RELATIVE LEFSCHETZ ACTION AND BPS STATE COUNTING

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Abstract. In this paper, we propose a mathematical definition of a new "numerical invariants" of Calabi–Yau 3-folds from stable sheaves of dimension one, which is motivated by the Gopakumar-Vafa conjecture [GV] in M-theory. Moreover, we show that for any projective morphism $f : X \longrightarrow Y$ of normal projective varieties, there exists a natural $sl_2 \times sl_2$ action on the intersection cohomology group $IH(X, \mathbb{Q})$ which fits into the perverse Leray spectral sequence.

1. Introduction

Let X be a Calabi–Yau 3-fold with $\pi_1(X) = \{1\}$ and let us fix an ample line bundle $\mathcal{O}_X(1)$ on X.

For $\beta \in H_2(X, \mathbb{Z})$ and an integer $g \geq 0$, we denote the 0-point genus g Gromov–Witten invariants of X in the homology class β by

$$N_g(\beta) := [\overline{\mathcal{M}_{g,0}}(X, \beta)]^{virt} \in A_0(\overline{\mathcal{M}_{g,0}}(X, \beta)) \simeq \mathbb{Q}.$$

Recently, physicists Gopakumar and Vafa [GV] introduced the following remarkable formula for the generating function of Gromov–Witten invariants based on the string duality between Type IIA and M-theory.

Conjecture 1.1. ([GV]) *There should exist* **integers** $n_h(\beta)$ *such that*

(1)

$$\boxed{\sum_{g \geq 0, \beta \in H_2(X,\mathbb{Z})} N_g(\beta) q^{\beta} \lambda^{2g-2} = \sum_{k>0, h \geq 0, \beta \in H_2(X,\mathbb{Z})} n_h(\beta) \frac{1}{k} \left(2 \sin(\frac{k\lambda}{2}) \right)^{2h-2} q^{k\beta},}$$

where $q^{\beta} := \exp(-2\pi \int_{\beta} \omega)$, $\omega := c_1(\mathcal{O}_X(1))$.

They also proposed that integers $n_h(\beta)$ should be defined by the spin contents of the BPS states of M2-branes wrapped around the curves in X. More precisely, they expect that a suitable D-brane moduli space M_β and the natural support map $\pi_\beta : M_\beta \to S_\beta$ exist. By assuming the existence of an $(sl_2)_L \times (sl_2)_R$-action on some suitable cohomology group $H^*(M_\beta)$, they decompose $H^*(M_\beta)$ and rearrange it as

$$H^*(M_\beta) = \bigoplus_{h \geq 0} \left[(\frac{1}{2})_L \oplus 2(0)_L \right]^{\otimes h} \otimes R_h(\beta),$$

to define numerical invariants

$$n_h(\beta) := Tr_{R_h(\beta)}(-1)^{2H_R}.$$

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To complete their conjecture we need to define mathematically their integral "numerical invariants" of Calabi–Yau 3-folds by the moduli space of "D-branes" and to formulate their conjecture as an equivalence of the new invariants and Gromov–Witten invariants. For this purpose, we have to

(i) define the moduli space of D-branes,
(ii) prove the existence of an $(sl_2)_L \times (sl_2)_R$-action on a suitable cohomology on the above moduli space,
(iii) prove the Gopakumar–Vafa formula (1).

In this paper we deal with the first two steps and present nontrivial evidences for Gopakumar–Vafa conjecture. Especially, we can provide the answer of the problem (ii) using the intersection cohomology of the D-brane moduli spaces and the decomposition theorem due to [BBD]. As for the D-brane moduli space we propose a natural definition in section 3.

Here is the brief plan of this paper. In Section 2, we recall the general theory of perverse sheaves, and prove that for any projective morphism $f : X \to Y$ there exists a natural $(sl_2)_L \times (sl_2)_R$-action on intersection cohomology $IH^*(X)$. In Section 3 we consider a suitable moduli space M_β of semi-stable sheaves on Calabi–Yau 3-folds and define the support map $\pi_\beta : M_\beta \to S_\beta$. We propose M_β as the moduli space of D-branes and by applying the results in Section 2 to π_β, we obtain the numerical invariants $n_h(\beta)$. In Section 4, we provide some evidences for Gopakumar–Vafa conjecture.

In [BP], Bryan and Pandharipande proved the integrality of BPS invariants $n_h(\beta)$ coming from the formula (1) for Gromov–Witten invariants of some super-rigid curves in a Calabi–Yau 3-fold by evaluating the virtual fundamental classes. Also, we are informed that Fukaya–Ono [FO] proved the genus 0 part of this conjecture in the symplectic category.

2. Relative Lefschetz Action, — General Theory by BBD

In this section, we recall the definition of perverse sheaves and the intersection cohomology briefly. Since we only need the formulation of perverse sheaves and the relative hard Lefschetz theorem for perverse direct image sheaves (2.3), we shall not give any proof. For detail, we refer the readers to [BBD].

Let X be a normal complex algebraic variety. As in [2.2.1, [BBD]], we will only consider stratifications $X = \coprod_{i=1}^r X_i$ by equidimensional algebraic strata X_i.

Definition 2.1. **(Constructible Sheaves)**([2.2.1, [BBD]]).
A $\mathbb{C}_X$-module $\mathcal{F}$ is called *constructible* if there exists a stratification $X = \coprod_{i=1}^r X_i$ such that restrictions $\mathcal{F}|_{X_i}$ are local systems on X_i.

Let $D^b_c(\mathbb{C}_X)$ be the derived category of bounded complexes of $\mathbb{C}$-modules with constructible cohomology sheaves. Let $X = \coprod_{i=1}^r X_i$ be a stratification of X. In order to define perverse sheaves, we have to fix a perversity p. As in [2.1.16, [BBD]], it is convenient to take the auto-dual perversity which is defined for each strata $j : S \hookrightarrow X$ as

$$p(S) = -\dim_{\mathbb{C}} S.$$

Definition 2.2. **(Perverse sheaves)**
A *perverse* $\mathbb{C}_X$-module (with the middle perversity) is an object $K^\bullet \in D^b_c(\mathbb{C}_X)$ such that the following conditions are satisfied:

(i) (Support condition)

$$\dim_{\mathbb{C}} \mathrm{supp} H^i(K^\bullet) \le -i, \ i \in \mathbb{Z}.$$

(ii) (Support condition for Verdier dual)

$$\dim_{\mathbb{C}} \mathrm{supp} H^i(\mathbb{D}_X K^\bullet) \le -i, \ i \in \mathbb{Z},$$

where $\mathbb{D}_X$ is a Verdier dualizing functor. Let ${}^pD^{\le 0}(\mathbb{C}_X)$ (resp. ${}^pD^{\ge 0}(\mathbb{C}_X)$) be the subcategory of $D^b_c(\mathbb{C}_X)$ whose objects are complexes $K^\bullet \in D^b_c(\mathbb{C}_X)$ satisfying the support condition (resp. support condition for Verdier dual). Let us set

$$Perv(\mathbb{C}_X) := {}^pD^{\le 0}(\mathbb{C}_X) \cap {}^pD^{\ge 0}(\mathbb{C}_X).$$

Remark. The category of perverse $\mathbb{C}_X$-modules is an abelian category which is both Artinian and Noetherian. The simple objects are of the form

$$\iota_{!*}L[\dim_{\mathbb{C}} V] := \mathrm{Im}(\iota_! L \to \iota_* L)[\dim_{\mathbb{C}} V],$$

where $V \hookrightarrow X$ is the immersion of locally closed subvariety of X and L is a local system on V.

Theorem 2.1. **(Théorème 1.3.6** [BBD])
The inclusion ${}^pD^{\le 0}(\mathbb{C}_X) \hookrightarrow D^b_c(\mathbb{C}_X)$ *(resp.* ${}^pD^{\ge 0}(\mathbb{C}_X) \hookrightarrow D^b_c(\mathbb{C}_X)$*) gives a right (resp. left) adjoint functor* $\tau_{\le 0} : D^b_c(\mathbb{C}_X) \longrightarrow {}^pD^{\le 0}(\mathbb{C}_X)$, *(resp.* $\tau_{\ge 0} : D^b_c(\mathbb{C}_X) \longrightarrow {}^pD^{\le 0}(\mathbb{C}_X)$ *).*

Moreover,

$${}^pH^0 := \tau_{\ge 0}\tau_{\le 0} : D^b_c(\mathbb{C}_X) \to Perv(\mathbb{C}_X)$$

is a cohomology functor, which is called a perverse cohomology functor*.*

By using the perverse cohomology functor, we can define the perverse direct images.

Definition 2.3. (Perverse direct images functor)
Let $f : X \to Y$ be a morphism of normal algebraic varieties.

$${}^pR^kf_* : Perv(\mathbb{C}_X) \to Perv(\mathbb{C}_Y), \ K^\bullet \mapsto {}^pR^kf_*K^\bullet := {}^pH^0(Rf_*K^\bullet[k]).$$

The following theorems are the main results of [BBD].

Theorem 2.2. (Decomposition Theorem (Theorémè 6.2.5 [BBD]))
Let $f : X \to Y$ be a proper morphism of normal algebraic varieties and $K^\bullet \in Perv(\mathbb{C}_X)$ be a simple object of geometric origin. Then

$$Rf_*K^\bullet \simeq \bigoplus_k {}^pR^kf_*K^\bullet[-k].$$

Theorem 2.3. (Relative hard Lefschetz theorem (Theorémè 5.4.10, 6.2.10 [BBD]))
Let ω be the first Chern class of the relative ample line bundle for the projective morphism $f : X \to Y$. Then for $k \ge 0$, we have

$$\omega^k \wedge : {}^pR^{-k}f_*K^\bullet \simeq {}^pR^kf_*K^\bullet.$$

From Theorem 2.2 and 2.3, we can derive the following corollary.

Corollary 2.1. *Let $f : X \longrightarrow Y$ be a projective morphism between normal projective varieties. Moreover let ω_L and ω_R be the first Chern classes of a relatively ample invertible sheaf for $f : X \longrightarrow Y$ and an ample invertible sheaf of Y respectively. Then the perverse Leray spectral sequence*

$$^pE_2^{r,s} = {}^pH^r(Y, {}^pR^s f_*\mathbb{C}) \Rightarrow {}^pH^{r+s}(X, \mathbb{C}) = IH^{r+s}(X, \mathbb{C})$$

degenerates at E_2-term. Moreover two relative hard Lefschetz actions for $f : X \longrightarrow Y$ and $Y \longrightarrow \{point\}$ define actions on E_2-terms

$$\omega_L\wedge : {}^pE_2^{r,s} = {}^pH^r(Y, {}^pR^s f_*\mathbb{C}) \longrightarrow {}^pE_2^{r,s+2} = {}^pH^r(Y, {}^pR^{s+2} f_*\mathbb{C})$$

and

$$\omega_R\wedge : {}^pE_2^{r,s} = {}^pH^r(Y, {}^pR^s f_*\mathbb{C}) \longrightarrow {}^pE_2^{r+2,s} = {}^pH^{r+2}(Y, {}^pR^s f_*\mathbb{C}),$$

so that $(\omega_L)^s : {}^pE_2^{*,-s} \stackrel{\simeq}{\to} {}^pE_2^{*,s}$ and $(\omega_R)^r : {}^pE_2^{-r,*} \stackrel{\simeq}{\to} {}^pE_2^{r,*}$. These two actions commute to each other and define an $(sl_2)_L \times (sl_2)_R$-action on the intersection cohomology ring $IH^*(X, \mathbb{C})$.

A digression for the representation of sl_2.

We recall some fundamental facts on the representation of sl_2 and fix some notation.

It is well-known that the isomorphism class of the complex irreducible representations V of sl_2 can be determined by their dimension k.

Definition 2.4. The irreducible representation of sl_2 of dimension k is called the spin $\frac{k-1}{2}$ representation and it is denoted by

$$(\frac{k-1}{2}).$$

Note that for a non-negative half-integer $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, the spin j representation (j) has dimension $2j+1$.

Let E, F, H be the usual generators of sl_2 with the relation

$$[E, F] = 2H, \quad [H, E] = E, \quad [H, F] = -F.$$

For example, one may take a matrix representation

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the spin j representation (j), we can find an eigenvector $v \in V$ of H with $Fv = 0$. Then one can show that

$$(j) = < v, Ev, \cdots, E^{2j}v >_{\mathbb{C}}, \quad \text{and} \quad E^{2j+1}v = 0,$$
$$H(E^k v) = (-j+k)\cdot v$$

In this case, the element $E^k v$ has the spin $-j+k$, $0 \leq k \leq 2j$.

Let E be an elliptic curve, or a compact complex torus of dimension 1 and let ω be a Lefschetz operator induced by a Kähler class. Then we have an isomorphism $H^2(E, \mathbb{C}) = \omega \cdot H^0(E, \mathbb{C})$. The cohomology ring $H^*(E, \mathbb{C}) = H^0 \oplus H^1 \oplus H^2 \simeq \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}$ has the Lefschetz decomposition and it defines a representation on $H^*(E, \mathbb{C})$ of sl_2 as

$$(\frac{1}{2}) = H^0(E, \mathbb{C}) \oplus \omega \cdot H^0(E, \mathbb{C}) \simeq \mathbb{C} \oplus \omega\mathbb{C}, \quad 2 \cdot (0) = H^1(E, \mathbb{C}) \simeq \mathbb{C}^2.$$

Hence we have

$$H^*(E, \mathbb{C}) = (\frac{1}{2}) \oplus 2(0).$$

In general, for a compact complex torus A of complex dimension g, its complex cohomology ring has the Lefschetz decomposition induced by the Lefschetz operator ω which defines a representation of sl_2 on $H^*(A, \mathbb{C})$. It is obvious that as a representation of sl_2

$$H^*(A, \mathbb{C}) = [(\frac{1}{2}) \oplus 2 \cdot (0)]^{\otimes g}.$$

Definition 2.5. For each $g \geq 0$, we set

$$I_g = [(\frac{1}{2}) \oplus 2 \cdot (0)]^{\otimes g}.$$

Remark. For each irreducible representation (j) of spin j, we can find integers $\alpha_r \in \mathbb{Z}$ so as

$$(j) = \oplus_{r=0}^{2j} \alpha_r I_r. \tag{2}$$

In fact, since I_{2j} contains $(j) = H^0 \oplus \omega H^0 \oplus \cdots \oplus \omega^{2j} H^0$ as an irreducible factor and the difference

$$I_{2j} - (j)$$

is a sum of irreducible representations of spin k where $k < j$. By induction with respect to j, we see that decomposition (2) holds.

For example, we see that $(0) = I_0, (\frac{1}{2}) = I_1 - 2I_0$, $(1) = I_2 - 4I_1 + 3I_0, \cdots$.

Corollary 2.1 says that for any projective morphism of projective normal varieties $f : X \to Y$ we can define an action of $(sl_2)_L \times (sl_2)_R$ on the intersection cohomology ring $IH^*(X, \mathbb{C})$. For any pairs (j_1, j_2) of non-negative half-integers, let us set

$$(j_1, j_2) = (j_1)_L \otimes (j_2)_R,$$

i.e., (j_1, j_2) is an irreducible representation of $(sl_2)_L \times (sl_2)_L$ of bi-spin (j_1, j_2). Let us consider the irreducible decomposition of $IH^*(X, \mathbb{C})$ defined by the relative Lefschetz action

$$IH^*(X, \mathbb{C}) = \oplus \alpha_{j_1, j_2} (j_1, j_2). \tag{3}$$

Moreover by the remark above, we can define the virtual decomposition of $IH^*(X, \mathbb{C})$ as

$$IH^*(X, \mathbb{C}) = \oplus_{h=0}^{l} I_h \otimes R_h \tag{4}$$

where R_j is a (virtual) representation of $(sl_2)_R$.

Summarizing the results, we obtain the following

Theorem 2.4. *Let $f : X \to Y$ be a projective morphism of normal complex projective varieties. Moreover let ω_L and ω_R be the first Chern classes of ample invertible sheaves on X and Y, respectively. Then the intersection cohomology ring $IH^*(X, \mathbb{C})$ has a natural $(sl_2)_L \times (sl_2)_R$ action induced by the relative Lefschetz action of ω_L and the Lefschetz action of ω_R on the E_2-terms of the perverse Leray spectral sequence for $f : X \to Y$. The representation of $IH^*(X, \mathbb{C})$ defines the irreducible decomposition and the virtual decomposition as*

$$IH^*(X, \mathbb{C}) = \oplus_{j_1, j_2} \alpha_{j_1, j_2} (j_1, j_2) = \oplus_{h=0}^{l} I_h \otimes R_h. \tag{5}$$

The following example shows that for a projective morphism $X \to Y$ the usual Leray spectral sequence does not detect the relative Lefschetz action and only the magic of the perverse sheaves can detect the natural $(sl_2)_L \times (sl_2)_R$-representation.

Example 2.1. (Blowing up of $\mathbb{P}^2$)
Let $\pi : \widehat{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blowing up at a point $p \in \mathbb{P}^2$.

For $\mathbb{C}_{\widehat{\mathbb{P}}^2}[2] \in Perv(\mathbb{C}_{\widehat{\mathbb{P}}^2})$, the usual direct image $R^*\pi_*\mathbb{C}_{\widehat{\mathbb{P}}^2}[2]$ can be given by:

$$R^j\pi_*(\mathbb{C}_{\widehat{\mathbb{P}}^2}[2]) = \begin{cases} \mathbb{C}_{\mathbb{P}^2} & \text{for} \quad j = -2 \\ \mathbb{C}_p & \text{for} \quad j = 0 \\ 0 & \text{otherwise} \end{cases}$$

The E_2-term of the ordinary Leray spectral sequence is given by

$$E^{i,j} = H^i(\mathbb{P}^2, R^j\pi_*(\mathbb{C}_{\widehat{\mathbb{P}}^2}[2])),$$

and more explicitly each term is given in the following table.

Table 1

$$\begin{array}{|lll} E^{4,-2} & E^{4,-1} & E^{4,0} \\ E^{3,-2} & E^{3,-1} & E^{3,0} \\ E^{2,-2} & E^{2,-1} & E^{2,0} \\ E^{1,-2} & E^{1,-1} & E^{1,0} \\ E^{0,-2} & E^{0,-1} & E^{0,0} \\ \hline \end{array} \quad = \quad \begin{array}{|ccc} \mathbb{C} & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{C} & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} \\ \hline \end{array}$$

$$\begin{array}{lll} \mathbb{H}^{-2}(\widehat{\mathbb{P}^2}, \mathbb{C}[2]) \simeq H^0(\widehat{\mathbb{P}^2}, \mathbb{C}) & \simeq & E^{0,-2} = \mathbb{C} \\ \mathbb{H}^0(\widehat{\mathbb{P}^2}, \mathbb{C}[2]) \simeq H^2(\widehat{\mathbb{P}^2}, \mathbb{C}) & \simeq & E^{2,-2} \oplus E^{0,0} \simeq \mathbb{C} \oplus \mathbb{C} \\ \mathbb{H}^2(\widehat{\mathbb{P}^2}, \mathbb{C}[2]) \simeq H^4(\widehat{\mathbb{P}^2}, \mathbb{C}) & \simeq & E^{4,-2} \simeq \mathbb{C} \end{array} \tag{6}$$

From this diagram, one can not detect a natural action of $(sl_2)_L \times (sl_2)_R$ on the cohomology ring $H^*(\widehat{\mathbb{P}^2}, \mathbb{C})$ because the Table 1 is not symmetric. On the other hand, the perverse direct image can be given as

$${}^pR^j\pi_*\mathbb{C}_{\widehat{\mathbb{P}}^2}[2] = \begin{cases} \mathbb{C}_{\mathbb{P}^2}[2] \oplus \mathbb{C}_p & \text{for} \quad j = 0 \\ 0 & \text{otherwise} \end{cases}$$

that is, the complex is the direct sum of $\mathbb{C}_{\mathbb{P}^2}[2] \oplus \mathbb{C}_p$ and concentrated on the degree 0 and moreover the relative Lefschetz action (= the left action of ω_L) on ${}^pR^*\pi_*\mathbb{C}_{\widehat{\mathbb{P}}^2}[2] = \mathbb{C}_{\mathbb{P}^2}[2] \oplus \mathbb{C}_p$ is trivial. Hence we have the decomposition

$$\mathbb{H}^*(\widehat{\mathbb{P}^2}, \mathbb{C}_{\widehat{\mathbb{P}}^2}[2]) \simeq \mathbb{H}^*(\mathbb{P}^2, {}^pR^*\pi_*\mathbb{C}_{\widehat{\mathbb{P}}^2}[2]) = \mathbb{H}^*(\mathbb{P}^2, \mathbb{C}_{\mathbb{P}^2}[2]) \oplus \mathbb{H}^*(p, \mathbb{C}_p)$$

where each decomposition factor has the right sl_2-action as

$$\begin{array}{ccccc} \mathbb{H}^{-2}(\mathbb{P}^2, \mathbb{C}_{\mathbb{P}^2}[2]) & \xrightarrow{\omega_R\wedge} & \mathbb{H}^0(\mathbb{P}^2, \mathbb{C}_{\mathbb{P}^2}[2]) & \xrightarrow{\omega_R\wedge} & \mathbb{H}^2(\mathbb{P}^2, \mathbb{C}_{\mathbb{P}^2}[2]) \quad (\text{spin } 1) \\ || & & || & & || \\ \mathbb{C} & \xrightarrow{\omega_R\wedge} & \mathbb{C}\cdot\omega_R & \xrightarrow{\omega_R\wedge} & \mathbb{C}\cdot\omega_R^2 \end{array}$$

and

$$\begin{array}{ccccc} \mathbb{H}^{-2}(p, \mathbb{C}_p) & \xrightarrow{\omega_R\wedge} & \mathbb{H}^0(p, \mathbb{C}_p) & \xrightarrow{\omega_R\wedge} & \mathbb{H}^2(p, \mathbb{C}_p) \quad (\text{spin } 0) \\ || & & || & & || \\ 0 & \xrightarrow{\omega_R\wedge} & \mathbb{C} & \xrightarrow{\omega_R\wedge} & 0. \end{array}$$

Noting that the action of ω_L is trivial, we see that the following isomorphisms of $(sl_2)_L \times (sl_2)_R$ representations

$$\mathbb{H}^*(\mathbb{P}^2, \mathbb{C}[2]) \simeq (0,1) \simeq (0)_L \otimes (1)_R, \quad \mathbb{H}^*(p, \mathbb{C}_p) \simeq (0,0) \simeq (0)_L \otimes (0)_R.$$

As a result, the $(sl_2)_L \times (sl_2)_R$ decomposition of $H^*(\widehat{\mathbb{P}^2}, \mathbb{C})$ is given by

$$\begin{aligned} H^*(\widehat{\mathbb{P}^2}, \mathbb{C}) &= \mathbb{H}^*(\mathbb{P}^2, \mathbb{C}[2]) \oplus \mathbb{H}^*(p, \mathbb{C}_p) \\ &\simeq (0)_L \otimes [(1)_R \oplus (0)_R] = I_0 \otimes [(1)_R \oplus (0)_R] \end{aligned}$$

For reader's convenience, we put a table of the E_2-terms of the perverse Leray spectral sequence. One can compare these with those of the ordinary one in Table 1:

TABLE 2

$$\begin{array}{c|ccc} & {}^pE^{2,-1} & {}^pE^{2,0} & {}^pE^{2,1} \\ & {}^pE^{1,-1} & {}^pE^{1,0} & {}^pE^{1,1} \\ \uparrow \omega_R \wedge & {}^pE^{0,-1} & {}^pE^{0,0} & {}^pE^{0,1} \\ & {}^pE^{-1,-1} & {}^pE^{1,0} & {}^pE^{-1,1} \\ & {}^pE^{-2,-1} & {}^pE^{-2,0} & {}^pE^{-2,1} \\ \hline & & \omega_L \wedge \rightarrow & \end{array} = \begin{array}{c|ccc} & 0 & \mathbb{C} \oplus 0 & 0 \\ & 0 & 0 & 0 \\ \uparrow \omega_R \wedge & 0 & \mathbb{C} \oplus \mathbb{C} & 0 \\ & 0 & 0 & 0 \\ & 0 & \mathbb{C} \oplus 0 & 0 \\ \hline & & \omega_L \wedge \rightarrow & \end{array}$$

3. D-BRANE MODULI SPACES

What is the mathematical definition of "D-branes wrapped around a cycle of dimension one" and the moduli space M of "D-branes"? Usually one may think them as cycles with flat $U(1)$-bundles (or equivalently holomorphic line bundles of degree 0) and the moduli of them.

This translation is sufficient in many cases, but since the cycles may have singularities, it is more useful for our purpose to regard D-branes as stable sheaves (Narasimhan–Seshadri theorem, Kobayashi–Hitchin correspondence).

Moreover, it is rather subtle to define the moduli of supports of sheaves and the natural support map

$$\pi : M \longrightarrow S \tag{7}$$

from the moduli space M of sheaves to the moduli space S of cycles. (cf. [LeP]).

For example, let us consider the following situations: Let X be a Calabi-Yau threefold and let $C \subset X$ be a smooth irreducible curve. We can consider the following two cases:

(i) n copies of D-branes wrapped around the cycle C once.
(ii) Large single D-brane wrapped around the cycle C n-times.

Mathematically, the first one corresponds to a sheaf of rank n on C and the second one corresponds to a sheaf of rank 1 on non-reduced scheme with the same topological space C (but multiplicity along C is n). Sometimes the above two objects have the same Hilbert polynomial and hence they define points in the same moduli space and the first one may be deformed to the second one algebraically.

Therefore, in order to make the support map (7) a morphism of algebraic schemes, one has to define natural multiplicities of irreducible components of the support of the corresponding sheaf. Since there seems to be no natural way to put the scheme structure on the supports of pure sheaves when the support has codimension greater than one, we consider the supports of sheaves with multiplicities as the algebraic cycles in the total space X. Hence the moduli space S can be considered as a subset of Chow varieties of X and the support map (7) can be considered as a generalization of the morphism from Hilbert scheme to Chow varieties.

Let us first recall the necessary background in the theory of moduli spaces of sheaves (cf. [HL]). Let Z be a Noetherian scheme and $\mathcal{E}$ be a coherent sheaf on Z.

Definition 3.1. The support of $\mathcal{E}$ is the close subset $\mathrm{Supp}(\mathcal{E}) = \{z \in Z | \mathcal{E}_z \neq 0\}$. $\mathrm{Supp}(\mathcal{E})$ becomes a closed reduced subscheme of X. Its dimension is called the dimension of the sheaf $\mathcal{E}$ and is denoted by $\dim(\mathcal{E})$.

Definition 3.2. A coherent sheaf $\mathcal{E}$ on a scheme Z is pure of dimension k if $\dim_{\mathbb{C}} \mathrm{Supp}(\mathcal{F}) = k$ for any nontrivial coherent subsheaf $\mathcal{F} \subset \mathcal{E}$.

Definition 3.3. Let Z be a projective scheme over $\mathbb{C}$ and L be an ample line bundle on Z, let $\mathcal{E}$ be a coherent sheaf which is pure of dimension d on Z. Let

$$P(\mathcal{E}, m) := \chi(Z, \mathcal{E}(m)) = \sum_{i=0}^{d} \alpha_i(\mathcal{E}) \frac{m^i}{i!}$$

be the Hilbert polynomial of $\mathcal{E}$. (Here $\mathcal{E}(m) := \mathcal{E} \otimes L^m$.) Then $p(\mathcal{E}, m) := P(\mathcal{E}, m)/\alpha_d(\mathcal{E})$ is called a reduced Hilbert polynomial of $\mathcal{E}$.

Definition 3.4. (**Stability**) Let $\mathcal{E}$ be a coherent sheaf which is pure of dimension d on a projective scheme Z. $\mathcal{E}$ is stable (resp. semistable) if for any proper subsheaf $\mathcal{F}$,

$$p(\mathcal{F}, m) < p(\mathcal{E}, m), \text{ for } m >> 0.$$
$$(\text{resp.} \quad p(\mathcal{F}, m) \leq p(\mathcal{E}, m), \text{ for } m >> 0).$$

Let X be a smooth projective scheme over $\mathbb{C}$ and let $\mathcal{E}$ be a coherent sheaf on X which is of pure of dimension 1. Let $\mathrm{Supp}(\mathcal{E})$ be the support of $\mathcal{E}$ and let $Y_1, \cdots, Y_l$ be the irreducible components of $\mathrm{Supp}(\mathcal{E})$, let v_i be the generic point of Y_i. Then the stalk $\mathcal{E}_{v_i} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X, v_i}$ is an Artinian module of finite length $length(\mathcal{E}_{v_i})$. We define an algebraic cycle $s(\mathcal{E})$ by

$$s(\mathcal{E}) := \sum_{i=1}^{l} length(\mathcal{E}_{v_i}) \cdot Y_i. \tag{8}$$

Moreover the homology class of $s(\mathcal{E})$ will be denoted by $[s(\mathcal{E})] \in H_2(X, \mathbb{Z})$.

We can define the following moduli spaces of semistable sheaves by the Simpson's construction (see, for example [HL]):

Definition 3.5. Let X be a smooth projective Calabi-Yau 3-fold defined over $\mathbb{C}$ and L an ample line bundle on X. For a positive integer d, let $M_d(X)$ be the moduli space of semistable sheaves $\mathcal{E}$ of pure dimension 1 on X with Hilbert polynomial

$$P(\mathcal{E}, m) = d \cdot m + 1 \tag{9}$$

It is known that $M_d(X)$ is a projective scheme over $\mathbb{C}$ (Theorem 4.3.4 [HL]). [1]

Let $\mathcal{E} \in M_d(X)$. It is easy to see that the degree d of $\mathcal{E}$ is given by the intersection number

$$d = L \cdot s(\mathcal{E}) = [L] \cdot [s(\mathcal{E})]$$

[1]From the view point of physics, we may consider $M_\beta = M_\beta(X)$ as the moduli space of D-branes by the physical discussion that the degeneracy of BPS states should be independent of the $U(1)$ flux (the degree of sheaves).

of L with the support cycle $s(\mathcal{E})$. (Here $[L]$ denote the homology class of the divisor associated to L). Let $Chow_d(X)$ denote the Chow variety parameterizing algebraic 1-cycles Y on X with $L \cdot Y = d$. It is known that $Chow_d(X)$ is a projective scheme over $\mathbb{C}$.

In the same way as (§5, Ch. 5, [M]), we can show that the natural map

$$\begin{array}{cccc} \pi_d : & M_d(X) & \longrightarrow & Chow_d(X). \\ & \mathcal{E} & \mapsto & s(\mathcal{E}) \end{array} \tag{10}$$

becomes a morphism of projective schemes.

For a homology cycle $\beta \in H_2(X, \mathbb{Z})$ with $L \cdot \beta = d$, we can define the closed subscheme

$$M_\beta(X) := \left\{ \mathcal{E} \left| \begin{array}{l} \text{a semistable sheaf on } X\text{, pure of} \\ \text{dimension 1 with } P(\mathcal{E}, m) = dm + 1 \\ \textit{and } [s(\mathcal{E})] = \beta \end{array} \right. \right\} / \text{isom.} \subset M_d(X). \tag{11}$$

$$Chow_\beta(X) := \{\gamma \in Chow_d(X),\ [\gamma] = \beta\} \subset Chow_d(X). \tag{12}$$

Then we can define a natural morphism

$$\pi_\beta : M_\beta(X) \longrightarrow Chow_\beta(X) \tag{13}$$

by $\pi_\beta(\mathcal{E}) = s(\mathcal{E})$. Taking the normalization of $\tilde{M}_\beta(X) \longrightarrow M_\beta(X)$ and setting

$$S_\beta(X) = \text{the normalization of the image } \pi_\beta(M_\beta(X)) \ ,$$

from the univesal property of the normalization, we obtain a natural surjective morphism between normal projective varieties over $\mathbb{C}$:

$$\boxed{\pi_\beta : \tilde{M}_\beta(X) \longrightarrow S_\beta(X).} \tag{14}$$

Our proposal for definition of BPS invariants which may be consistent with the Gopakummar–Vafa conjecture 1.1 is that the moduli $\tilde{M}_\beta(X)$ of sheaves on X with the homology class of support cycle β should be the natural moduli of D-branes wrapping around a support cycle β. We state this as our conjecture.

Conjecture 3.1. *The morphism (14)*$)\pi_\beta : \tilde{M}_\beta(X) \longrightarrow S_\beta(X)$ *is the natural morphism from D-brane moduli space* $\tilde{M}_\beta(X)$ *to the moduli space* $S_\beta(X)$ *of support curves whose homology class is* β*.*

Now as suggested in [GV], applying Theorem 2.4 to the morphism (14), we obtain the following theorem and definition of BPS invariants.

Theorem 3.1. *Let* $\pi_\beta : \tilde{M}_\beta(X) \longrightarrow S_\beta(X)$ *be the projective morphism defined as in (14) and fix ample line bundles* L_1 *on* $\tilde{M}_\beta(X)$ *and* L_2 *on* $S_\beta(X)$ *respectively. Then there exists an* $(sl_2)_L \times (sl_2)_R$*-action on* $IH^*(\tilde{M}_\beta(X))$ *defined by the relative Lefschetz operator* ω_L *and by the Lefschetz operator* ω_R *of the base. The* $(sl_2)_L \times (sl_2)_R$*-action gives the decomposition of* $IH^*(\tilde{M}_\beta(X))$

$$IH^*(\tilde{M}_\beta(X)) = \bigoplus_{h \geq 0} I_h \otimes R_h(\beta) = \bigoplus_{h \geq 0} \left[(\frac{1}{2})_L \oplus 2(0)_L \right]^{\otimes h} \otimes R_h(\beta). \tag{15}$$

where we denote by $(j)_L$ *the spin-*j *representation of the relative Lefschetz* $(sl_2)_L$*-action and by* $R_h(\beta)$ *a (virtual) representation of the* $(sl_2)_R$*-action.*

Definition 3.6. By using the decomposition (15), we can define integers $n_h(\beta)$, which will be called *BPS invariant*, by the following formula:

$$\boxed{n_h(\beta) := Tr_{R_h(\beta)}(-1)^{2H_R}.} \tag{16}$$

Conjecture 3.2. *Integers $n_h(\beta)$ defined in* (16) *should be deformation invariants satisfying the conjecture* 1.1 *of Gopakumar–Vafa. In particular, $n_0(\beta)$ should be the holomorphic Casson invariants defined by Thomas* [T].

Since neither $\tilde{M}_\beta(X)$ or the morphism π_β may not be smooth in general, we cannot prove the existence of such an action on $H^*(\tilde{M}_\beta(X), \mathbb{C})$ by the usual Leray's spectral sequence. However, the "perverse" Leray spectral sequence tells us the origin of the $(sl_2)_L \times (sl_2)_R$-action on intersection cohomology $IH^*(\tilde{M}_\beta(X))$. (Note that if $\tilde{M}_\beta(X)$ is smooth, $IH^*(\tilde{M}_\beta(X)) = H^*(\tilde{M}_\beta(X))$).

4. Evidences

In Section 3, we gave a mathematical definition of BPS invariants $n_h(\beta)$ (cf. Definition 3.6). In this section, we will present several pieces of evidences supporting Conjecture 3.2 or Conjecture 1.1 by using Definition 3.6. Rigid rational curves, super-rigid elliptic curves and also some curves in rational elliptic surfaces will be considered.

4.1. Super-rigid curves in a Calabi–Yau 3-fold and conjectural local BPS invariants.

Let $C \subset X$ be a smooth irreducible curve in a Calabi-Yau 3-fold with the homology class $\beta = [C] \in H_2(X, \mathbb{Z})$. The curve $C \subset X$ is *rigid* if $H^0(C, N) = 0$ where $N = N_{C/X}$ be the normal bundle of C in X. Moreover $C \subset X$ is called *super-rigid* if, for all non-constant maps of nonsingular curves $\mu : C' \longrightarrow C$,

$$H^0(C', \mu^*(N)) = 0. \tag{17}$$

For a super-rigid curve C of genus 0 or 1, one can define the local contributions of C to the Gromov-Witten invariants $N_g(n[C]) = N_g(n\beta)$ ([P], [BP]). Denote corresponding local Gromov–Witten invariant by $N_g(n \cdot C)$. Define the *(conjectural) local BPS invariants* $n_g^{conj}(d \cdot C) \in \mathbb{Q}$ by (conjectural) Gopakummar–Vafa formula (1)

$$\sum_{g \geq 0, n \geq 0} N_g(n \cdot C) q^n \lambda^{2g-2} = \sum_{k > 0, g \geq 0, n \geq 0} n_g^{conj}(n \cdot C) \frac{1}{k} \left(2 \sin(\frac{k\lambda}{2})\right)^{2g-2} q^{kn}. \tag{18}$$

(Matching the coefficients of the two series yeilds equation determining $n_g^{conj}(n \cdot C)$ recursively in terms of $N_g(n \cdot C)$ (cf. Proposition 2.1 [BP]). Note that in the notation of [BP] one has $N_{g+h}(n \cdot C) = N_n^h(g)$ and $n_{g+h}^{conj}(n \cdot C) = n_n^h(g)$ where $g =$ genus of C).

For the case of $C = \mathbb{P}^1$, Faber and Pandharipande proved the following theorem for the generating function of local Gromov–Witten invariants $N_g(n \cdot \mathbb{P}^1)$:

Theorem 4.1. ([FP])

$$\sum_{g \geq 0, n \geq 1} N_g(n \cdot \mathbb{P}^1) q^n \lambda^{2g-2} = \sum_{k \geq 1} \frac{1}{k} \left(2 \sin(\frac{k\lambda}{2})\right)^{-2} q^k.$$

From this formula for Gromov–Witten invariants, the conjectural local BPS invariants $n_g^{conj}(n\cdot\mathbb{P}^1)$ can be given by (cf. [BP])

$$n_g^{conj}(n\cdot\mathbb{P}^1)=\begin{cases}1 & \text{for } g=0 \text{ and } n=1\\ 0 & \text{otherwise.}\end{cases}\tag{19}$$

Next let $E\subset X$ be a super-rigid elliptic curve. Pandharipande [P] showed the following:

Theorem 4.2. ([P])
Let $E\subset X$ be a super-rigid elliptic curve. One has

$$N_1(n\cdot E)=\frac{\sigma(n)}{n}=\sum_{i|n}\frac{1}{i}.$$

Moreover for all $g>1, n>0$, one has

$$N_g(n\cdot E)=0.$$

Therefore the LHS of conjectural formula (1) are given by

$$\sum_{n\geq 1}N_1(n\cdot E)q^n=-\sum_{n\geq 1}\log(1-q^n)=\sum_{n\geq 1,k\geq 1}\frac{1}{k}q^{kn}.$$

Again the conjecture (1) with Theorem 4.2 reads

$$n_g^{conj}(n\cdot E)=\begin{cases}1 & \text{for } g=1 \text{ and all } n\geq 1\\ 0 & \text{otherwise.}\end{cases}\tag{20}$$

4.2. Calculations of BPS invariants $n_g(d\cdot C)$ via moduli of sheaves.

Let X be a Calabi-Yau 3-fold and fix an ample line bundle $\mathcal{O}_X(1)$ on X with $\omega:=c_1(\mathcal{O}_X(1))$. Let $C\subset X$ be a super-rigid rational or elliptic curve of degree

$$d=\mathcal{O}_X(1)\cdot C=\int_{[C]}\omega.$$

Let $[C]\in H_2(X,\mathbb{Z})$ denote the homology class of $C\subset X$. In Definition 3.6, for a non-negative integer g, we define the BPS invariant $n_g(n\cdot[C])$ by (16) with respect to the surjective projective morphism defined in (14)

$$\pi_{n\cdot[C]}:\tilde{M}_{n\cdot[C]}\longrightarrow S_{n\cdot[C]}.\tag{21}$$

Our next aim is to calculate (local) BPS invariants $n_g(d\cdot C)$ defined by (16) and to compare $n_g(d\cdot C)$ with $n_g^{conj}(d\cdot C)$.

In order to consider the local BPS invariant $n_g(d\cdot C)$, let us consider the subset (or more explicitly the subfunctor) of $M_{n[C]}(X)$ defined by

(22)
$$M_{n\cdot C}(X):=\left\{\ \mathcal{E}\ \left|\ \begin{array}{l}\text{a stable coherent } \mathcal{O}_X\text{-sheaf, pure of dimension 1}\\ \text{with } supp(\mathcal{E})=C \text{ and } P(\mathcal{E},m)=ndm+1.\end{array}\right.\right\}/\text{isom.}$$

Note that $\mathcal{E}\in M_{n\cdot C}(X)$ implies that the cycle theoretic support $s(\mathcal{E})$ of $\mathcal{E}$ is given by $n\cdot C$.

In Proposition 4.3 and 4.4, we will show that $M_{n\cdot C}(X)$ is a smooth irreducible component of $M_{n\cdot[C]}(X)$ or the empty set. Therefore if $M_{n\cdot[C]}(X)$ is not empty set, the image of the map

$$\pi:M_{n\cdot C}(X)\longrightarrow S_{n\cdot[C]}(X).\quad \pi(\mathcal{E})=s(\mathcal{E})=n\cdot C$$

consists of just one point $\{s(\mathcal{E}) = n \cdot C\} \simeq \operatorname{Spec} \mathbb{C}$. We remark that since $M_{n \cdot C}(X)$ is smooth, we will not distingusih $M_{n \cdot C}(X)$ and its normalization $\tilde{M}_{n \cdot C}(X)$.

Let us set $M_{n \cdot C} = M_{n \cdot C}(X)$ and consider the natural morphism $\pi : M_{n \cdot C} \longrightarrow \operatorname{Spec} \mathbb{C}$. By using the decomposition of the intersection cohomology ring $IH^*(M_{n \cdot C}, \mathbb{C})$ with respect to the morphism π (cf. Theorem 3.1), we obtain the decomposition into $(sl_2)_L \times (sl_2)_R$-representations as

$$IH^*(M_{n \cdot C}, \mathbb{C}) = \oplus_{h \geq 0} I_h \otimes R_h(n \cdot C).$$

Now the local BPS invariants $n_h(d \cdot C)$ are given by (cf. Theorem 3.1)

$$n_h(n \cdot C) = Tr_{R_h(n \cdot C)}(-1)^{2H_R}.$$

Now let us consider the moduli space $M_{n \cdot C}(X)$ for a super-rigid rational or elliptic curve $C \subset X$.

Proposition 4.3. *Let X be a Calabi-Yau 3-fold, $C \subset X$ a smooth rigid rational curve (i.e., $C = \mathbb{P}^1$ and $N = N_{C/X} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$). Let ω be the class of an ample line bundle L on X and set $d = L \cdot C = \int_\beta \omega$. Let $\mathcal{E}$ be an element of the moduli space $M_{n \cdot C}(X) = M_{n \cdot \mathbb{P}^1}(X)$, that is, an isomorphism class of a stable $\mathcal{O}_X$-coherent sheaf with $supp(\mathcal{E}) = C$ and $P(\mathcal{E}, m) = ndm + 1$ for some positive integer n. Then we have*

$$n = 1, \quad \text{and} \quad \mathcal{E} = \mathcal{O}_C.$$

Moreover we have isomorphisms of schemes

$$M_{n \cdot \mathbb{P}^1} \simeq \begin{cases} \{\mathcal{O}_C\} \simeq \operatorname{Spec} \mathbb{C} \quad \textit{(one point)} & \textit{if } n = 1 \\ \emptyset & \textit{otherwise.} \end{cases} \tag{23}$$

Proof. [2] Let $\mathcal{E}$ be a stable coherent $\mathcal{O}_X$ sheaf with $supp(\mathcal{E}) = C$ and $P(\mathcal{E}, m) = ndm + 1$. Since $\dim H^0(X, \mathcal{E}) \geq P(\mathcal{E}, 0) = 1$, there exists a non-trivial section s of $\mathcal{E}$ which defines a morphism of $\mathcal{O}_X$-sheaves

$$s : \mathcal{O}_X \longrightarrow \mathcal{E}.$$

Let J be the kernel of the morphism s and I the ideal sheaf of $C \subset X$. Since $supp(\mathcal{E}) = C$ and C is an irreducible reduced curve, we see that $J \subset I$. On the other hand, the morphism s defines an injection

$$\varphi_s : \mathcal{O}_X/J \hookrightarrow \mathcal{E}. \tag{24}$$

First assume that $J = I$. Then we have an injection $\varphi_s : \mathcal{O}_C \hookrightarrow \mathcal{E}$. If $\mathcal{O}_C \subsetneqq \mathcal{E}$, the stability condition implies that

$$p(\mathcal{O}_C, m) = \frac{1}{d}\chi(\mathcal{O}_C \otimes L^m) < p(\mathcal{E}, m) = \frac{1}{nd}\chi(\mathcal{E} \otimes L^m)$$

or

$$m + \frac{1}{d} < m + \frac{1}{nd}.$$

Since $d > 0$ and $n \geq 1$, this gives the contradiction. Hence this implies that if $I = J$ $\mathcal{E} = \mathcal{O}_C$ and $n = 1$.

Next let us consider the case $J \subsetneqq I$. We have an integer $k \geq 1$ such that $I^{k+1} \subseteqq J$ and $I^k \nsubseteqq J$. Then $(J + I^k)/J$ is a non-trivial subsheaf of $\mathcal{O}_X/J$ and

[2] We thank Kota Yoshioka for suggesting us the following proof and also the proof of Proposition 4.4.

hence $(J+I^k)/J \subseteq \mathcal{E}$. By using the isomorphism $(J+I^k)/J \simeq I^k/J\cap I^k$ and $I^{k+1} \subseteq J \cap I^k$, we see that there exists a surjection

$$I^k/I^{k+1} \simeq S^k(\mathcal{O}_C(1)^{\oplus 2}) \twoheadrightarrow I^k/J\cap I^k.$$

(Note that $I/I^2 \simeq N^\vee \simeq \mathcal{O}_C(1)^{\oplus 2}$. Here $\mathcal{O}_C(1)$ is the ample generator of $\mathrm{Pic}(C) = \mathbb{Z}$.) Consequently, one obtains the non-trivial morphism

$$\mathcal{O}_C(k) \longrightarrow \mathcal{E},$$

which again contradicts the stability of $\mathcal{E}$. Thus $\mathcal{E} \simeq O_C$ and $n=1$.

We have proved that the set of $\mathbb{C}$-valued point $M_{n\cdot\mathbb{P}^1}$ consists of $\mathcal{O}_C$ if $n=1$ and is empty if $n>1$. In order to see that $M_{1\cdot\mathbb{P}^1}$ is smooth, it suffices to show that the Zariski tangent space $T_{[\mathcal{O}_C]}$ of $M_{1\cdot\mathbb{P}^1}$ at $\mathcal{O}_E$ is 0. By general theory, we have the isomorphism

$$T_{[\mathcal{O}_C]} \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C). \tag{25}$$

Hence it suffices to show that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \simeq \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{O}_C) \simeq H^1(C, \mathcal{O}_C) \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \simeq 0. \tag{26}$$

Recall the exact sequence induced by local to global spectral sequence

$$\begin{aligned} 0 \to H^1(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_C,\mathcal{O}_C)) &\to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_C,\mathcal{O}_C) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C,\mathcal{O}_C)) \\ &\to H^2(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_C,\mathcal{O}_C)) \end{aligned} \tag{27}$$

From the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

and the isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_C) \simeq \mathcal{O}_C$, we obtain the isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(I, \mathcal{O}_C) \simeq \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C).$$

Since $\mathcal{H}om_{\mathcal{O}_X}(I, \mathcal{O}_C) \simeq \mathcal{H}om_{\mathcal{O}_C}(I/I^2, \mathcal{O}_E)$, we finally obtain the isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \simeq \mathcal{H}om_{\mathcal{O}_C}(I/I^2, \mathcal{O}_C) \simeq N \tag{28}$$

where $N = N_{C/X}$ is the normal sheaf for $C \subset X$. Therefore we have

$$H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C)) \simeq H^0(\mathbb{P}^1, N) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) \simeq 0.$$

Hence the sequence (27) implies (26).

□

Let $E \subset X$ be a super-rigid elliptic curve. Then it is known that the normal bundle N of E in X is isomorphic to

$$N \simeq L \oplus L^{-1}$$

where L is a non-torsion element of the Picard group of E, the group of line bundle on E of degree 0 .

Let $M_{n,1}(E)$ denote the moduli space of stable $\mathcal{O}_E$-locally free sheaves of rank n of degree 1. Fix an ample line bundle L on X and denote it by $\omega = c_1(L)$ and set $d = L\cdot E = \int_{[E]} \omega$. We denote by $M_{n\cdot E}$ the relevant moduli space of stable $\mathcal{O}_X$

sheaves $\mathcal{E}$ with $supp(\mathcal{E}) = E$ and $P(\mathcal{E}, m) = \chi(\mathcal{E} \otimes L^m) = ndm + 1$. Let $\iota : E \hookrightarrow X$ be the natural inclusion. Then we have a natural push-forward morphism

$$\begin{array}{rcl} \iota_* : M_{n,1}(E) & \longrightarrow & M_{n \cdot E} \\ \mathcal{F} & \mapsto & \iota_*(\mathcal{F}). \end{array} \tag{29}$$

Atiyah ([Theorem 7, [A]]) showed that $M_{n,1}(E)$ is isomorphic to $M_{1,1}(E)$ via the map $\mathcal{E} \mapsto \det \mathcal{E}$. Hence fixing a line bundle A of degree 1 and identifying $M_{1,1}(E)$ with $M_{1,0}(E) \simeq E$, we have an isomorphism

$$M_{n,1}(E) \simeq E.$$

Proposition 4.4. *Notation being as above, the map ι_* induces the isomorphism of schemes:*

$$\iota_* : M_{n,1}(E) \simeq M_{n \cdot E}.$$

Hence

$$M_{n \cdot E} \simeq E. \tag{30}$$

and $M_{n \cdot E}$ is a smooth irreducible component of $M_{n \cdot [E]}(X)$.

Proof. Since ι_* is an injective morphism, we only have to prove the surjectivity of ι_*. Let $\mathcal{E}$ be a stable $\mathcal{O}_X$-coherent sheaf of pure dimension 1 with $supp(\mathcal{E}) = E$ and $P(\mathcal{E}, m) = \chi(\mathcal{E} \otimes L^m) = ndm + 1$. Then we have to prove that there exists a $\mathcal{O}_C$-coherent sheaf $\mathcal{F} \in M_{n,1}(E)$ such that

$$\mathcal{E} = \iota_*(\mathcal{F}). \tag{31}$$

We prove the claim above (31) by induction with respect to n. For $n = 1$, let $\mathcal{E}$ be a $\mathcal{O}_X$ coherent sheaf in $M_{n \cdot E}$. Since $\dim H^0(X, \mathcal{E}) \geq \chi(\mathcal{E}) = P(\mathcal{E}, 0) = 1$, we have a non-trivial section $s \in H^0(X, \mathcal{E})$ which defines a non-trivial homomorphism

$$s : \mathcal{O}_X \longrightarrow \mathcal{E}.$$

Then setting $J = \ker s$, we have an inclusion of $\mathcal{O}_X$-coherent sheaves:

$$\varphi_s : \mathcal{O}_X/J \hookrightarrow \mathcal{E}.$$

Let $I = I_E$ be the ideal sheaf of $E \subset X$. In the same argument as in the proof of Proposition 4.3, one can shows that $J = I$ or for some $k \geq 1$ there exists a surjection

$$I^k/I^{k+1} = S^k(L \oplus L^{-1}) \twoheadrightarrow \mathcal{O}_X/J.$$

In each case, one can obtain an exact sequence of $\mathcal{O}_X$-sheaves

$$0 \to \iota_*(\mathcal{G}) \to \mathcal{E} \to \mathcal{E}/\iota_*(\mathcal{G}) \to 0 \tag{32}$$

where $\mathcal{G}$ is a line bundle on E of degree 0. The additivity of Hilbert polynomial $P(\mathcal{E}, m) = P(\mathcal{G}, m) + P(\mathcal{E}/\iota_*(\mathcal{G}), m)$ with $P(\mathcal{G}, m) = dm$ implies that $P(\mathcal{E}/\iota_*(\mathcal{G}), m) = 1$. Hence there exists a closed point $x \in E$ such that

$$\mathcal{E}/\iota_*(\mathcal{G}) \simeq \mathcal{O}_{X,x}/m_x \simeq \mathbb{C}_x.$$

(Here m_x is the maximal ideal sheaf of x.) Therefore one obtains the exact sequence of $\mathcal{O}_X$–sheaves

$$0 \to \iota_*(\mathcal{G}) \to \mathcal{E} \to \mathbb{C}_x \to 0 \tag{33}$$

Next we show that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{C}_x, \iota_*(\mathcal{G})) \simeq \mathrm{Ext}^1_{\mathcal{O}_E}(\mathbb{C}_x, \mathcal{G}). \tag{34}$$

If one can show (34), we can conclude that the extension $\mathcal{E}$ of $\mathcal{O}_X$-coherent sheaves can be written as $\mathcal{E} = \iota_*(\mathcal{F}_1)$ where $\mathcal{F}_1$ is an $\mathcal{O}_E$-coherent sheaf on E with an extension of $\mathcal{G}$ and $\mathbb{C}_x$. It is easy to see that $\mathcal{F}_1 \in M_{1,1}(E)$ and this proves our claim (31) for $n = 1$.

Next let us show the assertion (34).

We consider the following exact sequence followed from the local to global spectral sequence.

(35)

$$\begin{aligned} 0 \to H^1(\mathcal{H}om_{\mathcal{O}_X}(\mathbb{C}_x, \iota_*(\mathcal{G}))) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{C}_x, \iota_*(\mathcal{G})) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, \iota_*(\mathcal{G})) \\ \to H^2(\mathcal{H}om_{\mathcal{O}_X}(\mathbb{C}_x, \iota_*(\mathcal{G}))) \end{aligned}$$

One can easily see that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathbb{C}_x, \iota_*(\mathcal{G})) \simeq \mathcal{H}om_{\mathcal{O}_E}(\mathbb{C}_x, \mathcal{G}) = 0$$
$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathbb{C}_x, \iota_*(\mathcal{G})) \simeq \mathcal{E}xt^1_{\mathcal{O}_E}(\mathbb{C}_x, \mathcal{G}) \simeq \mathbb{C}_x$$

Then this implies that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathbb{C}_x, \iota_*(\mathcal{G})) \simeq H^0(E, \mathcal{E}xt^1_{\mathcal{O}_E}(\mathbb{C}_x, \mathcal{G})) \simeq \mathrm{Ext}^1_{\mathcal{O}_E}(\mathbb{C}_x, \mathcal{G}) \simeq H^0(E, \mathbb{C}_x) \simeq \mathbb{C}.$$

which proves the claim (34) and completes the proof for $n = 1$.

Next we assume that the assertion (31) is true for $\mathcal{E}' \in M_{k \cdot E}$ $1 \leq k \leq n-1$. Take $\mathcal{E} \in M_{n \cdot E}$ with $n \geq 2$. Then by the same argument there exists an invertible sheaf $\mathcal{G}$ of degree 0 on E and an injective homomorphism $\iota_*(\mathcal{G}) \subset \mathcal{E}$. Set $\mathcal{E}' = \mathcal{E}/\iota_*(\mathcal{G})$. Since $P(\mathcal{G}, m) = md$ and $P(\mathcal{E}', m) = ndm+1$, we see that $P(\mathcal{E}', m) = (n-1)dm+1$. Moreover again from the stability of $\mathcal{E}$ it is easy to see that $\mathcal{E}'$ is a stable $\mathcal{O}_X$-coherent sheaf of pure dimension 1 with support E if $n \geq 2$. Therefore $\mathcal{E}' \in M_{(n-1) \cdot E}$. By the assumption of induction, we have $\mathcal{F}' \in M_{n-1,1}(E)$ such that $\mathcal{E}' = \iota_*(\mathcal{F}')$.

Next we see that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}'), \iota_*(\mathcal{G})) \simeq \mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{F}', \mathcal{G}). \tag{36}$$

If (36) is true, the extension $\mathcal{E}$ of $\iota_*(\mathcal{F}')$ and $\iota_*(\mathcal{G})$ can be written as in (31) and this completes the proof.

In order to prove (36), we again use the following exact sequence

(37)

$$\begin{aligned} 0 \to H^1(\mathcal{H}om_{\mathcal{O}_X}(\iota_*(\mathcal{F}'), \iota_*(\mathcal{G}))) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}'), \iota_*(\mathcal{G})) \to H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}'), \iota_*(\mathcal{G})) \\ \to H^2(\mathcal{H}om_{\mathcal{O}_X}(\iota_*(\mathcal{F}'), \iota_*(\mathcal{G}))). \end{aligned}$$

Again we easily see that

$$\mathcal{H}om_{\mathcal{O}_X}(\iota_*(\mathcal{F}'), \iota_*(\mathcal{G})) \simeq \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}', \mathcal{G}). \tag{38}$$

From the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0,$$

and the isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_E, \mathcal{O}_E) \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_E)$, we obtain the isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(I, \mathcal{O}_E) \simeq \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_E, \mathcal{O}_E).$$

Since $\mathcal{H}om_{\mathcal{O}_X}(I, \mathcal{O}_E) \simeq \mathcal{H}om_{\mathcal{O}_E}(I/I^2, \mathcal{O}_E)$, we finally obtain the isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_E, \mathcal{O}_E) \simeq \mathcal{H}om_{\mathcal{O}_E}(I/I^2, \mathcal{O}_E) \simeq N \tag{39}$$

where $N = N_{E/X}$ is the normal sheaf for $E \subset X$.

Moreover we have the isomorphism

(40)
$$\mathcal{E}xt^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}'), \iota_*(\mathcal{G})) \simeq \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_E, \mathcal{O}_E) \otimes \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}', \mathcal{G}) \simeq N \otimes \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}', \mathcal{G}).$$

(Note that the first isomorphism follows from the locally freeness of $\mathcal{F}'$ and $\mathcal{G}$.) Since $\mathcal{F}'$ is a stable $\mathcal{O}_E$-sheaf with $\deg \mathcal{F}' = 1 > 0$ and $\deg \mathcal{G} \otimes N = 0$, we can conclude that

$$H^0(E, N \otimes \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}', \mathcal{G})) = Hom(\mathcal{F}', \mathcal{G} \otimes N) = \{0\}. \tag{41}$$

From (38), (40) and (41), the sequence (37) gives the isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}'), \iota_*(\mathcal{G})) \simeq H^1(E, \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}', \mathcal{G})) \simeq \mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{F}', \mathcal{G}) \tag{42}$$

as required in (36).

We have proved that $i_* : M_{n,1}(E) \longrightarrow M_{n \cdot E}$ gives the isomorphism of set of $\mathbb{C}$-valued points. Since $M_{n,1}(E) \simeq E$ is smooth one dimensional scheme, if we prove the Zariski tangent space at each point of $M_{n \cdot E}$ is one dimensional, i_* gives an isomorphism of schemes. For each element $\mathcal{E} \in M_{n \cdot E}$, we have a locally free $\mathcal{O}_E$-sheaf $\mathcal{F}$ of rank n of degree 1 such that $\mathcal{E} = \iota_*(\mathcal{F})$. The Zariski tangent space $T_{[\mathcal{E}]}$ at $\mathcal{E}$ in $M_{n \cdot E}$ is given by

$$T_{[\mathcal{E}]} = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) = \mathrm{Ext}^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}), \iota_*(\mathcal{F})).$$

We shall prove

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}), \iota_*(\mathcal{F})) \simeq \mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{F}, \mathcal{F}) \simeq \mathbb{C}. \tag{43}$$

(The last isomorphism follows from the fact that $M_{n,1}(E)$ is smooth and one dimensional.) From the similar exact sequence as (37), if we show

$$H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}), \iota_*(\mathcal{F}))) = 0, \tag{44}$$

we have the isomorphism

$$\begin{aligned} &\mathrm{Ext}^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}), \iota_*(\mathcal{F})) \simeq H^1(X, \mathcal{H}om_{\mathcal{O}_X}(\iota_*(\mathcal{F}), \iota_*(\mathcal{F}))) \simeq \\ &H^1(E, \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}, \mathcal{F})) \simeq \mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{F}, \mathcal{F}). \end{aligned} \tag{45}$$

as required. For (44), we again recall the isomorphism (cf. (40))

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}), \iota_*(\mathcal{F})) \simeq \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_E, \mathcal{O}_E) \otimes \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}, \mathcal{F}) \simeq N \otimes \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}, \mathcal{F}),$$

which induces the isomorphism

$$H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\iota_*(\mathcal{F}), \iota_*(\mathcal{F}))) \simeq H^0(E, N \otimes \mathcal{H}om_{\mathcal{O}_E}(\mathcal{F}, \mathcal{F})).$$

Since $N = L \oplus L^{-1}$, the non-vanishing of (44) implies the existence of a non-trivial isomorphism of $\mathcal{O}_E$-sheaf

$$\phi^+ : \mathcal{F} \longrightarrow \mathcal{F} \otimes L \quad \text{or} \quad \phi^- : \mathcal{F} \longrightarrow \mathcal{F} \otimes L^{-1}.$$

Since $\deg L = 0$, the stability of $\mathcal{F}$ again implies that $\phi^\pm$ is an isomorphism if it is non-trivial. Taking the highest power of $\phi^\pm$, we obtain the isomorphism

$$\wedge^n \phi^\pm : \wedge^n \mathcal{F} \xrightarrow{\simeq} (\wedge^n \mathcal{F}) \otimes L^{\pm n}.$$

This implies that $L^{\pm n} \simeq \mathcal{O}_E$, which contradicts to the fact that L is a non-torsion element of $\mathrm{Pic}(E)$. Thus we have the vanishing (44), and hence (43).

□

By Proposition 4.3 and 4.4, we can calculate our local BPS invariants as follows. Note that our local BPS invariants coincide with the conjectural local BPS invariants predicted by Conjecture 1.1 and the calculations of Gromov–Witten invariants (cf. Theorem 4.1 and 4.2).

Proposition 4.5. *Let $C \simeq \mathbb{P}^1 \subset X$ be a rigid rational curve in a smooth Calabi-Yau 3-fold. Then our local BPS invariants for C are given as follows:*

$$n_g(n \cdot \mathbb{P}^1) = \begin{cases} 1 & \textit{for } g = 0 \textit{ and } n = 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{46}$$

Proposition 4.6. *Let $E \simeq \mathbb{P}^1 \subset X$ be a super-rigid elliptic curve in a smooth Calabi-Yau 3-fold. Then our local BPS invariants for C are given as follows:*

$$n_g(n \cdot E) = \begin{cases} 1 & \textit{for } g = 1 \textit{ and all } n \geq 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{47}$$

Proof of Proposition 4.5 and 4.6.

For the case of $C = \mathbb{P}^1$, $M_{1 \cdot \mathbb{P}^1} \simeq \operatorname{Spec} \mathbb{C}$ for $n = 1$ and $M_{n \cdot \mathbb{P}^1}$ is empty if $n > 1$. Then the $sl_2 \times sl_2$ decomposition of

$$IH^*(M_{1 \cdot \mathbb{P}^1}) = (0)_L \otimes (0)_R \simeq (I_1)^0 \otimes (0).$$

Hence $R_0(1 \cdot \mathbb{P}^1) = (0)_R$ and $R_h(1 \cdot \mathbb{P}^1) = \emptyset$ for $h > 0$. Therefore we have

$$n_0(\mathbb{P}^1) = Tr_{R_0(1 \cdot \mathbb{P}^1)}(-1)^{2H_R} = (-1)^0 = 1.$$

Moreover $R_h(n \cdot \mathbb{P}^1) = \emptyset$ unless $n = 1$ and $h = 0$. Hence we have the assertion for $C = \mathbb{P}^1$.

Next let us consider the case of a super-rigid elliptic curve $E \subset X$. In this case, for all integer $n \geq 1$, $M_{n \cdot E}(X)$ is always isomorphic to E. Moreover the image of the support map

$$\pi : M_{n \cdot E}(X) \longrightarrow Chow_{[n \cdot E]}$$

is just one point $\{n \cdot E\}$. Hence we can identify it with the structural morphism $\pi : M_{n \cdot E} \longrightarrow \operatorname{Spec} \mathbb{C}$. Moreover from Proposition 4.4, we have

$$M_{n \cdot E} \simeq E.$$

Hence the $sl_2 \times sl_2$ decomposition for the morphism π is given by

$$IH^*(M_{n \cdot E}) \simeq H^*(E, \mathbb{C}) \simeq I_1 \otimes (0)_R.$$

Therefore we have $R_1(n \cdot E) \simeq (0)_R$ and $R_h(n \cdot E)$ is empty for $h \neq 1$. Therefore

$$n_1(n \cdot E) = 1$$

and

$$n_h(n \cdot E) = 0 \quad \text{otherwise.}$$

$\square$

4.3. **Rational Elliptic Surfaces in Calabi–Yau 3-folds.**

Next, we shall calculate our local BPS invariants for special homology class of a rational elliptic surface S in a smooth Calabi–Yau 3-fold X based on our mathematical Definition 3.6. We should remark that in [HST] by a physical argument (holomorphic anomaly equation) and explicit calculations of some part of Gromov–Witten invariants we calculated a part of the left hand side of (1). Then using Jacobi triple product formula, we found that the corresponding right hand side of the conjectural formula (1) can be obtained from the $sl_2 \times sl_2$ spin version of Göttsche's formula for the Poincaré polynomial of the Hilbert schemes $S^{[g]}$ of g-points on a rational elliptic surface S.

In what follows, we shall prove that the above physical arguments can be justified in our mathematical definition of BPS invariants in (3.6) using the moduli of sheaves.

Let $f : S \longrightarrow \mathbb{P}^1$ be a rational elliptic surface with a section $\sigma : \mathbb{P}^1 \longrightarrow S$, and $C = \sigma(\mathbb{P}^1)$ the image of the section. (Note that C is a smooth rational curve with $C^2 = -1$ and hence C is rigid in S.) We denote by F the divisor class of a fiber of f. We consider the case when S lies in a smooth Calabi-Yau 3-fold X as a divisor. (For example, see [HSS], [S]). For a non-negative integer g, consider the homology class

$$\beta_g = [C + gF] \in H_2(X, \mathbb{Z}). \tag{48}$$

In order to consider the BPS invariant $n_h(\beta_g) = n_h([C + gF])$, we have to take account of all curves which are homological equivalent to β_g. Instead, we will consider the local BPS invariants $n_h(C + gF)$ which are defined by the moduli of sheaves whose support are curves in the rational surface S. Since $C \subset S \subset X$ is a rigid rational curve both in X, this makes sense.

More precisely, let us consider the linear system $|C + gF|$ which consists of effective divisors linearly equivalent to the divisor $C + gF$ of S. Then it is easy to see that $|C + gF| \simeq C + |gF| \simeq \mathbb{P}^g$.

Let us fix an ample line bundle L on X. For simplicity, we assume that L can be chosen as[3]

$$L_{|S} = C + kF, \quad k >> 0. \tag{49}$$

and set $d = L \cdot (C + gF)$.

Then the moduli space which we should consider is given by

$$M_{C+gF} := \left\{ \ \mathcal{E} \ \left| \begin{array}{l} \text{a semistable sheaf on } X, \text{ pure of} \\ \text{dimension 1 with } P(\mathcal{E}, m) = dm + 1 \\ and\ s(\mathcal{E}) \in |C + gF| \end{array} \right. \right\} / \text{isom}. \tag{50}$$

Then we obtain the natural support map

$$\begin{array}{rcc} \pi : M_{C+gF} & \longrightarrow & |C + gF| \\ \mathcal{E} & \mapsto & \pi(\mathcal{E}) := s(\mathcal{E}). \end{array} \tag{51}$$

[3]We do not know that this assumption is always true, but we know examples of $S \subset X$ which satisfies the condition (49). See [HSS].

The local BPS invariant $n_h(C+gF)$ is defined by using the relative Lefschetz decomposition

$$IH^*(M_{C+gF}, \mathbb{C}) = \bigoplus_{h \geq 0} I_h \otimes R_h(C+gF)$$

as

$$n_h(C+gF) := Tr_{R_h(C+gF)}(-1)^{2H_R}. \tag{52}$$

The following is our main theorem (cf. [HST]).

Theorem 4.7.

$$\sum_{g\geq 0, h\geq 0} n_h(C+gF)\left(2\sin\frac{\lambda}{2}\right)^{2h-2} q^g$$
$$= \frac{1}{(e^{-\sqrt{-1}\lambda/2} - e^{\sqrt{-1}\lambda/2})^2} \prod_{n\geq 1} \frac{1}{(1-e^{-\sqrt{-1}\lambda}q^n)^2(1-e^{\sqrt{-1}\lambda}q^n)^2(1-q^n)^8}.$$

Fix an ample divisor L as in (49). Let $M^{ss}(r, c_1, \chi)$ be the moduli space of semistable sheaves $\mathcal{E}$ on S with rank $r(\mathcal{E}) = r$, $c_1(\mathcal{E}) = c_1$ and $\chi(\mathcal{E}) = \chi$ with respect to L. Moreover we assume that all closed fibers of rational elliptic surface $f: S \longrightarrow \mathbb{P}^1$ are integral. The following proposition is due to Kota Yoshioka (cf. [Y1]).

Proposition 4.8. *Under the assumption as above, we have the following.*

(i) *We have an isomorphism* $M^{ss}(0, C+gF, 1) \simeq M_{C+gF}$.
(ii) *Fourier–Mukai transform induces an isomorphism*

$$M^{ss}(0, C+gF, 1) \simeq M^{ss}(1, 0, 1-g) \simeq S^{[g]},$$

where $S^{[g]} := Hilb^g(S)$ *is the Hilbert scheme of* g *points on* S. *Hence we have an isomorphism*

$$M_{C+gF} \simeq S^{[g]}.$$

(iii) *Moreover the support map* $\pi: M_{C+gF} \longrightarrow |C+gF| \simeq \mathbb{P}^g$ *can be identified with the natural map*

$$\pi: S^{[g]} \longrightarrow Sym^g(\mathbb{P}^1) \simeq \mathbb{P}^g \tag{53}$$

given by the composite map of Hilbert–Chow morphism $S^{[g]} \longrightarrow Sym^g(S)$ *and the natural map* $Sym^g(f): Sym^g(S) \longrightarrow Sym^g(\mathbb{P}^1)$.

From Proposition 4.8, in order to show Theorem 4.7, we only have to determine the relative Lefschetz decomposition of $IH^*(S^{[g]}, \mathbb{C}) \simeq H^*(S^{[g]}, \mathbb{C})$ with respect to the natural morphism

$$\pi: S^{[g]} \longrightarrow \mathbb{P}^g. \tag{54}$$

We recall here well-known Göttsche's formula for the Poincaré polynomial of the Hilbert schemes $Y^{[g]} = Hilb^g(Y)$ for a smooth projective surface Y. Let $b_i(Z)$ be the i-th Betti number of a smooth projective variety Z. Let us define the shifted Poincaré polynomial by

$$P_t(Z) = t^{-\dim Z} \cdot \left(\sum_{k\geq 0} b_k(Z) t^k\right),$$

where $\dim Z$ is the complex dimension of Z. Note that $P_{t^{-1}}(Z) = P_t(Z)$. The following formula is proved by Göttsche [G1]. For a proof see [G1], [G-S] or [N].

Theorem 4.9. *For a smooth projective surface Y, the generating function of shifted Poincaré polynomials of the Hilbert schemes $Y^{[g]}$ of g-points is given by the formula*

$$\sum_{g \geq 0} P_t(Y^{[g]}) q^g = \prod_{n \geq 1} \frac{(1 + t^{-1} q^n)^{b_1(Y)} (1 + t\ q^n)^{b_3(Y)}}{(1 - t^{-2} q^n)^{b_0(Y)} (1 - t^2 q^n)^{b_4(Y)} (1 - q^n)^{b_2(Y)}}. \tag{55}$$

For a rational elliptic surface $f : S \longrightarrow \mathbb{P}^1$, it is easy to see that $b_1(S) = b_3(S) = 0$ and $b_2(S) = 10$. Therefore the formula (55) is reduced to

$$\sum_{g \geq 0} P_t(S^{[g]}) q^g = \prod_{n \geq 1} \frac{1}{(1 - t^{-2} q^n)(1 - t^2 q^n)(1 - q^n)^{10}}.$$

Recall that an ample line bundle L' on $S^{[g]}$ defines the usual Lefschetz sl_2 action on the cohomology ring $H^*(S^{[g]}, \mathbb{C})$. Denote by H the usual weight operator for the sl_2 action. Then the Poincaré polynomial can be expressed as the character of the representation:

$$P_t(S^{[g]}) = Tr_{H^*(S^{[g]}, \mathbb{C})} (-1)^{2H} t^{2H}.$$

The relative Lefschetz action on $H^*(S^{[g]}, \mathbb{C})$ with respect to the morphism $\pi : S^{[g]} \longrightarrow \mathbb{P}^g$ (cf. (54)) determines the representation of $(sl_2)_L \times (sl_2)_R$ and let us consider its character

$$P_{t_L, t_R}(S^{[g]}) = Tr_{H^*(S^{[g]}, \mathbb{C})} (-1)^{2H_L + 2H_R} t_L^{2H_L} t_R^{2H_R}.$$

We have the following generalization of Göttsche's formula.

Theorem 4.10. *Let $f : S \longrightarrow \mathbb{P}^1$ be a rational elliptic surface. Assume that all closed fibers are integral. Then we have*

$$\sum_{g \geq 0} P_{t_L, t_R}(S^{[g]}) q^g = \prod_{n \geq 1} \frac{1}{(1 - (t_L t_R)^{-1} q^n)(1 - (t_L t_R) q^n)} \times \prod_{n \geq 1} \frac{1}{(1 - (t_L t_R^{-1}) q^n)(1 - (t_L^{-1} t_R) q^n)(1 - q^n)^8}. \tag{56}$$

For a proof of Theorem 4.10, we need the following

Proposition 4.11. *Let $f : S \longrightarrow \mathbb{P}^1$ be a rational elliptic surface. Assume that all closed fibers are integral*[4].

(i) *Set $E^{r,s} := {}^p H^r(\mathbb{P}^1, {}^p R^s f_* \mathbb{C}_S[2])$. Then we have isomorphisms:*

$$\begin{array}{|ccc} E^{1,-1} & E^{1,0} & E^{1,1} \\ E^{0,-1} & E^{0,0} & E^{0,1} \\ E^{-1,-1} & E^{-1,0} & E^{-1,1} \\ \hline \end{array} \quad = \quad \begin{array}{|ccc} \mathbb{C} & 0 & \mathbb{C} \\ 0 & \mathbb{C}^8 & 0 \\ \mathbb{C} & 0 & \mathbb{C} \\ \hline \end{array}.$$

Moreover the $sl_2 \times sl_2$ decomposition of cohomology ring $H^*(S, \mathbb{C}[2])$ can be obtained as

$$H^*(S, \mathbb{C}[2]) \simeq (\frac{1}{2})_L \otimes (\frac{1}{2})_R + 8 \cdot (0)_L \otimes (0)_R.$$

Therefore we have

$$P_{t_L, t_R}(S) = (t_L t_R)^{-1} + t_L (t_R)^{-1} + (t_L)^{-1} t_R + t_L t_R + 8.$$

[4] This condition is just for simplicity. Even if we do not assume that all closed fibers are integral, we can obtain the statement. See remark below.

(ii) *Let $Sym^n(f) : Sym^n(S) \longrightarrow Sym^n(\mathbb{P}^1) \simeq \mathbb{P}^n$ be the natural morphism between n-th symmetric products of S and $\mathbb{P}^1$. Let $P_{t_L,t_R}(Sym^n(S))$ denote the character of $sl_2 \times sl_2$ representation of $IH^*(Sym^n(S), \mathbb{C}[2n]) \simeq H^*(Sym^n(S), \mathbb{C}[2n])$*[5]*. Then we have the following formula:*

$$\sum_{n\geq 0} P_{t_L,t_R}(Sym^n(S))q^n = \frac{1}{(1-(t_Lt_R)^{-1}q)(1-(t_Lt_R)q)} \times \frac{1}{(1-(t_Lt_R^{-1})q)(1-(t_L^{-1}t_R)q)(1-q)^8}. \tag{57}$$

Proof of Proposition 4.11. Since all fibers of f are integral, we have

$$ {}^pR^{-1}f_*\mathbb{C}_S[2] \simeq \mathbb{C}_{\mathbb{P}^1}[1], \quad {}^pR^1f_*\mathbb{C}_S[2] \simeq \mathbb{C}_{\mathbb{P}^1}[1]. \tag{58}$$

Hence, we see that $E^{r,s} \simeq \mathbb{C}$ for $(r, s) = (\pm 1, \pm 1)$. Moreover since the Leray spectral sequence degenerates and $\dim E^{0,0} + \dim E^{-1,1} + \dim E^{1,-1} = \dim \mathbb{H}^0(S, \mathbb{C}_S[2]) = \dim H^2(S, \mathbb{C}) = 10$, we have $\dim E^{0,0} = 8$. (Note that $E^{-1,1} \simeq H^0(\mathbb{P}^1, R^2f_*\mathbb{C}) \simeq \mathbb{C}$ is the space of the class of a fiber of f and $E^{1,-1} = H^2(\mathbb{P}^1, R^0f_*\mathbb{C}) \simeq \mathbb{C}$ is the space of the class of the section). Therefore, the relative Lefschetz action and hence $sl_2 \times sl_2$ decomposition of $H^*(S, \mathbb{C}[2])$ with respect to $f : S \longrightarrow \mathbb{P}^1$ are determined by this Leray spectral sequence. For the $sl_2 \times sl_2$-decomposition of the intersection cohomology group $IH^*(Sym^n(S), \mathbb{C}[2n]) \simeq H^*(Sym^n(S), \mathbb{C}[2n])$ of the symmetric power $Sym^n(S)$, we recall that $IH^*(Sym^n(S), \mathbb{C}) \simeq H^*(Sym^n(S), \mathbb{C})$ is $\mathfrak{S}_n$-invariant part of $H^*(S^n, \mathbb{C}) = (H^*(S, \mathbb{C}))^{\otimes n}$. Moreover the $sl_2 \times sl_2$-decomposition of $(H^*(S, \mathbb{C}))^{\otimes n}$ is nothing but the one induced by $H^*(S, \mathbb{C})$. Hence we have the formula (57). □

Remark. In the proof above, we do not have to assume that all fibers of $f : S \longrightarrow \mathbb{P}^1$ are integral. First we allways have ${}^pR^{-1}f_*\mathbb{C}_S[2] \simeq R^0f_*\mathbb{C}_S \simeq \mathbb{C}_{\mathbb{P}^1}$ (up to a shift) by the connectivity of fibers. Then by the relative Lefschetz theorem we have the isomorphism ${}^pR^{-1}f_*\mathbb{C}_S[2] \simeq {}^pR^1f_*\mathbb{C}_S[2]$. This shows that the isomorphism (58) is still true, and so are the all assertions of Proposition 4.11. Note that, if some fibers of f are reducible, the usual higher direct image sheaf $R^2f_*\mathbb{C}_S$ becomes a direct sum of $\mathbb{C}_{\mathbb{P}^1}$ and skyscraper sheaves supported on points corresponding to reducible fibers. Hence for usual higher direct image sheaves, the relative hard Lefschetz theorem $R^0f_*\mathbb{C}_S \simeq R^2f_*\mathbb{C}_S$ does not holds if some fibers of f are reducible, contray to the case of perverse higher direct image sheaves.

Proof of Theorem 4.10.

In [6.2, [N]], Nakajima gives a proof of Göttsche's formula using the perverse sheaf and the fact that the Hilbert–Chow morphism $S^{[n]} \longrightarrow Sym^n(S)$ is semismall. (See also [G-S].)

Consider the Hilbert–Chow morphism $S^{[n]} \to \mathrm{Sym}^n(S)$ and stratification of

$$\mathrm{Sym}^n(S) = \bigcup_{\nu} \mathrm{Sym}^n_{\nu} S,$$

[5]Since $Sym^n(S)$ has only quotient singularities, the intersection cohomology groups are isomorphic to the ordianry cohomology groups

$$\mathrm{Sym}^n_\nu S := \{\sum_{i=1}^k \nu_i[x_i] \in \mathrm{Sym}^g S \mid x_i \neq x_j \text{ for all } i\},\ \nu_1 \geq \nu_2 \geq \cdots \geq \nu_k,$$

defined by the partitions ν of n. For a partition ν, define $\alpha_i := \sharp\{l|\nu_l = i\}$. Then we can define

$$\mathrm{Sym}^\nu(S) := \mathrm{Sym}^{\alpha_1}S \times \cdots \times \mathrm{Sym}^{\alpha_n}S.$$

Let $2l(\nu)$ be the complex dimension of $\mathrm{Sym}^\nu(S)$. Then one can see (cf. [(6.13), [N]]):

$$H^{i+2n}(S^{[n]}, \mathbb{C}) = \oplus_\nu H^{i+2l(\nu)}(\mathrm{Sym}^\nu(S), \mathbb{C}). \tag{59}$$

Since $sl_2 \times sl_2$-decompositions of both side of (59) are compatible with each other, by the same argument as in [6.2, [N]] together with the formula (57), we can show the formula (56).

□

Proof of Theorem 4.7.

Now, we shall prove Theorem 4.7.

Let

$$H^*(S^{[g]}, \mathbb{C}) := \oplus_{h\geq 0} I_h \otimes R_h(C + gF)$$

be the $sl_2 \times sl_2$-decomposition of $H^*(S^{[g]}, \mathbb{C})$. In the left action, the character of $I_1 = (\frac{1}{2})_L + 2(0)_L$ is given by $(-t_L - t_L^{-1} + 2)$ and hence the character of $I_h = (I_1)^{\otimes h}$ is given by $(-t_L - t_L^{-1} + 2)^h$.

Since the character of this decomposition is given by $P_{t_L t_R}(S^{[g]})$ in (56),

$$\begin{aligned} P_{t_L,t_R}(S^{[g]})_{|t_R=1} &= \textstyle\sum_{h\geq 0} Tr_{R_h(C+gF)}(-1)^{2H_R}(-t_L - t_L^{-1} + 2)^h \\ &= \textstyle\sum_{h\geq 0} n_h(C + gF)(-t_L - t_L^{-1} + 2)^h \end{aligned} \tag{60}$$

Then setting $t_L = e^{\sqrt{-1}\lambda}$, we have

$$(-t_L - t_L^{-1} + 2) = -e^{\sqrt{-1}\lambda} - e^{-\sqrt{-1}\lambda} + 2 = \left(2\sin(\frac{\lambda}{2})\right)^2.$$

From Theorem 4.10, we obtain

$$\begin{aligned} &\sum_{g\geq 0, h\geq 0} n_h(C + gF)\left(2\sin\frac{\lambda}{2}\right)^{2h-2} q^g \\ =&\frac{1}{(e^{-\sqrt{-1}\lambda/2} - e^{\sqrt{-1}\lambda/2})^2} \sum_{g\geq 0} P_{t_L = e^{\sqrt{-1}\lambda}, t_R=1}(S^{[g]})q^g \\ =&\frac{1}{(e^{-\sqrt{-1}\lambda/2} - e^{\sqrt{-1}\lambda/2})^2} \prod_{n\geq 1} \frac{1}{(1 - e^{-\sqrt{-1}\lambda}q^n)^2(1 - e^{\sqrt{-1}\lambda}q^n)^2(1 - q^n)^8}. \end{aligned}$$

□

Remark. It is clear from the proof that Theorem 4.7 also holds for other elliptic surfaces in a Calabi–Yau manifold. In particular, if we consider an elliptic K3 surface, we have the same results as that of Kawai–Yoshioka [KY]. They considered the Abel–Jacobi map and counted the number of BPS states from D0-D2 system. They defined the moduli space of D0-D2 system as relative Hilbert schemes of d-points $\mathcal{C}_h^{[d]} \to |C_h| \simeq \mathbb{P}^h$, $d \geq 0$ where $C_h \subset K3$ is a curve of genus h and $\mathcal{C}_h^{[1]} \simeq \mathcal{C}_h$

is the universal family over $|C_h|$.

$$\sum_{h,d}\chi(\mathcal{C}_h^{[d]})q^h y^{d+1-h} = \frac{1}{(y^{1/2}-y^{-1/2})^2}\prod_{n\geq 1}\frac{1}{(1-yq^n)^2(1-y^{-1}q^n)^2(1-q^n)^{20}}.$$

On the other hand, we use the relative Lefschetz action on the relative Jacobian and counted the spin contents of BPS states from M2 brane. The coincidence of these results is very natural since the original physical theory is equivalent.

If we allow some physical arguments (holomorphic anomaly equation), we have the nontrivial evidence of Gopakumar–Vafa conjecture. Let us write the generating functions of Gromov–Witten invariants as

$$Z_{g;n}(q) := \sum_d N_{g,d;n}q^d,\ \ N_{g,d;n} := \sum_{(\beta,\sigma)=d,(\beta,F)=n} N_g(\beta),\ \ n\geq 1,$$

where $N_g(\beta)\in\mathbb{Q}$ are genus g Gromov–Witten invariants for $\beta\in H_2(S,\mathbb{Z})$ defined by

$$N_g(\beta) := \int_{[\overline{\mathcal{M}}_{g,0}(S,\beta)]^{virt}} c_{\mathrm{top}}(R^1\pi_*\mu^* N_{S/X}).$$

Explicitly, $Z_{0;1}(q)$ is given by ([HSS])

$$Z_{0;1}(q) = E_4(q)\prod_{k\geq 1}\frac{1}{(1-q^k)^{12}}. \tag{61}$$

For $Z_{g;n}(q)$, we have the following conjecture in mathematics suggested by some arguments in physics (cf. [HST]).

Conjecture 4.1. (**Holomorphic anomaly equation** [HST])

(i) $Z_{g;n}(q)$ *has the following expression*

$$Z_{g;n}(q) = \frac{P_{2g+6n-2}(E_2(q),E_4(q),E_6(q))}{\prod_{k\geq 1}(1-q^k)^{12n}},$$

where $P_{2g+6n-2}(E_2(q),E_4(q),E_6(q))$ *is a homogeneous polynomial of weight* $2g+6n-2$ *and* $E_*(q)$ *are Eisenstein series of weight* $*$.

(ii) $P_{2g+6n-2}(E_2,E_4,E_6)$ *satisfies the following equation:*

$$\frac{\partial P_{2g+6n-2}}{\partial E_2} = \frac{1}{24}\sum_{g=g'+g''}\sum_{s=1}^{n-1} s(n-s)P_{2g'+6s-2}P_{2g''+6(n-s)-2} + \frac{n(n+1)}{24}P_{2(g-1)+6n-2}.$$

We can solve the holomorphic anomaly equation easily (cf. [HST]) and, if $n=1$, the generating function of $Z_{g;1}(q)$ may be summarized to

$$\sum_{g\geq 0} Z_{g;1}(q)\lambda^{2g} = Z_{0;1}(q)\exp\left(2\sum_{k\geq 1}\frac{\zeta(2k)}{k}E_{2k}(q)\left(\frac{\lambda}{2\pi}\right)^{2k}\right).$$

By the famous Jacobi's triple product formula, we have

$$\lambda^{-2}\exp\left(2\sum_{k\geq 1}\frac{\zeta(2k)}{k}E_{2k}(q)\left(\frac{\lambda}{2\pi}\right)^{2k}\right)$$
$$=\frac{1}{(e^{-\sqrt{-1}\lambda/2}-e^{\sqrt{-1}\lambda/2})^2}\prod_{n\geq 1}\frac{(1-q^n)^4}{(1-e^{\sqrt{-1}\lambda}q^n)^2(1-e^{-\sqrt{-1}\lambda}q^n)^2}.$$

Multiplying $Z_{0;1}(q)$ both sides, we can easily verify the Gopakumar–Vafa conjecture, which was given in our previous paper [HST].

Finally we remark that our horomorphic anomaly equation suffices to determine $Z_{g;n}(q)$ recursively for all g and n, see [HST] for details. Here we present the first few solutions for $n=1,2$;

$$Z_{1,1}(q)=\frac{E_2(q)E_4(q)}{\prod_{n\geq 1}(1-q^n)^{12}},\quad Z_{2,1}(q)=\frac{E_4(q)(5E_2(q)^2+E_4(q))}{1440\prod_{n\geq 1}(1-q^n)^{12}}$$
$$Z_{3,1}(q)=\frac{E_4(q)(35E_2(q)^3+21E_2(q)E_4(q)+4E_6(q))}{362880\prod_{n\geq 1}(1-q^n)^{12}},$$

$$\begin{aligned}
Z_{0,2}(q)= &\ \frac{E_2(q)E_4(q)^2+2E_4(q)E_6(q)}{\prod_{n\geq 1}(1-q^n)^{24}},\\
Z_{1,2}(q)= &\ \frac{10E_2(q)^2E_4(q)^2+9E_4(q)^3+24E_2(q)E_4(q)E_6(q)+5E_6(q)^2}{1152\prod_{n\geq 1}(1-q^n)^{24}},\\
Z_{2,2}(q)= &\ \big(190E_2(q)^3E_4(q)^2+417E_2(q)E_4(q)^3+540E_2(q)^2E_4(q)E_6(q)+\\
&\ 356E_4(q)^2E_6(q)+225E_2(q)E_6(q)^2\big)\frac{1}{207360\prod_{n\geq 1}(1-q^n)^{24}},\\
Z_{3,2}(q)= &\ \big(2275E_2(q)^4E_4(q)^2+8925E_2(q)^2E_4(q)^3+3540E_4(q)^4+\\
&\ 7560E_2(q)^3E_4(q)E_6(q)+14984E_2(q)E_4(q)^2E_6(q)+\\
&\ 4725E_2(q)^2E_6(q)^2+4071E_4(q)E_6(q)^2\big)\frac{1}{34836480\prod_{n\geq 1}(1-q^n)^{24}}.
\end{aligned}$$

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