{on } U.$$

Now by the continuity of u, boundedness of U and the Sobolev embedding theorem there exist $C, C' > 0$ and $k, k' \in \mathbb{N}_0$ such that for any $v \in C_c^\infty(U)$

$$\begin{aligned}|\phi_v(f)| &= |(u, a^*(x,D)v)_{L^2}| \\ &\le C \sup\Big\{|\partial^\alpha a^*(x,D)v(x)|;\ |\alpha| \le k,\ x \in U\Big\} \\ &\le C' \|a^*(x,D)v\|_{H^{k'}}.\end{aligned}$$

This implies the claim.

Step 3. Now by the Baire category theorem there exist $g \in X$, $l \in \mathbb{N}_0$ and $\epsilon > 0$ such that

$$\Big\{h \in X;\ \|h - g\|_{H^l} \le \epsilon\Big\} \subset X_{n,k}.$$

Thus for any $v \in C_c^\infty(U)$ and $f \in X$ with $\|f\|_{H^l} \le \epsilon$

$$|\phi_v(f)| \le |\phi_v(f+g)| + |\phi_v(g)| \le 2n\|a^*(x,D)v\|_{H^k},$$

which in turn implies for any $v \in C_c^\infty(U)$ and $f \in X$

$$|(f,v)_{L^2}| \le 2n\epsilon^{-1}\|f\|_{H^l}\|a^*(x,D)v\|_{H^k}.$$

Hence it follows that for any $v \in C_c^\infty(U)$

$$\|v\|_{H^{-l}} \le 2n\epsilon^{-1}\|a^*(x,D)v\|_{H^k},$$

and the assertion 1 is verified.

2. Assume that there exist $U \subset \mathbb{R}^d$, $s, t \in \mathbb{R}$ and $c > 0$ as in the assertion 2. Take any $f \in H^t(\mathbb{R}^d)$. Define

$$\phi_f \colon L \to \mathbb{C};\quad L = a^*(x,D)C_c^\infty(U),$$

as, for any $w = a^*(x,D)v \in L$,

$$\phi_f(w) = (v,f)_{L^2}.$$

Note it is well-defined since $a^*(x,D)\colon H^{-t}(\mathbb{R}^d) \to H^{-s}(\mathbb{R}^d)$ is injective. Since

$$|\phi_f(w)| \le \|v\|_{H^{-t}}\|f\|_{H^t} \le C\|w\|_{H^{-s}}\|f\|_{H^t},$$

we can extend ϕ_f to $\widetilde{\phi}_f \in (H^{-s}(\mathbb{R}^d))^*$ by the Hahn–Banach theorem. Then we can write for some $u \in H^s(\mathbb{R}^d)$

$$\widetilde{\phi}_f = (\cdot, u)_{L^2},$$

and hence for any $w = a^*(x,D)v \in L$

$$(v,f)_{L^2} = \widetilde{\phi}_f(w) = (w,u)_{L^2} = (a^*(x,D)v, u)_{L^2} = (v, a(x,D)u)_{L^2}.$$

Thus the assertion 2 is verified. □

◦ **Topic: Derivative loss**

We present a refinement of local solvability for reference.

Definition. $a(x,D)$ is **locally solvable at** $x_0 \in \mathbb{R}^d$ **with derivative loss** $\mu \ge 0$ if for any $s \in \mathbb{R}$ there exists a neighborhood $U \subset \mathbb{R}^d$ of x_0 such that for any $f \in H^s(\mathbb{R}^d)$ there exists $u \in H^{s+m-\mu}(\mathbb{R}^d)$ satisfying

$$a(x,D)u = f \quad \text{on } U.$$

Remark. 1. If $a(x,D)$ is locally solvable at x_0 with derivative loss $\mu \ge 0$, then it is locally solvable at x_0.

2. The smaller μ gets, the stronger the above property gets, since we have to seek for u in a smaller Sobolev space.

§ 5.2 Examples

◦ Elliptic PDOs

Theorem 5.2. Assume $a(x,D)$ is elliptic at $x_0 \in \mathbb{R}^d$. Then there exist a neighborhood $U \subset \mathbb{R}^d$ of x_0 and $c > 0$ such that for any $v \in C_c^\infty(U)$

$$\|a^*(x,D)v\|_{L^2} \geq c\|v\|_{H^m}.$$

In particular, $a(x,D)$ is locally solvable at x_0.

Proposition 5.3 (Poincaré inequality). For any $k \in \mathbb{N}_0$ there exist $C, C' > 0$ such that for any bounded open subset $U \subset \mathbb{R}^d$ and any $u \in C_c^\infty(U)$

$$\|u\|_{H^k} \leq C(\operatorname{diam} U)\||D|u\|_{H^k} \leq C'(\operatorname{diam} U)\|u\|_{H^{k+1}},$$

where $\operatorname{diam} U$ denotes the diameter of U.

Proof. The latter inequality is obvious, and we verify only the former one. We may let $0 \in U$ by translation. Then for any $u \in C_c^\infty(U)$ we can estimate we can estimate

$$\begin{aligned}\|u\|_{H^k}^2 &\leq C_1 \sum_{|\alpha|\leq k} (\mathrm{i}[D_1, x_1]D^\alpha u, D^\alpha u)_{L^2} \\ &\leq C_1 \sum_{|\alpha|\leq k} \mathrm{i}\Big[(x_1 D^\alpha u, D_1 D^\alpha u)_{L^2} - (D_1 D^\alpha u, x_1 D^\alpha u)_{L^2}\Big] \\ &\leq 2C_1(\operatorname{diam} U) \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^2}\|D_1 D^\alpha u\|_{L^2} \\ &\leq C_2(\operatorname{diam} U)\|u\|_{H^k}\||D|u\|_{H^k}.\end{aligned}$$

Thus we obtain the assertion. □

Remark. It is obvious from the above proof that the assertion extends for any $U \subset \mathbb{R}^d$ bounded only in one direction.

Proof of Theorem 5.2. The assertion is obvious for $m = 0$, and we may let $m \geq 1$. By the assumption we can find $c_1, R > 0$ and $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ in a neighborhood of } x_0,$$

and that for any $(x,\xi) \in \mathbb{R}^{2d}$ with $|\xi| \geq R$

$$\chi(x)^2|a(x,\xi)|^2 + (1-\chi(x)^2)|\xi|^{2m} \geq c_1|\xi|^{2m}.$$

Then by the Gårding inequality we obtain for any $v \in H^m(\mathbb{R}^d)$

$$\|\chi a^*(x,D)v\|_{L^2}^2 \geq c_2\|v\|_{H^m}^2 - C_1\|v\|_{H^{m-1}}\|v\|_{H^m}.$$

Next, by the Poincaré inequality, if we take a sufficiently small neighborhood $U \subset \mathbb{R}^d$ of x_0, then for any $v \in C_c^\infty(U)$

$$\|a^*(x,D)v\|_{L^2}^2 \geq c_3\|v\|_{H^m}^2.$$

Thus we obtain the assertion. □

◦ PDOs of principal type

We shall denote the principal symbol of $a(x,D)$ by p, i.e.,

$$p(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

Definition. $a(x,D)$ is of **principal type** at $x_0 \in \mathbb{R}^d$ if

$$\partial_\xi p(x_0,\xi) \neq 0 \ \text{ for any } \xi \in \mathbb{R}^d \setminus \{0\} \text{ with } p(x_0,\xi) = 0.$$

Remarks. 1. The condition says, even if ellipticity is lost, a configuration component of the Hamilton vector field is alive.

2. Suppose $m = 0$. Then $a(x, D)$ is of principal type at $x_0 \in \mathbb{R}^d$ if and only if it is elliptic there, since a PDO of order 0 is just a multiplication operator.

3. Suppose $m \neq 0$. Then $a(x, D)$ is of principal type at $x_0 \in \mathbb{R}^d$ if and only if

$$\partial_\xi p(x_0, \xi) \neq 0 \quad \text{for any } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In fact, if $p(x_0, \xi) \neq 0$, then $\partial_\xi p(x_0, \xi) \neq 0$, since

$$\xi \cdot \partial_\xi p(x_0, \xi) = mp(x_0, \xi)$$

due to Euler's homogeneous function theorem.

Theorem 5.4. Let $m \neq 0$, and assume $a(x, D)$ is of principal type at x_0.

1. There exist $C, \delta > 0$ such that for any neighborhood U of x_0 with $\operatorname{diam} U < \delta$ and $u \in C_c^\infty(U)$

$$\|u\|_{H^{m-1}}^2 \leq C(\operatorname{diam} U)\left(\|a(x, D)u\|_{L^2}^2 + \|a^*(x, D)u\|_{L^2}^2\right).$$

2. In addition, assume p is real or purely imaginary in a neighborhood of x_0. Then there exist a neighborhood U of x_0 and $c > 0$ such that for any $u \in C_c^\infty(U)$

$$\|a^*(x, D)u\|_{L^2} \geq c\|u\|_{H^{m-1}}.$$

In particular, $a(x, D)$ is locally solvable at x_0

Proof. We may let $x_0 = 0$ by translation. In addition, we denote for any $r > 0$

$$B_r = \left\{x \in \mathbb{R}^d;\ |x| < r\right\}.$$

1. Step 1. For simplicity let us write

$$A = a(x, D), \quad Q_j = \mathrm{i}[A, x_j] = (\partial_{\xi_j} a)(x, D) \quad \text{for } j = 1, \ldots, d.$$

Note, although $x_j \notin \Psi^\infty_{\rho,\delta}(\mathbb{R}^d)$, the above symbol calculus is valid since A is a PDO. We will use such properties of PDOs below, too, without mentioning. We shall compute and bound

$$\sum_{j=1}^d (Q_j^* Q_j u, u) = \sum_{j=1}^d \|Q_j u\|_{L^2}^2$$

from above and below for any $u \in C_c^\infty(B_\epsilon)$ with small $\epsilon > 0$.

Step 2 (Bound from below). By the assumption there exist $\delta > 0$ and $c > 0$ such that for any $(x, \xi) \in B_{2\delta} \times \mathbb{R}^d$

$$|\partial_\xi p(x, \xi)|^2 \geq 4c|\xi|^{2m-2}.$$

Take any $\chi \in C_c^\infty(B_{2\delta})$ such that $\chi = 1$ on B_δ, and then

$$\chi(x)|\partial_\xi p(x, \xi)|^2 + 4c(1 - \chi(x))|\xi|^{2m-2} \geq 4c|\xi|^{2m-2},$$

so that we can apply the Gårding inequality. Noting

$$\sum_{j=1}^d Q_j^* \chi Q_j - \chi|\partial_\xi p|^2(x, D) \in S^{2m-3}(\mathbb{R}^d),$$

we can find $c_1, C_1 > 0$ such that for any $u \in C_c^\infty(B_\delta)$

$$\sum_{j=1}^d (Q_j^* Q_j u, u) \geq 2c_1\|u\|_{H^{m-1}}^2 - C_1\|u\|_{H^{m-2}}\|u\|_{H^{m-1}}.$$

Now we use the Poincaré inequality. Let $\delta > 0$ be smaller if necessary, and we obtain for any $u \in C_c^\infty(B_\delta)$

$$\sum_{j=1}^{d} (Q_j^* Q_j u, u) \geq c_1 \|u\|_{H^{m-1}}^2.$$

Step 3 (Bound from above). On the other hand, we can compute

$$\begin{aligned}\|Q_j u\|_{L^2}^2 &= \mathrm{i}((Ax_j - x_j A)u, Q_j u) \\ &= \mathrm{i}(x_j Q_j^* u, A^* u) + \mathrm{i}([Q_j^*, x_j]u, A^* u) \\ &\quad + \mathrm{i}(x_j u, [A^*, Q_j]u) - \mathrm{i}(x_j A u, Q_j u).\end{aligned}$$

Here we express, using a finite number of some PDOs R_k, S_k of order $m-1$, as

$$[A^*, Q_j] = \sum_k R_k^* S_k,$$

and then

$$\begin{aligned}\|Q_j u\|_{L^2}^2 = {}& \mathrm{i}(x_j Q_j^* u, A^* u) + \mathrm{i}([Q_j^*, x_j]u, A^* u) - \mathrm{i}(x_j A u, Q_j u) \\ & + \sum_k \mathrm{i}([R_k, x_j]u, S_k u) + \sum_k \mathrm{i}(x_j R_k u, S_k u).\end{aligned}$$

By the Cauchy–Schwarz inequality, the Sobolev boundedness and the Poincaré inequality we obtain for any $\epsilon > 0$ and $u \in C_c^\infty(B_\epsilon)$

$$\begin{aligned}\|Q_j u\|_{L^2}^2 &\leq \epsilon C_2 \|u\|_{H^{m-1}} \|A^* u\|_{L^2} + C_2 \|u\|_{H^{m-2}} \|A^* u\|_{L^2} \\ &\quad + \epsilon C_2 \|Au\|_{L^2} \|u\|_{H^{m-1}} + C_2 \|u\|_{H^{m-2}} \|u\|_{H^{m-1}} \\ &\quad + \epsilon C_2 \|u\|_{H^{m-1}}^2 \\ &\leq \epsilon C_3 \Big(\|Au\|_{L^2}^2 + \|A^* u\|_{L^2}^2 + \|u\|_{H^{m-1}}^2 \Big).\end{aligned}$$

Step 4. Let $\delta > 0$ be from Step 2. Then by Steps 1–3 it follows that for any $\epsilon \in (0, \delta)$ and $u \in C_c^\infty(B_\epsilon)$

$$(c_1 - \epsilon C_3) \|u\|_{H^{m-1}}^2 \leq \epsilon C_3 \Big(\|Au\|_{L^2}^2 + \|A^* u\|_{L^2}^2 \Big).$$

Let $\delta > 0$ be even smaller if necessary, and the assertion 1 follows.

2. If p is real/purely imaginary, then $a(x, D) \mp a^*(x, D)$ is a PDO of order $m - 1$, respectively. Then by the assertion 1 for any $\epsilon \in (0, \delta)$ and $u \in C_c^\infty(B_\epsilon)$

$$\|u\|_{H^{m-1}}^2 \leq \epsilon C_4 \left(\|a^*(x, D)u\|_{L^2}^2 + \|u\|_{H^{m-1}}^2 \right).$$

Letting $\epsilon \in (0, \delta)$ be small enough, we obtain the asserted bound. This bound and Theorem 5.1.2 imply the local solvability. We are done. □

○ **Topic: Conditions (Ψ) and (P)**

Definition. Let $U \subset \mathbb{R}^d$ be open, and let $p \in C^\infty(U \times (\mathbb{R}^d \setminus \{0\}))$.

1. We say p satisfies **condition** (Ψ) if for any $(x, \xi) \in p^{-1}(0)$ there exists a neighborhood $\Omega \subset U \times \mathbb{R}^n$ of (x, ξ) such that for $z = 1$ or i the following holds:

   (a) $H_{\mathrm{Re}(zp)}$ does not vanish on Ω;

   (b) Along any null bicharacteristic of $\mathrm{Re}(zp)$ on Ω, $\mathrm{Im}(zp)$ does not change sign from negative to positive.

2. We say p satisfies **condition** (P) if both p and $\bar{p}$ satisfy condition (Ψ).

Remarks. 1. For a ΨDO, or PDO, of principal type local solvability is practically characterized by condition (Ψ), or (P), respectively. However, in this course, we will present simpler characterizations under some *non-degeneracy* assumption.

2. Conditions (P) and (Ψ) are equivalent for the principal symbol of a PDO since it is a homogeneous polynomial in ξ.

Problem. 1. Verify the equivalence of conditions (P) and (Ψ) for a homogeneous polynomial in ξ.

2. Check the principal symbols from Theorems 5.2 and 5.4.2 satisfy conditions (P) and (Ψ).

§ 5.3 Characterization under Non-Degeneracy

◦ **A necessary condition**

Theorem 5.5. Assume $a(x,D)$ is locally solvable at $x_0 \in \mathbb{R}^d$. Then there exists a neighborhood $U \subset \mathbb{R}^d$ of $x_0 \in \mathbb{R}^d$ for which **Hörmander's condition** holds, i.e.,

$$\{\bar{p}, p\}(x,\xi) = 0 \quad \text{for any } (x,\xi) \in U \times \mathbb{R}^d \text{ with } p(x,\xi) = 0.$$

Proof. For the proof refer to Theorem 6.1.1 of "Linear Partial Differential Operators" by L. Hörmander. We omit it. □

Remark. Suppose there exists $(x_0,\xi_0) \in U \times \mathbb{R}^d$ such that

$$\{\bar{p}, p\}(x_0,\xi_0) \neq 0, \quad p(x_0,\xi_0) = 0.$$

Then we would be able to construct a **quasi-mode** for $a^*(x,D)$, or a family of functions $v = v(h)$, $h \in (0,1]$, on U such that

$$\|v(h)\| = 1, \quad \|a^*(x,D)v(h)\| \leq C_N h^N \text{ for any } N \in \mathbb{N},$$

which dissatisfies the inequality from Theorem 5.1.1.

In fact, multiplying i on p if necessary, we may let

$$(\partial_\xi \operatorname{Re} \bar{p})(x_0, \pm\xi_0) \neq 0.$$

On the other hand, $(x_0,\xi_0') = (x_0,\xi_0)$ or $(x_0,-\xi_0)$ satisfies

$$\begin{aligned}\Big(H_{\operatorname{Re}\bar{p}}(\operatorname{Im}\bar{p})\Big)(x_0,\xi_0') &= \{\operatorname{Re}\bar{p}, \operatorname{Im}\bar{p}\}(x_0,\xi_0') \\ &= \frac{\mathrm{i}}{2}\{\bar{p},p\}(x_0,\xi_0') \\ &< 0\end{aligned}$$

since $\{\bar{p},p\}$ is of odd degree in ξ. This implies that, along a null bicharactristic of $\operatorname{Re}\bar{p}$, $\operatorname{Im}\bar{p}$ changes sign at (x_0,ξ_0') from positive to negative. Thus we could construct a quasi-mode for $a^*(x,D)$ that lives in an arbitrarily small conic neighborhood of (x_0,ξ_0'), cf. Theorem 4.8 and Corollary 4.10. See also condition (P).

◦ **A sufficient condition**

Definition. $a(x,D)$ is **principally normal** at $x_0 \in \mathbb{R}^d$ if there exists a neighborhood $U \subset \mathbb{R}^d$ of x_0 and $q \in C^\infty(U \times (\mathbb{R}^d \setminus \{0\}))$ homogeneous of degree $m-1$ in ξ such that

$$\{\bar{p}, p\} = 2\mathrm{i}\,\mathrm{Re}(\bar{q}p) \quad \text{on } U \times (\mathbb{R}^d \setminus \{0\}).$$

Remarks. 1. Let $p = p_1 + \mathrm{i}p_2$ and $q = q_1 + \mathrm{i}q_2$ with p_1, p_2, q_1, q_2 being real-valued. Then the above condition is expressed as

$$\{\bar{p}, p\} = 2\mathrm{i}(q_1 p_1 + q_2 p_2).$$

This says $\{\bar{p}, p\}$ vanishes with the same order as p does. In particular, Hörmander's condition holds automatically.

2. If $a(x,D)$ is principally normal, so is $a^*(x,D)$.

Theorem 5.6. Let $m \neq 0$, and assume $a(x,D)$ is of principal type and principally normal at $x_0 \in \mathbb{R}^d$. There exist a neighborhood U of x_0 and $c > 0$ such that for any $v \in C_c^\infty(U)$

$$\|a^*(x,D)v\|_{L^2} \geq c\|v\|_{H^{m-1}}.$$

In partiulcar, $a(x,D)$ is locally solvable at x_0.

Proof. As in the proof of Theorem 5.4, we may let $x_0 = 0$. We also use the notation B_r there.

Step 1. We first show there exist $C_1, \delta > 0$ such that for any $u \in C_c^\infty(B_\delta)$

$$\|a(x,D)u\|_{L^2}^2 \leq C_1\left(\|a^*(x,D)u\|_{L^2}^2 + \|u\|_{H^{m-1}}^2\right).$$

In fact, by the assumption there exist $\delta > 0$ and $q \in C^\infty(B_{2\delta} \times (\mathbb{R}^d \setminus \{0\}))$ homogeneous of degree $m-1$ in ξ such that

$$\{\bar{p}, p\} = 2\mathrm{i}\,\mathrm{Re}\,(\bar{q}p) \quad \text{on } B_{2\delta} \times (\mathbb{R}^d \setminus \{0\}).$$

Fix any $\chi \in C_c^\infty(B_{2\delta})$ with $\chi = 1$ on B_δ, and then for any $u \in C_c^\infty(B_\delta)$

$$\|Au\|_{L^2}^2 = \|A^*u\|_{L^2}^2 + (\chi[A^*, A]\chi u, u).$$

If we modify q smoothly in a neighborhood of $\xi = 0$, then we can find $R \in \Psi^{2m-2}(\mathbb{R}^d)$ such that

$$\chi[A^*, A]\chi = QA^* + AQ^* + R; \qquad Q = \chi q(x,D)\chi.$$

Now by the Cauchy–Schwarz inequality and the Sobolev boundedness we obtain for any $u \in C_c^\infty(B_\delta)$

$$\begin{aligned}\|Au\|_{L^2}^2 &= \|A^*u\|_{L^2}^2 + (A^*u, Q^*u) + (Q^*u, A^*u) + (Ru, u)\\ &\leq \|A^*u\|_{L^2}^2 + C_4\|A^*u\|_{L^2}\|u\|_{H^{m-1}} + \|u\|_{H^{m-1}}^2\\ &\leq C_2\Big(\|A^*u\|_{L^2}^2 + \|u\|_{H^{m-1}}^2\Big).\end{aligned}$$

Hence the claim is verified.

Step 2. By Theorem 5.4.1 and Step 1 there exist $C_3, \delta' > 0$ such that for any $\epsilon \in (0, \delta')$ and $u \in C_c^\infty(B_\epsilon)$

$$\|u\|^2_{H^{m-1}} \le \epsilon C_3\Big(\|a^*(x,D)u\|^2_{L^2} + \|u\|^2_{H^{m-1}}\Big).$$

If we fix sufficiently small ϵ, then for any $u \in C_c^\infty(B_\epsilon)$

$$\|u\|_{H^{m-1}} \le C_4\|a^*(x,D)u\|_{L^2}.$$

Thus we obtain the assertion. □

○ **Characterization**

Theorem 5.7. Let $x_0 \in \mathbb{R}^d$, and assume the vectors

$$\partial_\xi \operatorname{Re} p(x_0, \xi), \quad \partial_\xi \operatorname{Im} p(x_0, \xi)$$

are linearly independent for any $\xi \in \mathbb{R}^d \setminus \{0\}$ with $p(x_0, \xi) = 0$. Then the following conditions are equivalent:

1. $a(x,D)$ is locally solvable at x_0.
2. $a^*(x,D)$ is locally solvable at x_0.
3. Hörmander's condition holds in some neighborhood of x_0.
4. $a(x,D)$ is principally normal at x_0.

Remarks. 1. By the assumption it automatically follows that both $a(x,D)$ and $a^*(x,D)$ are of principal type at x_0.

2. The assertion does not extend to a general PDO of principal type without non-degeneracy. In fact, for local solvability, the principal normality is not necessary, and Hörmander's condition is not sufficient either.
3. The principal symbol from Theorem 5.4.2 is *degenerate* in the sense that it does not satisfy the assumption.
4. See also Conditions (P) and (Ψ), and the subsequent remarks.

Proof. If $m = 0$, then $a(x,D)$ is merely a multiplication operator non-vanishing at x_0 by the assumption. Hence we may let $m \neq 0$.

4 ⇒ (1 and 2). This follows by Theorem 5.6.

(1 or 2) ⇒ 3. This follows by Theorem 5.5.

3 ⇒ 4. Step 1. We are going to construct q as in the definition of principal normality. Note the construction reduces to that on $|\xi| = 1$ by homogeneity, and further to that in a neighborhood of each (x_0, ξ) with $|\xi| = 1$ by partition-of-unity arguments. If $p(x,\xi) \neq 0$, we can actually take

$$q(x,\xi) = \frac{\{\bar{p}, p\}(x,\xi)}{2\mathrm{i}\bar{p}(x,\xi)},$$

and hence it suffices to find q for $p(x,\xi) = 0$.

Step 2. Let $\xi_0 \in \mathbb{R}^d \setminus \{0\}$ satisfy $p(x_0, \xi_0) = 0$. It suffices to find a neighborhood $\Omega \subset \mathbb{R}^{2d} \setminus 0$ of (x_0, ξ_0) and $q \in C^\infty(\Omega)$ such that

$$\{\bar{p}, p\} = 2\mathrm{i}\,\mathrm{Re}(\bar{q}p).$$

By the assumption there exists a neighborhood Ω of (x_0, ξ_0) and local coordinates $X\colon \Omega \to \mathbb{R}^{2d}$ such that

$$X_1(x, \xi) = \mathrm{Re}\, p(x, \xi), \quad X_2(x, \xi) = \mathrm{Im}\, p(x, \xi).$$

Then by Taylor's theorem we can find $q_1, \dots, q_{2d} \in C^\infty(\Omega)$ such that

$$\frac{1}{2\mathrm{i}}\{\bar{p}, p\}(x, \xi) = \frac{1}{2\mathrm{i}}\{\bar{p}, p\}(x_0, \xi_0) + q_1 X_1 + \cdots + q_{2d} X_{2d}.$$

However, by Hörmander's condition we have

$$\{\bar{p}, p\}(x_0, \xi_0) = 0.$$

Moreover, by Hörmander's condition again

$$q_3 = \cdots = q_{2d} = 0 \quad \text{for } X_1 = X_2 = 0,$$

so that, letting Ω be smaller if necessary, we can further find $\tilde{q}_1, \tilde{q}_2 \in C^\infty(\Omega)$ such that

$$\frac{1}{2\mathrm{i}}\{\bar{p}, p\} = \tilde{q}_1 X_1 + \tilde{q}_2 X_2.$$

Therefore it suffices to take $q = \tilde{q}_1 + \mathrm{i}\tilde{q}_2$. We are done. □