Bäcklund Transformations of the Sixth Painlevé Equation in Terms of Riemann-Hilbert Correspondence

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Abstract

It is well known that the sixth Painlevé equation $\mathrm{P_{VI}}$ admits a group of Bäcklund transformations which is isomorphic to the affine Weyl group of type $\mathrm{D}_4^{(1)}$. Although various aspects of this symmetry have been discussed by many authors, there still remains a basic problem yet to be considered, that is, the problem of characterizing the Bäcklund transformations in terms of Riemann-Hilbert correspondence. In this direction, we show that the Bäcklund transformations are just the pull-back of very simple transformations on the moduli of monodromy representations by the Riemann-Hilbert correspondence. This result gives a natural and clear picture of the Bäcklund transformations.

Key words: Bäcklund transformation, the sixth Painlevé equation, Riemann-Hilbert correspondence, isomonodromic deformation, affine Weyl group of type $\mathrm{D}_4^{(1)}$.

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1 Introduction

It is well known that the sixth Painlevé equation $\mathrm{P_{VI}}$ admits a group of Bäcklund transformations which is isomorphic to the affine Weyl group of type $\mathrm{D}_4^{(1)}$. Various aspects of this unexpectedly large symmetry have been discussed by many authors, e.g. Okamoto [21], Arinkin and Lysenko [2], Noumi and Yamada [18]. See also Conte and Musette [3], Fokas and Yorsos [5], Manin [16], Sakai [25], Watanabe [30] and others. However, there still seems to remain a basic problem yet to be considered with a special attention, namely, the problem of characterizing the Bäcklund transformations in terms of Riemann-Hilbert correspondence. This problem naturally arises from the work of Iwasaki [11, 12], which exploited the standpoint of studying $\mathrm{P_{VI}}$ based on a geometry of moduli spaces of monodromy representations via the Riemann-Hilbert correspondence; this standpoint had still previously been hinted at in Iwasaki [9, 10].

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The Riemann-Hilbert correspondence is a map from a moduli of Fuchsian differential equations to a moduli of monodromies, associating to each Fuchsian equation its monodromy representation. On the other hand, $\mathrm{P_{VI}}$ is a differential equation on the moduli of Fuchsian equations that describes the isomonodromic deformation. Moreover, a Bäcklund transformation is a discrete transformation on the moduli of Fuchsian equations that commutes with the solution flow of $\mathrm{P_{VI}}$. Therefore, it is very natural to ask what kind of discrete transformation on the moduli of monodromies is induced from a Bäcklund transformation by the Riemann-Hilbert correspondence. It is expected that the induced transformation looks simpler than the original, that is, a Bäcklund transformation looks simpler when viewed as a transformation on the moduli of monodromies. So the story should proceed in the other way round: Start with some simple transformations on the moduli of monodromies, pull them back via the Riemann-Hilbert correspondence to the moduli of Fuchsian equations and obtain the Bäcklund transformations.

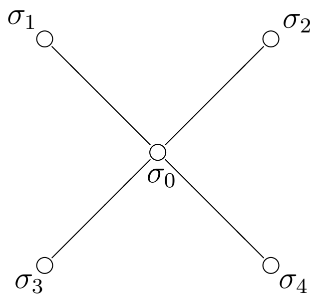
The aim of this paper is to verify that the speculation above is true; the Bäcklund transformations are just the pull-back of almost identity transformations on the moduli of monodromies, namely, of those transformations which may alter local monodromy data but do not change "global monodromy data". The affine Weyl group structure of the Bäcklund transformations will also be transparent from our point of view. We hope that our result and viewpoint give a natural and clear picture of the Bäcklund transformations for $\mathrm{P_{VI}}$.

We remark that there is a geometric way to understand Bäcklund transformations of $\mathrm{P_{VI}}$ by means of families of spaces of initial conditions constructed by Okamoto [19]. As explained in [17, 27, 25, 22], there exists a family of open algebraic surfaces parametrized by the 4-dimensional space $\mathcal{K}$ of local exponents, which correspond to the spaces of initial conditions of $\mathrm{P_{VI}}$. Then the affine Weyl group $W(\mathrm{D}_4^{(1)})$ acts on $\mathcal{K}$ in a natural way and the actions can be lifted to birational transformations of the total space of the family of Okamoto spaces. Saito and Umemura [23] pointed out that a Bäcklund transformation corresponding to a reflection of $W(\mathrm{D}_4^{(1)})$ is nothing but a flop whose center is a family of (-2)-rational curves contained in Okamoto spaces lying over the reflection hyperplanes in $\mathcal{K}$. Moreover, Saito and Terajima [24] clarified the relation between (-2)-curves in Okamoto spaces and Riccati solutions of $\mathrm{P_{VI}}$. Since a flop of family of algebraic surfaces appears as a simultaneous resolution of rational double points, one can expect that transformations of reflection type are related to the simultaneous resolutions of versal deformations of rational double points.

On the other hand, Arinkin and Lysenko [1] introduced the moduli space of $SL(2)$-bundles with connections on $\mathbb{P}^1$ parametrized by the local exponents and describe the Bäcklund transformations in [2]. From the viewpoint of the theory of isomonodromic deformations of flat connections (Fuchsian connections), the moduli space should corresponds to Okamoto spaces. Unfortunately, they treated the moduli space mostly as stack and restricted the parameter space to the complement of all reflection hyperplanes in order to avoid the reducible connections. In a forthcoming paper [8], we shall construct moduli spaces of stable parabolic connections over $\mathbb{P}^1$ for all parameters. By using our moduli spaces, we can give a more geometric and conceptual picture of the Riemann-Hilbert correspondence and the Bäcklund transformations of $\mathrm{P_{VI}}$.

2 Sketch of Main Result

We roughly state the main result of this paper. Here we are content with a sketch of outline, since a complete statement is possible only after all the necessary ingredients are prepared.



$$C = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Figure 1: Dynkin diagram and Cartan matrix of type $\mathrm{D}_4^{(1)}$

Detailed explanations will be supplied in the subsequent sections.

The sixth Painlevé equation $\mathrm{P_{VI}}$ is a second order nonlinear ordinary differential equation

$$\begin{aligned} q_{xx} &= \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-x}\right)q_x^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{q-x}\right)q \\ &+ \frac{q(q-1)(q-x)}{2x^2(x-1)^2}\left\{\kappa_4^2-\kappa_1^2\frac{x}{q^2}+\kappa_2^2\frac{x-1}{(q-1)^2}+(1-\kappa_3^2)\frac{x(x-1)}{(q-x)^2}\right\} \end{aligned} \tag{2.1}$$

for an unknown function $q = q(x)$, with complex parameters $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. $\mathrm{P_{VI}}$ is written $\mathrm{P_{VI}}(\kappa)$ when the dependence on parameters κ should be indicated explicitly. In describing the Bäcklund transformations, it is convenient to think of the parameter space as an affine space

$$\mathcal{K} = \{\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 \,:\, 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1\}. \tag{2.2}$$

We realize the affine Weyl group of type $\mathrm{D}_4^{(1)}$ as an affine reflection group acting on $\mathcal{K}$,

$$W(\mathrm{D}_4^{(1)}) = \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle,$$

where σ_i is the reflection in the hyperplane $\kappa_i = 0$. In terms of the Cartan matrix $C = (c_{ij})$ of type $\mathrm{D}_4^{(1)}$ (see Figure 1), the reflection σ_i is expressed as

$$\sigma_i(\kappa_j) = \kappa_j - \kappa_i c_{ij}. \tag{2.3}$$

A remarkable fact is that the affine Weyl group $W(\mathrm{D}_4^{(1)})$ lifts up to a transformation group of $\mathrm{P_{VI}}$. Namely, each reflection σ_i admits a lift s_i that is a transformation of $\mathrm{P_{VI}}(\kappa)$ to $\mathrm{P_{VI}}(\sigma_i(\kappa))$, and the correspondence $\sigma_i \mapsto s_i$ induces an isomorphism between $W(\mathrm{D}_4^{(1)})$ and

$$G = \langle s_0, s_1, s_2, s_3, s_4 \rangle. \tag{2.4}$$

The group G is called the group of Bäcklund transformations for $\mathrm{P_{VI}}$. The explicit form of s_i will be given in §4 after some Hamiltonian formalisms for $\mathrm{P_{VI}}$ are introduced in §3.

Let us turn to a monodromy problem. $\mathrm{P_{VI}}(\kappa)$ is the isomonodromic deformation equation for a class of second order linear Fuchsian differential equations on $\mathbb{P}^1$ having four regular singular points with prescribed local exponents, where κ is used to assign local exponents. We denote by $\mathcal{E}_t(\kappa)$ the moduli of relevant Fuchsian equations with regular singular points at $t = (t_1, t_2, t_3, t_4)$. The precise setting of $\mathcal{E}_t(\kappa)$ will be mentioned in §7. Let $\mathcal{R}_t(a)$ be the moduli of monodromy representations $\pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}) \to SL_2(\mathbb{C})$, up to conjugacy, having prescribed local monodromy data $a = (a_1, a_2, a_3, a_4)$, where a_i is defined to be the trace of the monodromy matrix M_i around the singular point t_i, namely,

$$a_i = \mathrm{Tr}\, M_i \qquad (i = 1, 2, 3, 4). \tag{2.5}$$

Then the monodromy map or the Riemann-Hilbert correspondence

$$\mathrm{RH} : \mathcal{E}_t(\kappa) \to \mathcal{R}_t(a) \tag{2.6}$$

is defined by associating to each Fuchsian equation its monodromy representation class. In our setting, which will be detailed in §7, the correspondence of parameters $\kappa \mapsto a$ is given by

$$a_i = 2\cos\pi\kappa_i \quad (i = 1, 2, 3), \qquad a_4 = -2\cos\pi\kappa_4, \tag{2.7}$$

The minus sign for a_4 is not a misprint; a_4 is distinguished for a reason to be explained in §7.

In Iwasaki [11, 12], the representation space $\mathcal{R}_t(a)$ is realized as an affine cubic surface. Let us recall this construction. We introduce variables $x = (x_1, x_2, x_3)$ by

$$x_i = \mathrm{Tr}(M_j M_k) \qquad \text{for} \quad \{i, j, k\} = \{1, 2, 3\}. \tag{2.8}$$

They are referred to as *global monodromy data*, since they carry a global information about monodromy; the product $M_j M_k$ is the monodromy matrix along a "global" loop surrounding the two singular points t_j and t_k simultaneously. Let $f(x, \theta)$ be a polynomial defined by

$$f(x, \theta) = x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1x_1 - \theta_2x_2 - \theta_3x_3 + \theta_4, \tag{2.9}$$

where the coefficients $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ are given by

$$\theta_i = \begin{cases} a_ia_4 + a_ja_k & (i = 1, 2, 3), \\ a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4 & (i = 4). \end{cases} \tag{2.10}$$

Then the representation space $\mathcal{R}_t(a)$ can be identified with an affine cubic surface

$$\mathcal{S}(\theta) = \{\, x = (x_1, x_2, x_3) \in \mathbb{C}^3 \;:\; f(x, \theta) = 0\}. \tag{2.11}$$

Therefore, the Riemann-Hilbert correspondence (2.6) is recast into

$$\mathrm{RH} : \mathcal{E}_t(\kappa) \to \mathcal{S}(\theta), \tag{2.12}$$

where the correspondence of parameters $\kappa \mapsto \theta$ is defined through (2.7) and (2.10). As a solution to the Riemann-Hilbert problem, the map (2.12) is an analytic isomorphism onto a Zariski open subset of $\mathcal{S}(\theta)$. The following simple but fundamental observation is due to Terajima [29].

Lemma 2.1 *Viewed as functions of κ, the coefficients θ are $W(\mathrm{D}_4^{(1)})$-invariants.*

The proof is just by calculations. Note that the local monodromy data $a = (a_1, a_2, a_3, a_4)$ are invariants of the reflections $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, but not of σ_0, so that the unexpected switch $a \mapsto \theta$ has been necessary to obtain invariants of all five reflections σ_i. In this sense, the "true" local monodromy data might be attributed to θ rather than to a. Thanks to Lemma 2.1, each Bäcklund transformation $s \in G$ induces an automorphism r of $\mathcal{S}(\theta)$ such that the diagram

$$\begin{array}{ccc} \mathcal{E}_t(\kappa) & \xrightarrow{\ s\ } & \mathcal{E}_t(\sigma(\kappa)) \\ {\scriptstyle \mathrm{RH}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{RH}} \\ \mathcal{S}(\theta) & \xrightarrow[\ r\ ]{} & \mathcal{S}(\theta) \end{array} \tag{2.13}$$

is commutative, where $\sigma \in W(\mathrm{D}_4^{(1)})$ is the transformation of parameters κ underlying the Bäcklund transformation s. Now a natural question occurs to us:

Problem 2.2 What is the transformation r ?

As one may expect naturally, this problem has a very simple solution:

Theorem 2.3 (Main Theorem) *The transformation r in diagram* (2.13) *is just the identity; that is to say, the Bäcklund transformations are those transformations which cover the identity transformation on the moduli of monodromies through the Riemann-Hilbert correspondence.*

The main result is far from trivial to the effect that the Bäcklund transformations, viewed as automorphisms of the cubic surface $\mathcal{S}(\theta)$, are distinguished to be the *identity* from many other automorphisms. In fact, $\mathcal{S}(\theta)$ admits a large number of automorphisms, even if the class is limited to algebraic automorphisms. In this respect, Iwasaki [11, 12] showed that the Riemann-Hilbert correspondence transforms the nonlinear monodromy of $\mathrm{P_{VI}}$ into a modular group action on $\mathcal{S}(\theta)$ realized as a polynomial automorphism group on it. So $\mathcal{S}(\theta)$ admits at least infinitely many algebraic automorphisms labeled by the elements of a modular group. It may be said that the Riemann-Hilbert correspondence is such a map that sends a transcendental object on the phase space of $\mathrm{P_{VI}}$ (the nonlinear monodromy of $\mathrm{P_{VI}}$) to an algebraic object on the space of monodromies (a polynomial automorphism group on it), while collapsing an algebraic object on the former (the Bäcklund transformations) to a trivial object on the latter (the identity).

In this paper we shall present an analytic proof of Theorem 2.3, which is very simple in its essential idea (see §8) but requires some elaborate calculations in technicalities (see §9). It is desirable that there exists an alternative geometrical proof of the theorem.

This paper is organized as follows. Some Hamiltonian formalisms for $\mathrm{P_{VI}}$ are presented in §3, which enable us to describe the group of Bäcklund transformations in a symmetrical way in §4. The Hamiltonian systems are characterized as isomonodromic deformation equations of second order Fuchsian differential equations with four regular singular points in §5. The Bäckulnd transformations s_1, s_2, s_3, s_4 are constructed as elementary gauge transformations in §6. The Riemann-Hilbert correspondence is formulated in §7. With these preliminaries, our main result, Theorem 2.3, is established in §8. The proof is based on the idea of coalescence of regular singular points along isomonodromic deformation. Here a key observation is that, by a coalescence procedure, a Fuchsian equation with four singular points degenerates into a Fuchsian equation with three singular points, and the difference of the two local exponents at the coalescent singular point is an invariant of the Bäcklund transformations; see Lemma 8.2.

In order for the idea to work, a certain technical lemma is needed concerning accumulation points of trajectories of the Hamiltonian system. This lemma is established in §9 (Lemma 9.1). The finial section, §10, is devoted to heuristics on finding Bäcklund transformations from our point of view. It provides a new way of discovering the hidden Bäcklund transformation s_0.

3 Hamiltonian Systems

In discussing Bäcklund transformations and monodromy problems related to $\mathrm{P_{VI}}$, it is convenient to use a Hamiltonian system equivalent to the original single equation (2.1). For this purpose, we shall employ the following three systems:

(H1) a Hamiltonian system with single time variable; see (3.1);

(H4) a completely integrable Hamiltonian system with four time variables; see (3.2);

(H3) a completely integrable Hamiltonian system with three time variables; see (3.6).

We will flexibly use one or another of them depending upon contexts and purposes.

The first system (H1) is the most conventional one which often appears in the literature:

$$\frac{\partial q}{\partial x} = \frac{\partial h}{\partial p}, \qquad \frac{\partial p}{\partial x} = -\frac{\partial h}{\partial q}, \tag{3.1}$$

where the Hamiltonian $h = h(q, p, x, \kappa)$ is given by

$$\begin{aligned} x(x-1)h &= q(q-1)(q-x)p^2 - \{(\kappa_3 - 1)q(q-1) + \kappa_1(q-1)(q-x) + \kappa_2 q(q-x)\}p \\ &\quad + \kappa_0(\kappa_0 + \kappa_4)(q-x). \end{aligned}$$

Indeed, equation (2.1) is recovered from system (3.1) by eliminating the variable p.

A more symmetric description is feasible in terms of the second system (H4):

$$\frac{\partial q}{\partial t_i} = \frac{\partial H_i}{\partial p}, \qquad \frac{\partial p}{\partial t_i} = -\frac{\partial H_i}{\partial q}, \qquad (i = 1, 2, 3, 4), \tag{3.2}$$

with four time variables $t = (t_1, t_2, t_3, t_4)$, the Hamiltonians $H_i = H_i(q, p, t, \kappa)$ being given by

$$\begin{aligned} (t_{ij}t_{ik}t_{il})H_i &= (q_i q_j q_k q_l)p^2 - \{(\kappa_i - 1)q_j q_k q_l + \kappa_j q_i q_k q_l + \kappa_k q_i q_j q_l + \kappa_l q_i q_j q_k\}p \\ &\quad + \kappa_0 q_i\{(\kappa_i - 1)q_i + (\kappa_j + \kappa_0)q_j + (\kappa_k + \kappa_0)q_k + (\kappa_l + \kappa_0)q_l\}, \end{aligned} \tag{3.3}$$

with $q_i = q - t_i$, $t_{ij} = t_i - t_j$ and $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We remark that (H1) is recovered from (H4) by a symplectic reduction. Indeed, we can observe that the diagonal action on $t = (t_1, t_2, t_3, t_4)$ of the Möbius transformations lifts symplectically up to system (3.2) and the associated symplectic reduction takes (3.2) into (3.1) having the cross ratio

$$x = \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_2)(t_3 - t_4)} \tag{3.4}$$

as the only time variable. (H3) is obtained from (H4) by letting t_4 tend to infinity. Indeed,

$$H_i(q, p, t, \kappa) \to h_i(q, p, t, \kappa) \quad (i = 1, 2, 3), \qquad H_4(q, p, t, \kappa) \to 0 \quad \text{as} \quad t_4 \to \infty, \tag{3.5}$$

and system (3.2) reduces to the Hamiltonian system

$$\frac{\partial q}{\partial t_i} = \frac{\partial h_i}{\partial p}, \qquad \frac{\partial p}{\partial t_i} = -\frac{\partial h_i}{\partial q}, \qquad (i = 1, 2, 3). \tag{3.6}$$

with three time variables $t = (t_1, t_2, t_3)$, where the Hamiltonians $h_i = h_i(q, p, t, \kappa)$ are given by

$$(t_{ij} t_{ik}) h_i = (q_i q_j q_k) p^2 - \{(\kappa_i - 1) q_j q_k + \kappa_j q_k q_i + \kappa_k q_i q_j\} p + \kappa_0 (\kappa_0 + \kappa_4) q_i, \tag{3.7}$$

with $\{i, j, k\} = \{1, 2, 3\}$. The symplectic reduction mentioned above amounts to taking the normalization $t_1 = 0$, $t_2 = 1$, $t_3 = x$, $t_4 = \infty$. Then system (3.6) with $i = 3$ yields (3.1).

4 Group of Bäcklund Transformations

Now we can state (recall) the explicit form of the Bäcklund transformations. For a symmetric description, we shall represent it in terms of the Hamiltonian system with four time variables (3.2). Necessary modifications for the systems (3.1) and (3.6) are a routine work. The Bäcklund transformation s_i corresponding to the reflection σ_i is expressed as

$$\begin{aligned}
&s_0 \begin{cases} s_0(t_j) &= t_j, \\ s_0(q) &= q + \dfrac{\kappa_0}{p}, \\ s_0(p) &= p, \end{cases} \qquad (i = 0), \\
&s_i \begin{cases} s_i(t_j) &= t_j, \\ s_i(q) &= q, \\ s_i(p) &= p - \dfrac{\kappa_i}{q - t_i}, \end{cases} \qquad (i = 1, 2, 3, 4).
\end{aligned} \tag{4.1}$$

Noumi and Yamada [18] expressed (4.1) in a unified manner by introducing variables

$$q_i = \begin{cases} p & (i = 0), \\ q - t_i & (i = 1, 2, 3, 4). \end{cases} \tag{4.2}$$

We remark that the fourth variable q_4 was degenerating in [18], since they employed system (3.1) as their representation of $\mathrm{P_{VI}}$, for which $t_4 = \infty$ and hence $q_4 = \infty$ was not observable. In any case, in terms of the variables q_i, formula (4.1) together with (2.3) is expressed as

$$s_i(\kappa_j) = \kappa_j - \kappa_i c_{ij}, \qquad s_i(t_j) = t_j, \qquad s_i(q_j) = q_j + \frac{\kappa_i}{q_i} u_{ij}, \tag{4.3}$$

where $C = (c_{ij})$ is the Cartan matrix indicated in Figure 1 and u_{ij} is defined by

$$u_{ij} = \{q_i, q_j\} \qquad (i, j = 0, 1, 2, 3, 4).$$

Here $\{f, g\}$ denotes the Poisson bracket,

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}.$$

Explicitly, the matrix $U = (u_{ij})$ is given by

$$U = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 4.1 When $t_4 = \infty$, formula (4.1) with $i = 4$ should be interpreted as

$$s_4(t_j) = t_j, \qquad s_4(q) = q, \qquad s_4(p) = p.$$

Namely, the lift s_4 of the reflection σ_4 acts on (t, q, p) trivially.

The Bäcklund transformations s_1, s_2, s_3, s_4 are not so difficult to understand; as will be seen in §6, they are none other than elementary gauge transformations of Fuchsian differential equations. Much more difficult and hence intriguing is the transformation s_0, whose existence is a sort of mystery. Hence the main body of this paper is devoted to understanding s_0.

The symmetry of $\mathrm{P_{VI}}$ with respect to the Bäcklund transformations is stated as follows.

Theorem 4.2 *The Hamiltonian system* (3.2) *is invariant under the action of the Bäcklund transformation group G, that is, the action of G commutes with the Hamiltonian vector fields*

$$X_i = \frac{\partial}{\partial t_i} + \frac{\partial H_i}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H_i}{\partial q}\frac{\partial}{\partial p}. \qquad (i = 1, 2, 3, 4) \tag{4.4}$$

Proof. The proof is by a direct check of condition (4.5) in the following lemma. $\square$

Lemma 4.3 *The transformation s_i commutes with the vector field X_j if and only if*

$$s_i(H_j) - H_j + \delta_{ij}\frac{\kappa_i}{q_i} \quad \textit{is independent of} \quad (q, p), \tag{4.5}$$

where δ_{ij} is Kronecker's delta symbol.

Proof. We prove the lemma for $i = 1, 2, 3, 4$. The case $i = 0$ can be treated in a similar manner. Write $\bar{t}_j = s_i(t_j)$, $\bar{q} = s_i(q)$, $\bar{p} = s_i(p)$ and $\bar{H}_j = s_i(H_j)$. The transformation rule (4.1) reads

$$t_j = \bar{t}_j, \qquad q = \bar{q}, \qquad p = \bar{p} + \frac{\kappa_i}{\bar{q}_i},$$

where $\bar{q}_i = \bar{q} - \bar{t}_i$. Applying the chain rule of partial differentiations, we have

$$\frac{\partial}{\partial \bar{t}_j} = \frac{\partial}{\partial t_j} + \delta_{ij}\frac{\kappa_i}{q_i^2}\frac{\partial}{\partial p}, \qquad \frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial q} - \frac{\kappa_i}{q_i^2}\frac{\partial}{\partial p}, \qquad \frac{\partial}{\partial \bar{p}} = \frac{\partial}{\partial p},$$

and hence

$$\begin{aligned} s_i(X_j) &= \frac{\partial}{\partial \bar{t}_j} + \frac{\partial \bar{H}_j}{\partial \bar{p}}\frac{\partial}{\partial \bar{q}} - \frac{\partial \bar{H}_j}{\partial \bar{q}}\frac{\partial}{\partial \bar{p}} \\ &= \left(\frac{\partial}{\partial t_j} + \delta_{ij}\frac{\kappa_i}{q_i^2}\frac{\partial}{\partial p}\right) + \frac{\partial \bar{H}_j}{\partial p}\left(\frac{\partial}{\partial q} - \frac{\kappa_i}{q_i^2}\frac{\partial}{\partial p}\right) - \left(\frac{\partial \bar{H}_j}{\partial q} - \frac{\kappa_i}{q_i^2}\frac{\partial \bar{H}_j}{\partial p}\right)\frac{\partial}{\partial p} \\ &= \frac{\partial}{\partial t_j} + \frac{\partial \bar{H}_j}{\partial p}\frac{\partial}{\partial q} - \left(\frac{\partial \bar{H}_j}{\partial q} - \delta_{ij}\frac{\kappa_i}{q_i^2}\right)\frac{\partial}{\partial p} \end{aligned}$$

Therefore, we have $s_i(X_j) = X_j$ if and only if

$$\frac{\partial \bar{H}_j}{\partial p} = \frac{\partial H_j}{\partial p}, \qquad \frac{\partial \bar{H}_j}{\partial q} - \delta_{ij}\frac{\kappa_i}{q_i^2} = \frac{\partial H_j}{\partial q}.$$

The last condition is equivalent to the assertion (4.5). The proof is complete. □

5 Isomonodromic Deformation

We discuss isomonodromy problems related to $\mathrm{P_{VI}}$. Corresponding to the Hamiltonian systems with four time variables (3.2) and with three time variables (3.6), we set up two classes of Fuchsian differential equations, namely, (5.1) and (5.4) respectively. Then we consider their isomonodromic deformations. The Riemann-Hilbert correspondence for (5.4) will be formulated in §7; a similar formulation for (5.1) is possible, but omitted.

For the Hamiltonian system (3.2), we consider second order Fuchsian differential equations

$$\frac{d^2 f}{dz^2} - u_1(z)\frac{df}{dz} + u_2(z) f = 0 \tag{5.1}$$

on $\mathbb{P}^1$ with six singular points at $z = t_1, t_2, t_3, t_4, q, \infty$ such that

(A1) $z = t_i$ is a regular singular point with exponents 0 and κ_i;

(A2) $z = q$ is an apparent singular point with exponents 0 and 2;

(A3) $z = \infty$ is a removable singular point, that is, (5.1) can be converted into a differential equation without singular point at $z = \infty$ by some transformation of the form $f = z^{-\kappa_0} g$.

Here we recall the notion of an apparent singular point: A regular singular point q of a second order Fuchsian differential equation is said to be *resonant* if the difference of the two exponents at q is an integer. A resonant singular point q falls into two cases; one is the generic case where a solution basis at q involves the logarithmic function, and the other is the nongeneric case without logarithmic term in any solution basis. In the latter case, q is called an *apparent* singular point. We remark that a removable singular point mentioned in (A3) is nothing but an apparent singular point such that the difference of exponents is one. We also remark that the number κ_0 in (A3) is uniquely determined, since equation (5.1) must satisfy Fuchs' relation

$$2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1. \tag{5.2}$$

Note that the affine linear relation (5.2) is a source of the parameter space $\mathcal{K}$ in (2.2).

By conditions (A1), (A2), (A3), the coefficients of equation (5.1) must be of the form

$$u_1(z) = \frac{1}{z-q} + \sum_{i=1}^{4} \frac{\kappa_i - 1}{z - t_i}, \qquad u_2(z) = \frac{p}{z-q} - \sum_{i=1}^{4} \frac{H_i}{z - t_i}, \tag{5.3}$$

where (q, p, t, κ) are free parameters, while H_i is a function of (q, p, t, κ), whose explicit form can be sought out by Frobenius' method in the theory of Fuchsian differential equations; the result is that $H_i = H_i(q, p, t, \kappa)$ must be the Hamiltonians (3.3) of the system (3.2).

It is well known that Painlevé-type equations arise from isomonodromic deformations of linear ordinary differential equations; see e.g. Fuchs [6], Schlesinger [26], Jimbo, Miwa and Ueno [13], Fokas and Its [4]. In the original case of $\mathrm{P_{VI}}$, we have the following:

Theorem 5.1 *The isomonodromic deformation of the Fuchsian differential equations* (5.1) *is described by the completely integrable Hamiltonian system* (3.2)*.*

Writing $\mathrm{P_{VI}}$ as a Hamiltonian system is due to Malmquist [15], Okamoto [20] and others. Iwasaki [9, 10] exploited an intrinsic (topological) reasoning for the Hamiltonian structure.

Similarly, for the Hamiltonian system (3.6), we consider Fuchsian differential equations

$$\frac{d^2 f}{dz^2} - v_1(z)\frac{df}{dz} + v_2(z) f = 0 \tag{5.4}$$

on $\mathbb{P}^1$ with five singular points at $z = t_1, t_2, t_3, q, \infty$ such that

(B1) $z = t_i$ is a regular singular point with exponents 0 and κ_i;

(B2) $z = q$ is an apparent singular point with exponents 0 and 2;

(B3) $z = \infty$ is a regular singular point with exponents κ_0 and $\kappa_4 + \kappa_0$.

By conditions (B1), (B2), (B3), the coefficients of equation (5.4) must be of the form

$$v_1(z) = \frac{1}{z-q} + \sum_{i=1}^{3} \frac{\kappa_i - 1}{z - t_i}, \qquad v_2(z) = \frac{p}{z-q} - \sum_{i=1}^{3} \frac{h_i}{z - t_i}, \tag{5.5}$$

where (q, p, t, κ) are free parameters, while $h_i = h_i(q, p, t, \kappa)$ are the Hamiltonians (3.7) of the system (3.6). In view of (3.5), formula (5.5) is obtained by taking the limit $t_4 \to 0$ in (5.3). The counterpart of Theorem 5.1 for equations (5.4) is stated as follows.

Theorem 5.2 *The isomonodromic deformation of the Fuchsian differential equation* (5.4) *is described by the completely integrable Hamiltonian system* (3.6)*.*

6 Gauge Transformations

For $i = 1, 2, 3, 4$, we can easily construct the Bäcklund transformation s_i as an elementary gauge transformation of the Fuchsian equation (5.1). Indeed, by the gauge transformation,

$$f = (z - t_i)^{\kappa_i} \bar{f} \qquad (i = 1, 2, 3, 4), \tag{6.1}$$

equation (5.1) is transformed into another Fuchsian equation, say,

$$\frac{d^2 \bar{f}}{dz^2} - \bar{u}_1(z)\frac{d\bar{f}}{dz} + \bar{u}_2(z)\bar{f} = 0,$$

whose coefficients $\bar{u}_1$ and $\bar{u}_2$ must be of the same form as (5.3), say,

$$\bar{u}_1(z) = \frac{1}{z-\bar{q}} + \sum_{j=1}^{4} \frac{\bar{\kappa}_j - 1}{z - \bar{t}_j}, \qquad \bar{u}_2(z) = \frac{\bar{p}}{z-\bar{q}} - \sum_{j=1}^{4} \frac{\bar{H}_j}{z - \bar{t}_j}, \tag{6.2}$$

where $\bar{H}_j = H_j(\bar{q}, \bar{p}, \bar{t}, \bar{\kappa})$. On the other hands, substituting (6.1) into (5.1) implies that

$$\bar{u}_1 = u_1 - \frac{2\kappa_i}{z - t_i}, \qquad \bar{u}_2 = u_2 - \frac{\kappa_i}{z - t_i} u_1 + \frac{\kappa_i(\kappa_i - 1)}{(z - t_i)^2}. \tag{6.3}$$

Then substituting (5.3) into (6.3) and comparing the result with (6.2), we have

$$\bar{\kappa}_j = \kappa_j - \kappa_i c_{ij}, \qquad \bar{t}_j = t_j, \qquad \bar{q} = q, \qquad \bar{p} = p - \frac{\kappa_i}{q - t_i},$$

which is none other than the transformation s_i in (4.1) for $i = 1, 2, 3, 4$.

7 Riemann-Hilbert Correspondence

Let us formulate the Riemann-Hilbert correspondence for Fuchsian equations of the form (5.4). It is convenient to define it in such a manner that monodromy matrices have values in the special linear group $SL_2(\mathbb{C})$. To this end, by applying the gauge transformation

$$f = \phi F \qquad \text{with} \qquad \phi = (z-q)\prod_{i=1}^{3}(z-t_i)^{\kappa_i/2}, \tag{7.1}$$

equation (5.4) should be normalized into a Fuchsian equation

$$\frac{d^2F}{dz^2} - V_1(z)\frac{dF}{dz} + V_2(z)F = 0 \tag{7.2}$$

with singular points at $z = t_1,\ t_2,\ t_3,\ q,\ \infty$ satisfying the following conditions:

(C1) $z = t_i$ is a regular singular point with exponents $\pm\kappa_i/2$;

(C2) $z = q$ is an apparent singular point with exponents ± 1;

(C3) $z = \infty$ is a regular singular point with exponents $(3 \pm \kappa_4)/2$.

Note that the coefficients of equation (7.2) and those of (5.4) are related by

$$V_1 = v_1 - 2\frac{\phi'}{\phi}, \qquad V_2 = v_2 - \frac{\phi'}{\phi}v_1 + \frac{\phi''}{\phi}.$$

Equation (7.2) is called the *normal form* of (5.4) and the two are often identified. Then the (normalized) monodromy of (5.4) will be defined to be the monodromy of its normal form (7.2).

Let $\mathcal{E}_t(\kappa)$ be the set of all Fuchsian equations of the form (5.4), or equivalently of their normal form (7.2), with a prescribed value of parameters κ and location of singular points $t = (t_1, t_2, t_3, t_4)$, where $t_4 = \infty$. Associating (q, p) to equation (5.4) yields an identification

$$\mathcal{E}_t(\kappa) \cong (\mathbb{C} \setminus \{t_1, t_2, t_3\}) \times \mathbb{C} \ :\ (q,p)\text{-space}, \tag{7.3}$$

through which $\mathcal{E}_t(\kappa)$ is thought of as a complex manifold. In an algebro-geometrical framework developed by Inaba, Iwasaki and Saito [8], the space $\mathcal{E}_t(\kappa)$ is a Zariski open chart of a certain moduli space $\mathcal{M}_t(\kappa)$ of rank two stable parabolic bundles with Fuchsian connections on $\mathbb{P}^1$.

We proceed to moduli of monodromies. Let γ_i be a simple loop encircling the singular point t_i as in Figure 2, γ_4 being a loop around $t_4 = \infty$. Let M_i be the monodromy matrix along the loop γ_i of the Fuchsian equation (7.2). In view of (C1) and (C3), the matrix M_i has eigenvalues

$$\exp(\pm\pi\sqrt{-1}\kappa_i) \quad \text{for} \quad i = 1,2,3; \qquad -\exp(\pm\pi\sqrt{-1}\kappa_4) \quad \text{for} \quad i = 4. \tag{7.4}$$

Hence $\det M_i = 1$, viz., $M_i \in SL_2(\mathbb{C})$, and the trace $a_i = \operatorname{Tr} M_i$ is expressed as (2.7). Let $\mathcal{R}_t(a)$ be the space of monodromy representations of $\pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\})$ into $SL_2(\mathbb{C})$, up to conjugacy, whose monodromy matrices along the loop γ_i have trace a_i. Then the Riemann-Hilbert correspondence $\mathrm{RH} : \mathcal{E}_t(\kappa) \to \mathcal{R}_t(a)$ in (2.6) is defined by associating to each Fuchsian equation (5.4) the $SL_2(\mathbb{C})$-monodromy representation class of its normal form (7.2), where two parameters κ and a are related by (2.7). Then, composed with a natural map $\mathcal{R}_t(a) \to \mathcal{S}(\theta)$,

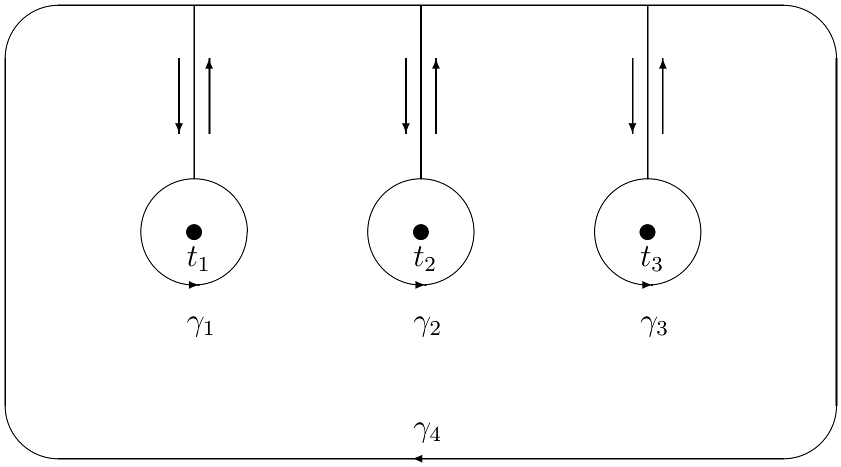


Figure 2: The loops γ_1, γ_2, γ_3, γ_4

where $\mathcal{S}(\theta)$ is the cubic surface in (2.11), the Riemann-Hilbert correspondence is realized more concretely as a map into the cubic surface, namely, as the map $\mathrm{RH} : \mathcal{E}_t(\kappa) \to \mathcal{S}(\theta)$ in (2.12).

The construction above can be made relatively over the parameter spaces. Let $\mathcal{E}_t$ be the space of all Fuchsian equations of the form (5.4), or equivalently of their normal form (7.2), where the regular singular points are fixed at $t = (t_1, t_2, t_3, t_4)$, $t_4 = \infty$, but the parameters κ may vary. Let $\pi_1 : \mathcal{E}_t \to \mathcal{K}$ be the natural projection associating to each equation in $\mathcal{E}_t$ its parameters κ. Note that $\mathcal{E}_t(\kappa)$ is a fiber of this projection. The space $\mathcal{E}_t$ is a Zariski open subset of the family $\mathcal{M}_t$ of moduli spaces $\mathcal{M}_t(\kappa)$ over $\mathcal{K}$, which is constructed in Inaba, Iwasaki and Saito [8]. Let $\mathcal{R}_t$ be the space of all monodromy representations $\pi_1(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, t_4\}) \to SL_2(\mathbb{C})$ up to conjugacy and let $\pi_2 : \mathcal{R}_t \to \mathbb{C}^4_a$ be the natural projection associating to each representation in $\mathcal{R}_t$ its local monodromy data a. Recall that $\mathcal{R}_t$ is the categorical quotient of the diagonal adjoint action of $SL_2(\mathbb{C})$ on $SL_2(\mathbb{C})^3$ and that $\pi_2 : \mathcal{R}_t \to \mathbb{C}^4_a$ is a family of affine cubic surfaces defined by the equation $f(x, \theta(a)) = 0$ in (2.9) with the relation $\theta = \theta(a)$ in (2.10). Moreover, let

$$\mathcal{S} = \{\, (x, \theta) \in \mathbb{C}^3 \times \mathbb{C}^4 \;:\; f(x, \theta) = 0 \,\},$$

and $\pi_3 : \mathcal{S} \to \mathbb{C}^4_\theta$ be the natural projection down to parameters θ. Note that $\pi : \mathcal{R}_t \to \mathbb{C}^4_a$ is obtained by the pullback of the family $\pi_3 : \mathcal{S} \to \mathbb{C}^4_\theta$ by the finite morphism $\mathbb{C}^4_a \to \mathbb{C}^4_\theta$. Now the (relative) Riemann-Hilbert correspondence RH is formulated as a commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_t \arrow[r, "\mathrm{RH}"] \arrow[d, "\pi_1"'] & \mathcal{R}_t \arrow[r, "\varphi"] \arrow[d, "\pi_2"] & \mathcal{S} \arrow[d, "\pi_3"] \\
\mathcal{K} \arrow[r, "u"'] & \mathbb{C}^4_a \arrow[r, "v"'] & \mathbb{C}^4_\theta,
\end{tikzcd}$$

where the maps $u : \kappa \mapsto a$ and $v : a \mapsto \theta$ are defined by (2.7) and (2.10); or as its contraction:

$$\begin{tikzcd}
\mathcal{M}_t \arrow[r, "\mathrm{RH}"] \arrow[d, "\pi_1"'] & \mathcal{S} \arrow[d, "\pi_3"] \\
\mathcal{K} \arrow[r, "vu"'] & \mathbb{C}^4_\theta,
\end{tikzcd} \tag{7.5}$$

8 Coalescence of Regular Singular Points

A principal idea for establishing our main result, Theorem 2.3, is to consider a coalescence of two regular singular points of Fuchsian equations, along an isomonodromic deformation. We first discuss the coalescence process only and then take the isomonodromic deformation into account. For this purpose we shall work with Fuchsian equations (5.4). Of course, working with Fuchsian equations (5.1) would lead us to the same conclusion; the latter choice would allow a more symmetrical discussion, but require somewhat heavier calculations. Here we employ (5.4), preferring simpler calculations at the cost of a minor symmetry breaking.

Lemma 8.1 *By the coalescence $t_k \to t_j$ of the singular points t_j and t_k, equation* (5.4) *with* (5.5) *degenerates into a Fuchsian equation with singular points at $z = t_i,\ t_j,\ q,\ \infty$,*

$$\frac{d^2 f}{dz^2} - w_1(z)\frac{df}{dz} + w_2(z) f = 0, \tag{8.1}$$

whose coefficients $w_1(z)$ and $w_2(z)$ are expressed as

$$w_1(z) = \frac{1}{z-q} + \frac{\kappa_i - 1}{z-t_i} + \frac{\kappa_j + \kappa_k - 2}{z-t_j}, \tag{8.2}$$

$$w_2(z) = \frac{p}{z-q} - \frac{L}{z-t_i} + \frac{M}{z-t_j} + \frac{N}{(z-t_j)^2}, \tag{8.3}$$

where L, M and N are given by

$$\begin{aligned} t_{ij}^2 L &= q_i q_j^2 p^2 - \{(\kappa_i - 1)q_j + (\kappa_j + \kappa_k) q_i\} q_j p + \kappa_0(\kappa_0 + \kappa_4) q_i, \\ t_{ij}^2 M &= q_i q_j^2 p^2 - \{q_i^2 + (\kappa_j + \kappa_k - 2) q_i q_j + \kappa_i q_j^2\} p + \kappa_0(\kappa_0 + \kappa_4) q_i, \\ t_{ij} N &= q_i q_j^2 p^2 - \{(\kappa_j + \kappa_k - 1) q_i + \kappa_i q_j\} q_j p + \kappa_0(\kappa_0 + \kappa_4) q_j. \end{aligned}$$

Proof. Formula (8.2) readily follows from the first formula of (5.5) by letting $t_k \to t_j$. Next we shall show formula (8.3). In view of (3.7), we notice that there is a relation $h_i + h_j + h_k = p$ and that L, M and L are defined in such a manner that

$$L = \lim_{t_k \to t_j} h_i, \qquad M = L - p, \qquad N = -\lim_{t_k \to t_j} t_{jk} h_j.$$

In particular, we have $h_j + h_k = p - h_i \to p - L = -M$ as $t_k \to t_j$. Hence we have

$$\begin{aligned} && \frac{h_i}{z-t_i} &\to \frac{L}{z-t_i}, \\ \frac{h_j}{z-t_j} + \frac{h_k}{z-t_k} &= \frac{h_j + h_k}{z-t_k} + \frac{t_{jk} h_j}{(z-t_j)(z-t_k)} &&\to -\frac{M}{z-t_j} - \frac{N}{(z-t_j)^2}, \end{aligned}$$

as $t_k \to t_j$. Therefore the second formula of (5.5) leads to (8.3). The proof is complete. $\square$

Note that the local exponents at $z = t_i,\ q,\ \infty$ and the apparentness of q are preserved in the process of coalescence (5.4) $\to$ (8.1). Let λ_1 and λ_2 be the exponents of (8.1) at the coalescent singularity $z = t_j$, which are the roots of the quadratic equation $\lambda^2 - (\kappa_j + \kappa_k - 1)\lambda + N = 0$. We are interested in its discriminant, that is, the squared difference between λ_1 and λ_2:

$$\varDelta = (\lambda_1 - \lambda_2)^2 = (\kappa_j + \kappa_k - 1)^2 - 4N. \tag{8.4}$$

Then the following lemma will play a key role in establishing our main theorem.

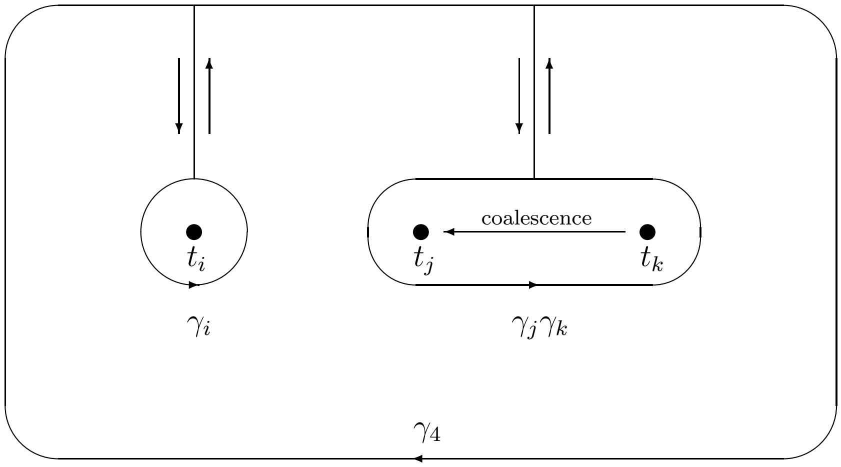


Figure 3: Coalescence of singular points $t_k \to t_j$

Lemma 8.2 (Key Lemma) *The discriminant Δ is G-invariant.*

Proof. Since t_{ij} is G-invariant, it is sufficient to show the G-invariance of $D = -t_{ij}\Delta$. Using $t_{ij} = q_j - q_i$ we have $D = (\kappa_j + \kappa_k - 1)^2(q_i - q_j) + 4t_{ij}N$. Since $t_{ij}N$ is a polynomial of (q_i, q_j, p, κ), so is D; explicitly, $D = D(q_i, q_j, p, \kappa)$ is given by

$$\begin{aligned} D(q_i, q_j, p, \kappa) &= (\kappa_j + \kappa_k - 1)^2(q_i - q_j) \\ &+ 4q_j[q_i q_j p^2 - \{(\kappa_j + \kappa_k - 1)q_i + \kappa_i q_j\}p + \kappa_0(\kappa_0 + \kappa_4)]. \end{aligned} \tag{8.5}$$

For the moment, we think of (q_i, q_j, p, κ) as independent variables, namely, we do not assume the relations (4.2) and (5.2), and put $q_k = q_j$. Then a direct check shows that

$$\begin{aligned} s_0(D) - D &= -4\kappa_0(2q_j p + \kappa_0)\eta, \\ s_i(D) - D &= 4\kappa_i q_j \eta, \\ s_j(D) - D &= 4\kappa_j q_j \eta, \\ s_k(D) - D &= 4\kappa_k q_j \eta, \\ s_4(D) - D &= 0, \end{aligned}$$

where $\eta = 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 - 1$. Here we must take $q_k = q_j$ and Remark 4.1 into account in the evaluations of $s_k(D) - D$ and $s_4(D) - D$, respectively. Hence, under Fuchs' relation (5.2), D is an invariant of s_0, s_1, s_2, s_3, s_4 and so is an invariant of G. The proof is complete. $\square$

The normalization procedure (5.4) $\to$ (7.2) passes through the coalescence process, leading to a parallel normalization of equation (8.1). The corresponding gauge transformation is

$$f = \psi F \qquad \text{with} \qquad \psi = (z - q)(z - t_i)^{\kappa_i/2}(z - t_j)^{(\kappa_j + \kappa_k)/2}, \tag{8.6}$$

by which equation (8.1) is normalized into a Fuchsian equation

$$\frac{d^2F}{dz^2} - W_1(z)\frac{dF}{dz} + W_2(z)F = 0, \tag{8.7}$$

having singular points at $z = t_i$, t_j, q, ∞, such that

(D1) $z = t_i$ is a regular singular point with exponents $\pm\kappa_i/2$;

(D2) $z = q$ is an apparent singular point with exponents ± 1;

(D3) $z = \infty$ is a regular singular point with exponents $(3 \pm \kappa_4)/2$;

(D4) $z = t_j$ is a regular singular point with exponents $(-1 \pm \sqrt{\varDelta})/2$.

It follows from (D4) that the trace of the monodromy matrix around $z = t_j$ is $-2\cos\pi\sqrt{\varDelta}$.

We proceed to taking the isomonodromic deformation into account and shall complete the proof of Theorem 2.3, leaving a certain technical issue to the next section.

Proof of Theorem 2.3. The notation of §2 will be retained in the subsequent discussions, except that the transformation r in diagram (2.13) will be written s upon identifying r and s. Establishing Theorem 2.3 amounts to showing that

$$s(x_i) = x_i \qquad (i = 1, 2, 3), \tag{8.8}$$

where $x_i = x_i(q, p, t, \kappa)$, which was defined in (2.8), is a holomorphic function of (q, p, t, κ) and $s(x_i)$ is understood to be $x_i(s(q), s(p), s(t), s(\kappa))$. It is sufficient to show (8.8) for the generators $s = s_j$ of the Bäcklund transformation group G. But this claim for $j = 1, 2, 3, 4$ is trivial, since s_j in this case is a gauge transformation as seen in §6, which does not change monodromy representation and hence the value of x_i. So the substantial part of the proof is to verify the claim for $j = 0$, though the reasoning below will carry over for $j = 1, 2, 3, 4$ as well. Moreover, we have only to establish it for generic values of κ; the validity of (8.8) for each κ in a dense open subset leads to the validity for every κ due to the continuous dependence of $x_i = x_i(t, q, p, \kappa)$ upon κ. As such a dense open condition on κ, we employ

$$1 - \kappa_j - \kappa_k \in \mathbb{C} \setminus \mathbb{R} \qquad (1 \le j < k \le 3), \tag{8.9}$$

for a certain technical reason to be explained in Lemma 8.3 below.

The idea of *coalescence along isomonodromic deformation* proceeds as follows. Given a point $(q, p) \in (\mathbb{C} \setminus \{t_1, t_2, t_3\}) \times \mathbb{C}$, let $(q(t), p(t))$ be the solution trajectory to the Hamiltonian system (3.6) starting from the initial point (q, p). Since (3.6) describes the isomonodromic deformation of Fuchsian equations (7.2), their monodromy matrix along the loop $\gamma_j\gamma_k$ (see Figure 3),

$$M_jM_k = (M_jM_k)(q(t), p(t), t, \kappa),$$

is independent of t on the trajectory $(q(t), p(t))$. Assume that the trajectory admits an accumulation point $(\bar{q}, \bar{p})$ as t_k tends to t_j along a curve. As t_k tends to t_j in such a manner, the Fuchsian equation (7.2) degenerates into (8.7), where (q, p) in (8.7) should be replaced by $(\bar{q}, \bar{p})$. In this process, the role of the matrix M_jM_k changes from being the global monodromy matrix of (7.2) along the loop $\gamma_j\gamma_k$ into being the local monodromy matrix of (8.7) at the coalescent singular point $z = t_j$. The former role gives $x_i = \mathrm{Tr}(M_jM_k)$, while the latter yields $\mathrm{Tr}(M_jM_k) = -2\cos\pi\sqrt{\varDelta}$ by virtue of the remark after (D4). Therefore we have

$$x_i = -2\cos\pi\sqrt{\varDelta}. \tag{8.10}$$

We can apply the same procedure with the initial point (q, p) replaced by $(s(q), s(p))$. Since $(s(q(t)), s(p(t)))$ has an accumulation point $(s(\bar{q}), s(\bar{p}))$, the same reasoning as above yields

$$s(x_i) = -2\cos\pi\sqrt{s(\varDelta)}. \tag{8.11}$$

Thanks to Lemma 8.2, which asserts that $\varDelta$ is an s-invariant, (8.10) and (8.11) lead to the desired equality (8.8). This completes the proof of Theorem 2.3.

For the true end of the proof, however, there is an extra issue yet to be argued, namely, the existence of the accumulation point $(\bar{q}, \bar{p})$. More precisely, the existence must be assured in such a manner that $(\bar{q}, \bar{p})$ is located in a general position. To understand what this means, we should notice that the arguments leading to Lemmas 8.1 and 8.2 are valid only under the condition that the apparent singular point q is different from the other singular points t_i, t_j, ∞ and that p is a finite complex number. In the current situation, this condition should be applied to $(q,p) = (\bar{q}, \bar{p})$ as well as to $(q,p) = (s(\bar{q}), s(\bar{p}))$. For the generators $s = s_0$, s_1, s_2, s_3, s_4 of G, formula (4.1) implies that the condition applied to them is expressed as

$$\bar{q} \in \mathbb{C} \setminus \{t_i, t_j\}, \qquad \bar{p} \in \mathbb{C} \setminus \{0\}, \qquad \bar{q} + \frac{\kappa_0}{\bar{p}} \in \mathbb{C} \setminus \{t_i, t_j\}. \tag{8.12}$$

Thus an accumulation point $(\bar{q}, \bar{p})$ is said to be in a general position if it satisfies condition (8.12). Intuitively the existence of such a point is quite likely, but logically nontrivial. It is at this stage that the generic condition (8.9) on κ is used to provide the following:

Lemma 8.3 *Suppose that $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathcal{K}$ satisfies condition* (8.9)*. Then there exists an open subset V of $(\mathbb{C} \setminus \{t_1, t_2, t_3\}) \times \mathbb{C}$ such that for every $(q,p) \in V$, the solution trajectory $(q(t), p(t))$ to the Hamiltonian system* (3.6) *starting from the initial point (q,p) admits an accumulation point $(\bar{q}, \bar{p})$ satisfying condition* (8.12) *as t_k tends to t_j along a curve.*

Lemma 8.3 guarantees the existence of an accumulation point $(\bar{q}, \bar{p})$ in a general position only when the initial point (q,p) belongs to an open subset V. But this is enough for proving (8.8). Indeed, by Lemma 8.3, equation (8.8) is valid at least for $(q,p) \in V$. Then the unicity theorem for holomorphic functions implies that it remains valid for every (q,p), since $x_i = x_i(q,p,t,\kappa)$ is holomorphic in (q,p) and the space $\mathcal{E}_t(\kappa) \cong (\mathbb{C} \setminus \{t_1, t_2, t_3\}) \times \mathbb{C}$ is connected. To establish Theorem 2.3, it only remains to prove Lemma 8.3. This final task will be done in §9. □

9 Accumulation Points in a General Position

We shall establish Lemma 8.3 for $j = 1$ and $k = 3$, namely, for the coalescence process $t_3 \to t_1$; due to the symmetry in t_i, t_j, t_k, the other cases can be treated in a similar manner. For this purpose, we may work with the Hamiltonian system (3.1) with single time variable x instead of the system (3.6) with three time variables $t = (t_1, t_2, t_3)$. In view of (3.4), letting $t_3 \to t_1$ means $x \to 0$. With these remarks, Lemma 8.3 is reduced to the following:

Lemma 9.1 *Suppose that $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathcal{K}$ satisfies a generic condition*

$$1 - \kappa_1 - \kappa_3 \in \mathbb{C} \setminus \mathbb{R}. \tag{9.1}$$

Then there exists a 2-parameter family of solutions to the Hamiltonian system (3.1)*,*

$$(q(x,c), p(x,c)), \qquad c = (c_1, c_2) \in U, \tag{9.2}$$

where U is an open subset of $\mathbb{C}^2$, such that the following conditions are satisfied:

(1) *The correspondence* $c = (c_1, c_2) \mapsto (q(x_0, c), p(x_0, c))$ *defines a biholomorphic map of* U *onto an open subset* V *of* $(\mathbb{C} \setminus \{0,1\}) \times \mathbb{C}$*, where* x_0 *is a point in* $\mathbb{C} \setminus \{0,1\}$*.*

(2) *The solution* (9.2) *admits an accumulation point* $(q(c), p(c))$ *as* $x \to 0$ *such that*

$$q(c) \in \mathbb{C} \setminus \{0,1\}, \qquad p(c) \in \mathbb{C} \setminus \{0\}, \qquad q(c) + \frac{\kappa_0}{p(c)} \in \mathbb{C} \setminus \{0,1\}. \tag{9.3}$$

Note that condition (9.3) corresponds to (8.12) in the last section. For a proof of Lemma 9.1, we utilize a result by Takano [28] and Kimura [14], who established a reduction theorem and constructed a 2-parameter family of solutions to $\mathrm{P_{VI}}$ around its fixed singular points. We recall their construction in a manner suitable for our purpose. Put

$$\begin{aligned} E(r,\rho) &= \{\, (x, Q, P) \in \mathbb{C}^3 \,:\, |x| < r,\ |Q| < \rho,\ |xP| < \rho,\ |QP| < \rho \,\}, \\ E(\rho) &= \{\, (Q, P) \in \mathbb{C}^2 \,:\, |Q| < \rho,\ |QP| < \rho \,\}. \end{aligned}$$

Then the following lemma is an easy consequence of Theorems 1 and 2 in Takano [28].

Lemma 9.2 *Suppose that* $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathcal{K}$ *satisfies condition* (9.1)*. Then there exist positive constants* r*,* $\rho > 0$ *and a unique canonical transformation*

$$q = b(x, Q, P), \qquad p = a(x, Q, P)$$

that reduces system (3.1) *into a Hamiltonian system with Hamiltonian*

$$h_0(x, Q, P) = \{(QP)^2 + (1 - \kappa_1 - \kappa_3)(QP)\}/x, \tag{9.4}$$

where $b(x, Q, P)$ *and* $a(x, Q, P)$ *are holomorphic functions in* $E(r, \rho)$ *such that*

$$|b(0, Q, P) - Q| \le M|Q|^2, \qquad |a(0, Q, P) - P| \le M \qquad \text{for} \quad (Q, P) \in E(\rho), \tag{9.5}$$

with some positive constant M*; we may and shall assume that* $M > 2$*.*

The Hamiltonian system with Hamiltonian (9.4) has a first integral QP and is settled as

$$Q(x, c) = c_1 x^{\lambda}, \quad P(x, c) = c_2 x^{-\lambda} \qquad \text{with} \quad \lambda = \lambda(c) = 1 - \kappa_1 - \kappa_3 + 2c_1c_2, \tag{9.6}$$

where $c = (c_1, c_2) \in \mathbb{C}^2$ is an arbitrary constant. Then Lemma 9.2 asserts that

$$q(x, c) = b(x, Q(x, c), P(x, c)), \qquad p(x, c) = a(x, Q(x, c), P(x, c)), \tag{9.7}$$

yields a solution to system (3.1), provided that $(x, Q(x, c), P(x, c)) \in E(r, \rho)$, namely,

$$0 < |x| < r, \qquad |Q(x, c)| < \rho, \qquad |xP(x, c)| < \rho, \qquad |c_1c_2| < \rho.$$

In terms of $\arg x$ and $\log |x|$, the second and third conditions are expressed as in Table 1, the exhibition of which is divided into five cases according to the values of $\mathrm{Re}\,\lambda$.

Proof of Lemma 9.1. By assumption (9.1), we can choose a number ρ_0 so that

$$0 < \rho_0 < \min\{\rho,\ |\mathrm{Im}(1 - \kappa_1 - \kappa_3)|/2,\ 1\}. \tag{9.8}$$

case	λ	$\arg x$	$\log\lvert x\rvert$
1	$\mathrm{Re}\,\lambda > 1$	$(\mathrm{Im}\,\lambda)(\arg x) < L_0$	$L_2(\arg x) < \log\lvert x\rvert < L_1(\arg x)$
2	$\mathrm{Re}\,\lambda = 1$	$(\mathrm{Im}\,\lambda)(\arg x) < L_0$	$\log\lvert x\rvert < L_1(\arg x)$
3	$0 < \mathrm{Re}\,\lambda < 1$	no constraints	$\log\lvert x\rvert < \min\{L_1(\arg x),\, L_2(\arg x)\}$
4	$\mathrm{Re}\,\lambda = 0$	$(\mathrm{Im}\,\lambda)(\arg x) > L_0$	$\log\lvert x\rvert < L_2(\arg x)$
5	$\mathrm{Re}\,\lambda < 0$	$(\mathrm{Im}\,\lambda)(\arg x) > L_0$	$L_1(\arg x) < \log\lvert x\rvert < L_2(\arg x)$

$$\begin{aligned} L_0 &= (\mathrm{Re}\,\lambda)\log(\rho/|c_2|) + (\mathrm{Re}\,\lambda - 1)\log(\rho/|c_1|), \\ L_1(\alpha) &= \frac{(\mathrm{Im}\,\lambda)\alpha + \log(\rho/|c_1|)}{\mathrm{Re}\,\lambda}, \\ L_2(\alpha) &= \frac{(\mathrm{Im}\,\lambda)\alpha - \log(\rho/|c_2|)}{\mathrm{Re}\,\lambda - 1}. \end{aligned}$$

Table 1: The conditions $|Q(x,c)| < \rho$ and $|xP(x,c)| < \rho$

Out of later necessity, if $\kappa_0 \neq 0$, we take ρ_0 so as to satisfy an additional condition

$$\rho_0 < \frac{|\kappa_0|}{2}, \tag{9.9}$$

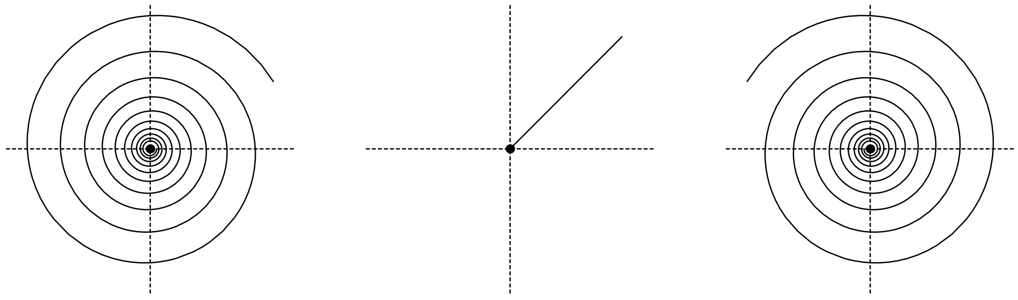
which should be neglected if $\kappa_0 = 0$. As the open subset U mentioned in (9.2), we put

$$U = \{\, c = (c_1, c_2) \in \mathbb{C}^2 \,:\, 0 < |c_1 c_2| < \rho_0 \,\}. \tag{9.10}$$

For each $c \in U$, let $D(c)$ be the set of those points on the universal covering of $0 < |x| < r$ which satisfy the conditions in Table 1. By (9.6), (9.8) and (9.10), one has $\mathrm{Im}\,\lambda(c) \neq 0$, which implies that $D(c)$ is a nonempty domain that contains a curve tending to the origin $x = 0$. Then the solution $(q(x,c), p(x,c))$ in (9.7) makes sense on the domain $D(c)$.

For each $c \in U$, we shall show that $(q(x,c), p(x,c))$ admits an accumulation point $(q(c), p(c))$ satisfying condition (9.3) as $x \to 0$ along a curve in $D(c)$. Let μ be a number such that

$$0 < \mu < \frac{|c_1 c_2|}{M(|\kappa_0| + 8)}. \tag{9.11}$$



$(\operatorname{Re}\lambda)(\operatorname{Im}\lambda) < 0$ $\qquad$ $\operatorname{Re}\lambda = 0$ $\qquad$ $(\operatorname{Re}\lambda)(\operatorname{Im}\lambda) > 0$

Figure 4: The curve γ: a spiral or a line segment

and consider a curve in the x-plane defined by

$$\gamma = \left\{ x \,:\, (\operatorname{Re}\lambda)\log|x| - (\operatorname{Im}\lambda)(\arg x) = \log\frac{\mu}{|c_1|},\ |x| < \frac{\mu\rho}{|c_1c_2|} \right\}$$

Note that (9.8), (9.10), (9.11) and $M > 2$ imply $\mu < \rho$. Along the curve γ one has

$$|Q(x,c)| = \mu, \qquad |P(x,c)| = \frac{|c_1c_2|}{\mu}. \tag{9.12}$$

In particular, $|Q(x,c)| < \rho$ and $|xP(x,c)| < \rho$, and hence the curve γ lies in the domain $D(c)$. The shape of γ is indicated in Figure 4; if $\operatorname{Re}\lambda \neq 0$, it is a spiral curve winding around and tending to the origin, while if $\operatorname{Re}\lambda = 0$, it is a line segment terminating at the origin.

In view of (9.12), as x tends to the origin along the curve γ, there exists an accumulation point $(Q(c), P(c)) \in E(\rho)$ of $(Q(x,c), P(x,c))$ such that $Q(c)P(c) = c_1c_2$ and

$$|Q(c)| = \mu, \qquad |P(c)| = \frac{|c_1c_2|}{\mu}. \tag{9.13}$$

Accordingly, as $x \to 0$ along γ, $(q(x,c), p(x,c))$ has an accumulation point $(q(c), p(c))$ with

$$q(c) = b(0, Q(c), P(c)), \qquad p(c) = a(0, Q(c), P(c)).$$

We shall show that $(q(c), p(c))$ satisfies the desired property (9.3). Hereafter $(Q(c), P(c))$ and $(q(c), p(c))$ will be abbreviated to (Q, P) and (q, p) respectively. The arguments below are rather technical, but the idea itself is very simple: If $\mu > 0$ is sufficiently small, then (9.13) implies that Q is sufficiently small but not zero and $P = c_1c_2/Q$ is sufficiently large but finite. Now (9.5) means that (q, p) equals (Q, P) up to a first order error term. Then one can show that q is small but not zero, p is large but finite and also $q + \kappa_0/p$ is small but not zero.

We verify the first and second conditions of (9.3). Note that (9.8), (9.10) and (9.11) yield

$$M < \frac{|c_1c_2|}{2\mu} < \frac{1}{2\mu}, \qquad \mu < \frac{1}{2}.$$

Then (9.5) and (9.13), combined with these inequalities, lead to

$$\begin{array}{lllllllll}
|q| & \le & |Q|+|q-Q| & \le & |Q|+M|Q|^2 & = & (1+M\mu)\mu & < & \dfrac{3\mu}{2} & < & \dfrac{3}{4}, \\
|q| & \ge & |Q|-|q-Q| & \ge & |Q|-M|Q|^2 & = & (1-M\mu)\mu & > & \dfrac{\mu}{2} & > & 0, \\
|p| & \ge & |P|-|p-P| & \ge & |P|-M & = & \dfrac{|c_1c_2|}{\mu}-M & > & \dfrac{|c_1c_2|}{2\mu} & > & 0,
\end{array}$$

which verifies the first and second conditions of (9.3) as desired.

We proceed to verify the third condition of (9.3). We may assume $\kappa_0 \neq 0$; for otherwise the third condition is the same as the first one. We observe that

$$|p| > \frac{|c_1c_2|}{2\mu}, \qquad |c_1c_2| < \frac{|\kappa_0|}{2}, \qquad \mu < \frac{|c_1c_2|}{2|\kappa_0|}. \tag{9.14}$$

Indeed, the first one is already seen in the last paragraph, the second one is obtained from (9.9) and (9.10), and the last one follows from (9.11) and $M > 2$, respectively. Let

$$R = \left(q+\frac{\kappa_0}{p}\right) - \left(1+\frac{\kappa_0}{c_1c_2}\right)Q = (q-Q) + \left(\frac{\kappa_0}{p}-\frac{\kappa_0}{P}\right) = (q-Q) + \frac{\kappa_0(P-p)}{pP}.$$

By applying (9.5), (9.13), (9.14) and (9.11) successively, R is estimated as

$$\begin{array}{lllll}
|R| & \le & |q-Q| + \dfrac{|\kappa_0||p-P|}{|p||P|} & \le & M|Q|^2 + \dfrac{|\kappa_0|M}{|p||P|} \\
 & < & M\mu^2 + \dfrac{2|\kappa_0|M\mu^2}{|c_1c_2|^2} & = & \dfrac{M\mu^2(|c_1c_2|^2+2|\kappa_0|)}{|c_1c_2|^2} \\
 & < & \dfrac{M\mu^2|\kappa_0|(|\kappa_0|+8)}{4|c_1c_2|^2} & < & \dfrac{|\kappa_0|\mu}{4|c_1c_2|}.
\end{array}$$

On the other hand, the second inequality in (9.14) leads to

$$\begin{array}{lllllll}
\left|1+\dfrac{\kappa_0}{c_1c_2}\right| & \le & \dfrac{|\kappa_0|}{|c_1c_2|}+1 & < & \dfrac{|\kappa_0|}{|c_1c_2|}+\dfrac{|\kappa_0|}{2|c_1c_2|} & = & \dfrac{3|\kappa_0|}{2|c_1c_2|}, \\
\left|1+\dfrac{\kappa_0}{c_1c_2}\right| & \ge & \dfrac{|\kappa_0|}{|c_1c_2|}-1 & > & \dfrac{|\kappa_0|}{|c_1c_2|}-\dfrac{|\kappa_0|}{2|c_1c_2|} & = & \dfrac{|\kappa_0|}{2|c_1c_2|}.
\end{array}$$

These preliminary estimates, (9.12) and the third inequality in (9.14) yield

$$\begin{array}{lll}
\left|q+\dfrac{\kappa_0}{p}\right| & = & \left|\left(1+\dfrac{\kappa_0}{c_1c_2}\right)Q+R\right| \le \left|1+\dfrac{\kappa_0}{c_1c_2}\right||Q|+|R| < \dfrac{3|\kappa_0|\mu}{2|c_1c_2|}+\dfrac{|\kappa_0|\mu}{4|c_1c_2|} = \dfrac{7|\kappa_0|\mu}{4|c_1c_2|} < \dfrac{7}{8}, \\
\left|q+\dfrac{\kappa_0}{p}\right| & = & \left|\left(1+\dfrac{\kappa_0}{c_1c_2}\right)Q+R\right| \ge \left|1+\dfrac{\kappa_0}{c_1c_2}\right||Q|-|R| > \dfrac{|\kappa_0|\mu}{2|c_1c_2|}-\dfrac{|\kappa_0|\mu}{4|c_1c_2|} = \dfrac{|\kappa_0|\mu}{4|c_1c_2|} > 0.
\end{array}$$

Hence the third condition of (9.3), i.e., assertion (2) of Lemma 9.1 is verified. The proof of assertion (1) is omitted, since it is a standard inverse function argument, in which the open set U will be replaced by a smaller one, if necessary. The proof is complete. $\square$

10 Finding Bäcklund Transformations

The Bäcklund transformations s_i have been found by Okamoto [21], Arinkin and Lysenko [2], Noumi and Yamada [18] and others by various methods. In any case, s_1, s_2, s_3, s_4 are easy to find, since they are elementary gauge transformations as constructed in §6. But things are different with the transformation s_0; it cannot be a gauge transformation of rank two differential equations, since it does change local monodromy data. Now we wish to propose another way of finding s_0 from our point of view.

Our idea is to revisit Lemma 8.2 together with formula (8.5), which is a key observation in this paper. It asserts that the discriminant Δ in (8.4), or equivalently the function D in (8.5), is G-invariant. Here we write $D = D(q,p,t,\kappa)$, since D is a polynomial of (q,p,t,κ) with $t_j = t_k$. An essence of the reasoning in §8 is the implication that if there is an invariance relation

$$D(Q,P,t,\sigma_0(\kappa)) = D(q,p,t,\kappa), \tag{10.1}$$

then one has $x_i(Q,P,t,\sigma_0(\kappa)) = x_i(q,p,t,\kappa)$ for the global monodromy data $x_i = \mathrm{Tr}(M_jM_k)$. Now our principle — a Bäcklund transformation should be a pull-back of the identity transformation on the moduli of global monodromy data — suggests that relation (10.1) would give us a Bäcklund transformation $(q,p) \mapsto (Q,P)$. So it must be promising to find out (Q,P) as a function of (q,p) satisfying (10.1). This thought brings our attention to the difference

$$E = E(Q,P;q,p;t,\kappa) = D(Q,P,t,\sigma_0(\kappa)) - D(q,p,t,\kappa). \tag{10.2}$$

Consider E as a polynomial of (t_i,t_j) and let E_{mn} denote the coefficient of the term $t_i^m t_j^n$ in E. If $E \equiv 0$, each coefficient E_{mn} must vanish. We especially take a look at E_{12} and E_{11}:

$$\begin{cases} E_{12} &=& 4(p-P)(p+P), \\ E_{11} &=& 4p(\kappa_j+\kappa_k-1-2qp)+4P(\kappa_i+\kappa_4+2QP). \end{cases}$$

As is easily seen, the system of equations $E_{12} = E_{11} = 0$ has two solutions

$$Q = q + \frac{\kappa_0}{p}, \qquad P = p, \tag{10.3}$$

$$Q = q + \frac{\kappa_0+\kappa_i+\kappa_4}{p}, \qquad P = -p. \tag{10.4}$$

The first solution (10.3) is none other than the Bäcklund transformation s_0 we seek. In this case we can check that substituting (10.3) into (10.2) yields $E \equiv 0$.

On the other hand, the second solution (10.4) does not imply $E \equiv 0$ in general. Indeed, (10.4) leads to $E_{02} = 4(2\kappa_i - \kappa_j - \kappa_k + 1)p$, and hence E does not vanish unless κ satisfies

$$2\kappa_i - \kappa_j - \kappa_k + 1 = 0. \tag{10.5}$$

Assume that (10.5) is the case. Then, substituting (10.4) and (10.5) into (10.2), we have $E = 2\kappa_i(\kappa_4-\kappa_i)(\kappa_4+\kappa_i)/p$. Therefore, (10.4) yields $E \equiv 0$ if and only if κ also satisfies

$$\kappa_i(\kappa_4-\kappa_i)(\kappa_4+\kappa_i) = 0. \tag{10.6}$$

We wonder whether the transformation (10.4) has any meaning under the restriction of parameters, (10.5) and (10.6). But this point will not be touched in this paper.

It is worth considering whether there is any other Bäcklund transformation than those are already known. As to this question, the following proposition shows that the Bäcklund transformations are exhausted by the known ones, even if the class is enlarged to the analytic category.

Proposition 10.1 *Let $\pi : \mathcal{M}_t \to \mathcal{K}$ be the family of moduli spaces of stable parabolic bundles with connections in §7. Let σ be an analytic automorphism of $\mathcal{K}$ such that $\theta(\kappa) = \theta(\sigma(\kappa))$ for $\kappa \in \mathcal{K}$ and let s be a bimeromorphic automorphism of $\mathcal{M}_t$ such that $\sigma\pi = \pi s$. Assume that for general values of $\kappa \in \mathcal{K}$, the analytic isomorphisms $s_\kappa : \mathcal{M}_t(\kappa) \to \mathcal{M}_t(\sigma(\kappa))$ induce the identity on $\mathcal{S}(\theta(\kappa))$ via the Riemann-Hilbert correspondence* (7.5). *Then $\sigma \in W(\mathrm{D}_4^{(1)})$ and s is a known Bäcklund transformation, the unique lift of σ.*

Proof. From the invariant-theoretical argument in Terajima [29], it is not difficult to see that the analytic quotient $\mathcal{K}/W(\mathrm{D}_4^{(1)})$ is biholomorphic to the complex 4-space $\mathbb{C}_\theta^4$ with coordinates $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. In particular, distinct $W(\mathrm{D}_4^{(1)})$-orbits in $\mathcal{K}$ have distinct values of θ. This observation shows that any analytic automorphism σ of $\mathcal{K}$ such that $\theta(\kappa) = \theta(\sigma(\kappa))$ for $\kappa \in \mathcal{K}$ is necessarily an element of $W(\mathrm{D}_4^{(1)})$. Then clearly the transformation s is obtained as the unique lift of σ relative to the Riemann-Hilbert correspondence. $\square$

We conclude this paper by putting some questions to ourselves. We were able to characterize the Bäcklund transformations in a natural manner in terms of the Riemann-Hilbert correspondence based on the following nice observation: The difference of the two local exponents at a coalescent regular singular point happens to be an invariant of the Bäcklund transformations (Lemma 8.2). Does this phenomenon occur just by chance or more universally? Do similar phenomena occur for other Painlevé equations than $\mathrm{P_{VI}}$ or for Garnier systems? If so, do they help us find Bäcklund transformations for those equations?

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