CLASSIFICATION OF OKAMOTO–PAINLEVÉ PAIRS

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Abstract. In this paper, we introduce the notion of an Okamoto–Painlevé pair (S, Y) which consists of a compact smooth complex surface S and an effective divisor Y on S satisfying certain conditions. Though spaces of initial values of Painlevé equations introduced by K. Okamoto gives examples of Okamoto–Painleve pairs, we find a new example of Okamoto–Painlevé pairs not listed in [Oka]. We will give the complete classification of Okamoto–Painlevé pairs.

0. Introduction

In this paper, we will introduce the notion of an Okamoto–Painlevé pair (S, Y), which is defined as follows:

Definition 0.1. (Cf. Definition 2.1). *Let S be a compact smooth complex surface and $Y = \sum_{i=1}^r a_i Y_i$ an effective divisor on S. We say that a pair (S, Y) is an Okamoto–Painlevé pair if it satisfies the following conditions:*

(i) *There exists a meromorphic 2-form ω on S such that $(\omega) = -Y$, that is, ω has the pole divisor Y (counting multiplicities) and has no zero outside Y.*

(ii) *For all i $(1 \leq i \leq r)$, $Y \cdot Y_i = \deg[Y]|_{Y_i} = 0$.*

(iii) *Let us set $D := Y_{red} = \sum_{i=1}^r Y_i$. Then $S - D$ contains $\mathbf{C}^2$ as a Zariski open set.*

(iv) *Set $F = S - \mathbf{C}^2$ where $\mathbf{C}^2$ is the same Zariski open set as in (ii). Then F is a (reduced) divisor with normal crossings.*

Historically, Okamoto [Oka] introduced *the space $M_J(t)$ of initial values* for each Painlevé equation of type P_J $(J = I, \dots, VI)$ with the time parameter t, which is a noncompact complex surface. We can obtain a nice compactification $\overline{M}_J(t)$ of $M_J(t)$ so that $(\overline{M}_J(t), D_J(t))$ becomes an Okamoto–Painlevé pair, where $D_J(t) = \overline{M}_J(t) - M_J(t)$ (cf. [ST] and [MMT]). Conversely, for each Okamoto–Painlevé pair one can associate a Hamiltonian system via the deformation theory of pairs, and such a Hamiltonian system is equivalent to a differential equation of Painlevé type (cf. [Sa–T]).

The main purpose of this paper is the classification of Okamoto–Painlevé pairs. We will classify Okamoto–Painlevé pairs (S, Y) into seven types according to the configuration of the divisor $Y \in |-K_S|$. (See Theorem 2.1). One can easily see that the configuration of the divisor Y is same as one of singular fibers of elliptic surfaces [Kod1]. According to the Kodaira's classification, the seven configurations can be denoted by II^*, III^*, IV^*, I_3^*, I_2^*, I_1^*, I_0^*, or in the notation of the dual graph of Y they can be denoted by

$$\widetilde{E}_8, \widetilde{E}_7, \widetilde{E}_6, \widetilde{D}_7, \widetilde{D}_6, \widetilde{D}_5, \widetilde{D}_4,$$

respectively.

It is known that there is a one-to-one correspondence between Okamoto–Painlevé pairs of the six types $\widetilde{E}_8, \widetilde{E}_7, \widetilde{E}_6, \widetilde{D}_7, \widetilde{D}_6, \widetilde{D}_5, \widetilde{D}_4$ and Painlevé equations of six types P_I, P_{II}, P_{III}, P_{IV}, P_V, P_{VI}, respectively (cf. [Oka], [MMT] and [ST]). Though an Okamoto–Painlevé pair of type $\widetilde{D}_7$ did not appear in the list of Okamoto [Oka], we can obtain a Hamiltonian system associated to the pairs. Recently, Okamoto informed us that he found Painlevé equations for an Okamoto–Painlevé pair of type $\widetilde{D}_7$ which are special cases of P_{III}. Therefore, we denote this Painlevé equation by P_{III}^*. In §4,

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we will construct an Okamoto–Painlevé pair of type $\widetilde{D}_7$ by blowing-up of $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ with centers on the anti-canonical divisor $-K_{\mathbf{F}_0}$.

Let us discuss main ideas of classfying Okamoto–Painlevé pairs. For an Okamoto–Painlevé pair (S, Y), the surface S is the compactification of $\mathbf{C}^2$, namely S includes $\mathbf{C}^2$ as a Zariski open set. Let $F = S - \mathbf{C}^2$. By the definition of Okamoto–Painlevé pairs, F is a normal crossing divisor. Each component of Y_{red} is also a component of F, namely,

$$Y_{red} = \sum_{i=1}^{r} Y_i \subset \sum_{i=1}^{l} F_i = F,$$

hence Y_{red} is also a normal crossing divisor. One can easily see that the divisor Y has the same configuration of a singular fiber of an elliptic surface, and the number of irreducible components of F is 10. Therefore, the number of the irreducible components of Y_{red} is less than or equal to 10. Moreover, the configuration of F must be a tree, so is Y_{red}. These arguments show that the configuration of Y is one of the types $\widetilde{E}_{r-1}$ for $r = 9, 8, 7$ and $\widetilde{D}_{r-1}$ for $r = 5, \ldots, 10$.

Now we will consider the classification of the configurations of normal clossing divisors F. The divisor F can be obtained by adding some components to Y_{red}. We call an irreducible component of $F - Y_{red}$ an *additional component*. We will classify all of configurations of F, and this gives the complete classification of Okamoto–Painlevé pairs and also the configurations of F.

The complete list of all normal crossing divisors F is given in §5. Each of those is a divisor with ten components including one of the seven Kodaira types. All configurations of F in the list can be transformed into the anti-canonical divisors of $\mathbf{P}^2, \mathbf{F}_0$ and $\mathbf{F}_2$ by blowings-up and blowings-down.

Let us discuss the relation between our results and results of Sakai in [Sak]. Sakai defined the notion of *generalized Halphen surfaces*. By definition, a generalized Halphen surface S is a compact complex surface satisfying the condition of (i) and (ii) of Definition 0.1. He related generalized Halphen surfaces to the discrete Painlevé equations via Cremona transformations. The Painlevé differential equations can be obtained as limits of discrete Painlevé equations. Most essential extra conditions of an Okamoto–Painlevé pair (S, Y) are that S is a compactification of $\mathbf{C}^2$ and F is a normal crossing divisor. Our classification gives more direct correspondences between Okamoto–Painlevé pairs and Painlevé differential equations (cf. [Sa–T]).

1. Preliminary

Let S be a compact complex surface and let $-K_S$ denote the anti-canonical divisor class of S. For any divisor D on S, we denote by $[D]$ linear equivarence class of D, and by $|D| = |[D]|$ the linear space of effective divisors C, such that $C \sim D$. Here "$\sim$" means the linear equivalence of divisors. In this paper, we often identify the divisor class $[D]$ with the isomorphism class of the corresponding line bundle $[D]$ or the correponding invertible sheaf $\mathcal{O}_S(D)$.

Let us assume that there exists a normal crossing effective divisor $Y \in |-K_S|$. Moreover, assume that every irreducible component of Y is a smooth rational curve. Set $Y = \sum_{i=1}^n m_i C_i$. Consider the blowing-up $\pi : \widetilde{S} \to S$ with center of $P \in C_{i_0}$, and let C_i' be the strict transform of C_i by π, and E the exceptional curve of π. Note that $m_i \geq 0$. By a standard calculation, we can obtain the following lemma.

Lemma 1.1. *If* $p \in C_{i_0} \setminus \bigcup_{i \neq i_0} (C_{i_0} \cap C_i)$, *then*

$$-K_{\widetilde{S}} = [\sum_{i=1}^n m_i C_i' + (m_{i_0} - 1)E],$$

$$(C_i')^2 = (C_i)^2 \quad (i \neq i_0),$$

and

$$(C_{i_0}')^2 = (C_{i_0})^2 - 1.$$

If $p \in C_{i_0} \cap C_{i_1}$ *for some* i_1 $(\neq i_0)$, *then*

$$-K_{\widetilde{S}} = [\sum_{i=1}^{n} m_i C_i' + (m_{i_0} + m_{i_1} - 1)E],$$

$$(C_i')^2 = (C_i)^2 \quad (i \neq i_0, i_1),$$

and

$$(C_i')^2 = (C_i)^2 - 1 \quad (i = i_0, i_1).$$

Let $\mathbf{F}_n$ denote the Hirzebruch surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n))$. We denote by f the class of fiber of $\mathbf{F}_n$, s_0 the class of the minimal section of $\mathbf{F}_n$ respectively. We also set $s_\infty = s_0 + nf$. Then, the self-intersection numbers are given by

$$f^2 = 0, \quad (s_0)^2 = -n, \quad (s_\infty)^2 = n.$$

It is well-known that $\mathbf{F}_n$ can be obtained by performing several blowings-up and blowing-downs to $\mathbf{P}^2$.

Now we consider the anti-canonical divisor class of $\mathbf{P}^2$ and $\mathbf{F}_n$. Let h denote the class of lines in $\mathbf{P}^2$. Then, they are given as

$$\begin{aligned} -K_{\mathbf{P}^2} &= 3h, \\ -K_{\mathbf{F}_n} &= 2s_0 + (n+2)f = 2s_\infty - (n-2)f. \end{aligned}$$

In this paper, we often use the formula $-K_{\mathbf{F}_0} = 2s_0 + 2f$ and $-K_{\mathbf{F}_2} = 2s_\infty$. Note that $-K_{\mathbf{F}_0}$ and $-K_{\mathbf{F}_2}$ have complements in $\mathbf{F}_0$ and $\mathbf{F}_2$ respectively which contain $\mathbf{C}^2$ as Zariski open sets.

2. Okamoto–Painlevé Pairs

Now let us give the definition of Okamoto–Painlevé pairs.

Definition 2.1. *Let* S *be a compact smooth complex surface and* $Y = \sum_{i=1}^{r} a_i Y_i$ *an effective divisor on* S. *We say that a pair* (S, Y) *is an Okamoto–Painlevé pair if it satisfies the following conditions:*

(i) *There exists a meromorphic* 2*-form* ω *on* S *such that* $(\omega) = -Y$, *that is,* ω *has the pole divisor* Y (*counting multiplicities*) *and has no zero outside* Y.

(ii) *For all* i $(1 \leq i \leq r)$, $Y \cdot Y_i = \deg[Y]|_{Y_i} = 0$.

(iii) *Let us set* $D := Y_{red} = \sum_{i=1}^{r} Y_i$. *Then* $S - D$ *contains* $\mathbf{C}^2$ *as a Zariski open set.*

(iv) *Set* $F = S - \mathbf{C}^2$ *where* $\mathbf{C}^2$ *is the same Zariski open set as in* (ii). *Then* F *is a* (*reduced*) *divisor with normal crossings.*

Remark 2.1. (i) *The meromorphic* 2*-form* ω *as above induces a holomorphic symplectic structure on* $S - D$.

(ii) *Let* K_S *denote the canonical divisor class of* S. *The condition* (i) *means that* $K_S = [-Y]$ *or* $-K_S = [Y]$.

(iii) *The condition* (v) *implies that the reduced part* $D = Y_{red}$ *of* Y *is also a divisor with normal crossings.*

(iv) *The spaces of initial values of Painlevé equations can be written as* $S - D$ *for some Okamoto–Painlevé pair* (S, Y). (See [Oka], [ST] and [MMT]).

The following is the main theorem of this paper.

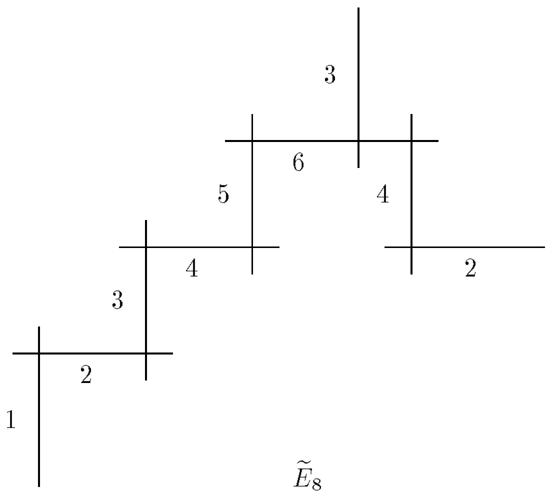
Theorem 2.1. *Let* (S, Y) *be an Okamoto–Painlevé pair. Then we have the following assertions.*

(i) *The surface* S *is a projective rational surface.*

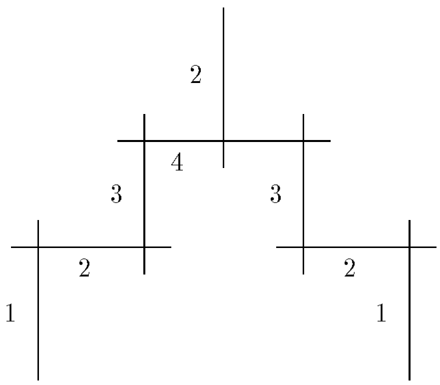
(ii) *The configuration of* Y *counting multiplicities is in the list of Kodaira's classification of singular fibers of elliptic surfaces. More precisely, they are given by one of the following lists.*

(iii) *All Okamoto–Painlevé pairs can be obtained by blowings-up and blowings-down of* $(\mathbf{P}^2, 3H)$, *where* H *is a line in* $\mathbf{P}^2$.

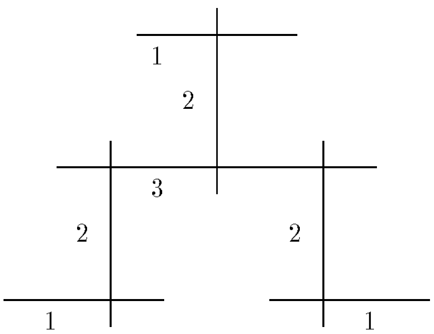
Y	$\widetilde{E}_8$	$\widetilde{E}_7$	$\widetilde{D}_7$	$\widetilde{D}_6$	$\widetilde{E}_6$	$\widetilde{D}_5$	$\widetilde{D}_4$
Kodaira's notation	II^*	III^*	I_3^*	I_2^*	IV^*	I_1^*	I_0^*
Painlevé equation	P_I	P_{II}	P_{III}^*	P_{III}	P_{IV}	P_V	P_{VI}



$\widetilde{E}_8$



$\widetilde{E}_7$



$\widetilde{E}_6$

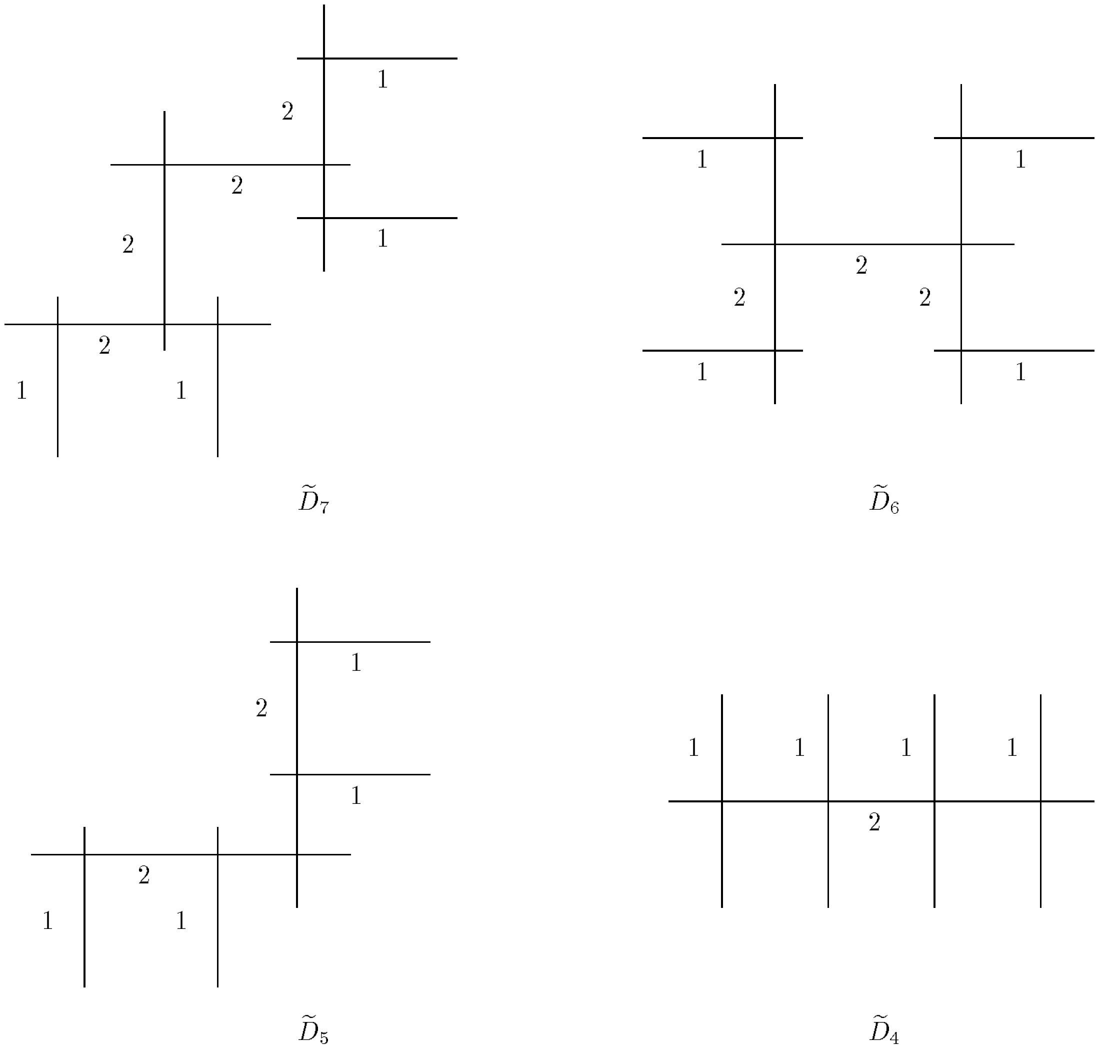


Figure 1.

In Figure 1, each line denotes $\mathbf{P}^1$ whose self-intersection number is equal to -2, and the number next to each line denotes the multiplicity of corresponding component in $-K_S$.

Let us start our proof with the following easy lemmas.

Lemma 2.1. *Let (S, Y) be an Okamoto–Painlevé pair, then we have:*
(i) $Y_i \cong \mathbf{P}^1$ *for* i $(1 \leq i \leq r)$.
(ii) *The following conditions are equivarent for* i $(1 \leq i \leq r)$.
(a) $Y \cdot Y_i = 0$.
(b) $(Y_i)^2 = -2$.

Proof. (i) It is sufficient to show that the component F_i of F is isomorphic to $\mathbf{P}^1$. We will show that $H^1(F_i, \mathbf{Z}) = 0$.

We have the following isomorphisms of cohomology groups with $\mathbf{Z}$-coefficients by Poincaré duality.

$$\begin{array}{rcl} H^i(S,F;\mathbf{Z}) & \cong & H^i_c(\mathbf{C}^2,\mathbf{Z}) \\ & \cong & H_{4-i}(\mathbf{C}^2,\mathbf{Z}) \\ & \cong & \begin{cases} 0 & i=0,1,2,3 \\ \mathbf{Z} & i=4 \end{cases} \end{array}$$

On the other hand, consider the long exact sequence of cohomology groups for pair (S,F)

$$\begin{array}{rlllll} 0 & \to & H^0(F,\mathbf{Z}) & \to & H^0(S,\mathbf{Z}) & \to & H^0(S,F;\mathbf{Z})=0 \\ & \to & H^1(F,\mathbf{Z}) & \to & H^1(S,\mathbf{Z}) & \to & H^1(S,F;\mathbf{Z})=0 \\ & \to & H^2(F,\mathbf{Z}) & \to & H^2(S,\mathbf{Z}) & \to & H^2(S,F;\mathbf{Z})=0 \\ & \to & H^3(F,\mathbf{Z}) & \to & H^3(S,\mathbf{Z}) & \to & H^3(S,F;\mathbf{Z})=0 \\ & \to & H^4(F,\mathbf{Z}) & \to & H^4(S,\mathbf{Z}) & \to & H^4(S,F;\mathbf{Z})\cong\mathbf{Z} \\ & \to & \dots. & & & & \end{array}$$

Then we have

$$H^i(S,\mathbf{Z}) \cong H^i(F,\mathbf{Z}) \quad (i=0,1,2,3). \tag{1}$$

Especially,

$$H^3(S,\mathbf{Z}) \cong H^3(F,\mathbf{Z}) = 0.$$

By Poincaré duality again, we have

$$H^1(S,\mathbf{Z}) \cong H^3(S,\mathbf{Z}) = 0. \tag{2}$$

Now we consider an irreducible component F_i of F. Let $F' = \bigcup_{k\neq i} F_k$. Then $F_i \cap F'$ consists of a finite set of points. And we have the Mayer–Vietoris exact sequence

$$\begin{array}{rlllll} 0 & \to & H^0(F,\mathbf{Z}) & \to & H^0(F_i,\mathbf{Z})\oplus H^0(F',\mathbf{Z}) & \to & H^0(F_i\cap F',\mathbf{Z}) \\ & \to & H^1(F,\mathbf{Z}) & \to & H^1(F_i,\mathbf{Z})\oplus H^1(F',\mathbf{Z}) & \to & H^1(F_i\cap F',\mathbf{Z}) \\ & \to & \dots. & & & & \end{array} \tag{3}$$

By (1) and (2), we have $H^1(F,\mathbf{Z})=0$. Since $H^1(F_i\cap F',\mathbf{Z})=0$, we have $H^1(F_i,\mathbf{Z})\oplus H^1(F',\mathbf{Z})=0$. Therefore, $H^1(F_i,\mathbf{Z})=0$.

If the irreducible component F_i is a singular nodal curve, then $H^1(F_i,\mathbf{Z})\neq 0$. So F_i is nonsingular, namely $F_i\cong\mathbf{P}^1$.

(ii) Since Y_i is a nonsingular rational curve, by the adjunction formula, we have $K_S\cdot Y_i+(Y_i)^2=-2$. Since $K_S=[-Y]$, the conditions (a) and (b) are equivalent to each other.

Lemma 2.2.

$$H^2(F,\mathbf{Z}) \cong \bigoplus_{i=1}^{l} H^2(F_i,\mathbf{Z}) \cong \bigoplus_{i=1}^{l} \mathbf{Z}[F_i].$$

Proof. Let $F'=\bigcup_{k\neq 1}F_k$. We see that $H^1(F_1\cap F',\mathbf{Z})=H^2(F_1\cap F',\mathbf{Z})=0$. From the exact sequence (3), we have

$$H^2(F,\mathbf{Z}) \cong H^2(F_1,\mathbf{Z})\oplus H^2(F',\mathbf{Z}).$$

Using the argument inductively, we obtain the assertion.

Lemma 2.3. *The configuration of F is a tree,* that is, the dual graph of F is connected and contains no cycles.

Proof. Since $H^0(S,\mathbf{Z})\cong H^0(F,\mathbf{Z})$, the connectivity of S implies the connectivity of F. To show that F contains no cycles, it is sufficient to show that there are no irreducible components $F_{i_1},\dots,F_{i_m}$ $(m\geq 3)$ such that

$$\begin{array}{l} F_{i_k}\cap F_{i_{k+1}}\neq\emptyset \quad (k=1,\dots,m-1), \\ F_{i_m}\cap F_{i_1}\neq\emptyset. \end{array}$$

We assume that there exist such $F_{i_1},\dots,F_{i_m}$. Let $F^{(1)}=F_{i_1}\cup\dots\cup F_{i_m}$ and $F^{(2)}=F_{i_1}\cup\dots\cup F_{i_{m-1}}$. Since $F^{(1)}$ and $F^{(2)}$ are connected, we have $H^0(F^{(1)},\mathbf{Z})\cong H^0(F^{(2)},\mathbf{Z})\cong\mathbf{Z}$. $F^{(2)}\cap F_{i_m}$ consists of at least two points, and let ν be the number of the points of $F^{(2)} \cap F_{i_m}$. Then we have $H^0(F^{(2)} \cap F_{i_m}, \mathbf{Z}) \cong \mathbf{Z}^\nu$ $(\nu \geq 2)$. Note that $H^0(F_{i_m}, \mathbf{Z}) \cong H^0(\mathbf{P}^1, \mathbf{Z}) \cong \mathbf{Z}$. We consider the Mayer–Vietoris exact sequence

$$\begin{array}{ccccccc} 0 & \to & \underbrace{H^0(F^{(1)}, \mathbf{Z})}_{\cong \mathbf{Z}} & \to & \underbrace{H^0(F^{(2)}, \mathbf{Z}) \oplus H^0(F_{i_m}, \mathbf{Z})}_{\cong \mathbf{Z} \oplus \mathbf{Z}} & \to & \underbrace{H^0(F^{(2)} \cap F_{i_m}, \mathbf{Z})}_{\cong \mathbf{Z}^\nu} \\ & \to & H^1(F^{(1)}, \mathbf{Z}) & \to & H^1(F^{(2)}, \mathbf{Z}) \oplus H^1(F_{i_m}, \mathbf{Z}) & \to & \cdots. \end{array}$$

If we assume $H^1(F^{(1)}, \mathbf{Z}) = 0$ then ν must be one, which contradicts the fact that $\nu \geq 2$. So we have $H^1(F^{(1)}, \mathbf{Z}) \neq 0$. Now let $F^{(3)}$ be the union of irreducible components of F which do not belong to $F^{(1)}$. Namely, $F = F^{(1)} \cup F^{(3)}$. We consider the exact sequence

$$\underbrace{H^1(F, \mathbf{Z})}_{=0} \to H^1(F^{(1)}, \mathbf{Z}) \oplus H^1(F^{(3)}, \mathbf{Z}) \to H^1(F^{(1)} \cap F^{(3)}, \mathbf{Z}).$$

Since $F^{(1)} \cap F^{(3)}$ consists of a finite set of points, we have $H^1(F^{(1)} \cap F^{(3)}, \mathbf{Z}) = 0$, so $H^1(F^{(1)}, \mathbf{Z}) \oplus H^1(F^{(3)}, \mathbf{Z}) = 0$. Therefore we have $H^1(F^{(1)}, \mathbf{Z}) = 0$, which is a contradiction. This completes the proof.

Proof of Theorem 2.1. (i) Since S is a compactification of $\mathbf{C}^2$, we see that S is a rational surface by Theorem 5 of Kodaira [Kod2].

(ii) Let (S, Y) be an Okamoto–Painlevé pair. We know that S is a projective rational surface and the configuration of the dual graph of $F = S - \mathbf{C}^2$ is a tree. Let $F = \sum_{i=1}^l F_i$ be the irreducible decomposition of F, where l denotes the number of the irreducible components of F. Let $[F_i]$ denote the class in $H^2(S, \mathbf{Z})$ dual to the class of $F_i \subset S$. From Lemma 2.2 and (1), we obtain the isomorphism

$$\bigoplus_{i=1}^l \mathbf{Z}[F_i] \cong H^2(F, \mathbf{Z}) \xrightarrow{\cong} H^2(S, \mathbf{Z}).$$

This implies that $\{[F_i] \mid 1 \leq i \leq l\}$ forms an integral basis of $H^i(S, \mathbf{Z})$.

From the condition (iii) of Definition 0.1, we have

$$Y^2 = \sum_{i=1}^r a_i (Y \cdot Y_i) = 0.$$

Since $K_S = -Y$, we know that $(K_S)^2 = (-Y)^2 = Y^2 = 0$. By Noether's formula, we have

$$e(S) + (K_S)^2 = 12\chi(S, \mathcal{O}_S),$$

and $\chi(S, \mathcal{O}_S) = 1$. Then we obtain

$$e(S) = 2 - 2b_1(S) + b_2(S) = 12.$$

Since $b_1(S) = 0$, we have $b_2(S) = 10$, and hence

$$\boxed{l = (\text{the number of the components of } F) = 10.}$$

Since $D = \sum_{i=1}^r Y_i \subset F = \sum_{i=1}^{10} F_i$, we have

$$r \leq 10.$$

Since $F = \sum_{i=1}^{10} F_i$ is a connected divisor with normal crossings, the divisor $D = Y_{red} = \sum_{i=1}^r Y_i$ is also a connected divisor with normal crossings. To see that $D = Y_{red}$ is connected, it is sufficient to show that Y is connected. By Serre duality, we have $H^1(S, \mathcal{O}_S) \cong H^1(S, \mathcal{O}(K_S))$. Hence we have $H^1(S, \mathcal{O}(K_S)) = 0$. By the exact sequence

$$0 \to \mathcal{O}(K_S) \to \mathcal{O}_S \to \mathcal{O}_Y \to 0,$$

we obtain the long exact sequence

$$\begin{array}{ccccccc} 0 & \to & \underbrace{H^0(S,\mathcal{O}(K_S))}_{=0} & \to & H^0(S,\mathcal{O}_S) & \to & H^0(S,\mathcal{O}_Y) \\ & \to & \underbrace{H^1(S,\mathcal{O}(K_S))}_{=0} & \to & H^1(S,\mathcal{O}_S) & \to & \dots . \end{array}$$

Therefore we have $H^0(S,\mathcal{O}_Y) \cong H^0(S,\mathcal{O}_S) \cong \mathbf{C}$. This shows that Y is connected.

Now we prove that the divisor $Y = \sum_{i=1}^r a_i Y_i$ has one of the configurations in the list of singular fibers of elliptic surfaces (cf. Theorem 6.2, [Kod1]). First let us show that the greatest common divisor of $\{a_i\}_{i=1}^r$ is equal to one. Since S is a rational surface with $b_2(S) = 10$, S is not relatively minimal. Hence S contains an exceptional curve E of the first kind, that is, $E \cong \mathbf{P}^1$ and $E^2 = -1$. Note that $K_S = [-Y]$. By the adjunction formula, we have $K_S \cdot E + E^2 = -2$ and this implies that $K_S \cdot E = -1$ and equivalently $Y \cdot E = 1$. Since $K_S = [-Y]$ and $Y \cdot Y_i = 0$ for every irreducible component Y_i of Y, we know that E is not a component of Y hence $E \cdot Y_i \geq 0$. On the other hand, since we have

$$1 = Y \cdot E = \sum_{i=1}^r a_i (Y_i \cdot E),$$

there exists an irreducible component Y_i with $a_i = 1$. Under the condition, in order to see that the configuration of Y is one in the list of the singular fibers of elliptic surfaces in [Kod1], we can follow the proof of Theorem 6.2 of [Kod1]. In fact, only the following conditions are needed to determine all of the configuration of singular fibers of elliptic surfaces:

(1) $D = Y_{red}$ is connected.
(2) $K_S \cdot Y = 0$.
(3) $Y \cdot Y_i = 0$, for i $(1 \leq i \leq r)$.

We have proved the first assertion, and the third assertion follows from Definition 0.1. The second assertion follows from the third assertion because $K_S = [-Y]$. Hence the configuration of Y is in the list of Kodaira. Next, we note that $D = Y_{red}$ must be a divisor with normal crossings and the dual graph of D must be a tree. Then Y must be one of the types of $\widetilde{E}_{r-1}$ for $r = 9, 8, 7$ and $\widetilde{D}_{r-1}$ for $r = 5, \dots, 10$. At this moment, we see that each irreducible component Y_i of $D = Y_{red}$ is a smooth rational curve with $(Y_i)^2 = -2$.

We will show that the configurations of types $\widetilde{D}_9$ and $\widetilde{D}_8$ can not occur. Let us set $\Lambda_Y = \sum_{i=1}^r \mathbf{Z}[Y_i]$ and $\Lambda_F = \sum_{j=1}^{10} \mathbf{Z}[F_j] \cong \mathrm{Pic}(S) \cong H^2(S,\mathbf{Z})$. Then we have the inclusion map

$$\iota : \Lambda_Y \hookrightarrow \Lambda_F \cong H^2(S,\mathbf{Z}).$$

By this inclusion map ι, Λ_Y can be considered as a sublattice of $H^2(S,\mathbf{Z})$ and $H^2(S,\mathbf{Z})$ is generated by $[F_j]$ $(1 \leq j \leq 10)$. The intersection matrix $I_F := ((F_i \cdot F_j))_{1 \leq i,j \leq 10}$ is the unimodular matrix with the signature $(b_+, b_-) = (1, 9)$. On the other hand, the intersection matrix

$$I_Y = ((Y_i \cdot Y_j))_{1 \leq i,j \leq r}$$

has a null eigenvalue corresponding to Y because $Y^2 = 0$. This means that the rank of Λ_Y is strictly less than ten, that is,

$$r < 10.$$

This proves that the configuration of type $\widetilde{D}_9$ does not occur. If Y is of type $\widetilde{D}_8$, then $r = 9$ and hence we can write $F = \sum_{i=1}^9 Y_i + F_{10}$. Since F is connected and the dual graph of F is a tree, we see that F_{10} intersects only one irreducible component Y_i for some i $(1 \leq i \leq 9)$. In this case, $\Lambda_F = H^2(S,\mathbf{Z})$ is generated by $Y_1, \dots, Y_9$ and F_{10} where the dual graph of $Y_1, \dots, Y_9$ is of type $\widetilde{D}_8$. Now by a direct calculation we see that the intersection matrix of $\{Y_1, \dots, Y_9, F_{10}\}$ can not have the determinant -1. Hence $\widetilde{D}_8$ does not occur. Then we see that the dual graph of Y must be one of the following types:

$$\widetilde{E}_8, \widetilde{E}_7, \widetilde{E}_6, \widetilde{D}_7, \widetilde{D}_6, \widetilde{D}_5, \widetilde{D}_4.$$

Conversely, if Y is one of the types as above, we can construct Okamoto–Painlevé pair (S, Y) by blowing up and blowing down of $(\mathbf{P}^2, Y = 3H)$ where H denotes a line in $\mathbf{P}^2$. For detail, see §5. This proves our theorem.

Remark 2.2. We will classify not only the configurations of Y but also F of an Okamoto–Painlevé pair (S, Y). By using the result, one can show that $S - D$ is covered by a finite number of Zariski open sets $\{U_i\}$ each of which is isomorphic to $\mathbf{C}^2$ if Y is not of type $\widetilde{E}_8$. (See [ST] and [MMT]).

Example 2.1. We consider the case where the configuration of Y is of type $\widetilde{D}_4$. Namely, the corresponding Painlevé equation is type P_{VI}. For more details, see [ST]. At first, we take the Hirzebruch surface $\Sigma^{(2)}_{(\varepsilon)}$ which is obtained by gluing four copies of $\mathbf{C}^2$ via following identification. Note that $\Sigma^{(2)}_{(0)} = \mathbf{F}_2$.

$$U_i = \operatorname{Spec} \mathbf{C}[x_i, y_i] \cong \mathbf{C}^2 \ (i = 0, 1, 2, 3)$$

$$\begin{aligned} x_0 &= x_1, & y_0 &= 1/y_1, \\ x_0 &= 1/x_2, & y_0 &= x_2(\varepsilon - x_2 y_2), \\ x_2 &= x_3, & y_2 &= 1/y_3 \end{aligned} \tag{4}$$

We consider a fiber space $(\Sigma^{(2)}_{(\varepsilon)} \times B_{VI}, \pi, B_{VI})$, where $B_{VI} = \mathbf{C} \setminus \{0, 1\}$. Let us take

$$\varepsilon = (\kappa_0 + \kappa_1 + \kappa_t - 1 + \kappa_\infty)/2,$$

where κ_ν $(\nu = 0, 1, t, \infty)$ are complex constants in the Hamiltonian function H_{VI} (cf. [ST]). For any parameter $t \in B_{VI}$, we define a divisor $D^{(0)}(t)$ on $\Sigma^{(2)}_{(\varepsilon)} \times t$:

$$D^{(0)}(t) = \{(x_1, y_1, t) \in U_1 \times t \mid y_1 = 0\} \cup \{(x_3, y_3, t) \in U_3 \times t \mid y_3 = 0\}.$$

Note that $(D^{(0)}(t))^2 = 2$, and $2D^{(0)}(t) \in |-K_{\Sigma^{(2)}_{(\varepsilon)}}|$. And we take four points $a^{(0)}_\nu(t) \in D^{(0)}(t)$ $(\nu = 0, 1, t, \infty)$:

$$\begin{aligned} a^{(0)}_\nu(t) &= \{(x_1, y_1, t) = (\nu, 0, t)\} \in U_1 \cap D^{(0)}(t) \quad (\nu = 0, 1, t), \\ a^{(0)}_\infty(t) &= \{(x_3, y_3, t) = (0, 0, t)\} \in U_3 \cap D^{(0)}(t). \end{aligned}$$

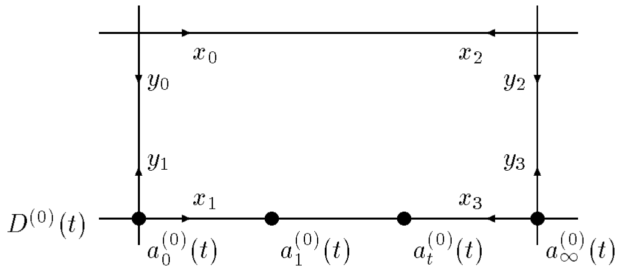


Figure 2.

We perform blowings-up to $\Sigma^{(2)}_{(\varepsilon)} \times t$ at $a^{(0)}_\nu(t)$ for all $t \in B_{VI}$, and let $D^{(1)}_\nu(t)$ be the exceptional curves of the blowings-up at $a^{(0)}_\nu(t)$ for $\nu = 0, 1, t, \infty$. We can take four coordinate systems (z_ν, w_ν) around the points at infinity of the exceptional curves $D^{(1)}_\nu(t)$ $(\nu = 0, 1, t, \infty)$, where

$$\begin{aligned} (z_\nu, w_\nu) &= ((x_1 - \nu) y_1^{-1}, y_1) \quad (\nu = 0, 1, t), \\ (z_\infty, w_\infty) &= (x_3 y_3^{-1}, y_3). \end{aligned}$$

Note that we have $w_\nu = 0$ on $D_\nu^{(1)}(t)$ for $\nu = 0, 1, t, \infty$. In order to perform the second blowings-up, let us take four points $a_\nu^{(1)}(t)$ for $\nu = 0, 1, t, \infty$.

$$a_\nu^{(1)}(t) = \{(z_\nu, w_\nu, t) = (\kappa_\nu, 0, t)\} \in D_\nu^{(1)}(t) \ (\nu = 0, 1, t, \infty),$$

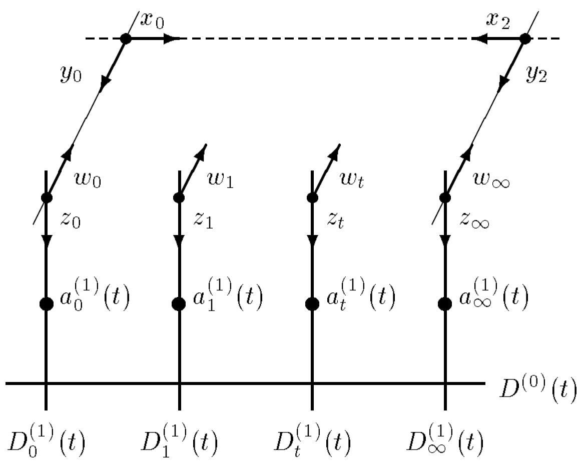


Figure 3.

Let us perform blowings-up at $a_\nu^{(1)}(t)$, and denote $D_\nu^{(2)}(t)$ for the exceptional curves, respectively. We take four coordinate systems (Z_ν, W_ν) around the points at infinity of $D_\nu^{(2)}(t)$ for $\nu = 0, 1, t, \infty$, where

$$\begin{aligned}(Z_\nu, W_\nu) &= ((x_1 y_1^{-1} - \kappa_0) y_1^{-1}, y_1), \quad \nu = 0, 1, t, \\ (Z_\infty, W_\infty) &= ((x_3 y_3^{-1} - \kappa_\infty) y_3^{-1}, y_3).\end{aligned}$$

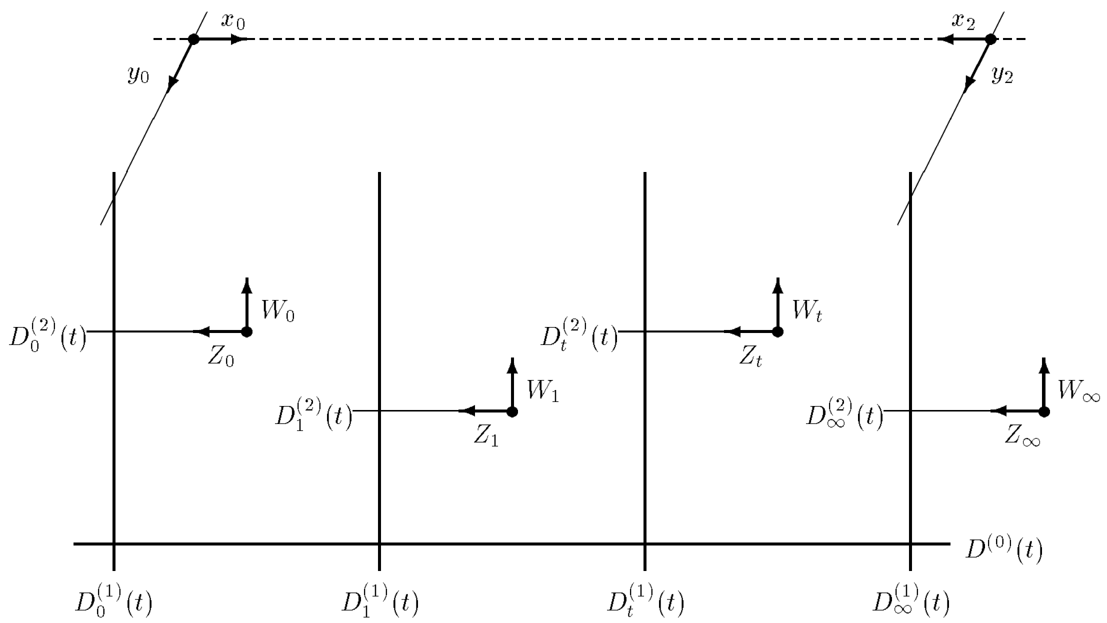


Figure 4.

For the strict transform of $D_\nu^{(i)}(t)$ by the blowing-up, we also denote by $D_\nu^{(i)}(t)$, respectively. Let $S(t) \to \Sigma_{(\varepsilon)}^{(2)} \times t$ be the composition of above eight blowings-up for the parameter t. Then, we see that the configuration of the divisor

$$D(t) := 2D^{(0)}(t) + \sum_{\nu=0,1,t,\infty} D_\nu^{(1)}(t)$$

on $S(t)$ is of type $\widetilde{D}_4$. And we see that the complements of $D(t)$ in $S(t)$ is covered by six Zariski open sets

$$\begin{aligned}&\operatorname{Spec} \mathbf{C}[Z_\nu, W_\nu] \quad (\nu = 0, 1, t, \infty),\\ &\operatorname{Spec} \mathbf{C}[x_0, y_0],\\ &\operatorname{Spec} \mathbf{C}[x_2, y_2].\end{aligned}$$

Note that this example corresponds to the Okamoto–Painlevé pair of type $\widetilde{D}_4$–(2) in our classification. (See §5).

Example 2.2. We will construct the space of initial values of the Painlevé equation of type P_I for an example of the Okamoto–Painlevé pair of type $\widetilde{E}_8$, As we remarked in Remark 2.2, the spaces of initial values of P_I can not be covered by some Zariski open sets each of which is isomorphic to $\mathbf{C}^2$. For any $t \in B_I$, we consider the Hirzebruch surface $\Sigma(t) := \Sigma_{(0)}^{(2)} = \mathbf{F}_2$, and take two curves $D_0(t) = \{y_0 = 0\} \cup \{y_2 = 0\}$ and $D_0'(t) = \{x_2 = 0\} \cup \{x_3 = 0\}$ on $\Sigma(t)$. Furthermore, we have to consider the multiplicities of each component of the anti-canonical divisor. For the surface $\Sigma(t)$, we can take the anti-canonical divisor $2D_0(t) + 4D_0'(t) \in |-K_{\Sigma(t)}|$. Let us perform blowing-up at the point $a_0(t) = \{(x_3, y_3) = (0, 0)\}$.

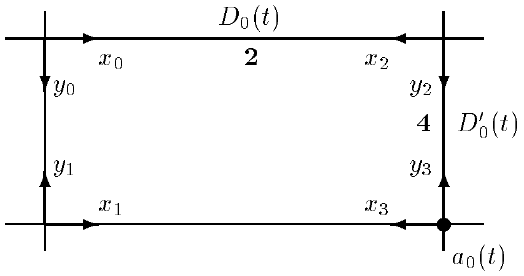


Figure 5.

Let D_1 denote the exceptional curve, and take two coordinate systems (z_1, W_1) and (Z_1, w_1) on D_1 which satisfies $D_1 = \{z_1 = 0\} \cup \{w_1 = 0\}$ and $Z_1 = W_1^{-1}$. Note that

$$\begin{aligned}(z_1, W_1) &= (x_3, x_3^{-1} y_3),\\ (Z_1, w_1) &= (x_3 y_3^{-1}, y_3).\end{aligned}$$

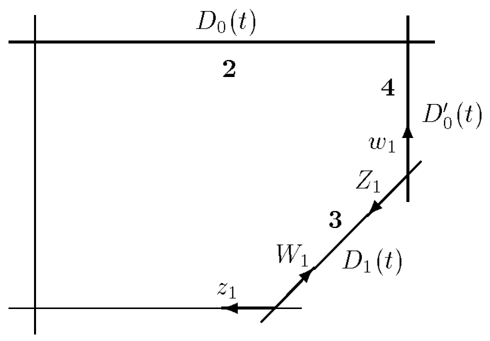


Figure 6.

Next, let us perform the blowing-up at the point $a_1(t) = \{(Z_1, w_1) = (0,0)\}$, and denote the exceptional curve by $D_2(t)$. Take two coordinate systems (z_2, W_2) and (Z_2, w_2) on D_2 such that $D_2 = \{z_2 = 0\} \cup \{w_2 = 0\}$ and $Z_2 = W_2^{-1}$. Note that

$$\begin{array}{rcl}(z_2, W_2) & = & (x_3 y_3^{-1}, x_3^{-1} y_3^2), \\ (Z_2, w_2) & = & (x_3 y_3^{-2}, y_3).\end{array}$$

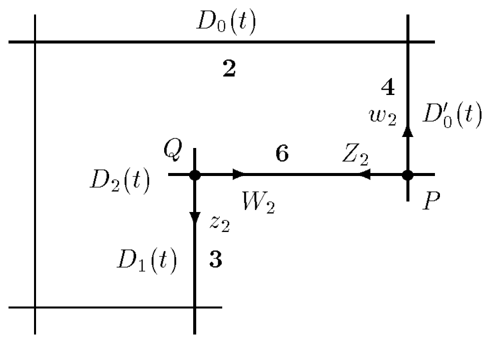


Figure 7.

Let $P = P(t) = \{(Z_2, w_2) = (0,0)\}$ and $Q = Q(t) = \{(z_2, W_2) = (0,0)\}$. Let us perform blowing-up at the point $a_2(t) = \{(Z_2, w_2) = (1/4, 0)\} = \{(z_2, W_2) = (0, 4)\}$, and denote the exceptional curve by $D_3(t)$. Note that we can consider this blowing-up by using the coordinate system either (Z_2, w_2) or (z_2, W_2).

At first, we consider the blowing-up at P with the coordinate system (Z_2, w_2). Take two coordinate systems (z_3, W_3) and (Z_3, w_3) on $D_3(t)$ such that $D_3(t) = \{z_3 = 0\} \cup \{w_3 = 0\}$ and $Z_3 = W_3^{-1}$. Then we have

$$\left\{\begin{array}{rcl} z_3 & = & x_3 y_3^{-2} - 1/4, \\ W_3 & = & (x_3 y_3^{-2} - 1/4)^{-1} y_3, \end{array}\right.$$

and

$$\left\{\begin{array}{rcl} Z_3 & = & (x_3 y_3^{-2} - 1/4) y_3^{-1}, \\ w_3 & = & y_3. \end{array}\right.$$

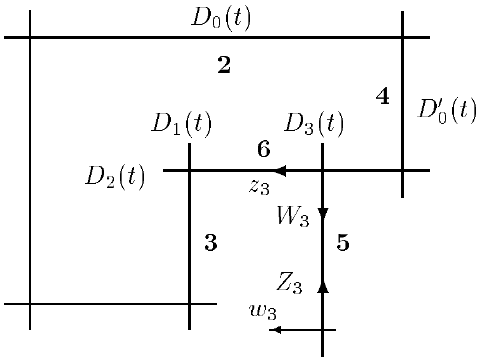


Figure 8.

In the same way, we have to perform five more blowings-up, and take the coordinate systems (z_i, W_i) and (Z_i, w_i) which satisfies that $D_i(t) = \{z_i = 0\} \cup \{w_i = 0\}$ and $Z_i = W_i^{-1}$, where $D_i(t)$ is the exceptional curve of each blowing-up $(i = 4, 5, 6, 7)$. Similarly, let $a_i(t)$ be the center of each blowing-up. Then we have to take $a_i(t)$ as follows. For more details, see [Oka].

$$\begin{aligned}
a_3(t) &= \{(Z_3, w_3) = (0,0)\}, \\
a_4(t) &= \{(Z_4, w_4) = (0,0)\}, \\
a_5(t) &= \{(Z_5, w_5) = (0,0)\}, \\
a_6(t) &= \{(Z_6, w_6) = (t/2,0)\}, \\
a_7(t) &= \{(Z_7, w_7) = (1/2,0)\}.
\end{aligned}$$

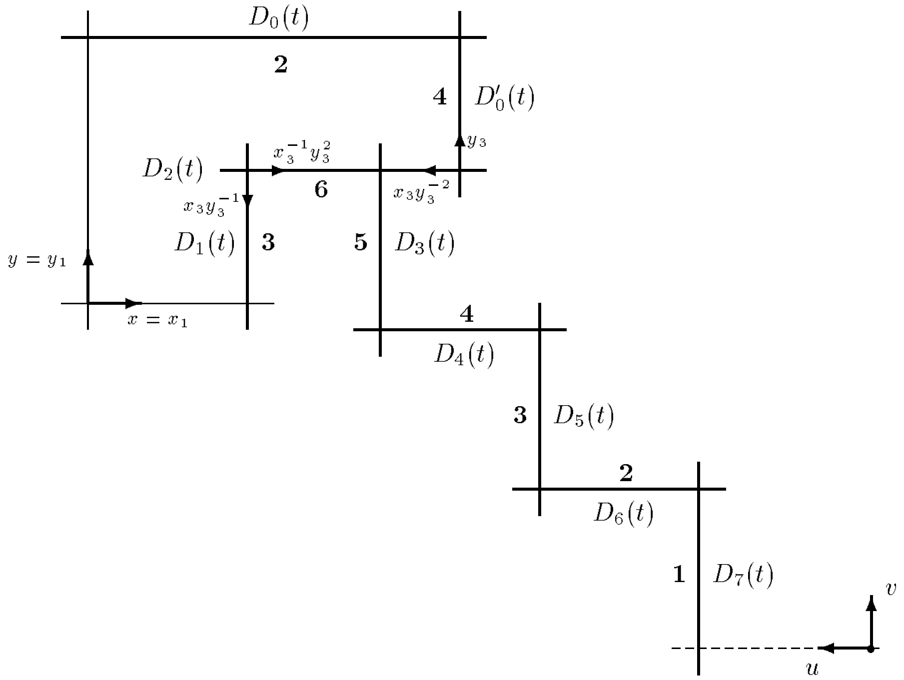


Figure 9.

Let $Y = 2D_0 + 4D'_0 + 3D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6 + D_7$, and $D = Y_{red}$. For simplicity, we rewrite (x_1, y_1) as (x, y), and set $U = \mathrm{Spec}\,\mathbf{C}[x, y]$. Note that

$$(U; (x, y)) \subset S - D.$$

Now we consider the curve D_8. Let us take the coordinate system (u, v) around the point at infinity of D_8. Let U' be the coordinate system $(U''; (u, v)) \cong \mathbf{C}^2$. Note that $D_8 \cap U' = \{u = 0\}$. By considering the coordinate transformations via above eight blowings-up, we have the following coordinate transformation between $(U; (x, y))$ and $(U'; (u, v))$:

$$u = x^{15}y^{-8} - \frac{1}{4}x^{12}y^{-6} - \frac{t}{2}x^4y^{-2} + \frac{1}{2}x^2y^{-1}, \tag{5}$$

$$v = -x^{-2}y. \tag{6}$$

By calculating exterior derivations of u and v, we have

$$du \wedge dv = x^{12}y^{-8}dx \wedge dy.$$

On the other hand, by solving the system of equations (5) and (6), we have

$$x = A^{-1}v^{-2},$$
$$y = -A^{-2}v^{-3},$$

where

$$A = uv^6 + \frac{1}{2}v^5 + \frac{t}{2}v^4 + \frac{1}{4}.$$

So,

$$x^{12}y^{-8} = A^4 = (uv^6 + \frac{1}{2}v^5 + \frac{t}{2}v^4 + \frac{1}{4})^4.$$

Therefore, we have

$$dx \wedge dy = \frac{du \wedge dv}{(uv^6 + \frac{1}{2}v^5 + \frac{t}{2}v^4 + \frac{1}{4})^4}.$$

On the other hand, note that we are denoting (x_1, y_1) by (x, y). By using $x_1 = x_3^{-1}$ and $y_1 = -x_3^{-2}y_3$ (see (4)), from (5) and (6), we see that

$$\begin{aligned} u &= x_3 y_3^{-8} - \frac{1}{4}y_3^{-6} - \frac{t}{2}y_3^{-2} - \frac{1}{2}y_3^{-1}, \\ v &= y_3. \end{aligned}$$

Then we have

$$uv^6 + \frac{1}{2}v^5 + \frac{t}{2}v^4 + \frac{1}{4} \quad = \quad x_3 y_3^{-2}.$$

From Figure 9, the curve $C = \{uv^6 + (1/2)v^5 + (t/2)v^4 + (1/4) = 0\}$ coincides with the component $D_0'(t)$ of the divisor Y. Note that the order of the pole at C of the 2-form $dx_0 \wedge dy_0$ and the multiplicity of $D_0'(t)$ in Y are both equal to four. Namely the affine chart $(U'; (u, v)) \cong \mathbf{C}^2$ intersects Y, and satisfies that $U' \cap Y = D_0'(t)$.

Next, let us perform the third blowing-up (the blowing-up with center of $a_2(t)$), by using the coordinate system whose origin is $Q(t)$, namely (z_2, W_2). For simplicity, let us use the same notations (Z_i, w_i), (z_i, W_i), or D_i as above. But we denote (u', v') for the coordinate system around the point at infinity of D_8. And let U'' be the coordinate system $(U''; (u', v')) \cong \mathbf{C}^2$. Now we have the following results:

$$\begin{aligned} u' &= x^{-9}y^8 - 4x^{-6}y^6 - \frac{t}{2}x^{-2}y^2 + \frac{1}{2}x^{-1}y \\ &= x_3^{-7}y_3^8 - 4x_3^{-6}y_3^6 - \frac{t}{2}x_3^{-2}y_3^2 - \frac{1}{2}x_3^{-1}y_3 \\ v' &= -xy^{-1} \\ &= -x_3 y_3^{-1} \end{aligned}$$

By the similar calculation, we have

$$dx \wedge dy = dx_1 \wedge dy_1 = \frac{du' \wedge dv'}{(u'(v')^6 + \frac{1}{2}(v')^5 + \frac{t}{2}(v')^4 + 4)^3},$$

and

$$u'(v')^6 + \frac{1}{2}(v')^5 + \frac{t}{2}(v')^4 + 4 \quad = \quad x_3^{-1}y_3^2.$$

From Figure 9, the curve $C' = \{u'(v')^6 + (1/2)(v')^5 + (t/2)(v')^4 + 4 = 0\}$ coincides with the component $D_1(t)$ of the divisor Y. The order of the pole C' of the 2-form $dx_1 \wedge dy_1$ and the multiplicity of $D_1(t)$ in Y are both equal to three. We see that $U'' \cap Y = D_0'(t)$.

Consequently, for an Okamoto–Painlevé pair of type $\widetilde{E}_8$, $S - Y$ is covered by $(U; (x_1, y_1)) \cong \mathbf{C}^2$ and $(U'; (u, v)) - C \cong \mathbf{C}^2 - C$ (or $(U''; (u', v')) - C'$).

3. Okamoto–Painlevé pairs of non-elliptic type

Definition 3.1. An Okamoto–Painlevé pair (S, Y) is of *elliptic type* if there exists a fibration $f : S \longrightarrow \mathbf{P}^1$ of elliptic curves such that a scheme theoretic fiber $f^*(\infty)$ at ∞ is Y, that is, $f^*(\infty) = Y$. If (S, Y) is not of elliptic type, we call (S, Y) is of *non-elliptic type.*

A rational elliptic surface can be obtained by blowings-up of 9 base points of a cubic pencil on $\mathbf{P}^2$. Actually, one can obtain an Okamoto–Painlevé pairs (S, Y) of type $\tilde{E}_6$ by blowings-up of infintely near base points of a pencil of $3 \times line$ and a non-singular cubic curve. This example gives an elliptic fibration $f : S \longrightarrow \mathbf{P}^1$ with $f^*(\infty) = Y$, and hence (S, Y) is an Okamoto-Painlevé pair of elliptic type. On the other hand, if one blows up 8 (infinitely near) base points of the pencil and blows up a point on anti-canonical divisor which is not a base points of the pencil, one can not obtain an elliptic fibration, so (S, Y) becomes an Okamoto-Painlevé pair of non-elliptic type. For each type of Okamoto-Painlevé pair, one can obtain both elliptic type and non-elliptic type depending on the position of the points of blowings-up and blowings-down. Note that non-elliptic type is general in the moduli space of Okamoto-Painlevé pairs of each type.

The following porposition is shown in [Sa–T].

Proposition 3.1. *Let (S,Y) be an Okamoto–Painlevé pair. Then (S,Y) is of non-elliptic type if and only if $H^0(S-Y,\mathcal{O}^{alg}) \cong \mathbf{C}$. (Here $H^0(S-Y,\mathcal{O}^{alg}) \cong \mathbf{C}$ means that all of algebraic regular functions on $S-Y$ are constant functions.)*

4. Construction of type $\widetilde{D}_7$

Now we will construct the Okamoto–Painlevé pair of type $\widetilde{D}_7$ by blowing-up of $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ on $-K_{\mathbf{F}_0} = 2s_0 + 2f$. In the following figure two numbers near each solid line denotes the multiplicity and the self-intersection number in $-K_S$. The broken lines denote (-1)-curves, whose multiplicities in F are zero. Namely, they are *additional components* of F.

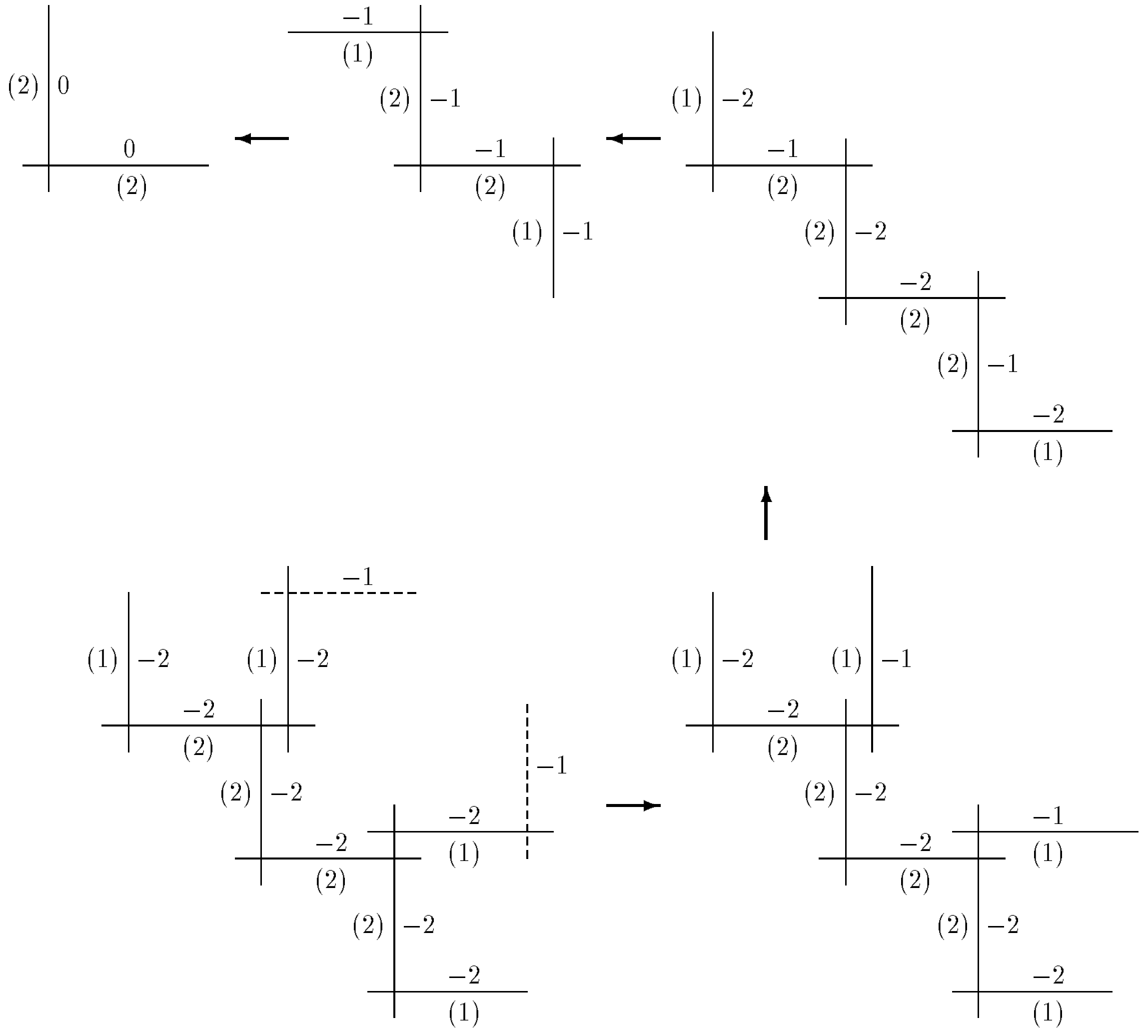


Figure 10.

5. Additional Components

By definition, each Okamoto–Painlevé pair (S, Y) contains the affine plain $\mathbf{C}^2$ as a Zariski open set and we set

$$F = S - \mathbf{C}^2.$$

We can see F as a divisor with ten components which is obtained by adding some irreducible smooth rational curves to Y_{red}. We call such curves *additional components.*

Now we will consider F counting the multiplicity m_j for the component F_j $(j = 1, \dots, 10)$. We set $F = \sum_{j=1}^{10} m_j F_j$ and $Y = \sum_{i=1}^{r} a_i Y_i$. Note that $r < 10$. If $F_j = Y_i$, then we set $m_j := a_i$. And if F_j is an additional component, then we set $m_j = 0$.

Lemma 5.1. *Let C be an additional component of F which intersects F_j. Then we have $C^2 = m_j - 2$*

Proof. Note that F_j which satisfies $m_j > 0$ and which C intersects exists uniquely. By the adjunction formula, we have $C^2 = C \cdot Y - 2$. Since $C \cdot Y = m_j$, our assertion is proved.

Now we assume that $F = \sum_{i=1}^{10} F_i$ satisfies the following condition.

$(*)$ Both $(F_1)^2$ and $(F_2)^2$ are even numbers, and both F_1 and F_2 intersect only F_3.

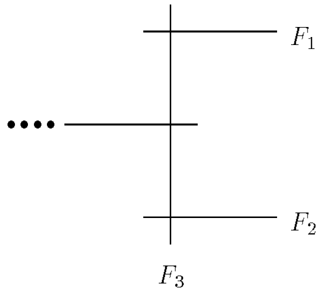


Figure 11.

Lemma 5.2. *There exists no Okamoto–Painlevé pairs which satisfy $(*)$.*

Proof. We consider the determinant of the intersection matrix of F.

$$\det I_F \quad = \quad \begin{vmatrix} (F_1)^2 & 0 & 1 & 0 & \cdots & 0 \\ 0 & (F_2)^2 & 1 & 0 & \cdots & 0 \\ 1 & 1 & (F_3)^2 & * & \cdots & * \\ 0 & 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & * & \cdots & * \end{vmatrix}$$

We see that $\det I_F$ must be an even number, if we expand the determinant. Namely $\det I_F \neq -1$. This contradicts the fact that I_F is a unimodular matrix with the signature $(b_+, b_-) = (1, 9)$. This completes the proof.

Lemma 5.3. *There is not more than one additional component which intersects the components of F whose multiplicities are ≥ 2.*

Proof. We consider additional components C_1 and C_2. We assume that C_1 and C_2 intersect F_1 and F_2 respectively, and that $m_1 \geq 2$ and $m_2 \geq 2$. Note that $C_1 \cdot C_2 = 0$. If $C_1 \cdot C_2 \neq 0$, then F includes a cycle. From Lemma 5.1, it follows that $(C_1)^2 = m_1 - 2$ and $(C_2)^2 = m_2 - 2$. Note that $(C_1)^2 \geq 0$ and $(C_2)^2 \geq 0$. We will prove by separating into some cases.

[i] *The case where $m_1 \geq 3$ or $m_2 \geq 3$.*
For example, we assume $m_1 \geq 3$. We have $(C_1)^2 > 0$ and $C_1 \cdot C_2 = 0$. Therefore, the Hodge index theorem shows that $(C_2)^2 < 0$ or $C_2 \equiv 0$. This is a contradiction.

[ii] *The case where $m_1 = m_2 = 2$.*

(1) *The case where another additional component intersects C_1 or C_2.*
In this case, we suppose that C_3 intersects C_1 as an example. Then we have $(C_3)^2 = -2$, $C_1 \cdot C_3 = 1$ and $C_2 \cdot C_3 = 0$. Now we consider a divisor $A = nC_1 + C_3$ for $n \in \mathbf{Z}_{\geq 2}$. We have $A^2 = 2n - 2 > 0$ and $A \cdot C_2 = 0$. By applying the Hodge index theorem again, it follows that $(C_2)^2 < 0$ or $C_2 \equiv 0$. Thus we obtained a contradiction.

(2) *The case where another additional component intersect neither C_1 nor C_2.*
In this case, C_1 and C_2 can not intersect the same component of F by Lemma 5.2. Hence we see that C_1 and C_2 intersects the different components. Let us denote them by F_1 and F_2 respectively. Note that $(C_1)^2 = (C_2)^2 = 0$, $C_1 \cdot F_1 = 1$, $C_2 \cdot F_1 = 0$ and $(F_1)^2 = -2$. Now we consider a divisor $B = nC_1 + F_1$ for $n \in \mathbf{Z}_{\geq 2}$. Then we have $B^2 = 2n - 2 > 0$ and $B \cdot C_2 = 0$. This contradicts the Hodge index theorem. This completes the proof.

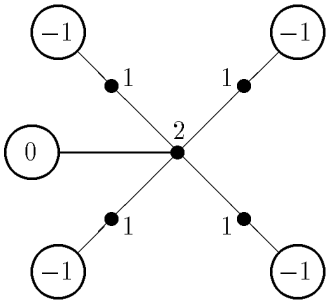
By using the above lemmas, we will be able to give the complete list of configurations of F and $Y \subset F$.

Moreover, one can also check that each pattern in the following list can be transformed into $(\mathbf{P}^2, -K_{\mathbf{P}^2} = 3h)$, $(\mathbf{F}_0, -K_{\mathbf{F}_0} = 2s_0 + 2f)$ or $(\mathbf{F}_2, -K_{\mathbf{F}_2} = 2s_\infty)$ by performing blowing-up and blowing-down to S on $-K_S$. Since $(\mathbf{F}_0, -K_{\mathbf{F}_0} = 2s_0 + 2f)$ or $(\mathbf{F}_2, -K_{\mathbf{F}_2} = 2s_\infty)$ can be transformed into $(\mathbf{P}^2, -K_{\mathbf{P}^2} = 3h)$ by birational transformations, as a consequence, each pattern can be transformed into $(\mathbf{P}^2, -K_{\mathbf{P}^2} = 3h)$ by performing blowing-up and blowing-down of (S, Y). (This gives a proof of the assertion (iii) of Theorem 2.1.) This also shows that all of the configuration of F in the following list do exist. In fact, one can perform the inverse birationl transformation from $(\mathbf{P}^2, -3l)$ to (S, Y) or (S, F).

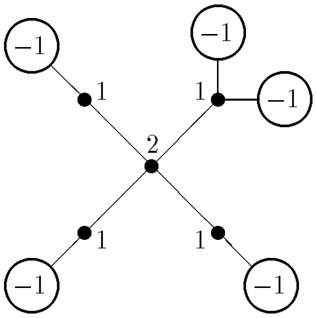
COMPLETE LIST OF CONFIGURATIONS OF F.

We will give the list of configurations of F in the form of dual graphs. In each figure, we denote by each vertex a component with the positive multiplicity in F, and by each circle an additional component. The number near each vertex means the multiplicity of the component in F, and the number inside each circle means the self-intersection number of the corresponding additional component. Note that the multiplicity of an additional component in F is zero.

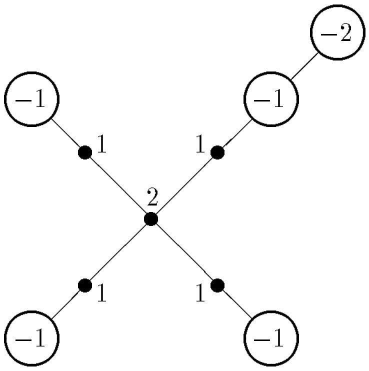
Type $\widetilde{D}_4$



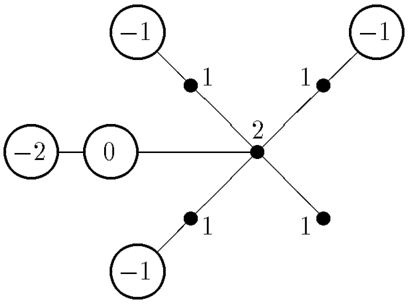
$\widetilde{D}_4$–(1)



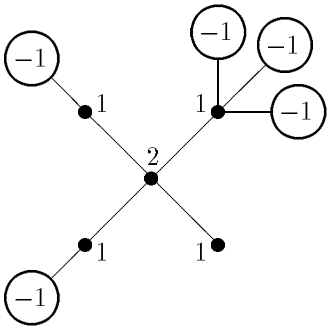
$\widetilde{D}_4$–(2)



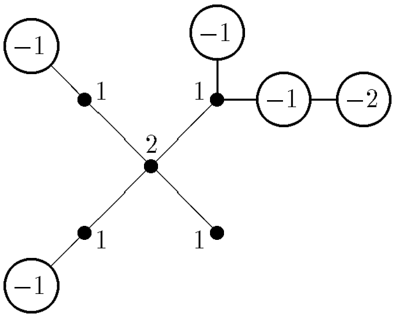
$\widetilde{D}_4$–(3)



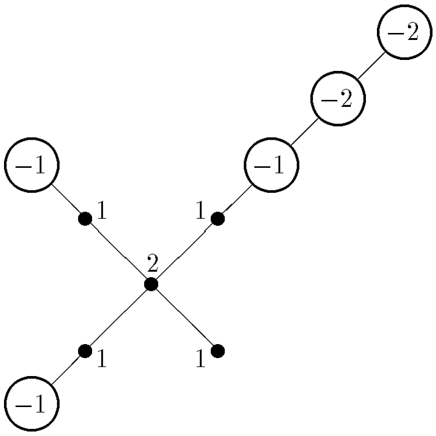
$\widetilde{D}_4$–(4)



$\widetilde{D}_4$–(5)

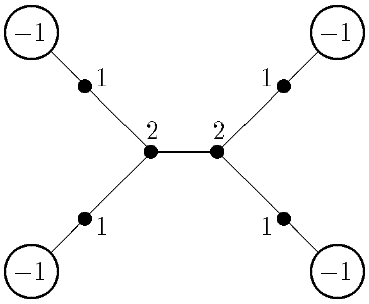


$\widetilde{D}_4$–(6)

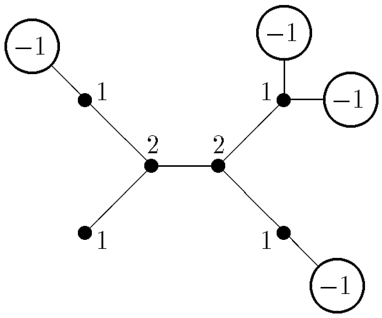


$\widetilde{D}_4$–(7)

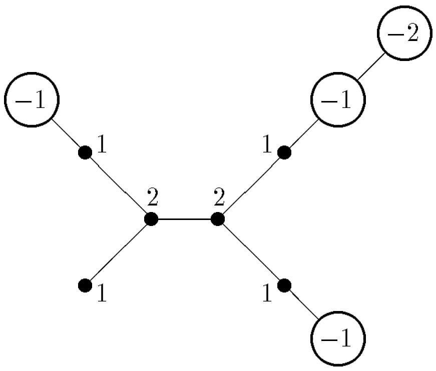
Type $\widetilde{D}_5$



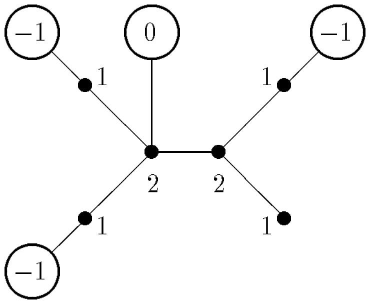
$\widetilde{D}_5$–(1)



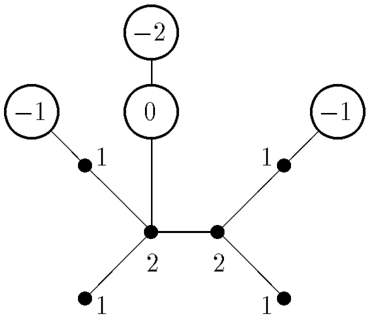
$\widetilde{D}_5$–(2)



$\widetilde{D}_5$–(3)



$\widetilde{D}_5$–(4)



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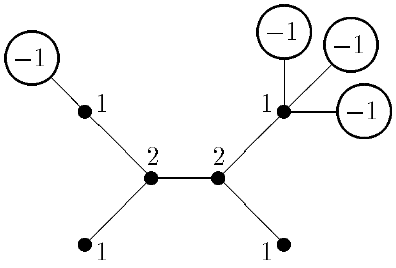
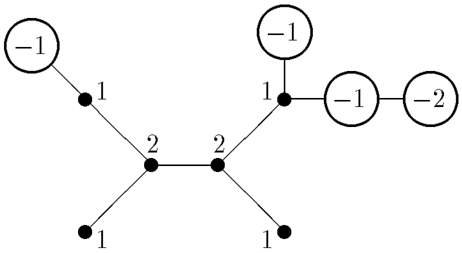
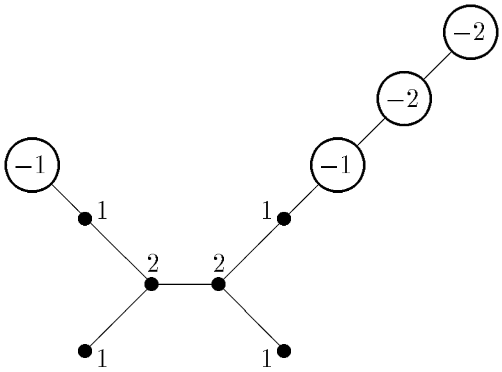


$\widetilde{D}_5$–(6)

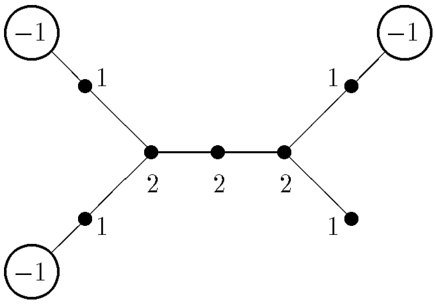


$\widetilde{D}_5$–(7)

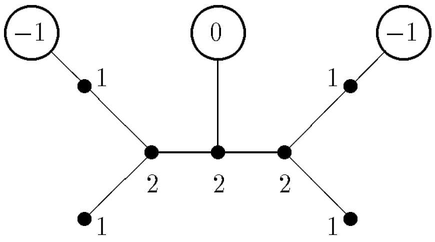


$\widetilde{D}_5$–(8)

Type $\widetilde{D}_6$

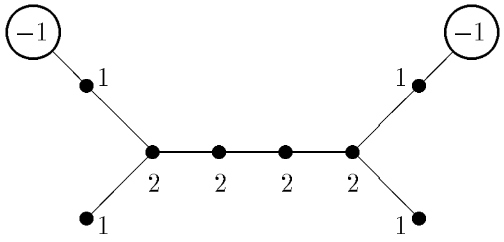


$\widetilde{D}_6$–(1)

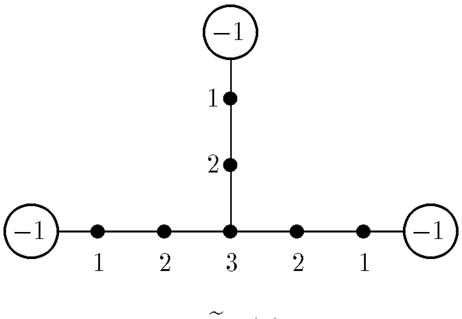


$\widetilde{D}_6$–(2)

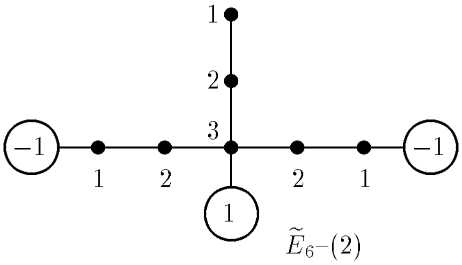
Type $\widetilde{D}_7$



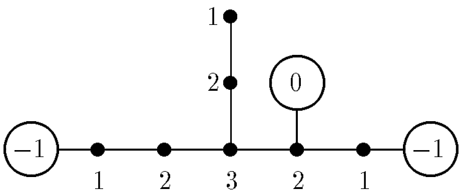
Type $\widetilde{E}_6$



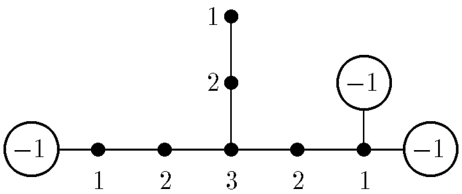
$\widetilde{E}_6$–(1)



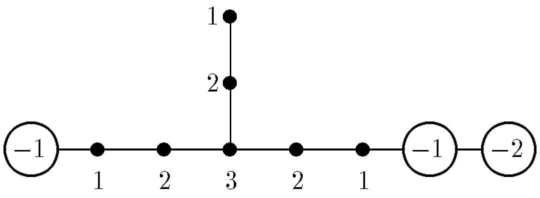
$\widetilde{E}_6$–(2)



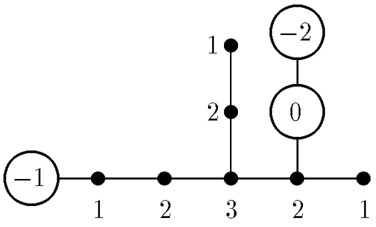
$\widetilde{E}_6$–(3)



$\widetilde{E}_6$–(4)

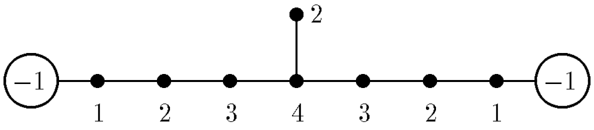


$\widetilde{E}_6$–(5)

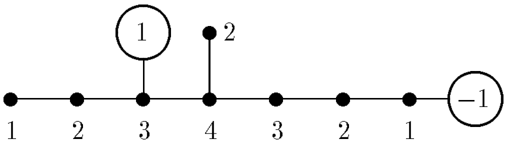


$\widetilde{E}_6$–(6)

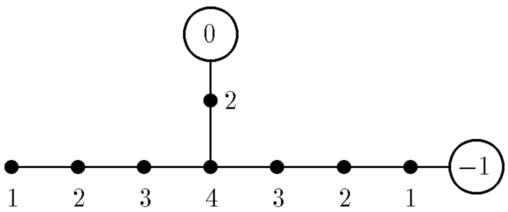
Type $\widetilde{E}_7$



$\widetilde{E}_7$–(1)

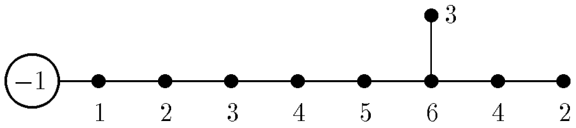


$\widetilde{E}_7$–(2)



$\widetilde{E}_7$–(3)

Type $\widetilde{E}_8$



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