DEFORMATION OF OKAMOTO–PAINLEVÉ PAIRS AND PAINLEVÉ EQUATIONS

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Abstract. In this paper, we introduce the notion of generalized rational Okamoto–Painlevé pair (S,Y) by generalizing the notion of the spaces of initial conditions of Painlevé equations. After classifying those pairs, we will establish an algebro-geometric approach to derive the Painlevé differential equations from the deformation of Okamoto–Painlevé pairs by using the local cohomology groups. Moreover the reason why the Painlevé equations can be written in Hamiltonian systems is clarified by means of the holomorphic symplectic structure on $S-Y$. Hamiltonian structures for Okamoto–Painlevé pairs of type $\tilde{E}_7(=P_{II})$ and $\tilde{D}_8(=P_{III}^{\tilde{D}_8})$ are calculated explicitly as examples of our theory.

0. Introduction

In the study of Painlevé equations, the spaces of initial conditions introduced by K. Okamoto [O1], [O2], [O3] have been playing essential roles. It is known that each Painlevé differential equation is equivalent to one of Hamiltonian systems whose Hamiltonians are given by the polynomials in two variables (x,y). (See Table 7 and 8 in §7). The space $(x,y)\in\mathbf{C}^2$ can be compactified and one can obtain a pair (S,Y) of complex projective surface S and an anti-canonical divisor $Y\in|-K_S|$ such that $S-Y_{red}$ becomes a space of initial conditions. In the study of the space of initial conditions as in [O1], [MMT], it became clear that after eliminating the singularities of Painlevé differential equation by blowings-up, the boundary divisor Y should have the same configuration as in the list of degenerate elliptic curves classified by Kodaira [Kod2]. This condition can be translated into the following conditions. Let $Y=\sum_{i=1}^r m_iY_i\in|-K_S|$ be the irreducible decomposition. Then Y is called *of canonical type* if and only if

$$\deg(-K_S)_{|Y_i}=\deg Y_{|Y_i}=Y\cdot Y_i=0\quad\text{for all } i.$$

In [Sa-Tak], we call such a pair (S,Y) an *Okamoto–Painlevé pair* if $S-Y_{red}$ contains $\mathbf{C}^2$ as a Zariski open set and $F=S-\mathbf{C}^2$ is a normal crossing divisor. One can verify that all compactifications of the spaces of initial conditions of known Painlevé equations satisfy these conditions (cf. [O1], [MMT]). Therefore, in this notation, the former studies of Painlevé equations

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establish the route in the direction:

$$\boxed{\text{Painlevé equations}} \quad \Longrightarrow \quad \boxed{\text{Okamoto–Painlevé pairs } (S,Y)}. \tag{1}$$

The main purpose of this paper is to establish the route backward, that is, the route in the following direction

$$\boxed{\text{Okamoto–Painlevé pairs } (S,Y)} \quad \Longrightarrow \quad \boxed{\text{Painlevé equations}}. \tag{2}$$

Our main tool here is the deformation theory of pairs (S, Y) (cf. [KS], [Kaw]) and local cohomology exact sequence (cf. [B-W]). The deformation theory was established by Kodaira–Spencer in [KS] in late 1950's. Kawamata [Kaw] generalized the deformation theory to compactified complex manifolds, or pairs of smooth compact complex manifolds and simple normal crossing subvarieties. Applying the deformation theory, we can see that the space of infinitesimal deformations of the Okamoto–Painlevé pair (S, Y) is isomorphic to the cohomology group $H^1(S, \Theta_S(-\log D))$ where $D = Y_{red}$. Looking at the restriction homomorphism

$$\mathsf{res} : H^1(S, \Theta_S(-\log D)) \longrightarrow H^1(S - D, \Theta_{S-D}),$$

we see that the kernel of the restriction map res has the important meaning, that is, the direction corresponding to the kernel of res is the infinitesimal deformation of (S, Y) which induces the trivial deformation on $S - D$. Roughly speaking, one can say that the Painlevé differential equations describe the deformations corresponding to the direction of the kernel of the restriction map.

To be precise, let us consider a family of Okamoto–Painlevé pairs $\mathcal{D} \hookrightarrow \mathcal{S} \longrightarrow \mathcal{B}_R$ with one-dimensional base space $\mathcal{B}_R$ with a coordinate t such that the Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ lies in the kernel of the restriction map res. Then by using the affine covering of $\mathcal{S} - \mathcal{D}$ and Čech cocycles, we can derive a system of ordinary differential equation. Note that this observation will be applicable to the higher dimensional cases.

From these observation, we see that the kernel of res corresponds to the directions of time variables in the Painlevé differential equation.

Furthermore, we can apply the local cohomology exact sequence to our settings and obtain the exact sequence (cf. [B-W], [Gr])

$$H^1_D(S, \Theta_S(-\log D)) \xrightarrow{\mu} H^1(S, \Theta_S(-\log D)) \xrightarrow{\mathsf{res}} H^1(S - D, \Theta_{S-D}).$$

Under the condition that (S, Y) is of non-fibered type, the map μ is injective, and hence, the local cohomology group $H^1_D(S, \Theta_S(-\log D))$ coincides with the kernel of res. Therefore, non-zero element of the local cohomology group $H^1_D(S, \Theta_S(-\log D))$ corresponds to a time variable of the Painlevé equation. For a generalized rational Okamoto–Painlevé pair (S, Y) of additive type, if Y_{red} is normal crossing divisor, Terajima [T] proved that the dimension of the local cohomology group is positive, hence, we can always obtain a differential equation.

In order to obtain an explicit differential equation for each type R from our setting, we need to construct a global family of generalized rational Okamoto–Painlevé pairs of type R over a parameter space $\mathcal{M}_R \times \mathcal{B}_R$ which is semiuniversal at each point. Moreover we need to introduce a good affine open covering of the total space such that the rational two form ω_S restricted to $S - Y_{red}$ has a canonical form. (See §5 and §6.)

The organization of this paper is as follows. In §1, we define the notion of generalized Okamoto–Painlevé pairs and recall the relation to generalized Halphen surfaces, which are studied by Sakai [Sakai]. We also classify generalized rational Okamoto–Painlevé pairs (S, Y) such that Y_{red} are normal crossing divisors. A generalized rational Okamoto–Painlevé pair (S, Y) is called of fibered type if there exists a structure of elliptic fibration $f : S \longrightarrow \mathbf{P}^1$ such that $f^*(\infty) = nY$ for some $n \geq 1$. We show that (S, Y) is not of fibered type if and only if regular algebraic functions on $S - Y_{red}$ are just constant functions. This fact is also important for later purpose. After recalling the theory of deformation of pairs in §2, we investigate the cohomology groups for generalized rational Okamoto–Painlevé pairs. In §3, we will apply the theory of local cohomology to our situation, and have the fundamental exact sequence (Proposition 3.1). Moreover, we state an important result, Theorem 3.1, which is proved in [T]. After reviewing the Kodaira–Spencer theory in §4, in §5, we will explain how one can construct the family of generalized rational Okamoto–Painlevé pairs and their open coverings. In §6, we will state our main theorem (Theorem 6.1), which states that from the special global deformation of generalized Okamoto–Painlevé pairs one can obtain the differential equations. Moreover the reason for the equations to be in Hamiltonian systems will be explained geometrically. In §8, we will derive the Painlevé equations from the deformations of Okamoto–Painlevé pairs of type $\tilde{E}_7$ and $\tilde{D}_8$.

Prior to our work here, in [SU], M.-H. Saito and H. Umemura essentially pointed out that the deformation of spaces of initial conditions describes Painlevé equation completely. In this sense, this paper is a continuation of [SU], though we have clarified the meaning of time variables by means of local cohomology groups in this paper.

The recent work due to Sakai [Sakai] introduce the following beautiful viewpoint: The geometry of certain rational surfaces with the symmetries induced by Cremona transformations describe the discrete Painlevé equations and the Painlevé equation can be obtained as a limit of the discrete Painlevé equations. We owe much to his beautiful paper [Sakai]. In particular, some of the explicit calculations are done by using his descriptions of the family of Okamoto–Painlevé pairs in [Sakai].

The works of Takano et al [MMT], [ST] is also essential to our work. In §8, we use the coordinate systems introduced by them.

The series of the work is started by [Sa-Tak], where we introduce the notion of Okamoto-Painlevé pair and classify Okamoto–Painlevé pairs (S, Y).

One of missing points in our work here is the theory of Bäcklund transformation of Okamoto–Painlevé pairs. In this direction, one should refer to a series of works of M. Noumi and Y. Yamada (e.g. [NY]), also Sakai's work [Sakai]. In [SU], the authors tried to understand the Bäcklund transformation by using the notion of "flip " or "flop" in the theory of the minimal models of higher dimensional varieties. We will investigate this point in future.

1. Generalized Okamoto–Painlevé Pairs

Let S be a complex projective surface. We denote by K_S the canonical line bundle or the canonical divisor class of S. Assume that the anti-canonical divisor class $-K_S$ is effective, that is, there exists an effective divisor $Y \in |-K_S|$. Geometrically, this is equivalent to the existence of a rational 2-form ω_Y on S whose corresponding divisor $(\omega_Y) = -(\omega_Y)_\infty = -Y$. Such a divisor Y is called an anti-canonical divisor of S as usual. Since ω_Y dose not vanish on $S - Y$, it induces a holomorphic symplectic structure on $S - Y$.

In [Sa-Tak], we introduce a notion of Okamoto–Painlevé pair (S, Y) which is a pair of complex projective surface S and an anti-canonical divisor Y satisfying certain conditions ([Definition 2.1 [Sa-Tak]]). Generalizing the notion, we will start this section with the following definition.

Definition 1.1. Let (S, Y) be a pair of a complex projective surface S and an anti-canonical divisor $Y \in |-K_S|$ of S. Let $Y = \sum_{i=1}^r m_i Y_i$ be the irreducible decomposition of Y. We call a pair (S, Y) a *generalized Okamoto–Painlevé Pair* if for all $i, 1 \leq i \leq r$,

$$Y \cdot Y_i = \deg Y_{|Y_i} = 0. \tag{3}$$

According to [Sa-Tak], we listed the additional conditions for Okamoto–Painlevé pairs besides the condition (3) as follows.

(1) Let us set $D := Y_{red} = \sum_{i=1}^r Y_i$. Then $S - D$ contains the complex affine plane $\mathbf{C}^2$ as a Zariski open set.
(2) Set $F = S - \mathbf{C}^2$ where $\mathbf{C}^2$ is the same Zariski open set as in (1). Then F is a (reduced) divisor with normal crossings. In particular, $D = Y_{red}$ is also a reduced divisor with normal crossings.

Under this definition, we proved the following classification of Okamoto–Painlevé pairs in [Sa-Tak]. We remark that an Okamoto–Painlevé pair of type $\tilde{D}_7$ did not appear in the classification of classical Painlevé equations [O1].

Theorem 1.1. ([Sa-Tak].) *Let (S, Y) be a generalized Okamoto-Painlevé pair and assume that $S - Y_{red}$ contains $\mathbf{C}^2$ and $F = S - \mathbf{C}^2$ is a reduced divisor with normal crossings. (That is, (S, Y) is an Okamoto–Painlevé pair in original sense.) Then we have the following assertions.*

(1) *The surface S is a projective rational surface.*
(2) *The configuration of Y counting with multiplicity is in the list of Kodaira's classification of singular fibers of elliptic surfaces (cf.* [Kod2]*). More precisely, it coincides with one in the following Table 1. (In Figure 1, each line denotes a smooth rational curve C with $C^2 = -2$ and the configuration of lines show how they intersect to each other. The number next to each line denotes the multiplicity of each curve in $Y = -K_S$.)*

Generalized Halphen surfaces

According to Sakai [§4, [Sakai]], we recall the following definition.

Definition 1.2. (1) Let S be a rational surface with an effective anti-canonical divisor $Y \in |-K_S|$. Let $Y = \sum_{i=1}^r m_i Y_i$ be the irreducible decomposition. The divisor Y is called *of canonical type* if

$$K_S \cdot Y_i = -Y \cdot Y_i = 0 \quad \text{for all} \quad i$$

(2) A rational surface S is called a generalized Halphen surface if S has an effective anti-canonical divisor Y of canonical type. A generalized Halphen surface S is called of index

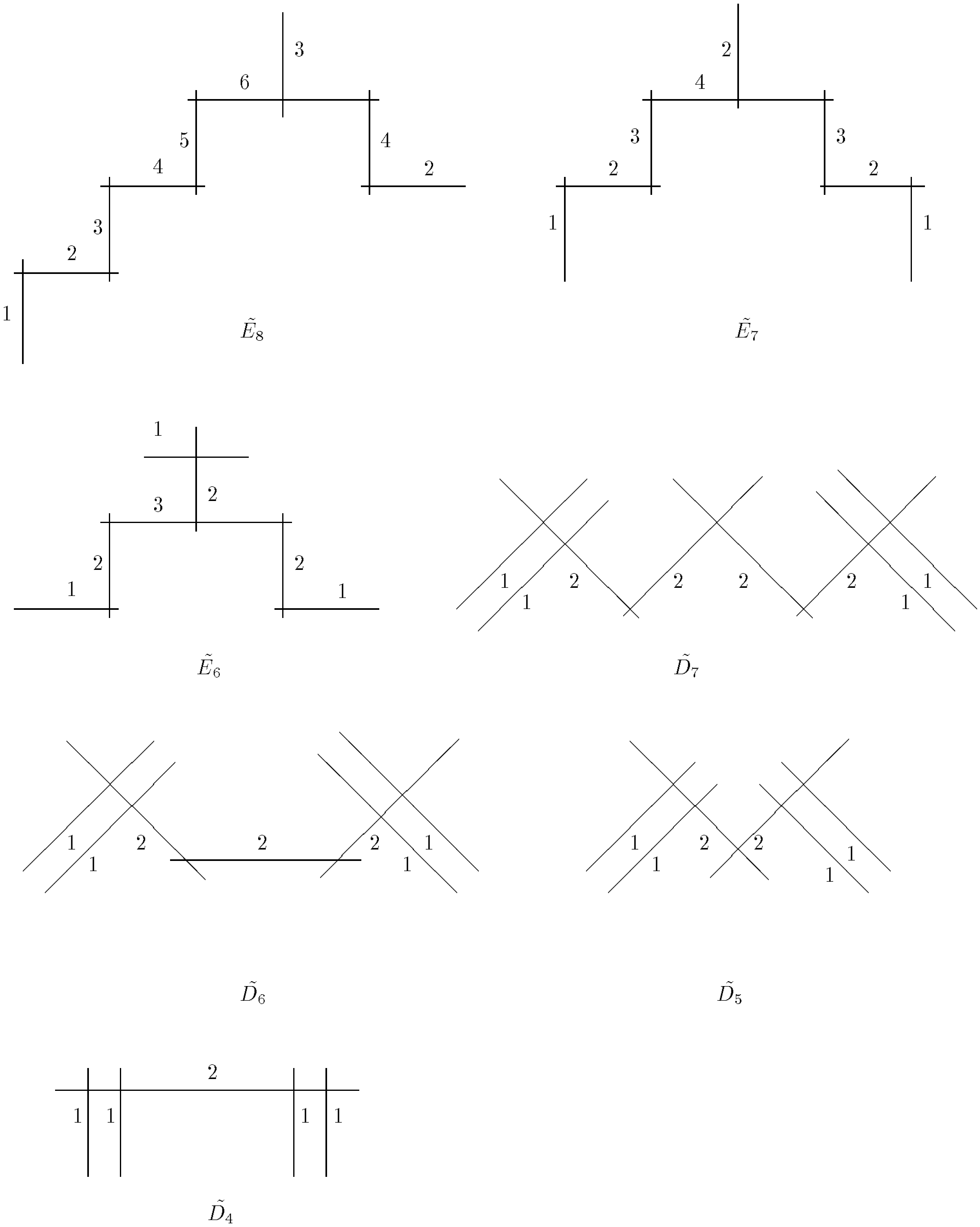


Figure 1

Y	$\tilde{E}_8$	$\tilde{E}_7$	$\tilde{D}_7$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{D}_4$
Kodaira's notation	II^*	III^*	I_3^*	I_2^*	IV^*	I_1^*	I_0^*
Painlevé equation	P_I	P_{II}	$P_{III}^{\tilde{D}_7}$	$P_{III} = P_{III}^{\tilde{D}_6}$	P_{IV}	P_V	P_{VI}

TABLE 1

one if

$$\dim |-K_S| = 1.$$

Remark 1.1. By Riemann-Roch theorem, it is easy to see that for a generalized Halphen surface S $\dim |-K_S| \leq 1$.

The following Proposition ensures that one can obtain a generalized Halphen surfaces from blowing-up of 9-points of $\mathbf{P}^2$.

Proposition 1.1. (*Proposition 2, §2,* [Sakai]). *Let S be a generalized Halphen surface, then there exists a birational morphism $\rho : S \longrightarrow \mathbf{P}^2$.*

Let (S, Y) be a generalized Okamoto-Painlevé pair such that S is a rational surface. Then S is a generalized Halphen surface with a specified anti-canonical divisor Y. As a corollary of Proposition 1.1, we obtain the following corollary.

Corollary 1.1. *Let (S, Y) be a generalized rational Okamoto–Painlevé pair. Then S can be obtained as 9 points blowing-up of* $\mathbf{P}^2$.

One can show that Y has a same configuration as one of Kodaira's degenerate elliptic curves for a generalized rational Okamoto–Painlevé pair (S, Y) (cf. Proof of Theorem 2.1 in [Sa-Tak]).

If S is a generalized Halphen surface of index one, the morphism associated to the linear system $|-K_S|$ induces an elliptic fibration $\varphi : S \longrightarrow \mathbf{P}^1$ with $\varphi^*(\infty) = Y$. (Here $\varphi^*(\infty)$ denotes the scheme theoretic fibers at $\infty \in \mathbf{P}^1$.) This leads us the following

Definition 1.3. A generalized Okamoto-Painlevé pair (S, Y) is called "of fibered type" if there exists an elliptic fibration $\varphi : S \longrightarrow \mathbf{P}^1$ such that $\varphi^*(\infty) = nY$ for some $n \geq 1$. If (S, Y) is not of fibered type, we call (S, Y) "*of non-fibered type* ".

Note that if (S, Y) is of fibered type and $\varphi : S \longrightarrow \mathbf{P}^1$ is elliptic surface with $\varphi^*(\infty) = nY$ with $n > 1$, $\varphi^*(\infty)$ is called a multiple fiber. This happens only when Y is of elliptic type or multiplicative type in the notation below.

In [Sakai], Sakai classified generalized Halphen surface S with $\dim |-K_S| = 0$. In the case when $\dim |-K_S| = 0$, the associated Okamoto-Painlevé pair (S, Y) with a unique member $Y \in |-K_S|$ is of non-fibered type and they can be classified by means of the configuration of Y.

Let $Y = \sum_{i=1}^{r} m_i Y_i$ be the irreducible decomposition of Y. Denote by $M(Y)$ the sublattice of $\mathrm{Pic}(S) \simeq H^2(S, \mathbf{Z})$ generated by the irreducible components $\{Y_i\}_{i=1}^{r}$. Here the bilinear form on $\mathrm{Pic}(S)$ is (-1) times the intersection form on $\mathrm{Pic}(S)$. Then $\{Y_i\}_{i=1}^{r}$ forms a root basis of $M(Y)$ and we denote by $R(Y)$ the type of the root system.

One can easily classify $R(Y)$ as in Table 3. Note that according to the type of Y, $R(Y)$ can be classified into three classes: *elliptic type* when Y is a smooth elliptic curve, *multiplicative type* when Y is a cycle of rational curves, *additive type* when the configuration of Y is tree. These types also correspond to the type of generalized Jacobean $\mathrm{Pic}^0(Y)$ of Y. If we denote by $(\mathrm{Pic}^0(Y))^0$ the component of identity of $\mathrm{Pic}^0(Y)$, we have the following correspondence (cf. Table 2).

$R(Y)$	$(\mathrm{Pic}^0(Y))^0$
elliptic type	smooth elliptic curve Y
multiplicative type	$\mathbf{G}_m \simeq \mathbf{C}^{\times}$
additive type	$\mathbf{G}_a \simeq \mathbf{C}$

TABLE 2

	$R(Y)$ (Kodaira type)
elliptic type	$\tilde{A}_0(= I_0)$
multiplicative type	$\tilde{A}_0{}^{*}(= I_1), \tilde{A}_1(= I_2), \cdots, \tilde{A}_7(= I_8), \tilde{A}_8(= I_9)$
additive type	$\tilde{A}_0^{**}(= II), \tilde{A}_1^{*}(= III), \tilde{A}_2^{*}(= IV)$ $\tilde{D}_4(= I_0^{*}), \cdots, \tilde{D}_8(= I_4)$ $\tilde{E}_6(= IV^{*}), \tilde{E}_7(= III^{*}), \tilde{E}_8(= II^{*})$

TABLE 3

Proposition 1.2. *Let (S, Y) be a generalized rational Okamoto–Painlevé pair such that Y_{red} is a divisor with only normal crossings. Then besides the list of Okamoto–Painlevé pairs in Table 1, we have a pair (S, Y) of type $\tilde{D}_8$ and also $\tilde{A}_r$ for $0 \leq r \leq 8$ and $\tilde{A}_0{}^{*}$. Here for $\tilde{A}_0$, Y is a smooth elliptic curve (Kodaira I_0-type) and for $\tilde{A}_0{}^{*}$, Y is a rational curve with a node (Kodaira I_1-type).*

We list up generalized rational Okamoto–Painlevé pairs with normal crossing divisor Y_{red} in Table 4.

Y	$\tilde{E}_8$	$\tilde{D}_8$	$\tilde{E}_7$	$\tilde{D}_7$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{D}_4$	$\tilde{A}_{r-1}$ $2 \leq r \leq 9$	$\tilde{A}_0$ $r=1$	$\tilde{A}_0{}^*$ $r=1$
Kodaira's notation	II^*	I_4^*	III^*	I_3^*	I_2^*	IV^*	I_1^*	I_0^*	I_r	I_0	I_1
$r = \sharp$ of comps. of Y	9	9	8	8	7	7	6	5	r	1	1

TABLE 4

Regular algebraic functions on $S - Y$.

Let (S, Y) be a generalized rational Okamoto-Painlevé pair. If (S, Y) is of fibered type, that is, if there exists an elliptic fibration $\varphi : S \longrightarrow \mathbf{P}^1$ with $\varphi(\infty)^* = nY$, pulling back non-constant regular algebraic functions on $\mathbf{P}^1 - \{\infty\} \simeq \mathbf{C}$, we have many non-constant regular functions on $S - Y$. We can prove the converse of this fact.

Proposition 1.3. *Let (S, Y) be a generalized rational Okamoto–Painlevé pair. The following conditions are equivalent to each other.*

(1) *(S, Y) is of non-fibered type.*

(2) *$H^0(S-Y, \mathcal{O}^{alg}) \simeq \mathbf{C}$, that is, all regular algebraic functions of $S-Y$ are constant functions.*

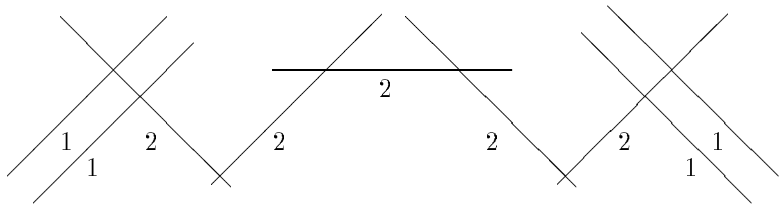
Proof. As we remarked as above, the implication (2) $\Rightarrow$ (1) is obvious. Assume that there exists a non-constant regular function f on $S - Y$. Then the morphism $f : S - Y \longrightarrow \mathbf{C}$ extends to a morphism

$$\overline{f} : S \longrightarrow \mathbf{P}^1.$$

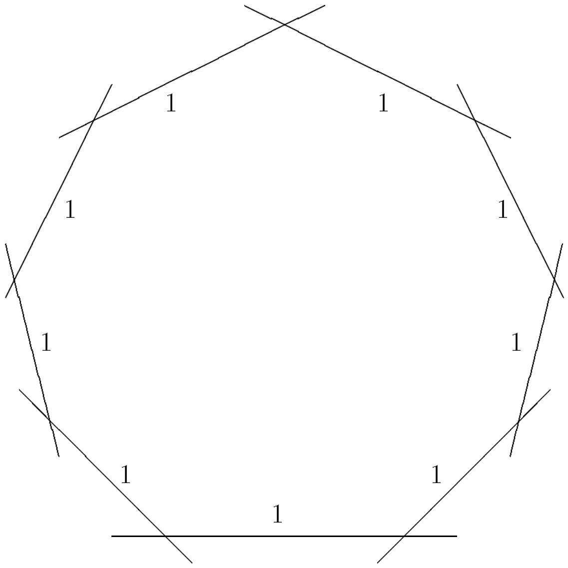
Set $Y' = \overline{f}^*(\infty)$. Since $\overline{f}$ is regular on $S - Y$, recalling the irreducible decomposition of $Y = \sum_{i=1}^r m_i Y_i$, we can write

$$Y' = \sum_{i=1}^{r} a_i Y_i$$

with $a_i \geq 0$. First we show that $a_i > 0$ for every $1 \leq i \leq r$. If $a_i = 0$ for some i, the configuration of Y'_{red} becomes a proper sub-graph of the configuration of Y_{red}. Since the graph of Y_{red} corresponds to an affine Dynkin diagram, one can easily see that Y' can be contracted to rational double points $\{p_1, \cdots, p_s\}$ and obtain a normal surface S' with normal singular points $\{p_1, \cdots, p_s\}$. Since $S - Y'$ and $S' - \{p_1, \cdots, p_s\}$ are isomorphic and $\overline{f}$ is regular on $S - Y' \simeq S' - \{p_1, \cdots, p_s\}$, $\overline{f}$ extends to a regular function on S'. Since S' is proper, $\overline{f}$ must be constant and hence f is also constant. This contradicts to the original assumption. Hence we see that $a_i > 0$ for all $1 \leq i \leq r$.



$\tilde{D}_8$



$\tilde{A}_8$

FIGURE 2

This implies that $Y'_{red} = Y_{red}$ and since Y_{red} is connected, so is Y'_{red}. Taking the Stein factorization if necessary, we may assume that all of the fiber of $\overline{f} : S \longrightarrow \mathbf{P}^1$ is connected and $\overline{f}^*(\infty)_{red} = Y'_{red} = Y_{red}$. (Note that S is a rational surface, hence the irregularity of S is zero.) We will show that general fiber of $\overline{f}$ is an elliptic curve. Since S is smooth and $\overline{f}$ has connected fibers, we only have to show that the virtual genus of Y' is one. Since $-K_S = Y$ and $Y' = \sum_{i=1}^r a_i Y_i$, we see that

$$K_S \cdot Y' = \sum_{i=1}^r a_i(-Y) \cdot Y_i = 0$$

by definition of Okamoto–Painlevé pair. Moreover since Y' is linear equivalent to a general fiber of $\overline{f}$, we see that $Y' \cdot Y_i = 0$ for all $1 \leq i \leq r$. Hence we see that $(Y')^2 = 0$. Then the virtual genus of Y' is given by

$$\pi(Y') = \frac{K_S \cdot Y' + (Y')^2}{2} + 1 = 1,$$

and this completes the proof of proposition. □

2. Deformation of generalized rational Okamoto–Painlevé pairs

In this section, we will recall necessary background of theory of infinitesimal deformation of Okamoto–Painlevé pairs. First, we will recall the general theory of deformation of pairs.

General Theory of Deformation of Pairs

Let (X, H) be a pair of a complex manifold X and a (reduced) normal crossing divisor and let $H = \sum_{i=1}^r H_i$ be the irreducible decomposition of H. By a technical reason we will assume that H is a simple normal crossing divisor, that is, each irreducible component H_i is a smooth divisor. We call such a pair (S, H) is a *non-singular pair* .

For such a non-singular pair (S, H), the normalization $\tilde{H}$ of H is given by the disjoint union $\coprod_{i=1}^r H_i$ of H_i's, and we denote by $\nu : \tilde{H} \longrightarrow H$ the normalization map.

First, we recall the general theory of deformation of a non-singular pair (X, H) due to Kawamata [Kaw]. (See also [SSU]).

Let $\Omega_X^1(\log H)$ denote the sheaf of germs of meromorphic one forms on X which have logarithmic poles along H. Moreover, we set

$$\Theta_X(-\log H) := \underline{\mathrm{Hom}}(\Omega_X^1(\log H), \mathcal{O}_X).$$

This is the sheaf of germs of regular vector fields which have logarithmic zero along H.

Definition 2.1. (Cf. [Definition 3, [Kaw]]) A deformation of a non-singular pair (X, H) is a 5-tuple $(\mathcal{X}, \mathcal{H}, \pi, B, \iota)$

(1) $\pi : \mathcal{X} \longrightarrow B$ is a proper smooth holomorphic map from a complex manifold $\mathcal{X}$ to a connected complex manifold B
(2) $\mathcal{H} = \sum_{i=1}^r \mathcal{H}_i$ is a simple normal crossing divisor of $\mathcal{X}$
(3) For a point $0 \in B$, we have an isomorphism $\iota : (\pi^{-1}(0), \pi^{-1}(0) \cap \mathcal{H}) = (\mathcal{X}_0, \mathcal{H}_0) \simeq (X, H)$.
(4) π is locally a projection of a product space as well as the restriction of it to $\mathcal{H}$, that is, for each $p \in \mathcal{X}$ there exists an open neighborhood U of p and an isomorphism $\varphi : U \longrightarrow V \times W$, where $V = \pi(U)$ and $W = \pi^{-1}(\pi(p))$, such that the following diagram

$$\begin{tikzcd} U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & V \times W \arrow[dl, "pr_1"] \\ & V & \end{tikzcd}$$

is commutative and $\varphi(U \cap \mathcal{H}) = V \times (W \cap \mathcal{H})$.

The deformation of a pair is often denoted by the following diagram:

$$\begin{array}{ccc} \mathcal{X} & \hookleftarrow & \mathcal{H} \\ \pi \downarrow & \swarrow & \\ B & & \end{array}$$

For a deformation $\pi : \mathcal{X} \longrightarrow B$ of complex manifold $X = \mathcal{X}_0$, we can define the Kodaira–Spencer class

$$\rho : T_0(B) \longrightarrow H^1(X, \Theta_X).$$

Similarly, for a deformation of a pair (X, H) as above, we can define the Kodaira–Spencer map

$$\rho : T_0(B) \longrightarrow H^1(X, \Theta_X(-\log H)).$$

As for the existence of Kuranishi space of local semiuniversal deformation of a pair, we have the following theorem due to Kawamata [Cor. 4, [Kaw]].

Theorem 2.1. *For each pair (X, H) of a compact complex manifold X and a normal crossing divisor H, there exist a germ of a complex variety (B, o) and the semiuniversal deformation of (X, H)*

$$\begin{array}{ccc} \mathcal{X} & \hookleftarrow & \mathcal{H} = \sum_{i=1}^r \mathcal{H}_i \\ \pi \downarrow & \swarrow \varphi & \\ B & & \end{array}.$$

Moreover if

$$H^2(X, \Theta_X(-\log H)) = \{0\},$$

the germ $(B, 0)$ is smooth and the Kodaira–Spencer map induces an isomorphism

$$T_0(B) \xrightarrow{\simeq} H^1(X, \Theta_X(-\log H)).$$

The following Lemma is well known and easy to verify.

Lemma 2.1. *Let $(X, H = \sum_{i=1}^l H_i)$ be as above, and let $\nu : \tilde{H} = \coprod_{i=1}^r H_i \longrightarrow H$ be the normalization map. Then we have exact sequences of sheaves:*

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log H) \xrightarrow{P.R.} \nu_*(\oplus_{i=1}^l \mathcal{O}_{H_i}) \longrightarrow 0 \tag{4}$$

$$0 \longrightarrow \Theta_X(-\log H) \longrightarrow \Theta_X \longrightarrow \nu_*(\oplus_{i=1}^l N_{H_i/X}) \longrightarrow 0 \tag{5}$$

Here the map $P.R. : \Omega_X^1(\log H) \longrightarrow \nu_*(\oplus_{i=1}^l \mathcal{O}_{H_i})$ is induced by the Poincaré residue and $N_{H_i/X} = \mathcal{O}_X(H_i)/\mathcal{O}_X$ denotes the normal bundle of the divisor $H_i \subset X$.

Deformation of generalized rational Okamoto–Painlevé pairs

Let (S, Y) be a generalized rational Okamoto–Painlevé pair. Recall that $Y = \sum_{i=1}^r m_i Y_i$ is the anti-canonical divisor $-K_S$. Moreover we set $D_i = Y_{red} = \sum_{i=1}^r Y_i$. From now on, we will calculate some cohomology groups of the pair (S, D) for Okamoto–Painlevé pair (S, Y) which we will use later.

Let $i : D \hookrightarrow S$ be the natural inclusion and $\nu : \tilde{D} = \coprod_{i=1}^r Y_i \to D$ the normalization map. Set $j = i \cdot \nu$. First, let us consider the following Gysin exact sequence

$$H^1(S, \mathbf{C}) \longrightarrow H^1(S - D, \mathbf{C}) \longrightarrow H^0(\tilde{D}, \mathbf{C}) \xrightarrow{j!} H^2(S, \mathbf{C}) \longrightarrow \cdots. \tag{6}$$

The following lemma is important.

Lemma 2.2. *Under the same notation as above, the Gysin map gives an injective homomorphism*

$$H^0(\tilde{D}, \mathbf{C}) \hookrightarrow H^2(S, \mathbf{C}).$$

Proof. Since

$$H^0(\tilde{D}, \mathbf{C}) = \oplus_{i=1}^r H^0(Y_i, \mathbf{C}) = \oplus_{i=1}^r \mathbf{C} \cdot \mathbf{1}_{Y_i},$$

and the image of the Gysin map of $\mathbf{1}_{Y_i}$ is just the divisor class $Y_i \in H^2(S, \mathbf{C}) \simeq \mathrm{Pic}(S) \otimes \mathbf{C}$, we only have to show that the classes $\{Y_i\}_{i=1}^r$ is lineally independent in $H^2(S, \mathbf{C})$. Looking at the intersection matrix of $\{Y_i\}_{i=1}^r$, we easily see that only possible linear relation is

$$Y = \sum_{i=1}^r m_i Y_i = \mathbf{0}.$$

On the other hand, $K_S = -Y$ and S has at least one (-1)-smooth rational curve E. By adjunction formula, we see that $Y \cdot E = -(K_S) \cdot E = 1$, hence we see that $\{Y_i\}_{i=1}^r$ is linearly independent. □

As a corollary to Lemma 2.2, we obtain:

Corollary 2.1. *For a generalized rational Okamoto–Painlevé pair (S, Y), we have the following.*

1. $H^1(S - D, \mathbf{C}) = 0$.
2. $H^0(S, \Omega^1_S(\log D)) = 0$.
3. $H^2(S, \Theta_S(-\log D)) = 0$.
4. $H^2(S, \Theta_S) = 0$.

Proof. Since S is a rational surface, we have $H^1(S, \mathbf{C}) = 0$. From the exact sequence (6) and Lemma 2.2, we have the first assertion. Then from the mixed Hodge theory, we have an inclusion

$$H^0(S, \Omega^1_S(\log D)) \hookrightarrow H^1(S - D, \mathbf{C}).$$

This proves the second assertion. For the third assertion, let us consider the Serre duality

$$H^2(S, \Theta_S(-\log D))^\vee \simeq H^0(S, \Omega^1_S(\log D) \otimes K_S).$$

Since $K_S = \mathcal{O}_S(-Y)$, we have an inclusion

$$H^0(S, \Omega^1_S(\log D) \otimes K_S) \hookrightarrow H^0(S, \Omega^1_S(\log D)) = \{0\},$$

This shows the third assertion. From the exact sequence (5), we obtain the exact sequence

$$\longrightarrow H^2(S, \Theta_S(-\log D)) \longrightarrow H^2(S, \Theta_S) \longrightarrow H^2(D, N_{D_i/S}) \longrightarrow 0.$$

Since $\dim D = 1$, $H^2(D, N_{D_i/S}) = 0$, hence, from the third assertion we obtain the fourth assertion.

The following geometric facts are very important for our purpose. (cf. [Lemma 3, [AL]], [SU]).

Proposition 2.1. *Let (S, Y) be a generalized rational Okamoto–Painlevé pair such that Y is a divisor with normal crossing and (S, Y) is not of fibered type.*

1. $H^0(S - D, \mathcal{O}^{alg}) \simeq \mathbf{C}$
2. $H^0(S - D, \Theta^{alg}_{S-D}) \simeq 0$. *Here Θ^{alg}_{S-D} denotes the sheaf of germs of algebraic regular infinitesimal automorphisms.*
3. $H^0(S, \Theta_S(H)) = 0$ *for any effective divisor H supported on D.*

(4) $H^0(S, \Theta_S(-\log D)(H)) = 0$ *for any effective divisor* H *supported on* D.

Proof. The first assertion follows from Proposition 1.3. Since on S there exists a non-zero rational 2-forms ω_S which is non-degenerate on $S-D$, ω_S induces an isomorphism $\Theta_{S-D} \simeq \Omega^1_{S-D}$. Hence it suffices to show that $H^0(S-D, \Omega^{1,alg}_{S-D}) = (0)$. Taking a section $\eta \in H^0(S-D, \Omega^{1,alg}_{S-D})$, we see that $d\eta/\omega_S$ is a regular holomorphic function on $S-D$, hence constant c (cf. Proposition 1.3). So this implies that $d\eta = c \cdot \omega_S$. On the other hand, we can easily see that ω_S is non-zero element in $H^2_{DR}(S-D, \mathbf{C})$, hence $d\eta = 0$. Hence it lies in $H^0(S-D, d\mathcal{O}^{alg}_{S-D})$. Since we have an isomorphism (cf. [3.1.7.1, [D]])

$$H^1(S-D, \mathbf{C}_{S-D}) \simeq \mathbf{H}^1_{DR}(S-D),$$

and $H^1(S-D, \mathbf{C}_{S-D}) = (0)$ from Corollary 2.1, (1), we see that η can be written as df for some $f \in H^0(S-D, \mathcal{O}^{alg}_{S-D})$. However since f is constant (Proposition 1.3), we see that $\eta = df = 0$. The last two assertions easily follow from the second assertion. □

The following proposition shows that the Kuranishi space of a generalized rational Okamoto–Painlevé pair (S, Y) is smooth and has dimension $10 - r$ where r denotes the number of irreducible components of Y.

Proposition 2.2. *Let* (S, Y) *be a generalized rational Okamoto–Painlevé pair such that* $D = Y_{red}$ *is a simple normal crossing divisor and* $Y \neq \tilde{A}_0$*-type. Then we have*

$$c_2(S) = \textit{topological Euler characteristic} = 12, \tag{7}$$

$$b_2(S) = \operatorname{rank} H^2(S, \mathbf{Z}) = 10, \tag{8}$$

$$\dim H^1(S, \Theta_S) = 10, \tag{9}$$

and

$$\dim H^1(S, \Theta_S(-\log D)) = 10 - r \tag{10}$$

where r *is the number of irreducible components of* Y. *Moreover, the Kuranishi space of the local deformation of the pair* (S, D) *is smooth and of dimension* $10 - r$.

Proof. First, from Noether's formula, we obtain

$$\chi(S, \mathcal{O}_S) = \frac{1}{12}((K_S)^2 + c_2(S)). \tag{11}$$

From the definition of a generalized rational Okamoto–Painlevé pair (S, Y), we have $K_S = -Y$ and $(K_S)^2 = Y^2 = 0$. Since S is rational, we have $\chi(S, \mathcal{O}_S) = 1$. Hence, we have

$$c_2(S) = 12.^1$$

Since $B_1(S) = 0$, the above equality implies that $B_2(S) = 12 - 2 = 10$.

From Riemann-Roch-Hirzeburuch formula, we obtain

$$\chi(S, \Theta_S) = \sum_{i=1}^{2} (-1)^i \dim H^i(S, \Theta_S) = \frac{1}{6}(7 \cdot (K_S)^2 - 5c_2(S)).$$

[1] Note that this also follows from the fact that S is a 9-points blown-up of $\mathbf{P}^2$.

Table of the deformation of generalized rational Okamoto–Painlevé pairs.

Y	$\tilde{E}_8$	$\tilde{D}_8$	$\tilde{E}_7$	$\tilde{D}_7$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{D}_4$	$\tilde{A}_{r-1}$, $r \geq 2$
Number of components of Y	9	9	8	8	7	7	6	5	r
$\dim H^1(S, \Theta_S(-\log D))$	1	1	2	2	3	3	4	5	$10-r$
Painlevé equation	P_I	$P_{III}^{\tilde{D}_8}$	P_{II}	$P_{III}^{\tilde{D}_7}$	$P_{III}^{\tilde{D}_6}$	P_{IV}	P_V	P_{VI}	none

Table 5

Then again from $(K_S)^2 = 0$ we obtain

$$\chi(S, \Theta_S) = -\frac{5}{6}c_2(S) = -\frac{5}{6} \times 12 = -10.$$

Moreover from Corollary 2.1, (4) and Proposition 2.1 we obtain $H^i(S, \Theta_S) = 0$ for $i = 0, 2$, and hence

$$\dim H^1(S, \Theta_S) = 10. \tag{12}$$

For $H^1(S, \Theta_S(-\log D))$, consider the exact sequence

$$0 \longrightarrow \Theta_S(-\log D) \longrightarrow \Theta_S \longrightarrow \nu_*(\oplus_{i=1}^r N_{Y_i/S}) \longrightarrow 0. \tag{13}$$

Remember $D = Y_{red} = \sum_{i=1}^r Y_i$. Since $K_S \cdot Y_i = 0$ and $Y_i \simeq \mathbf{P}^1$ by assumption, the adjunction formula shows that

$$\deg N_{Y_i/S} = -2 \quad \text{or} \quad N_{Y_i/S} \simeq \mathcal{O}_{\mathbf{P}^1}(-2).$$

Then since $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2)) = 0$, $H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2)) \simeq \mathbf{C}$ and $H^2(S, \Theta_S(-\log D)) = 0$, we have the following exact sequence

$$0 \longrightarrow H^1(S, \Theta_S(-\log D)) \longrightarrow H^1(S, \Theta_S) \longrightarrow \oplus_{i=1}^r \mathbf{C}[Y_i] \longrightarrow 0 \tag{14}$$

This implies that the assertion (10) holds. The last assertion follows from Theorem 2.1 and the fact that $H^2(S, \Theta_S(-\log D)) = 0$.

3. Local cohomology sequences and Time variables

Let (S, Y) be a generalized rational Okamoto–Painlevé pair and set $D = Y_{red}$. Moreover, in this section, we assume that

(1) (S, Y) is of non-fibered type and
(2) Y_{red} is a normal crossing divisor with at least two irreducible components, i.e. $(r \geq 2)$ so that all irreducible components of Y_{red} are smooth rational curves.

In what follows, $\mathcal{O}_S$ and $\mathcal{O}_{S-D}$ denote the sheaves of germs of algebraic regular functions on S and $S - D$ respectively. Moreover all sheaves of $\mathcal{O}_S$-modules are considered in algebraic category unless otherwise stated. Let us consider the following exact sequence of local cohomology groups ([Corollary 1.9, [Gr]])

$$H^0(S, \Theta_S(-\log D)) \to H^0(S - D, \Theta_S(-\log D)) \to H^1_D(\Theta_S(-\log D)) \to \tag{15}$$

$$H^1(S, \Theta_S(-\log D)) \overset{\mathrm{res}}{\to} H^1(S - D, \Theta_S(-\log D)) \quad . \tag{16}$$

Since (S, Y) is of non-fibered type, from (2), Proposition 2.1, we see that

$$H^0(S - D, \Theta_S(-\log D)) = H^0(S - D, \Theta_S) = \{0\}.$$

Hence, we have the following

Proposition 3.1. *For a generalized rational Okamoto–Painlevé pair of non-fibered type, we have the following exact sequence:*

$$0 \to \quad H^1_D(\Theta_S(-\log D)) \quad \to \quad H^1(S, \Theta_S(-\log D)) \quad \overset{\mathrm{res}}{\to} \quad H^1(S - D, \Theta_S(-\log D)) \tag{17}$$

The following theorem is proved in [T].

Theorem 3.1. *Let (S, Y) be a generalized rational Okamoto-Painlevé pair (S, Y) with the condition above. Moreover $D = Y_{red}$ is of additive type. Then we have*

$$\dim H^0(D, \Theta_S(-\log D) \otimes N_D) = 1. \tag{18}$$

Here we put $N_D = \mathcal{O}_S(D)/\mathcal{O}_S$.

Since we have a natural inclusion

$$H^0(D, \Theta_S(-\log D) \otimes N_D) \hookrightarrow H^1_D(\Theta_S(-\log D)),$$

we obtain

$$\dim H^1_D(\Theta_S(-\log D)) \geq 1. \tag{19}$$

This theorem plays an important role to understand the Painlevé equation related to (S, Y). We will not investigate the further structure of local cohomology here. Instead, we propose the following

Conjecture 3.1. *Under the same notation and assumption as in Theorem 3.1,*

$$H^1_D(\Theta_S(-\log D)) \simeq H^0(D, \Theta_S(-\log D) \otimes N_D) \simeq \mathbf{C}. \tag{20}$$

From the exact sequence (17), we see that the subspace $H^1_D(S, \Theta_S(-\log Y))$ of $H^1(S, \Theta_S(-\log Y))$ coincides with the kernel of μ. This implies that:

$$H^1_D(S, \Theta_S(-\log D)) \simeq \left\{ \begin{array}{l} \text{Infinitesimal deformations of } (S, D) \text{ whose restriction} \\ \text{to } S - D \text{ induces the trivial deformation} \end{array} \right\}.$$

In §6, we will show that any direction corresponding to a non-zero element of the local cohomology group $H^1_D(S, \Theta_S(-\log D))$ induces a differential equation (at least locally) by using Čech coboundaries.

At this moment, we can not prove Conjecture 3.1 in the full generality. However, we see that the one dimensional vector subspace $H^1(D, \Theta_S(-\log D) \otimes N_D)$ of $H^1_D(\Theta_S(-\log D)) \subset H^1(\Theta_S(-\log D))$ really corresponds to the time variable t in the known Painlevé equation. It is unlikely that we will have more time variables, so this gives an evidence of Conjecture 3.1.

Let us explain the strategy of proving Theorem 3.1 in [T]. Recall that

$$H^1_D(S, \Theta_S(-\log D)) = \varinjlim \operatorname{Ext}^1(\mathcal{O}_{nD}, \Theta_S(-\log D))$$

where $\mathcal{O}_{nD} = \mathcal{O}_S/\mathcal{O}_S(-nD)$ (cf. [Theorem 2.8, [Gr]]).

On the other hand, since $\Theta_S(-\log D)$ is a locally free $\mathcal{O}_S$-module, we see that

$$\mathcal{H}om(\mathcal{O}_{nD}, \Theta_S(-\log D)) = 0, \tag{21}$$

and

$$\mathcal{E}xt^1(\mathcal{O}_{nD}, \Theta_S(-\log D)) = \Theta_S(-\log D) \otimes N_{nD}, \tag{22}$$

where $N_{nD} = \mathcal{O}_S(nD)/\mathcal{O}_S$. By an argument using a spectral sequence, we see that

$$H^1_D(S, \Theta_S(-\log D)) = \varinjlim H^0(\Theta_S(-\log D) \otimes N_{nD}) \tag{23}$$

Hence, we have a natural inclusion

$$H^0(\Theta_S(-\log D) \otimes N_D) \hookrightarrow H^1_D(S, \Theta_S(-\log D)). \tag{24}$$

Lemma 3.1. *Let (S, Y) be a generalized rational Okamoto–Painlevé pair as above and set $D = Y_{red}$. Then we have the following exact sequences*

$$0 \longrightarrow \Theta_D \otimes N_D \longrightarrow \Theta_S \otimes N_D \longrightarrow \nu_*(\oplus_{i=1}^r N_{Y_i/S}) \otimes N_D \longrightarrow 0. \tag{25}$$

$$0 \longrightarrow \nu_*(\oplus_{i=1}^r N_{Y_i/S}) \longrightarrow \Theta_S(-\log D) \otimes N_D \longrightarrow \Theta_D \otimes N_D \longrightarrow 0. \tag{26}$$

Here Θ_D denotes the tangent sheaf of D and $\nu : \tilde{D} \longrightarrow D$ the normalization map.

Proof. The first exact sequence (25) follows from [(1.9), [B-W]].

Let us consider the following diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & \ker\lambda & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Theta_S(-\log D) & \longrightarrow & \Theta_S(-\log D) \otimes \mathcal{O}_S(D) & \longrightarrow & \Theta_S(-\log D) \otimes N_D & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow \lambda & \\
0 \longrightarrow & \Theta_S & \longrightarrow & \Theta_S \otimes \mathcal{O}_S(D) & \longrightarrow & \Theta_S \otimes N_D & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \nu_*(\oplus_{i=1}^r N_{Y_i/S}) & \xrightarrow{\mu} & \nu_*(\oplus_{i=1}^r N_{Y_i/S}) \otimes N_D & \longrightarrow & \operatorname{coker}\lambda & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & .
\end{array}$$

By the snake lemma, we obtain the exact sequence

$$0 \longrightarrow \ker\lambda \longrightarrow \nu_*(\oplus_{i=1}^r N_{Y_i/S}) \xrightarrow{\mu} \nu_*(\oplus_{i=1}^r N_{Y_i/S}) \otimes N_D \longrightarrow \operatorname{coker}\lambda \longrightarrow 0.$$

From a local consideration, we see that the map μ is the zero map, hence

$$\ker \lambda \simeq \nu_*(\oplus_{i=1}^r N_{Y_i/S}), \quad \nu_*(\oplus_{i=1}^r N_{Y_i/S}) \otimes N_D \simeq \operatorname{coker} \lambda.$$

Moreover since $\operatorname{Im} \lambda \simeq \ker[\Theta_S \otimes N_D \longrightarrow \nu_*(\oplus_{i=1}^r N_{Y_i/S}) \otimes N_D]$, from the exact sequence (25), we obtain the exact sequence (26).

□.

From the exact sequence (26), one can obtain

$$H^0(\oplus_{i=1}^r N_{Y_i/S}) \longrightarrow H^0(\Theta_S(-\log D) \otimes N_D) \longrightarrow H^0(\Theta_D \otimes N_D) \xrightarrow{\delta} H^1(\oplus_{i=1}^r N_{Y_i/S}). \tag{27}$$

where δ denotes the connected homomorphism.

Note that since $N_{Y_i/S} = \mathcal{O}_{Y_i}(-2)$, we have

$$H^0(\oplus_{i=1}^r N_{Y_i/S}) = \{0\}, \quad H^1(\oplus_{i=1}^r N_{Y_i/S}) \simeq \mathbf{C}^r.$$

Moreover, one can easily see that

$$\Theta_D \simeq \nu_*(\oplus \Theta_{Y_i}(-t_i)) \simeq \nu_*(\oplus \mathcal{O}_{Y_i}(2 - t_i))$$

where t_i is the number of intersections of Y_i with the other Y_js. On the other hand, since $D \cdot Y_i = t_i - 2$, we see that

$$H^0(\Theta_D \otimes N_D) \simeq H^0(\oplus_{i=1}^r \mathcal{O}_{Y_i}),$$

hence

$$H^0(\Theta_D \otimes N_D) \simeq \mathbf{C}^r. \tag{28}$$

From this, the connecting homomorphism δ

$$\delta : H^0(\Theta_D \otimes N_D) \longrightarrow \oplus_{i=1}^r H^1(N_{Y_i/S}) \tag{29}$$

can be identified with a linear map $\delta : \mathbf{C}^r \longrightarrow \mathbf{C}^r$ and we have an isomorphism

$$H^0(D, \Theta_S(-\log D) \otimes N_D) \simeq \ker \delta. \tag{30}$$

The following proposition is the main theorem of [T].

Proposition 3.2. *Let (S, Y) be as in Theorem 3.1. A matrix representation of the linear map $\delta : H^0(\Theta_D \otimes N_D) \longrightarrow \oplus_{i=1}^r H^1(N_{Y_i/S})$ in (29) is equal to the $\pm$ of the intersection matrix of $D = \sum_{i=1}^r Y_i$, that is,*

$$\delta = ((Y_i \cdot Y_j))_{1 \leq i,j \leq r}.$$

Since the intersection matrix $((Y_i \cdot Y_j))_{1 \leq i,j \leq r}$ has exactly one-dimensional kernel corresponding to the space of $Y = \sum_{1=1}^r m_i Y_i$, we have

$$H^0(D, \Theta_S(-\log D) \otimes N_D) \simeq \ker \delta = \mathbf{C}.$$

Remark 3.1. We have the following exact sequence of sheaves

$$0 \longrightarrow \Theta_S(-\log D) \longrightarrow \Theta_S(-\log D) \otimes \mathcal{O}_S(D) \longrightarrow \Theta_S(-\log D) \otimes N_D \longrightarrow 0. \tag{31}$$

Combining this with the local cohomology exact sequence (17) and the natural inclusion (24), we have the following commutative diagram

$$\begin{array}{ccccccc} \mathbf{C} \simeq & H^0(D, \Theta_S(-\log D) \otimes N_D) & \stackrel{\delta}{\longrightarrow} & H^1(\Theta_S(-\log D)) & \stackrel{\varphi}{\longrightarrow} & H^1(\Theta_S(-\log D) \otimes \mathcal{O}_S(D)) \\ & \downarrow & & \| & & \downarrow \\ 0 \to & H^1_D(\Theta_S(-\log D)) & \longrightarrow & H^1(\Theta_S(-\log D)) & \stackrel{\mathrm{res}}{\longrightarrow} & H^1(S-D, \Theta_{S-D}) \end{array} \tag{32}$$

4. Reviews on Kodaira–Spencer theory

In this section, we review on Kodaira–Spencer theory of complex analytic deformation. A main reference is [KS].

Let X be a compact complex manifold of dimension n. We can take a locally finite open covering $\{U_i\}_{i \in I}$ of X such that each open subset U_i admits local coordinates $\mathbf{z}_i = (z_i^1, \cdots, z_i^n)$:

$$X = \cup_{i \in I} U_i.$$

For a point $p \in U_i \cap U_j$, we have two local coordinates $\mathbf{z}_i(p)$ and $\mathbf{z}_j(p)$ whose coordinate transformation are given by

$$\mathbf{z}_i = (z_i^1, z_i^2, \cdots, z_i^n) = \mathbf{f}_{ij}(\mathbf{z}_j),$$

or more precisely for $\alpha = 1, \cdots, n$,

$$z_i^\alpha = f_{ij}^\alpha(z_j^1, \cdots . z_j^n). \tag{33}$$

Here $f_{ij}^\alpha(\mathbf{z}_j)$ are holomorphic functions defined on $U_i \cap U_j$. Note that one can give the compatibility conditions for $U_i \cap U_j \cap U_k \neq \emptyset$

$$f_{ik}^\alpha(\mathbf{z}_k) = f_{ij}^\alpha(f_{jk}^1(\mathbf{z}_k), \cdots, f_{jk}^n(\mathbf{z}_k)) \tag{34}$$

Complex structure of X can be deformed by changing these coordinate transformations keeping the compatibility conditions.

Let $\mathcal{B}$ be a complex manifold with a (global) coordinate system $(t_1, \cdots, t_m)$ and a specific marked point $\mathbf{0} = (0, \cdots, 0) \in \mathcal{B}$. We may think that $\mathcal{B}$ is an affine variety or a complex analytic small open ball around the origin.

Definition 4.1. A deformation of X with a parameter space $\mathcal{B} \ni (t_1, \cdots, t_m)$ is a proper smooth holomorphic map $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ such that the following diagram is commutative:

$$\begin{array}{cccc} \mathcal{X} & \stackrel{\iota}{\hookleftarrow} & X_0 & \simeq X \\ \pi \downarrow & & \downarrow & \\ \mathcal{B} & \ni & \mathbf{0} & . \end{array}$$

Definition 4.2. A deformation $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ of X is said to have *a finite covering relative to $\mathcal{B}$* if $\mathcal{X}$ is covered by $\{\tilde{U}_i = U_i \times \mathcal{B}\}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{X} & = & \cup_{i \in I}(U_i \times \mathcal{B}) \\ \pi \downarrow & & \downarrow \\ \mathcal{B} & = & \mathcal{B} \\ . & & \end{array}$$

Let us assume that $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ has a finite covering relative to $\mathcal{B}$ and take the local coordinate of $\tilde{U}_i = U_i \times \mathcal{B}$ by $(z_i^1, \cdots, z_i^n, t_1, \cdots, t_m)$. The coordinate transformation for $\tilde{U}_i \cap \tilde{U}_j$ is given by

$$z_i^{\alpha} = f_{ij}^{\alpha}(z_j^1, \cdots, z_j^n, t_1, \cdots, t_m).$$

We may assume that for $\mathbf{t} = \mathbf{0}$, we have $f_{ij}^{\alpha}(z_j^1, \cdots, z_j^n, 0, \cdots, 0) = f_{ij}^{\alpha}(z_j^1, \cdots, z_j^n)$.

Now we can introduce the *Kodaira–Spencer class* of the deformation $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ for each $\mathbf{t} \in \mathcal{B}$.

For simplicity we assume that $\mathcal{B}$ is one dimensional, hence $t = t_1$ is the global parameter of $\mathcal{B}$. Let h be a holomorphic function on an open subset V of $\mathcal{X}$. Then on $\tilde{U}_i \cap V$, h is a function in a local coordinate $h(z_i^1, \cdots, z_i^n, t)$. Assume that $\tilde{U}_i \cap \tilde{U}_j \cap V \neq \emptyset$. Regarding as

$$h(\mathbf{z}_i, t) = h(f_{ij}^1(\mathbf{z}_j, t), \cdots, f_{ij}^n(\mathbf{z}_j, t), t),$$

from the chain rule, we obtain

$$\left(\frac{\partial h}{\partial t}\right)_j = \left(\frac{\partial h}{\partial t}\right)_i + \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^{\alpha}(\mathbf{z}_j, t)}{\partial t} \frac{\partial h}{\partial z_i^{\alpha}}. \tag{35}$$

This implies that, as a vector field on $\tilde{U}_i \cap \tilde{U}_j$, we have the following identity:

$$\left(\frac{\partial}{\partial t}\right)_j = \left(\frac{\partial}{\partial t}\right)_i + \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^{\alpha}(\mathbf{z}_j, t)}{\partial t} \frac{\partial}{\partial z_i^{\alpha}}. \tag{36}$$

Let us set $\{\theta_{ij}(t)\}$ by

$$\theta_{ij}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^{\alpha}(\mathbf{z}_j, t)}{\partial t} \frac{\partial}{\partial z_i^{\alpha}}. \tag{37}$$

From the compatibility conditions for $\tilde{U}_i \cap \tilde{U}_j \cap \tilde{U}_k \neq \emptyset$

$$f_{ik}^{\alpha}(\mathbf{z}_k, t) = f_{ij}^{\alpha}(f_{jk}^1(\mathbf{z}_k, t), \cdots, f_{jk}^n(\mathbf{z}_k, t), t), \tag{38}$$

we obtain the identity

$$\theta_{ik}(t) = \theta_{ij}(t) + \theta_{jk}(t).$$

This implies that $\{\theta_{ij}(t)\}$ defines a Čech 1-cocycle with values in Θ_{X_t}, hence defines a cohomology class

$$\theta(t) \in H^1(X_t, \Theta_{X_t}),$$

which is called the *Kodaira–Spencer class* .

If the dimension of $\mathcal{B}$ is greater than one, we can define the cohomology class for each $\frac{\partial}{\partial t_{\mu}}$. More precisely one can define the *Kodaira-Spencer map*

$$\begin{aligned} \rho : T_{\mathbf{t}}(\mathcal{B}) \quad &\longrightarrow \quad H^1(X_t, \Theta_{X_t}) && (39)\\ v \quad &\mapsto \quad \rho(v) = \theta_v(\mathbf{t}) && (40) \end{aligned}$$

by

$$\theta_{v,ij} = \{\theta_{v,ij}(\mathbf{t}) = \sum_{\alpha=1}^{n} v(f_{ij}^{\alpha}(\mathbf{z}_j, \mathbf{t})) \frac{\partial}{\partial z_i^{\alpha}}\}.$$

Here for

$$v = \sum_{\mu=1}^{m} A_{\mu}(\mathbf{t}) \frac{\partial}{\partial t_{\mu}},$$

we set

$$v(f_{ij}^{\alpha}(\mathbf{z}_j, \mathbf{t})) = \sum_{\mu=1}^{m} A_{\mu} \frac{\partial f_{ij}^{\alpha}(\mathbf{z}_j, \mathbf{t})}{\partial t_{\mu}}.$$

Definition 4.3. A deformation $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ is called *locally trivial*, if for each point $\mathbf{t} \in \mathcal{B}$ there exists an open neighborhood I of $\mathbf{t}$ such that $\mathcal{X}_{|I} \longrightarrow I$ is complex analytically isomorphic to the product $\mathcal{X}_{\mathbf{t}} \times I$.

Proposition 4.1. ([KS]) *Let $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ be a deformation of a compact complex manifold with parameter space $\mathcal{B} \ni \mathbf{t} = (t_1, \cdots, t_m)$. If for every point $\mathbf{t} \in \mathcal{B}$ $\dim H^1(\mathcal{X}_{\mathbf{t}}, \Theta_{\mathcal{X}_{\mathbf{t}}})$ is constant and Kodaira–Spencer map ρ is the zero map, then $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ is a locally trivial fibration.*

For a proof in detail, we refer the reader to [Theorem 5.1, [KS]]. Since we will use the idea of the proof later, we explain an outline of the proof of theorem when $\dim \mathcal{B} = 1$. By replacing $\mathcal{B}$ with a neighborhood of $\mathbf{t} \in \mathcal{B}$, we may assume that a deformation $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ has a finite covering $\{\tilde{U}_i = U_i \times \mathcal{B}\}$ relative to $\mathcal{B}$. Then the Kodaira–Spencer class

$$\rho(\frac{\partial}{\partial t}) = \theta(t) \in H^1(\mathcal{X}_t, \Theta_{\mathcal{X}_t}).$$

is represented by Čech cocycles $\{\theta_{ij}(t)\}$ given in (37). Since $\theta(t)$ is cohomologus to zero, for each t we can find

$$\theta_i(t) \in \Gamma(\tilde{U}_i \cap \mathcal{X}_t, \Theta_{U_i}).$$

such that

$$\theta_{ij}(t) = \theta_j(t) - \theta_i(t) \quad \text{on} \quad \tilde{U}_i \cap \tilde{U}_j \cap \mathcal{X}_t$$

From (36), we obtain the following identities of vector fields on each $\tilde{U}_i \cap \tilde{U}_j \cap \mathcal{X}_t$

$$\left(\frac{\partial}{\partial t}\right)_j = \left(\frac{\partial}{\partial t}\right)_i + (\theta_j(t) - \theta_i(t)), \tag{41}$$

and hence

$$\left(\frac{\partial}{\partial t}\right)_j - \theta_j(t) = \left(\frac{\partial}{\partial t}\right)_i - \theta_i(t). \tag{42}$$

At this moment, it is not obvious that the dependence of

$$\theta_i(t) = \sum_{\alpha=1}^{n} \theta_i^{\alpha}(\mathbf{z}_i, t) \frac{\partial}{\partial z_i^{\alpha}} \tag{43}$$

with respect to t is in C^{∞} class. However under the condition that $\dim H^1(\mathcal{X}_t, \Theta_{\mathcal{X}_t})$ is constant on $\mathcal{B}$, one can prove that $\theta_i(t)$ can be chosen as a vector field on $\tilde{U}_i = U_i \times \mathcal{B}$ in C^{∞} class.

This implies that the vector field

$$\{\left(\frac{\partial}{\partial t}\right)_i - \theta_i(t)\}_{i \in I} \tag{44}$$

on $\tilde{U}_i$ can be glued together and defines a global C^{∞}–vector field, say, $\tilde{v}$ on the total space $\mathcal{X}$. We see that $\tilde{v}$ is a lift of vector field $\frac{\partial}{\partial t}$ by π. Then on each open set $\tilde{U}_i$, we can consider the ordinary differential equation

$$\frac{dz_i^{\alpha}}{dt} = -\theta_i^{\alpha}(\mathbf{z}_i, t) \quad \alpha = 1, \cdots, n. \tag{45}$$

And these set of differential equations can be patched together on whole $\mathcal{X}$. Starting from an initial conditions $(a_1, \cdots, a_n, t_0) \in \mathcal{X}_{t_0}$, the solution $(z_1(a_j, t), \cdots, z_n(a_j, t))$ of differential equation (45) defines a C^∞–curve which is transversal to each fiber $\mathcal{X}_t$. Then the whole solutions of (45) define a foliation on $\mathcal{X}$ and define C^∞-defeomorphisms $\varphi_t : \mathcal{X}_0 \overset{\simeq}{\to} \mathcal{X}_t$. Moreover, one can show that this defeomorphism φ_t is a complex biholomorphic morphism for each $t \in \mathcal{B}$.

This implies the following. If we have a family of compact complex manifolds $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ with a parameter $t \in \mathcal{B}$ such that the Kodaira–Spencer map ρ_t is zero, we will obtain a differential equation as in (45) defined on the total space $\mathcal{X}$.

Summarizing these, we have the following implications (cf. Figure 3).

Deformation $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ of complex manifolds with zero Kodaira–Spencer map

$\Downarrow$

There exists a vector field $\tilde{v}$ on $\mathcal{X}$ which is a lift of $\frac{\partial}{\partial t}$.

$\Downarrow$

Differential Equation on $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ given by $\tilde{v}$

$\Downarrow$

Local trivializations of the deformation $\mathcal{X} \longrightarrow \mathcal{B}$

.

FIGURE 3

5. Global Deformations of Okamoto–Painlevé pairs

Affine coverings and Symplectic Structures on $S - D$

Let (S, Y) be a generalized rational Okamoto–Painlevé pair. Then by definition, S has a rational 2-form ω_Y whose pole divisor is Y. Setting $D = Y_{red}$, the rational 2-form ω_Y induces a non-degenerate holomorphic 2-forms on the open surface $S - D$, hence induces a holomorphic symplectic structure on $S - D$.

In [O1], Okamoto introduced the space of initial conditions of Painlevé equation of each type, which can be written as $S - D$ for an Okamoto-Painlevé pair (S, Y). The main reason why Painlevé equations can be written as Hamiltonian systems is this holomorphic symplectic structure. For Painlevé equations P_J, $(J = II, III, IV, V, VI)$, Takano et al. [ST], [MMT] constructed a good family of Okamoto–Painlevé pairs $(S_{\boldsymbol{\alpha},t}, Y_{\boldsymbol{\alpha},t})$ depending on the time variable and a system of auxiliary parameters $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_s)$ appeared in each Painlevé equation.

Summarizing results in [ST], [MMT], let us explain the situation of spaces of initial conditions of classical Painlevé equations in the way of our setting. Let $R = R(Y)$ be a type of the root systems corresponding to a Painlevé equation. Then there exist an affine open subset $\mathcal{M}_R$ of $\mathbf{C}^s = \mathrm{Spec}\,\mathbf{C}[\boldsymbol{\alpha}] = \mathrm{Spec}\,\mathbf{C}[\alpha_1, \cdots, \alpha_s]$, an affine open subset $\mathcal{B}_R$ of $\mathbf{C} = \mathrm{Spec}\,\mathbf{C}[t]$ and the following deformation of non-singular pair

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{M}_R \times \mathcal{B}_R & & \end{array} \tag{46}$$

where $\mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ is a smooth family of rational surface and $\mathcal{D} \hookrightarrow \mathcal{S}$ is a normal crossing divisor. In order to relate this diagram to Okamoto–Painlevé pair, let $\Omega^2_{\mathcal{S}/\mathcal{M}_R \times \mathcal{B}_R}(*\mathcal{D})$ denote the sheaf of germs of relative rational two forms on $\mathcal{S}$ which have poles only along $\mathcal{D}$. There exists a section

$$\omega_{\mathcal{S}} \in \Gamma(\mathcal{S}, \Omega^2_{\mathcal{S}/\mathcal{M}_R \times \mathcal{B}_R}(*\mathcal{D}))$$

which induces a rational 2-form $\omega_{\mathcal{S}_{\boldsymbol{\alpha},t}}$ for each fiber $\mathcal{S}_{\boldsymbol{\alpha},t}$. The pole divisor $\omega_{\mathcal{S}}$ is denoted by $\mathcal{Y}$, and with suitable choice of $\omega_{\mathcal{S}}$ we may assume that each fiber $(\mathcal{S}_{\boldsymbol{\alpha},t}, \mathcal{Y}_{\boldsymbol{\alpha},t})$ is an Okamoto–Painlevé pair of type $R = R(Y)$ and $\mathcal{Y}_{red} = \mathcal{D}$. (Note that on $\mathcal{S} - \mathcal{D}$ the relative rational 2-form $\omega_{\mathcal{S}}$ is holomorphic and non-degenerate on each fiber $\mathcal{S}_{\boldsymbol{\alpha},t} - \mathcal{D}_{\boldsymbol{\alpha},t}$). Assuming that the family (46) is effectively parameterized and semiuniversal at each point of $\mathcal{M}_R \times \mathcal{B}_R$, or equivalently the Kodaira–Spencer map

$$\rho : T_{\boldsymbol{\alpha},t}(\mathcal{M}_R \times \mathcal{B}_R) \longrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \tag{47}$$

is an isomorphism at each point, we have the equality

$$\dim \mathcal{M}_R = \dim H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) - 1. \tag{48}$$

The dimensions of $\mathcal{M}_R$ for generalized rational Okamoto–Painlevé pairs.

$R = R(Y)$	$\tilde{E}_8$	$\tilde{D}_8$	$\tilde{E}_7$	$\tilde{D}_7$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{D}_4$
Painlevé equation	P_I	$P_{III}^{\tilde{D}_8}$	P_{II}	$P_{III}^{\tilde{D}_7}$	$P_{III}^{\tilde{D}_6}$	P_{IV}	P_V	P_{VI}
$s = s(R) = \dim \mathcal{M}_R$ (= $\sharp$ of auxiliary parameters.)	0	0	1	1	2	2	3	4

TABLE 6

(Note that the Okamoto–Painlevé pairs of type $\tilde{D}_8, \tilde{D}_7$ did not appear in the classical literatures (cf. [O1], [O2], [MMT]).)

More notably, Takano et al. [ST], [MMT] constructed an affine open covering $\{\tilde{U}_i\}_{i=1}^{l+m}$ of $\mathcal{S}$ for the classical Okamoto–Painlevé pair of Painlevé equation P_J $(J = II, \cdots, VI)$, which is relative to π and so that

$$\tilde{U}_i = \mathcal{M}_R \times \mathcal{B}_R \times U_i$$

where $U_i = \operatorname{Spec} \mathbf{C}[x_i, y_i] \simeq \mathbf{C}^2$. Moreover, we may assume that $\{\tilde{U}_i\}_{i=1}^{l}$ covers $\mathcal{S} - \mathcal{D}$ and for $1 \leq i \leq l$, we have

$$\omega_{\mathcal{S}|\tilde{U}_i} = dx_i \wedge dy_i \tag{49}$$

In this sense, the restricted morphism

$$\pi : \mathcal{S} - \mathcal{D} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$$

is a deformation of open symplectic surfaces.

By using the results in Appendix B of [Sakai], we can generalize the result of Takano, et al. as follows.

Proposition 5.1. *Let $R = R(Y)$ be one of types of the root systems appeared in Proposition 1.2 which is additive type, so that*

$$\dim H^0(D, \Theta_S(-\log D) \otimes N_D) = 1$$

for corresponding generalized rational Okamoto–Painlevé pair (S, Y) (cf. Theorem 3.1). Moreover denote by r the number of irreducible components of $D = Y_{red}$.

Let $\mathcal{M}_R$ be an affine open subscheme in $\mathbf{C}^s = \operatorname{Spec} \mathbf{C}[\alpha_1, \cdots, \alpha_s]$ of dimension $s = s(R) = 9 - r$ and $\mathcal{B}_R$ be an affine open subscheme of $\mathbf{C} = \operatorname{Spec} \mathbf{C}[t]$. Then there exists the following commutative diagram satisfying the conditions below.

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{M}_R \times \mathcal{B}_R & & . \end{array} \tag{50}$$

(1) *The above diagram is a deformation of non-singular pair of projective surfaces and normal crossing divisors in the sense of Definition 2.1*

(2) *There exists a rational relative 2-form*

$$\omega_{\mathcal{S}} \in \Gamma(\mathcal{S}, \Omega^2_{\mathcal{S}/\mathcal{M}_R \times \mathcal{B}_R}(\mathcal{Y}))$$

whose pole divisor is $\mathcal{Y}$ with $\mathcal{Y}_{red} = \mathcal{D}$.

(3) *If we denote by $\mathcal{Y}$ the pole divisor of $\omega_{\mathcal{S}}$, then for each point $(\boldsymbol{\alpha}, t) \in \mathcal{M}_R \times \mathcal{B}_R$, $(\mathcal{S}_{\boldsymbol{\alpha},t}, \mathcal{Y}_{\boldsymbol{\alpha},t})$ is a generalized Okamoto–Painlevé pair of non fibered type with type $R = R(Y)$. Moreover $\mathcal{Y}_{red} = \mathcal{D}$.*

(4) *The family is semiuniversal at a general point $(\boldsymbol{\alpha}, t) \in \mathcal{M}_R \times \mathcal{B}_R$, that is, the Kodaira–Spencer map*

$$\rho : T_{\boldsymbol{\alpha},t}(\mathcal{M}_R \times \mathcal{B}_R) \longrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \tag{51}$$

is an isomorphism for a general point $(\boldsymbol{\alpha}, t)$.

The Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ of t-direction ($= \mathcal{B}_R$) lies in the image of the natural map (cf. (32)):

$$\delta : \mathbf{C} \simeq H^0(\mathcal{D}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}) \otimes N_{\mathcal{D}_{\boldsymbol{\alpha},t}}) \hookrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})). \tag{52}$$

(5) *Let M_R and B_R denote the affine coordinate rings of $\mathcal{M}_R$ and $\mathcal{B}_R$ respectively so that $\mathcal{M}_R =$ $\mathrm{Spec}\, M_R$ and $\mathcal{B}_R = \mathrm{Spec}\, B_R$. (Note that M_R and B_R is obtained by some localization's of $\mathbf{C}[\alpha_1, \cdots, \alpha_s]$ and $\mathbf{C}[t]$ respectively.) There exists a finite affine covering $\{\tilde{U}_i\}_{i=1}^{l+k}$ of $\mathcal{S}$ relative to $\mathcal{M}_R \times \mathcal{B}_R$ such that there exists an isomorphism for each i*

$$\tilde{U}_i \simeq \mathrm{Spec}(M_R \otimes B_R)[x_i, y_i, \frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)}] \subset \mathrm{Spec}\, \mathbf{C}[\boldsymbol{\alpha}, t, x_i, y_i] \simeq \mathbf{C}^{s+3} \simeq \mathbf{C}^{12-r} \tag{53}$$

Here $f_i(x_i, y_i, \boldsymbol{\alpha}, t)$ is a polynomial in $(M_R \otimes B_R)[x_i, y_i]$. Moreover we may assume that $\mathcal{S} - \mathcal{D}$ can be covered by $\{\tilde{U}_i\}_{i=1}^{l}$. Moreover for each i the restriction of the rational two form $\omega_{\mathcal{S}}$ can be written as

$$\omega_{\mathcal{S}|\tilde{U}_i} = \frac{dx_i \wedge dy_i}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)^{m_i}} \tag{54}$$

(6) *For each pair i, j such that $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$, the coordinate transformation functions*

$$x_i = f_{ij}(x_j, y_j, \boldsymbol{\alpha}, t), \quad y_i = g_{ij}(x_j, y_j, \boldsymbol{\alpha}, t) \tag{55}$$

are rational functions in variables $x_j, y_j, \boldsymbol{\alpha}, t$.

Here we will give a sketch of the proof of Proposition 5.1. (See [Sa-Te] for explicit constructions). For a generalized rational Okamoto–Painlevé pair (S, Y), we see that S can be obtained as a blowing up of $\mathbf{P}^2$ at (possibly infinitely near) 9-points which lie on anti-canonical divisors. Then one can parameterize these 9-points in a suitable way, and this leads to a special time parameter t and other parameter $\alpha_1, \cdots, \alpha_s$, hence we obtain affine schemes $\mathcal{M}_R \subset \mathbf{C}^s$ and $\mathcal{B}_R \subset \mathbf{C}$ deleting the points which do not correspond to Okamoto–Painlevé pairs of non-fibered type with type R. (We can show that non-fiberedness and type of (S, Y) just depends on the parameters $\boldsymbol{\alpha}$ and not depends on $t \in B_R$.) Moreover we can construct a semiuniversal family $\pi : \mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ of rational surfaces by blowings–up of $\mathbf{P}^2 \times \mathcal{M}_R \times \mathcal{B}_R$. Moreover by these explicit constructions, we can obtain the affine coverings of the total space $\mathcal{S}$ as above.

Remark 5.1. We can construct a similar family of generalized Okamoto–Painlevé pairs of type $\tilde{A}_{r-1}$, $2 \leq r \leq 9$ (multiplicative type). However as proved in [T], we see that

$$H^1_Y(S, \Theta_S(-\log Y)) = \{0\}.$$

This result implies that we can not obtain a differential equation from the generalized Okamoto–Painlevé pair of type $\tilde{A}_r$ as in the way above.

6. From Global Deformations to Hamiltonian systems

In this section, we will explain how one can derive differential equations from the one dimensional global deformation of generalized rational Okamoto-Painlevé pairs of additive type which arises from the image of linear map

$$\delta : \mathbf{C} \simeq H^0(D, \Theta_S(-\log D) \otimes N_D) \longrightarrow H^1(S, \Theta_S(-\log D))$$

or more generally the kernel of the restriction map res in (17). (Recall that $\mathrm{Im}(\delta) \subset \ker(\mathsf{res})$). The idea of the proof is essentially same as in Kodaira–Spencer theory (Section 4). We will be able to obtain a global rational vector field $\tilde{v}$ on $\mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ whose poles are only supported on $\mathcal{D}$, which is a lift of $\frac{\partial}{\partial t}$. Hence the restriction of $\tilde{v}$ to $\mathcal{S} - \mathcal{D}$ gives a global regular algebraic vector field. Moreover we will show that such a lift $\tilde{v}$ of $\frac{\partial}{\partial t}$ is unique. The restriction of $\tilde{v}$ to the open affine subset $\tilde{U}_i$ of $\mathcal{S} - \mathcal{D}$ gives an explicit system of algebraic ODEs of 1st order.

The second aim of this section is to explain the reason why the systems obtained as above are Hamiltonian systems. This fact is known classically ([O1], [O2], [MMT]) and from the uniqueness of the ODE we can check the fact from explicit equations. In this section, we will show that global vector field $\tilde{v}$ preserves the relative 2 forms $\omega_{\mathcal{S}}$. This fact can be expressed (cf. Theorem 6.2) as

$$d_{\mathcal{S}/\mathcal{M}}(\tilde{v} \cdot (\omega_{\mathcal{S}} \wedge dt)) = 0. \tag{56}$$

In classical cases (that is, $R \neq \tilde{D}_7, \tilde{D}_8$), as in ([MMT]) we have at least one open embedding $\mathbf{C}^2 \subset S - D$ and its nice coordinates (x, y) such that $\omega_{\mathcal{S}|\mathbf{C}^2} = dx \wedge dy$. By using such nice coordinates, the equation (56) shows that the differential equation obtained as above becomes a Hamiltonian system with a polynomial Hamiltonian. However since we can not always cover $S - D$ by copies of $\mathbf{C}^2$ in case of $\tilde{D}_7, \tilde{D}_8$ and $\tilde{E}_8$ and the restriction of $\omega_{\mathcal{S}}$ to some open affine set looks like as in (54), we can not always written the systems in Hamiltonian systems in the affine coordinates.

We start to explain how to obtain the differential equations.

Let $R = R(Y)$ be one of types of additive affine root systems appeared in Proposition 1.2 and let

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{M}_R \times \mathcal{B}_R & & \end{array} \tag{57}$$

be a global deformation of generalized Okamoto–Painlevé pairs of type R as in Proposition 5.1. The total space $\mathcal{S}$ has a finite affine covering $\{\tilde{U}_i\}_{i=1}^{l+k}$ such that

$$\tilde{U}_i \simeq \mathrm{Spec}(M_R \otimes B_R)[x_i, y_i, \frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)}] \subset \mathrm{Spec}\, \mathbf{C}[\boldsymbol{\alpha}, t, x_i, y_i] \tag{58}$$

as in (53). Moreover, we may assume that $\mathcal{S} - \mathcal{D}$ can be covered by $\{\tilde{U}_i\}_{i=1}^{l}$, that is,

$$\mathcal{S} - \mathcal{D} = \cup_{i=1}^{l} \tilde{U}_i.$$

Let us recall that the coordinate transformations in (55) for $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$ are given by the rational functions

$$x_i = f_{ij}(x_j, y_j, \boldsymbol{\alpha}, t), \quad y_i = g_{ij}(x_j, y_j, \boldsymbol{\alpha}, t) \tag{59}$$

The Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ can be represented by the Čech 1-cocycles

$$\rho(\frac{\partial}{\partial t}) = \{ \ \theta_{ij} = \frac{\partial f_{ij}}{\partial t}\frac{\partial}{\partial x_i} + \frac{\partial g_{ij}}{\partial t}\frac{\partial}{\partial y_i} \in \Gamma(\tilde{U}_i \cap \tilde{U}_j, \Theta_{\mathcal{S}/\mathcal{M}_R \times \mathcal{B}_R}(-\log \mathcal{D})) \ \} \tag{60}$$

Recall that (cf. Proposition 5.1, 3) on $\mathcal{M}_R \times \mathcal{B}_R$ the corresponding Okamoto–Painlevé pairs $(\mathcal{S}_{\boldsymbol{\alpha},t}, \mathcal{Y}_{\boldsymbol{\alpha},t})$ are *of non-fibered type*.

From (52) of Proposition 5.1, we may assume that $\rho(\frac{\partial}{\partial t})$ is non-zero element of the image of δ:

$$\begin{array}{ccc} \mathbf{C} \simeq H^0(D, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}) \otimes N_D) & \subset & H^1_{\mathcal{D}_{\boldsymbol{\alpha},t}}(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})). \\ & \delta \searrow & \downarrow \\ & & H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \end{array} \tag{61}$$

Since the local cohomology group is the kernel of the natural restriction map (cf. Proposition 3.1)

$$\mathsf{res} : H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \longrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t} - \mathcal{D}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})), \tag{62}$$

the Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ is cohomologus to zero in $H^1(\mathcal{S}_{\boldsymbol{\alpha},t} - \mathcal{D}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}))$. More precisely, from (32), we see that the 1-cocycle $\rho(\frac{\partial}{\partial t})$ is cohomologus to zero in

$$H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}) \otimes \mathcal{O}_S(\mathcal{D}_{\boldsymbol{\alpha},t})). \tag{63}$$

Since dimensions of these cohomology groups are constant as functions of $(\boldsymbol{\alpha}, t)$, by an argument using the base change theorem, we can see that for $1 \leq i \leq l+k$ there exist rational vector fields

$$\theta_i(x_i, y_i, \boldsymbol{\alpha}, t) = \eta_i(x_i, y_i, \boldsymbol{\alpha}, t)\frac{\partial}{\partial x_i} + \zeta_i(x_i, y_i, \boldsymbol{\alpha}, t)\frac{\partial}{\partial y_i} \ \in \ \Gamma(\tilde{U}_i, \Theta_{\tilde{U}_i}(-\log \mathcal{D}) \otimes \mathcal{O}(\mathcal{D})) \tag{64}$$

such that

$$\theta_{ij}(x_i, y_i, \boldsymbol{\alpha}, t) = \theta_j(x_j, y_j, \boldsymbol{\alpha}, t) - \theta_i(x_i, y_i, \boldsymbol{\alpha}, t). \tag{65}$$

Note that for $1 \leq i \leq l$ one has $\tilde{U}_i \cap \mathcal{D} = \emptyset$. Hence for $1 \leq i \leq l$, $\theta_i(x_i, y_i, \boldsymbol{\alpha}, t)$ is a regular algebraic vector field, that is, $\eta_i(x_i, y_i, \boldsymbol{\alpha}, t)$ and $\zeta(x_i, y_i, \boldsymbol{\alpha}, t)$ are regular algebraic functions on $\tilde{U}_i$.

As in (36) of §4, for any pair i, j such that $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$, we have the identity on $\tilde{U}_i \cap \tilde{U}_j$

$$\left(\frac{\partial}{\partial t}\right)_j = \left(\frac{\partial}{\partial t}\right)_i + \theta_{ij}(\boldsymbol{\alpha}, t), \tag{66}$$

and hence we have

$$\left(\frac{\partial}{\partial t}\right)_j = \left(\frac{\partial}{\partial t}\right)_i + (\theta_j(x_j, y_j, \boldsymbol{\alpha}, t) - \theta_i(x_i, y_i, \boldsymbol{\alpha}, t)), \tag{67}$$

or

$$\left(\frac{\partial}{\partial t}\right)_j - \theta_j(x_j, y_j, \boldsymbol{\alpha}, t) = \left(\frac{\partial}{\partial t}\right)_i - \theta_i(x_i, y_i, \boldsymbol{\alpha}, t). \tag{68}$$

This means that the vector fields

$$\{\left(\frac{\partial}{\partial t}\right)_i - \theta_i(x_i, y_i, \boldsymbol{\alpha}, t)\}_{1 \leq i \leq l+k} \tag{69}$$

can be patched together and define a global rational vector field

$$\tilde{v} \in \Gamma(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D}) \otimes \mathcal{O}(\mathcal{D})).$$

Note that this global rational vector field $\tilde{v}$ is a lift of $\frac{\partial}{\partial t}$ via $\pi : \mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ and

$$\tilde{v}_{|\mathcal{S}-\mathcal{D}} \tag{70}$$

is a regular algebraic vector field. Moreover since for each $(\boldsymbol{\alpha}, t) \in \mathcal{M}_R \times \mathcal{B}_R$ we have $H^0(\mathcal{S}_{\boldsymbol{\alpha},t} - \mathcal{D}_{\boldsymbol{\alpha},t}, \Theta^{alg}_{\mathcal{S}_{\boldsymbol{\alpha},t}-\mathcal{D}_{\boldsymbol{\alpha},t}}) = 0$ (cf. Proposition 2.1), we can show the uniqueness of $\tilde{v}$ of lifting of $\frac{\partial}{\partial t}$ to the rational vector field to $\mathcal{S}$. Now we obtain the following

Theorem 6.1. *Let $R = R(Y)$, $\mathcal{S}, \mathcal{D}, \mathcal{M}_R \times \mathcal{B}_R$ be as above. Then there exists a unique global rational vector field*

$$\tilde{v} \in \Gamma(\mathcal{S}, \Theta(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{D}))$$

on $\mathcal{S}$ which is a lift of $\frac{\partial}{\partial t}$. The restriction $\tilde{v}_{\mathcal{S}-\mathcal{D}}$ to $\mathcal{S}-\mathcal{D}$ is a regular algebraic vector field. Moreover the restriction $\tilde{v}$ to each open covering $\tilde{U}_i$ for $1 \leq i \leq l$ (corresponding to the open coverings of $\mathcal{S} - \mathcal{D}$) can be written as

$$\tilde{v}_{|\tilde{U}_i} = \frac{\partial}{\partial t} - \theta_i = \frac{\partial}{\partial t} - \eta_i \frac{\partial}{\partial x_i} - \zeta_i \frac{\partial}{\partial y_i} \tag{71}$$

and defines a system of differential equations

$$\begin{cases} \dfrac{dx_i}{dt} = -\eta_i(x_i, y_i, \boldsymbol{\alpha}, t) \\ \dfrac{dy_i}{dt} = -\zeta_i(x_i, y_i . \boldsymbol{\alpha}, t) \end{cases} . \tag{72}$$

Here the functions η_i, ζ_i are regular algebraic functions on $\tilde{U}_i$.

Remark 6.1. (1) The argument above shows that there exists a differential equation as above at least locally for any direction corresponding to the kernel of the restriction map

$$\mathsf{res} : H^1(S, \Theta_S(-\log D)) \longrightarrow H^1(S - D, \Theta_S(-\log D)).$$

(2) Let us recall the so-called Painlevé property which is states as follows. *If $(x(t), y(t))$ is a local solution of (72) determined by an arbitrary initial conditions $(x_0 = x(t_0), y_0 = y(t_0)) \in \tilde{U}_i$ with fixed $t_0 \in \mathcal{B}_R$ then both solutions $x(t)$ and $y(t)$ can be meromorphic-ally continued along any curve in $\mathcal{B}_R$ with a starting point t_0.* For non-classical Okamoto–Painlevé pair, it is not clear that the differential equation in (72) has the Painlevé property. In general, the proof of Painlevé property for classical Painlevé equation is not so easy. We hope that there is an easy geometric proof of the Painlevé property for differential equation in (72).

It is well-known that each classical Painlevé differential equation P_J, $J = I, II, \cdots, VI$ is equivalent to a Hamiltonian system (H_J) whose Hamiltonian $H_J(x, y, \boldsymbol{\alpha}, t)$ is a polynomial in $(x, y) \in \mathbf{C}^2$ (cf. [O1], [MMT], §7).

$$(H_J) : \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H_J}{\partial y} \\ \dfrac{dy}{dt} = -\dfrac{\partial H_J}{\partial x} \end{cases} . \tag{73}$$

This fact can be checked by explicit calculations for each differential equations (cf. [Sa-Te]). However, the reason why the equations obtained in (72) become Hamiltonian systems is not clear.

In what follows, we will give a geometric explanation for this fact. For the purpose, we will introduce sheaves of infinitesimal symplectic automorphisms.

Lemma 6.1. *Let (S, Y) be a rational Okamoto–Painlevé pair of non-fibered type with the root system R of additive type and set $D = Y_{red}$ and $E = Y - D$. Fix a non-zero rational 2-form ω with the pole divisor Y. Then there exists a commutative diagram of exact sequences of sheaves*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_\omega & \longrightarrow & \Theta_S(-\log D) & \xrightarrow{d_\omega} & \Omega^2_S(Y) \simeq \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(D)_\omega & \longrightarrow & \Theta_S(-\log D) \otimes \mathcal{O}_S(D) & \xrightarrow{d_\omega} & \Omega^2_S(Y + D) \simeq \mathcal{O}_S(D) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma(D)_\omega/\Gamma_\omega & \longrightarrow & \Theta_S(-\log D) \otimes N_D & \xrightarrow{d_\omega} & \Omega^2_S(Y + D)/\Omega^2_S(D) \simeq N_D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{74}
$$

Here the map d_ω is given by $\theta \mapsto d_\omega(\theta) = d(\theta \cdot \omega)$ and Γ_ω and $\Gamma(D)_\omega$ are the kernels of d_ω's in the above diagram.

Proof. Since Y is the pole of the rational two form ω and $K_S = -Y$, we see that $\Omega^2_S(Y) \simeq \mathcal{O}_S$ and the contractions with ω give the isomorphisms

$$
\Theta_S(-\log D) \xrightarrow{\simeq} \Omega^1_S(\log D) \otimes \mathcal{O}_S(E), \; \Theta_S(-\log D) \otimes \mathcal{O}_S(D) \xrightarrow{\simeq} \Omega^1_S(\log D) \otimes \mathcal{O}_S(Y) \tag{75}
$$

Then we only have to show that the following two maps of sheaves are well-defined and surjective.

$$
\begin{array}{rcl}
\Omega^1_S(\log D) \otimes \mathcal{O}_S(E) & \xrightarrow{d} & \Omega^2_S(\log D) \otimes \mathcal{O}(E) \simeq \Omega^2_S(D + E) \simeq \Omega^2_S(Y) \\
\Omega^1_S(\log D) \otimes \mathcal{O}_S(Y) & \xrightarrow{d} & \Omega^2_S(\log D) \otimes \mathcal{O}(Y) \simeq \Omega^2_S(Y + D)
\end{array}
\tag{76}
$$

We can consider the problem analytic locally. At a point $p \in S - D$, the above two maps are just usual exterior derivatives and so they are well-defined and surjective. Let us consider the point $p \in D$. We have the local coordinates (x, y) near p such that the local equation of Y is given by $x^m y^n = 0$. First we consider the case of $m = 0$ and $n > 0$. Then the equations of D and E near p is given by $y = 0$ and y^{n-1}. Local generators of $\Omega^1_S(\log D) \otimes \mathcal{O}(E)$ and $\Omega^1_S(\log D) \otimes \mathcal{O}(Y)$ are given by

$$
\frac{1}{y^{n-1}} dx, \frac{1}{y^n} dy, \qquad \frac{1}{y^n} dx, \frac{1}{y^{n+1}} dy
$$

respectively. Moreover in case that both of n, m are positive, local generators of these sheaves are given by

$$
\frac{1}{x^m y^{n-1}} dx, \frac{1}{x^{m-1} y^n} dy, \quad \frac{1}{x^{m+1} y^n} dx, \frac{1}{x^m y^{n+1}} dy.
$$

By using these local generators, we see that the exterior derivatives in (76) are well-defined and surjective.

Definition 6.1. *The kernel* Γ_ω *of* $d_\omega : \Theta_S(-\log D) \longrightarrow \Omega^2_S(Y)$ *is called the sheaf of germs of local symplectic logarithmic vector fields with respect to* ω. *Note that* Γ_ω *depends on the choice of* ω.

Now let us consider the relative situation. Let $\pi : \mathcal{S} \longrightarrow \mathcal{M} \times \mathcal{B}$ be the global semiuniversal family of Okamoto–Painlevé pair of type R (cf. Proposition 5.1). Note that we have a relative rational 2-form $\omega_{\mathcal{S}}$ with the pole divisor $\mathcal{Y}$, that is, $\omega_{\mathcal{S}} \in \Gamma(\mathcal{S}, \Omega^2_{\mathcal{S}/\mathcal{M}\times\mathcal{B}}(\mathcal{Y}))$ and $\mathcal{D} = \mathcal{Y}_{red}$. Let dt denote the restriction of the canonical generator of $\Omega^1_{\mathrm{Spec}\,\mathbf{C}[t]}$ to $B \subset \mathrm{Spec}\,\mathbf{C}[t]$. It is easy to see that

$$\Psi = \omega_{\mathcal{S}} \wedge dt \tag{77}$$

defines a relative rational 3-form in $\Gamma(\mathcal{S}, \Omega^3_{\mathcal{S}/\mathcal{M}}(\mathcal{Y}))$ with the pole divisor $\mathcal{Y}$. (Note that Ψ defines an isomorphism $\mathcal{O}_{\mathcal{S}} \simeq \Omega^3_{\mathcal{S}/\mathcal{M}}(\mathcal{Y})$.) We shall prove the following fundamental theorem.

Theorem 6.2. *Let* $\tilde{v} \in \Gamma(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D}))$ *be the global rational vector field obtained in Theorem 6.1. Then we have*

$$d_{\mathcal{S}/\mathcal{M}}(\tilde{v} \cdot \Psi) = 0. \tag{78}$$

Without loss of generality, we may assume that $\mathcal{M} = \mathrm{Spec}\,\mathbf{C}$ in the following proof of Theorem 6.2.

In order to prove Theorem 6.2, we need some preparation. Let us consider the following commutative diagram of exact sequences of tangent sheaves. (Note that since $\pi : \mathcal{S} \longrightarrow \mathcal{B}$ is smooth and $\mathcal{D}$ is a divisor with normal crossings $\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})$ and $\Theta_{\mathcal{S}}(-\log \mathcal{D})$ are locally free $\mathcal{O}_{\mathcal{S}}$-sheaves.)
n

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}) & \longrightarrow & \Theta_{\mathcal{S}}(-\log \mathcal{D}) & \longrightarrow & \pi^*(\Theta_{\mathcal{B}}) & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})(\mathcal{D}) & \longrightarrow & \Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D}) & \longrightarrow & \pi^*(\Theta_{\mathcal{B}})(\mathcal{D}) & \longrightarrow 0 & . \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}) \otimes N_{\mathcal{D}} & \longrightarrow & \Theta_{\mathcal{S}}(-\log \mathcal{D}) \otimes N_{\mathcal{D}} & \longrightarrow & \pi^*(\Theta_{\mathcal{B}}) \otimes N_{\mathcal{D}} & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{79}$$

Lemma 6.2. *Let* $\pi : \mathcal{S} \longrightarrow \mathcal{B}$ *be the family of rational Okamoto–Painlevé pairs of type* R *of non-fibered type as above. Then we have the following:*

(1) $\pi_*(\mathcal{O}_{\mathcal{S}}) \simeq \pi_*(\mathcal{O}_{\mathcal{S}}(\mathcal{D})) \simeq \mathcal{O}_{\mathcal{B}}, \quad R^1\pi_*\mathcal{O}_{\mathcal{S}} = 0.$

(2) $\pi_*(N_{\mathcal{D}}) = 0.$

(3) $\pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})) \simeq \pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})(\mathcal{D})) \simeq 0.$

(4)
$$\mathcal{O}_{\mathcal{B}} \simeq \pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})) \otimes N_{\mathcal{D}}) \simeq \pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})) \otimes N_{\mathcal{D}}).$$

(5) *The connecting homomorphism*
$$\rho : \Theta_{\mathcal{B}} \longrightarrow R^1\pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}))$$
is the Kodaira–Spencer map and injective.

(6) $\pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})) = 0.$

Proof. Since all fibers of $\pi : \mathcal{S} \longrightarrow \mathcal{B}$ are rational surfaces of non-fibered type, (1) of Proposition 2.1 shows the first assertion, from which the second assertion follows. The third assertion follows from (3) of Proposition 2.1. The fourth and fifth assertion follows from (4) of Proposition 5.1. Taking the direct images of the first row of (79), the assertion (6) follows from the assertion (5). $\square$

Taking the direct images of the diagram (79), from Lemma 6.2, we obtain the commutative diagram:

$$\begin{array}{cccccc}
 & & & & & 0 \\
 & & & & & \downarrow \\
 & 0 & & 0 & & \pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})) \otimes N_{\mathcal{D}}) \simeq \mathcal{O}_{\mathcal{B}} \\
 & \downarrow & & \downarrow & & \downarrow \mu \\
 & 0 & \longrightarrow & \Theta_{\mathcal{B}} & \xrightarrow{\rho} & R^1\pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})) \\
 & \downarrow & & || & & \downarrow \lambda \\
0 \longrightarrow & \pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D})) & \xrightarrow{\tau} & \Theta_{\mathcal{B}} & \xrightarrow{\rho_1} & R^1\pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})(\mathcal{D})) \\
 & \downarrow \nu & & \downarrow & & \downarrow \\
\mathcal{O}_{\mathcal{B}} \simeq & \pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})) \otimes N_{\mathcal{D}}) & \longrightarrow & 0 & \longrightarrow & \\
 & \downarrow & & & &
\end{array} . \tag{80}$$

Lemma 6.3. (1) *The image of ρ and μ in (80) coincides, i.e.,*

$$\mu(\pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}) \otimes N_{\mathcal{D}}) = \rho(\Theta_{\mathcal{B}}) \subset R^1\pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})). \tag{81}$$

(2) *The sheaf map ρ_1 in (80) is the zero map and τ induces an isomorphism*

$$\tau : \pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D})) \xrightarrow{\simeq} \Theta_{\mathcal{B}}. \tag{82}$$

Hence the vector field $\tilde{v}$ in Theorem 6.1 is the lift of $\frac{\partial}{\partial t}$ via the natural isomorphism

$$\tilde{\tau} : H^0(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D})) \xrightarrow{\simeq} H^0(\mathcal{B}, \Theta_{\mathcal{B}}). \tag{83}$$

(3) *The map*

$$\nu : \pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D})) \longrightarrow \pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})) \otimes N_{\mathcal{D}}) \simeq \mathcal{O}_{\mathcal{B}}$$

is an isomorphism.

Proof. The first assertion follows from (52) of Proposition 5.1. Then we have the second assertion because

$$\rho_1(\Theta_{\mathcal{B}}) = \lambda(\rho(\Theta_{\mathcal{B}})) = \lambda(\mu(\pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}) \otimes N_{\mathcal{D}}))) = 0.$$

The rest of assertions follow from this easily. $\square$

By the same argument as in Lemma 6.1, we obtain the following commutative diagram of sheaves:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \tilde{\Gamma}_{\Psi} & \longrightarrow & \Theta_{\mathcal{S}}(-\log \mathcal{D}) & \xrightarrow{d_{\Psi}} & \Omega^3_{\mathcal{S}}(\mathcal{Y}) \simeq \mathcal{O}_S & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \tilde{\Gamma}(D)_{\Psi} & \xrightarrow{\varphi} & \Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D}) & \xrightarrow{d_{\Psi}} & \Omega^3_{\mathcal{S}}(\mathcal{Y}+\mathcal{D}) \simeq \mathcal{O}_{\mathcal{S}}(\mathcal{D}) & \longrightarrow 0 & . \\
 & & \downarrow \nu_1 & & \downarrow \nu & & \downarrow & & \\
0 \longrightarrow & & \tilde{\Gamma}(D)_{\Psi}/\tilde{\Gamma}_{\Psi} & \xrightarrow{\varphi_1} & \Theta_{\mathcal{S}}(-\log D) \otimes N_{\mathcal{D}} & \xrightarrow{d_{\Psi}} & \Omega^3_{\mathcal{S}}(\mathcal{Y}+\mathcal{D})/\Omega^3_{\mathcal{S}}(\mathcal{D}) \simeq N_{\mathcal{D}} & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{84}$$

Here d_{Ψ} can be defined for a local section θ of $\Theta_{\mathcal{S}}(-\log \mathcal{D})$ as

$$d_{\Psi}(\theta) = d_{\mathcal{S}}(\theta \cdot \Psi),$$

and $\tilde{\Gamma}_{\Psi}$ and $\tilde{\Gamma}(D)_{\Psi}$ are defined to be the kernels of d_{Ψ}'s in (84).

Theorem 6.2 follows from the following

Lemma 6.4. *The natural map φ in (84) induces an isomorphism*

$$\tilde{\varphi} : H^0(\mathcal{S}, \tilde{\Gamma}(D)_{\Psi}) \xrightarrow{\simeq} H^0(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D})). \tag{85}$$

In fact, since $\tilde{v} \in H^0(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D}))$, Lemma 6.4 implies that $\tilde{v} \in H^0(\mathcal{S}, \tilde{\Gamma}(D)_{\Psi})$ and by definition

$$d_{\Psi}(\tilde{v}) = d_{\mathcal{S}}(\tilde{v} \cdot \Psi) = 0.$$

Proof of Lemma 6.4. We only have to show that the natural morphism

$$\varphi : \pi_*(\tilde{\Gamma}(D)_{\Psi}) \longrightarrow \pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D})) \tag{86}$$

is an isomorphism.

Taking the direct images of the sheaves in (84), together with Lemma 6.2, we obtain the following commutative diagram:

$$\begin{array}{cccccccc}
 & & & & & & & 0 \\
 & & & & & & & \downarrow \\
 & 0 \longrightarrow & 0 & \longrightarrow & 0 & \xrightarrow{d_{\Psi}} & & \mathcal{O}_B \\
 & & \downarrow & & \downarrow & & & \downarrow \\
 & 0 \longrightarrow & \pi_*(\tilde{\Gamma}(D)_{\Psi}) & \xrightarrow{\varphi} & \pi_*(\Theta_{\mathcal{S}}(-\log \mathcal{D})(\mathcal{D})) \simeq \mathcal{O}_{\mathcal{B}} & \xrightarrow{d_{\Psi}} & & \mathcal{O}_{\mathcal{B}} \ . \\
 & & \downarrow \nu_1 & & \downarrow \nu & & & \downarrow \\
 & 0 \longrightarrow & \pi_*(\tilde{\Gamma}(D)_{\Psi}/\tilde{\Gamma}_{\Psi}) & \xrightarrow{\varphi_1} & \pi_*(\Theta_{\mathcal{S}}(-\log D) \otimes N_{\mathcal{D}}) \simeq \mathcal{O}_{\mathcal{B}} & \xrightarrow{d_{\Psi}} & & 0 \\
 & & \downarrow \mu_1 & & \downarrow & & & \downarrow \\
0 \longrightarrow & \mathcal{O}_{\mathcal{B}} \longrightarrow & R^1\pi_*(\tilde{\Gamma}_{\Psi}) & \longrightarrow & R^1\pi_*\Theta_{\mathcal{S}}(-\log \mathcal{D}) & \longrightarrow & & 0
\end{array} \tag{87}$$

Since we have proved that the map ν and φ_1 are isomorphism of sheaves, in order to show that φ is an isomorphism, we only have to show that the map

$$\mu_1 : \pi_*(\tilde{\Gamma}(D)_{\Psi}/\tilde{\Gamma}_{\Psi}) \longrightarrow R^1\pi_*(\tilde{\Gamma}_{\Psi}) \tag{88}$$

is the zero map.

Combining the commutative diagram (84) with (79), we obtain the the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \Gamma_{\Psi} & \longrightarrow & \Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}) & \xrightarrow{d_{\Psi}} & \Omega^3_{\mathcal{S}}(\mathcal{Y}) \simeq \mathcal{O}_S & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \Gamma(D)_{\Psi} & \longrightarrow & \Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{D}) & \xrightarrow{d_{\Psi}} & \Omega^3_{\mathcal{S}}(\mathcal{Y}+\mathcal{D}) \simeq \mathcal{O}_{\mathcal{S}}(\mathcal{D}) & \longrightarrow 0 & . \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & \Gamma(D)_{\Psi}/\Gamma_{\Psi} & \longrightarrow & \Theta_{\mathcal{S}/\mathcal{B}}(-\log D) \otimes N_{\mathcal{D}} & \xrightarrow{d_{\Psi}} & \Omega^3_{\mathcal{S}}(\mathcal{Y}+\mathcal{D})/\Omega^3_{\mathcal{S}}(\mathcal{D}) \simeq N_{\mathcal{D}} & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{89}
$$

Again, here Γ_{Ψ} and $\Gamma(D)_{\Psi}$ denote the kernels of d_{Ψ}'s in the above diagram.

Then it is easy to see the following diagrams are commutative:

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \Gamma_{\Psi} & \longrightarrow & \tilde{\Gamma}_{\Psi} & \longrightarrow & \pi^*(\Theta_{\mathcal{B}}) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \| & \\
0 \longrightarrow & \Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}) & \longrightarrow & \Theta_{\mathcal{S}}(-\log \mathcal{D}) & \longrightarrow & \pi^*(\Theta_{\mathcal{B}}) & \longrightarrow 0 \quad . \\
 & \downarrow & & \downarrow & & & \\
 & \mathcal{O}_{\mathcal{S}} & = & \mathcal{O}_{\mathcal{S}} & & & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}
\tag{90}
$$

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Gamma_{\Psi} & \longrightarrow & \tilde{\Gamma}_{\Psi} & \longrightarrow & \pi^*(\Theta_{\mathcal{B}}) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Gamma(\mathcal{D})_{\Psi} & \longrightarrow & \tilde{\Gamma}(\mathcal{D})_{\Psi} & \longrightarrow & \pi^*(\Theta_{\mathcal{B}})(\mathcal{D}) & \longrightarrow 0 \quad . \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Gamma(D)_{\Psi}/\Gamma_{\Psi} & \longrightarrow & \tilde{\Gamma}(D)_{\Psi}/\tilde{\Gamma}_{\Psi} & \longrightarrow & \pi^*(\Theta_{\mathcal{B}}) \otimes N_{\mathcal{D}} & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
\tag{91}
$$

By Lemma 6.2, we easily see the following.

$$
\begin{array}{l}
\pi_*(\Gamma_{\Psi}) \simeq \pi_*(\Gamma(D)_{\Psi}) = 0 \\
\pi_*(\tilde{\Gamma}_{\Psi}) = 0 \\
\pi_*(\Theta_{\mathcal{S}/\mathcal{D}}(-\log \mathcal{D}) \otimes N_{\mathcal{D}}) \simeq \pi_*(\Gamma(D)_{\Psi}/\Gamma_{\Psi}) \simeq \pi_*(\tilde{\Gamma}(D)_{\Psi}/\tilde{\Gamma}_{\Psi}) \simeq \mathcal{O}_{\mathcal{B}}
\end{array}
\tag{92}
$$

By using these facts, from (89) and (90), we obtain the commutative diagram:

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ & & \pi_*(\Gamma(D)_\Psi/\Gamma_\Psi) & \simeq & \pi_*(\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D})\otimes N_{\mathcal{D}}) \simeq \mathcal{O}_{\mathcal{B}} \\ & & \downarrow \mu_2 & & \downarrow \mu \\ 0 \longrightarrow \quad \mathcal{O}_{\mathcal{B}} & \longrightarrow & R^1\pi_*\Gamma & \longrightarrow & R^1\pi_*\Theta_{\mathcal{S}/\mathcal{B}}(-\log \mathcal{D}) \\ & & \uparrow \rho & & \uparrow \rho \\ & & \Theta_{\mathcal{B}} & = & \Theta_{\mathcal{B}} \\ & & \uparrow & & \uparrow \\ & & 0 & & 0 \end{array} \tag{93}$$

Since $\operatorname{Im}\mu = \operatorname{Im}\rho$ (cf. (81)), the diagram (93) shows that

$$\mu_2(\pi_*(\Gamma(D)_\Psi/\Gamma_\Psi)) = \rho(\Theta_{\mathcal{B}}). \tag{94}$$

On the other hand, from (91) and (92), we obtain

$$\begin{array}{ccccc} & & 0 & \longrightarrow & \pi_*(\tilde{\Gamma}(\mathcal{D})_\Psi) \\ & & \downarrow & & \downarrow \nu_1 \\ & 0 \longrightarrow & \pi_*(\Gamma(D)_\Psi/\Gamma_\Psi) & \overset{\phi}{\simeq} & \pi_*(\tilde{\Gamma}(D)_\Psi/\tilde{\Gamma}_\Psi) \\ & & \downarrow \mu_2 & & \downarrow \mu_1 \\ 0 \longrightarrow & \Theta_{\mathcal{B}} \xrightarrow{\rho} & R^1\pi_*(\Gamma_\Psi) & \xrightarrow{\phi_1} & R^1\pi_*\tilde{\Gamma}_\Psi \end{array} . \tag{95}$$

Now from (95) and (94), we obtain

$$\mu_1(\pi_*(\tilde{\Gamma}(D)_\Psi/\tilde{\Gamma}_\Psi)) = \mu_1(\phi(\pi_*(\Gamma(D)_\Psi/\Gamma_\Psi))) = \phi_1\mu_2((\pi_*(\Gamma(D)_\Psi/\Gamma_\Psi))) = \phi_1(\rho(\Theta_{\mathcal{B}})) = 0. \tag{96}$$

Therefore μ_1 is the zero map, which shows Lemma 6.4.

□.

In what follows, we shall see the consequence of Theorem 6.2.

Recall that $\mathcal{S} - \mathcal{D}$ is covered by $\{\tilde{U}_i\}_{i=1}^l$ and

$$\tilde{U}_i = \operatorname{Spec}(M_R \otimes B_R)[x_i, y_i, \frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)}] \subset \mathcal{M}_R \times \mathcal{B}_R \times \mathbf{C}^2,$$

and the restriction of the relative two form $\omega_{\mathcal{S}}$ to $\tilde{U}_i$ can be written as

$$\omega_{\mathcal{S}|\tilde{U}_i} = \frac{dx_i \wedge dy_i}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)^m}. \tag{97}$$

Let $\tilde{v}$ be the global rational vector fields obtained in Theorem (6.1) and let $\Psi = \omega_{\mathcal{S}} \wedge dt$ be the relative 3-form in $\Omega^3_{\mathcal{S}/\mathcal{M}}(\mathcal{Y})$ defined in (77). Then the contraction gives a global rational 2-form

$$\Omega := \tilde{v} \cdot \Psi \in \Gamma(\mathcal{S}, \Omega^2_{\mathcal{S}/\mathcal{M}}(\log \mathcal{D})(\mathcal{Y})). \tag{98}$$

On $\tilde{U}_i$, the restriction of 2-form Ω to $\tilde{U}_i$ is given by

$$\Omega_i := \Omega_{|\tilde{U}_i} = (\frac{\partial}{\partial t} - \theta_i) \cdot (\omega_{\mathcal{S}|\tilde{U}_i} \wedge dt) = \omega_{\mathcal{S}|\tilde{U}_i} - \theta_i \cdot \omega_{\mathcal{S}|\tilde{U}_i} \wedge dt. \tag{99}$$

Here, for $1 \leq i \leq l$, the contraction of θ_i and $\omega_{\mathcal{S}|\tilde{U}_i}$ is given by

$$\theta_i \cdot \omega_{\mathcal{S}|\tilde{U}_i} = \frac{1}{f_i^{m_i}}(\eta_i dy_i - \zeta_i dx_i).$$

Theorem 6.2 can be restated as follows.

Proposition 6.1.

$$d_{\mathcal{S}/\mathcal{M}}(\Omega) = d_{\mathcal{S}/\mathcal{M}}(\tilde{v}\cdot\Psi) = 0$$

For $\tilde{U}_i \subset \mathcal{S} - \mathcal{D}$ $(1 \leq i \leq l)$, we have

$$\begin{aligned} d_{\mathcal{S}/\mathcal{M}}(\Omega_{|\tilde{U}_i}) &= d_{\mathcal{S}/\mathcal{M}}\Big(\frac{dx_i \wedge dy_i}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)^{m_i}} - (\theta_i \cdot \omega_{\mathcal{S}}) \wedge dt\Big) \\ &= \Big(\frac{\partial}{\partial t}\left(\frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)^{m_i}}\right) dx_i \wedge dy_i - d_\pi(\theta_i \cdot \omega_{\mathcal{S}})\Big) \wedge dt = 0 \end{aligned}$$

where we set $d_\pi = d_{\mathcal{S}/\mathcal{M}\times\mathcal{B}}$.

Therefore, Proposition 6.1 implies the following

Corollary 6.1. *For each i, $1 \leq i \leq l$, we have the fundamental equation*

$$\frac{\partial}{\partial t}\left(\frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)^{m_i}}\right) dx_i \wedge dy_i - d_\pi(\theta_i \cdot \omega_{\mathcal{S}}) = 0 \tag{100}$$

Finally, we obtain the following result which explains the reason why ODE (72) becomes a Hamiltonian system in the classical cases.

Proposition 6.2. *For i, $1 \leq i \leq l$ such that*

$$\tilde{U}_i = \mathcal{M}_R \times \mathcal{B}_R \times \operatorname{Spec} \mathbf{C}[x_i, y_i] \simeq \mathcal{M}_R \times \mathcal{B}_R \times \mathbf{C}^2, \quad \omega_{\mathcal{S}|\tilde{U}_i} = dx_i \wedge dy_i, \tag{101}$$

we have

$$d_\pi(\theta_i \cdot dx_i \wedge dy_i) = d_\pi(\eta_i dy_i - \zeta_i dx_i) = 0.$$

Since $H^1_{DR}(\mathbf{C}^2) = 0$, we have a regular function $H_i(x_i, y_i, \boldsymbol{\alpha}, t) \in (M_R \otimes B_R)[x_i, y_i]$ such that

$$d_\pi H_i = \frac{\partial H_i}{\partial x_i} dx_i + \frac{\partial H_i}{\partial y_i} dy_i = -(\theta_i \cdot dx_i \wedge dy_i) = -\eta_i dy_i + \zeta_i dx_i.$$

From this, we have

$$-\eta_i = \frac{\partial H_i}{\partial y_i}, -\zeta_i = -\frac{\partial H_i}{\partial x_i}.$$

Therefore, the differential equation (72) can be written in the Hamiltonian system

$$\left\{ \begin{array}{rcl} \dfrac{dx_i}{dt} &=& \dfrac{\partial H_i}{\partial y_i} \\ \dfrac{dy_i}{dt} &=& -\dfrac{\partial H_i}{\partial x_i}. \end{array} \right. \tag{102}$$

Remark 6.2. (1) If $f_i(x_i, y_i, \boldsymbol{\alpha}, t)$ in (97) is independent of t, from the fundamental equation (100),we obtain

$$d_\pi\Big(\theta_i \cdot \frac{dx_i \wedge dy_i}{f_i(x_i, y_i, \boldsymbol{\alpha})^{m_i}}\Big) = 0.$$

Therefore, we may have a chance to obtain a regular function $H_i(x_i, y_i, \boldsymbol{\alpha}, t)$ on $\tilde{U}_i$ such that

$$d_\pi H_i = -\theta_i \cdot \frac{dx_i \wedge dy_i}{f_i(x_i, y_i, \boldsymbol{\alpha})^{m_i}}.$$

In this case, the differential equation in (72) can be written in

$$\begin{cases} \dfrac{dx_i}{dt} & = & (f_i)^{m_i} \cdot \dfrac{\partial H_i}{\partial y_i} \\ \dfrac{dy_i}{dt} & = & -(f_i)^{m_i} \cdot \dfrac{\partial H_i}{\partial x_i}, \end{cases} \tag{103}$$

or equivalently,

$$\begin{cases} \dfrac{1}{(f_i)^{m_i}} \dfrac{dx_i}{dt} & = & \dfrac{\partial H_i}{\partial y_i} \\ \dfrac{1}{(f_i)^{m_i}} \dfrac{dy_i}{dt} & = & -\dfrac{\partial H_i}{\partial x_i}. \end{cases} \tag{104}$$

(2) In general, we can not transform the differential equation in (72) into a Hamiltonian system in the global affine coordinates.

(3) Takano, et al. show that for any Okamoto–Painlevé pair (S, Y) of type $\tilde{D}_4(= P_{VI}), \tilde{D}_5(= P_V), \tilde{D}_6(= P_{III}), \tilde{E}_6(= P_{IV}), \tilde{E}_7(= P_{II})$, the open surface $S - Y_{red}$ is covered by a finite number of affine spaces $U_i = \mathbf{C}^2$ and regular 2-form $\omega_{\mathcal{S}|\tilde{U}_i}$ can be written as in $dx_i \wedge dy_i$. Hence from Proposition 6.2 we obtain the Hamiltonian systems for those Okamoto–Painlevé pairs on any affine chart U_i of $S - D$ as proved in [O1], [MMT]. Note that for an Okamoto–Painlevé pair (S, Y) of type $\tilde{D}_8$, $S - Y_{red}$ does not contain $\mathbf{C}^2$ (cf. Theorem 1.1 and Proposition 1.2). For explicit descriptions of $\tilde{E}_7$ and $\tilde{D}_8$, see §8.

We summarize our results in this section as follows (cf. Figure 4).

Deformation $\mathcal{D} \hookrightarrow \mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R \ni (\boldsymbol{\alpha}, t)$ of Okamoto–Painlevé pairs such that for any $(\boldsymbol{\alpha}, t) \in \mathcal{B}_R$ the Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ lies in the Kernel of the restriction map $\mathsf{res} : H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \longrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t} - \mathcal{D}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}))$

$\Downarrow$

$\boxed{\text{There exists a global holomorphic vector field } \tilde{v} \text{ on } \mathcal{S} - \mathcal{D} \text{ which is a lift of } \frac{\partial}{\partial t}}$

$\Downarrow$

$\boxed{\text{Differential Equations on } \pi : \mathcal{S} - \mathcal{D} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R \text{ defined by } \tilde{v}}$

$\Downarrow$ $\boxed{\text{Painlevé property}}$

$\boxed{\text{Local trivializations of the deformation } \mathcal{S} - \mathcal{D} \longrightarrow \mathcal{B}_R}$.

FIGURE 4

7. Painlevé Equations

Let us recall the classical Painlevé differential equations and Hamiltonian systems which are equivalent to the Painlevé equations ([IKSY], [T], [O1]).

Painlevé equations P_J $(J = I, II, \cdots, VI)$ are given in Table 7:

$$
\begin{aligned}
P_I: \quad \frac{d^2x}{dt^2} &= 6x^2 + t,\\
P_{II}: \quad \frac{d^2x}{dt^2} &= 2x^3 + tx + \alpha,\\
P_{III}: \quad \frac{d^2x}{dt^2} &= \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{1}{t}(\alpha x^2 + \beta) + \gamma x^3 + \frac{\delta}{x},\\
P_{IV}: \quad \frac{d^2x}{dt^2} &= \frac{1}{2x}\left(\frac{dx}{dt}\right)^2 + \frac{3}{2}x^3 + 4tx^2 + 2(t^2 - \alpha)x + \frac{\beta}{x},\\
P_V: \quad \frac{d^2x}{dt^2} &= \left(\frac{1}{2x} + \frac{1}{x-1}\right)\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{(x-1)^2}{t^2}\left(\alpha x + \frac{\beta}{x}\right) + \gamma\frac{x}{t} + \delta\frac{x(x+1)}{x-1},\\
P_{VI}: \quad \frac{d^2x}{dt^2} &= \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right),\\
&\quad + \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{x^2} + \gamma\frac{t-1}{(x-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(x-t)^2}\right].
\end{aligned}
$$

Table 7

Here x and t are complex variables, α, β, γ and δ are complex constants. It is known that each P_J is equivalent to a Hamiltonian system (cf. [O1], [IKSY], [MMT]):

$$
(H_J): \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H_J}{\partial y}, \\[2mm] \dfrac{dy}{dt} = -\dfrac{\partial H_J}{\partial x}, \end{cases} \tag{105}
$$

where the Hamiltonians H_J are given in Table 8.

Moreover the relations between the constants in the equations P_J and the Hamiltonians H_J are given in Table 9.

For the meaning of the constants in Table 9, see [IKSY], [O2]. Note that these constants are not effective parameters. In some cases, we can normalize these constants further by coordinate transformations. Moreover, the equivalence of P_J and (H_J) means that if we eliminate the variable y in (H_J) then we obtain (P_J).

$$
\begin{aligned}
H_I(x,y,t) &= \frac{1}{2}y^2 - 2x^3 - tx, \\
H_{II}(x,y,t) &= \frac{1}{2}y^2 - \left(x^2 + \frac{t}{2}\right)y - \left(\alpha + \frac{1}{2}\right)x, \\
H_{III}(x,y,t) &= \frac{1}{t}\left[2x^2y^2 - \left\{2\eta_\infty t x^2 + (2\kappa_0+1)x - 2\eta_0 t\right\}y + \eta_\infty(\kappa_0+\kappa_\infty)tx\right], \\
H_{IV}(x,y,t) &= 2xy^2 - \left\{x^2 + 2tx + 2\kappa_0\right\}y + \kappa_0 x, \\
H_V(x,y,t) &= \frac{1}{t}\left[x(x-1)^2y^2 - \left\{\kappa_0(x-1)^2 + \kappa_t x(x-1) - \eta t x\right\}y + \kappa(x-1)\right], \\
& \left(\kappa := \frac{1}{4}\left\{(\kappa_0+\kappa_t)^2 - \kappa_\infty^2\right\}\right), \\
H_{VI}(x,y,t) &= \frac{1}{t(t-1)}\left[x(x-1)(x-t)y^2 - \{\kappa_0(x-1)(x-t)\right. \\
& \left. + \kappa_1 x(x-t) + (\kappa_t - 1)x(x-1)\}y + \kappa(x-t)\right] \\
& \left(\kappa := \frac{1}{4}\left\{(\kappa_0+\kappa_1+\kappa_t-1)^2 - \kappa_\infty^2\right\}\right).
\end{aligned}
$$

TABLE 8

	α	β	γ	δ	number of aux. parameters
P_I	none	none	none	none	0
P_{II}	α	none	none	none	1
P_{III}	$-4\eta_\infty\kappa_\infty$	$4\eta_\infty(\kappa_0+1)$	$4\eta_\infty^2$	$-4\eta_0^2$	2
P_{IV}	$-\kappa_0+2\kappa_\infty+1$	$-2\kappa_0^2$	none	none	2
P_V	$\frac{1}{2}\kappa_\infty^2$	$-\frac{1}{2}\kappa_0^2$	$-\eta(1+\kappa_t)$	$-\frac{1}{2}\eta^2/2$	3
P_{VI}	$\frac{1}{2}\kappa_\infty^2$	$\frac{1}{2}\kappa_0^2$	$\frac{1}{2}\kappa_1^2$	$\frac{1}{2}\kappa_t^2$	4

TABLE 9

Remark 7.1. For the Painlevé equation of type P_{III}, we have the following remark by Sakai in [Sakai]. The Painlevé equation of type P_{III} as in Table 7 can be transformed into

$$\frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{x^2}{4t^2}(\gamma x + \alpha) + \frac{\beta}{4t} + \frac{\delta}{4x}. \tag{106}$$

If $\gamma\delta \neq 0$, then we can normalize $\gamma = -\delta = 4$ without loss of generality. In this case we obtain the Painlevé equation of type $P_{III}^{\tilde{D}_6}$:

$$P_{III}^{\tilde{D}_6}: \quad \frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} + \frac{x^2}{4t^2}(4x+\alpha) + \frac{\beta}{4t} - \frac{1}{x}. \tag{107}$$

If one of γ and δ equals to zero (not both), then we have $P_{III}^{\tilde{D}_7}$.

$$P_{III}^{\tilde{D}_7}: \quad \frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} - \frac{4x^2}{t^2} - \frac{1+2a}{t}. \tag{108}$$

Moreover when $\gamma = \delta = 0$, we have $P_{III}^{\tilde{D}_8}$:

$$P_{III}^{\tilde{D}_8}: \quad \frac{d^2x}{dt^2} = \frac{1}{x}\left(\frac{dx}{dt}\right)^2 - \frac{1}{t}\frac{dx}{dt} - \frac{x^2}{4t^2} - \frac{4}{t}. \tag{109}$$

These differential equations correspond to generalized rational Okamoto–Pailevé pairs of type $\tilde{D}_6, \tilde{D}_7, \tilde{D}_8$ respectively.

8. Examples

In this section, we will apply our methods for deriving the differential equation in (72) from the explicit deformations of Okamoto–Painlevé pairs. We shall give a full detail of the cases of $\tilde{E}_7(= P_{II})$ and $\tilde{D}_8(= P_{III}^{\tilde{D}_8})$. For other cases, see [Sa-Te].

Example 8.1. $\tilde{\mathbf{E}}_7$–type: In this case, we will use the Takano's description of the family of Okamoto–Painlevé pairs of type $\tilde{E}_7$ (cf. [Theorem 4, [MMT]]). We will not consider all of the family $\mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$, but consider the family $\mathcal{S} - \mathcal{D} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ which is constructed as follows (cf. [SU]). Let us set

$$\mathcal{M}_R = \mathrm{Spec}\, \mathbf{C}[\alpha] \simeq \mathbf{C}, \quad \mathcal{B}_R = \mathrm{Spec}\, \mathbf{C}[t] \simeq \mathbf{C}$$

and take three affine schemes $i = 1, 2, 3$

$$\tilde{U}_i = \mathrm{Spec}\, \mathbf{C}[\alpha, t, x_i, y_i] \simeq \mathbf{C}^4. \tag{110}$$

The family $\mathcal{S} - \mathcal{D} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ can be constructed by patching these affine schemes by the coordinate transformations given as follows (cf. [MMT]):

$$\begin{array}{llll}
x_0 & = \dfrac{1}{x_1} & & = \dfrac{1}{x_2} \\
y_0 & = x_1((-\alpha - \frac{1}{2}) - x_1 y_1) & & = 2x_2^{-2} + t + (\alpha - \frac{1}{2})x_2 - y_2 x_2^2 \\
x_1 & = x_2 & & = \dfrac{1}{x_0} \\
y_1 & = -\dfrac{2}{x_2^4} - \dfrac{t}{x_2^2} - \dfrac{2\alpha}{x_2} + y_2 & & = x_0((-\alpha - \dfrac{1}{2}) - x_0 y_0) \\
x_2 & = \dfrac{1}{x_0} & & = x_1 \\
y_2 & = 2x_0^4 + t x_0^2 + (\alpha - \dfrac{1}{2})x_0 - x_0^2 y_0 & & = \dfrac{2}{x_1^4} + \dfrac{t}{x_1^2} + \dfrac{2\alpha}{x_1} + y_1
\end{array}.$$

The Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})_{\mathcal{S}_t - \mathcal{D}_t}$ is given by the Čech 1-cocycle

$$\theta_{01} = 0, \quad \theta_{02} = \frac{\partial}{\partial y_0}, \quad \theta_{12} = -x_1^{-2}\frac{\partial}{\partial y_1}. \tag{111}$$

Setting

$$\begin{aligned}
\theta_0 &:= \left[-y_0 + x_0^2 + \frac{t}{2}\right] \frac{\partial}{\partial x_0} - \left[2x_0 y_0 + \alpha + \frac{1}{2}\right] \frac{\partial}{\partial y_0} && (112)\\
\theta_1 &:= \frac{1}{2}\left[-2 - t x_1^2 - x_1^3 - 2\alpha x_1^3 - 2x_1^4 y_1\right] \frac{\partial}{\partial x_1} && (113)\\
&\quad + \frac{1}{4}\left[(1 + 2\alpha + 4x_1 y_1)(t + x_1(1 + 2\alpha + 2x_1 y_1))\right] \frac{\partial}{\partial y_1} && (114)\\
\theta_2 &:= \frac{1}{2}(2 + t x_2^2 + (2\alpha - 1)x_2^3 - 2x_2^4 y_2)\frac{\partial}{\partial x_2}\\
&\quad + \frac{1}{4}(-1 + 2\alpha - 4x_2 y_2)(t + x_2(-1 + 2\alpha - 2x_2 y_2))\frac{\partial}{\partial y_2}, && (115)
\end{aligned}$$

we have the relations

$$\theta_{01} = \theta_1 - \theta_0, \quad \theta_{02} = \theta_2 - \theta_0, \quad \theta_{12} = \theta_2 - \theta_1,$$

that is,

$$\theta_0 = \theta_1, \quad \theta_2 = \theta_{02} + \theta_0.$$

Since on each $\tilde{U}_i$, the relative 2-forms $\omega_{\mathcal{S}-\mathcal{D}}$ is given by

$$\omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_i} = dx_i \wedge dy_i,$$

by Proposition 6.2 the 1-forms $\theta_i dx_i \wedge dy_i$ is exact form, hence there exists a polynomial $H_i(x_i, y_i, \alpha, t)$ satisfying

$$-\theta_i dx_i \wedge dy_i = d_\pi H_i.$$

The polynomials H_i are called the Hamiltonians and given by

$$\begin{aligned}
H_0(x_0, y_0, \alpha, t) &= \frac{1}{2}y_0^2 - \left(x_0^2 + \frac{t}{2}\right) y_0 - \left(\alpha + \frac{1}{2}\right) x_0, && (116)\\
H_1(x_1, y_1, \alpha, t) &= \frac{t x_1}{4} + \frac{\alpha t x_1}{2} + \frac{x_1^2}{8} + \frac{\alpha x_1^2}{2} + \frac{\alpha^2 x_1^2}{2} + y_1 + \frac{1}{2} t x_1^2 y_1 && (117)\\
&\quad + \frac{x_1^3 y_1}{2} + \alpha x_1^3 y_1 + \frac{x_1^4 y_1^2}{2},\\
H_2(x_2, y_2, \alpha, t) &= \frac{1}{8}\left[(1 - 2\alpha)^2 x_2^2 - 8y_2 - 4(-1 + 2\alpha)x_2^3 y_2 + 4x_2^4 y_2^2\right. && (118)\\
&\quad \left. - 2t x_2(1 - 2\alpha + 2x_2 y_2))\right]. && (119)
\end{aligned}$$

Hence the Hamiltonian system defined on $\tilde{U}_0$ is given by

$$\begin{cases} \dfrac{dx_0}{dt} = \dfrac{\partial H_0}{\partial y_0} = y_0 - x_0^2 - \dfrac{t}{2}, \\[2mm] \dfrac{dy_0}{dt} = -\dfrac{\partial H_0}{\partial x_0} = 2x_0 y_0 + \alpha + \dfrac{1}{2}. \end{cases} \qquad (120)$$

Eliminating y_0 in (120), we obtain

$$\frac{d^2 x_0}{dt^2} = 2x_0^3 + x_0 t + \alpha, \qquad (121)$$

which is the Painlevé equation P_{II} in Table 7.

Example 8.2. $\tilde{\mathbf{D}}_8$–type: The Okamoto–Painlevé pair (S, Y) of type $\tilde{D}_8$ did not appear in the former literatures [IKSY], [O1] explicitly. Since $S - Y_{red}$ does not contain $\mathbf{C}^2$ as a Zariski open set (cf. Theorem 1.1), the situation is a little bit different from the classical cases.

We can construct a family of generalized rational Okamoto–Painlevé pair of type $\tilde{D}_8$ $\pi : \mathcal{S} - \mathcal{D} \longrightarrow \mathcal{B}_R$ by blowings-ups as in Sakai [Sakai]. For detail, see [Sa-Te]. Here note that $\dim H^1(\mathcal{S}_t, \Theta_{\mathcal{S}_t}(-\log \mathcal{D}_t)) = 1$ and $\dim \mathcal{M}_R = 0$ and

$$\mathcal{B}_R = \mathrm{Spec}\, \mathbf{C}[t, t^{-1}] \simeq \mathbf{C}^{\times}.$$

The total space $\mathcal{S} - \mathcal{D}$ is covered by the three affine open sets:

$$\mathcal{S} - \mathcal{D} \ = \tilde{U}_0 \cup \tilde{U}_1 \cup \tilde{U}_2.$$

These affine open sets are given by:

$$\begin{aligned}
\tilde{U}_0 &= \mathrm{Spec}\, \mathbf{C}[x_0, y_0, \frac{1}{y_0}, t, t^{-1}] \ \cong (\mathbf{C}^2 - \{y_0 = 0\}) \times \mathbf{C}^{\times}, \\
\tilde{U}_1 &= \mathrm{Spec}\, \mathbf{C}[x_1, y_1, \frac{1}{F_1(x_1, y_1)}, t, t^{-1}] \ \cong \ (\mathbf{C}^2 - \{F_1(x_1, y_1) = 0\}) \times \mathbf{C}^{\times}, \\
\tilde{U}_2 &= \mathrm{Spec}\, \mathbf{C}[x_2, y_2, \frac{1}{F_2(x_2, y_2, t)}, t, t^{-1}] \ \cong \ \mathbf{C}^3 - \{F_2(x_2, y_2, t) = 0, \ t = 0\},
\end{aligned}$$

where

$$\begin{aligned}
F_1(x_1, y_1) &= 1 + x_1 {y_1}^2, & (122)\\
F_2(x_2, y_2, t) &= t - t y_2 + x_2 {y_2}^2. & (123)
\end{aligned}$$

The coordinate transformations are given as follows

$$\begin{aligned}
x_0 &= \frac{1}{y_1 F_1(x_1, y_1)} && = \frac{1}{y_2}, \\
y_0 &= \frac{1}{{y_1}^2 F_1(x_1, y_1)} && = {y_2}^2 \ (t + x_2 {y_2}^2 - t y_2), \\
x_1 &= \frac{{y_0}^2(-{x_0}^2 + y_0)}{{x_0}^4} && = y_2^6(t + x_2 y_2^2 - t y_2), \\
y_1 &= \frac{x_0}{y_0} && = \frac{1}{y_2^3(t(1 - y_2) + x_2 y_2^2)}, \\
x_2 &= x_0(-t x_0 + x_0^3 y_0 + t) && = \frac{1 + t y_1^4 F_1^3 (y_1^2 F_1 - 1)}{y_1^6 F_1^5}, \\
y_2 &= \frac{1}{x_0} && = y_1^2(1 + x_1 y_1^2).
\end{aligned}$$

The Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})_{\mathcal{S}_t - \mathcal{D}_t} \in H^1(\mathcal{S}_t - \mathcal{D}_t, \Theta_{\mathcal{S}_t}(-\log \mathcal{D}_t))$ is given by Čech cocycles

$$\theta_{01} = 0, \quad \theta_{02} = \frac{-1 + x_0}{{x_0}^3} \frac{\partial}{\partial y_0}, \quad \theta_{21} = \frac{-1 + y_2}{{y_2}^2} \frac{\partial}{\partial x_2}. \tag{124}$$

Since $\rho(\frac{\partial}{\partial t})_{\mathcal{S}_t - \mathcal{D}_t} = 0 \in H^1(\mathcal{S}_t - \mathcal{D}_t, \Theta_{\mathcal{S}_t}(-\log \mathcal{D}_t))$, we can obtain Čech coboundary $\{\theta_i \in \Gamma(\tilde{U}_i, \Theta_{\tilde{U}_i/\mathcal{B}_R}\}$ such that

$$\{\theta_{ij}\} = \{\theta_j - \theta_i\}.$$

In fact, we can choose the following holomorphic vector field θ_i on each open set $\tilde{U}_i$

$$\begin{cases} \theta_0 = \dfrac{t - {y_0}^2}{t\ y_0}\dfrac{\partial}{\partial x_0} + \dfrac{-2\ x_0\ y_0}{t}\dfrac{\partial}{\partial y_0}, \\ \theta_1 = \dfrac{f_1(x_1,y_1,t)}{t\ F_1(x_1,y_1)}\dfrac{\partial}{\partial x_1} + \dfrac{g_1(x_1,y_1,t)}{t\ F_1(x_1,y_1)}\dfrac{\partial}{\partial y_1}, \\ \theta_2 = \dfrac{f_2(x_2,y_2,t)}{t\ F_2(x_2,y_2,t)}\dfrac{\partial}{\partial x_2} + \dfrac{g_2(x_2,y_2,t)}{t\ F_2(x_2,y_2,t)}\dfrac{\partial}{\partial y_2}. \end{cases} \tag{125}$$

where

$$\begin{aligned} f_1(x_1,y_1,t) = &\quad -2y_1(t - 2{x_1}^2 + 5tx_1{y_1}^2 + 9t{x_1}^2{y_1}^4 + 7t{x_1}^3{y_1}^6 + 2t{x_1}^4{y_1}^8), \\ g_1(x_1,y_1,t) = &\quad 1 - x_1{y_1}^2 + t{y_1}^4 + 3tx_1{y_1}^6 + 3t{x_1}^2{y_1}^8 + t{x_1}^3{y_1}^{10}, \\ f_2(x_2,y_2,t) = &\quad -t^2 + 3tx_2 - 2t^3y_2 + tx_2y_2 - 2{x_2}^2y_2 + 7t^3{y_2}^2 - 8t^3{y_2}^3 - 8t^2x_2{y_2}^3, \\ &\quad +3t^3{y_2}^4 + 18t^2x_2{y_2}^4 - 10t^2x_2{y_2}^5 - 10t{x_2}^2{y_2}^5 + 11t{x_2}^2{y_2}^6 - 4{x_2}^3{y_2}^7, \\ g_2(x_2,y_2,t) = &\quad -t + t^2{y_2}^4 - 2t^2{y_2}^5 + t^2{y_2}^6 + 2tx_2{y_2}^6 - 2tx_2{y_2}^7 + {x_2}^2{y_2}^8. \end{aligned} \tag{126}$$

Then we actually have the following relation as required

$$\theta_0 = \theta_1, \quad \theta_2 = \theta_{02} + \theta_0.$$

Hence, we have the differential equation on $\mathcal{S} - \mathcal{D}$ as in Theorem 6.1, and on each open set $\tilde{U}_i, i = 0,1,2$ the differential equation can be written as follows (cf. (72)).

$$\text{On } \tilde{U}_0 \qquad \begin{cases} \dfrac{dx_0}{dt} = -\dfrac{t - {y_0}^2}{ty_0} = y_0\dfrac{\partial H_0}{\partial y_0} \\ \dfrac{dy_0}{dt} = \dfrac{2\ x_0\ y_0}{t} = -y_0\dfrac{\partial H_0}{\partial x_0}. \end{cases} \tag{127}$$

$$\text{On } \tilde{U}_1 \qquad \begin{cases} \dfrac{dx_1}{dt} = -\dfrac{f_1(x_1,y_1,t)}{t\ F_1(x_1,y_1)} = F_1(x_1,y_1)^2\dfrac{\partial H_1}{\partial y_1} \\ \dfrac{dy_1}{dt} = -\dfrac{g_1(x_1,y_1,t)}{t\ F_1(x_1,y_1)} = -F_1(x_1,y_1)^2\dfrac{\partial H_1}{\partial x_1}. \end{cases} \tag{128}$$

$$\text{On } \tilde{U}_2 \qquad \begin{cases} \dfrac{dx_2}{dt} = -\dfrac{f_2(x_2,y_2,t)}{t\ F_2(x_2,y_2,t)} \\ \dfrac{dy_2}{dt} = -\dfrac{g_2(x_2,y_2,t)}{t\ F_2(x_2,y_2,t)}. \end{cases} \tag{129}$$

Here H_0, H_1 are given by

$$H_0 = \left(-\frac{{x_0}^2}{t} + \frac{y_0}{t} + \frac{1}{y_0}\right), \tag{130}$$

$$H_1 = \left({y_1}^2 + x_1{y_1}^4 + \frac{x_1}{t(1 + x_1{y_1}^2)^2}\right). \tag{131}$$

Moreover on each $\tilde{U}_i$, $i = 0, 1, 2$, the relative 2-form $\omega_{\mathcal{S}-\mathcal{D}}$ are given by

$$\begin{aligned} \omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_0} &= \frac{1}{y_0}\, dx_0 \wedge dy_0 \\ \omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_1} &= \frac{1}{F_1(x_1, y_1)^2} dx_1 \wedge dy_1 \\ \omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_2} &= \frac{1}{F_2(x_2, y_2, t)}\, dx_2 \wedge dy_2. \end{aligned}$$

For each $i = 0, 1, 2$, consider the 1-form on $\tilde{U}_i$

$$\theta_i(\omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_i}).$$

Then since $\theta_i(\omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_i})$ does not depend on t for $i = 0, 1$, the fundamental equation (100) is reduced to

$$d_\pi(\theta_i(\omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_i})) = 0, \quad \text{for } i = 0, 1\ . \tag{132}$$

Though $\tilde{U}_i$ is not simply connected, we can integrate $\theta_i(\omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_i})$ and obtain H_i for $i = 0, 1$ defined in (130) and (131), that is,

$$d_\pi H_i = \theta_i(\omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_i}). \tag{133}$$

On the other hand, since $\theta_2(\omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_i})$ is really depend on t, the fundamental equation (100) becomes as follows.

$$\frac{\partial}{\partial t}\left(\frac{1}{F_2(x_2, y_2, t)}\right) dx_2 \wedge dy_2 - d_\pi(\theta_i(\omega_{\mathcal{S}-\mathcal{D}|\tilde{U}_i})) = 0. \tag{134}$$

This last equation is equivalent to the following equation, which one can check by hand.

$$\frac{\partial}{\partial t}\left(\frac{1}{F_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{f_2}{tF_2^2}\right) + \frac{\partial}{\partial y_2}\left(\frac{g_2}{tF_2^2}\right) = 0. \tag{135}$$

Eliminating x_0 from the differential equation (127), we obtain the differential equation

$$\frac{d^2 y_0}{dt^2} = \frac{1}{y_0}\left(\frac{dy_0}{dt}\right)^2 - \frac{1}{t}\frac{dy_0}{dt} + \frac{2y_0^2}{t^2} - \frac{2}{t}. \tag{136}$$

It is easy to see that this equation is equivalent to the equation $P_{III}^{\tilde{D}_8}$ in (109).

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