Painlevé equations and deformations of rational surfaces with rational double points*†

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In this paper we give an answer to the fundamental questions about the Painlevé equations. Where do the Bäcklund transformations come from? Our approach depends on the geometry of the projective surface constructed by Okamoto and reviewed in [U2].

The Painlevé equations were discovered around 1900 in the pursuit of special functions. Painlevé and Gambier classified algebraic differential equations $y'' = R(t, y, y')$ without movable singular points, where R is a rational function of t, y, y' and t is the independent variable. After the refinement procedure of throwing away among the differential equations those that are integrable by the so far known functions, they arrived at the list of the six Painlevé equations P_J $(I \leq J \leq VI)$. After the discovery of Painlevé equations, an unexpected but important feature of the Painlevé equations was revealed. The Painlevé equations have symmetries or the Bäcklund transformations. In fact, if we recall the motivation of the discovery, it is surprising that the Painlevé equations admit symmetries. So a natural question arise. Where do the Bäcklund transformations come from? Our answer is that they arise from the rational double points.

The Painlevé equation P_J is written in a Hamiltonian form

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H_J}{\partial p}, \\[2ex] \dfrac{dp}{dt} = -\dfrac{\partial H_J}{\partial q} \end{cases}$$

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for an appropriate polynomial H_J of q, p and t ($I \leq J \leq VI$). We also clarify where the polynomial H_J comes from and why it is uniquely determined. This point is a geometric interpretation of [ST] and [MMT]. It is interesting to notice that our theory provides us a lot of pairs (U, V) of algebraic varieties defined over $\mathbb{C}$ such that U and V are not isomorphic as algebraic varieties but they are isomorphic as complex manifolds (cf §11). (The idea in §11 is developed further in the direction of deformation theory of Okamoto–Painlevé pairs and its relation to Painlevé equations in [STT].)

We work with the second Painlevé equation P_{II} to illustrate the general case. Our argument is given in a form easily applicable to the other Painlevé equations (cf. [STT]). Through out the paper, the ground field in $\mathbb{C}$. The most natural setting seems to be, however, over the ring $\mathbb{Z}[\frac{1}{2}]$.

We should mention that a recent work of Sakai [Sakai] is also working on the discrete and differential Painlevé equations related to the geometry of rational surfaces and symmetries of affine Weyl groups.

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§1. Construction of a family of rational surfaces.

The second Painlevé equation is an ordinary differential equation

$$P_{II}(\alpha) \qquad y'' = 2y^3 + ty + \alpha$$

of the second order, where t is the independent variable, $y'' = d^2y/dt^2$ and $\alpha \in \mathbb{C}$ is a parameter. The extended affine Weyl group $\widehat{G}$ of type $A_1^{(1)}$ appears as the symmetry of $P_{II}(\alpha)$. Namely if y is a solution of $P_{II}(\alpha)$, then

$$T_+(y) = -y - \frac{\alpha + \frac{1}{2}}{y' + y^2 + \frac{t}{2}}$$

is a solution of $P_{II}(\alpha + 1)$,

$$T_-(y) = -y + \frac{\alpha - \frac{1}{2}}{y' - y^2 - \frac{t}{2}}$$

is a solution of $P_{II}(\alpha - 1)$ and

$$I(y) = -y$$

is a solution of $P_{II}(-\alpha)$. The automorphisms i, t_+, t_- of the affine line $\mathbb{A}^1$ with coordinate system α, i.e. $\mathbb{A}^1 = \mathrm{Spec}\ \mathbb{C}[\alpha]$, given by

$$t_+(\alpha) = \alpha + 1, \quad t_-(\alpha) = \alpha - 1, \quad i(\alpha) = -\alpha$$