THE MODULARITY CONJECTURE FOR RIGID CALABI–YAU THREEFOLDS OVER $\mathbb{Q}$

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Abstract. : We formulate the modularity conjecture for rigid Calabi–Yau threefolds defined over the field $\mathbb{Q}$ of rational numbers. We establish the modularity for the rigid Calabi–Yau threefold arising from the root lattice A_3. Our proof is based on geometric analysis.

1. The L–series of Calabi–Yau threefolds

Let $\mathbb{Q}$ be the field of rational numbers, and let $\bar{\mathbb{Q}}$ be its algebraic closure with Galois group $\mathcal{G} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Let X be a smooth projective threefold defined over $\mathbb{Q}$ or more generally over a number field.

Definition 1.1. X is a *Calabi–Yau* threefold if it satisfies the following two conditions:
(a) $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, and
(b) The canonical bundle is trivial, i.e., $K_X \simeq \mathcal{O}_X$.

The numerical invariants of Calabi–Yau threefolds.

Let X be a Calabi–Yau threefold defined over $\mathbb{Q}$, and let $\bar{X} = X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$.

(1) The (i, j)-th *Hodge* number $h^{i,j}(X)$ of X is defined by

$$h^{i,j}(X) = \dim_{\bar{\mathbb{Q}}} H^j(\bar{X}, \Omega^i_{\bar{X}}).$$

There are symmetries among Hodge numbers: $h^{i,j}(X) = h^{j,i}(X)$, and $h^{i,j}(X) = h^{3-j,3-i}(X)$.

The first identity follows from the complex conjugation operation and the latter from Serre duality on the Hodge cohomology groups. The condition (a) implies that $h^{1,0}(X) = h^{2,0}(X) = 0$, and the condition (b) that $h^{3,0}(X) = h^{0,3}(X) = 1$. The number $h^{2,1}(X)$ represents the number of deformations of complex structures on X, and $h^{1,1}(X)$ is the number of Hodge $(1, 1)$-cycles on X. The Hodge numbers are concocted to define the *Hodge diamond*:

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$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(2) The n-th *Betti* number of X is defined by

$$B_n(X) = \dim_{\mathbb{Q}_\ell} H^n_{\text{et}}(\bar{X}, \mathbb{Q}_\ell) = \dim_{\mathbb{C}} H^n(X \otimes \mathbb{C}, \mathbb{C})$$

where the cohomology groups are, respectively, the ℓ-adic étale and singular cohomology groups. The Hodge decomposition $H^n(X \otimes \mathbb{C}, \mathbb{C}) = \oplus_{i+j=n} H^j(X \otimes \mathbb{C}, \Omega^i_{X\otimes\mathbb{C}})$ ensures that

$$B_n(X) = \sum_{i+j=n} h^{i,j}(X) = \sum_{i=0}^{n} h^{i,n-i}(X).$$

Using the above formula and the Poincaré duality $B_n(X) = B_{2\dim(X)-n}$, we can compute the Betti numbers of Calabi–Yau threefolds:

$$B_0(X) = 1 = B_6(X), \quad B_1(X) = 0 = B_5(X),$$

$$B_2(X) = h^{1,1}(X) = h^{2,2}(X) = B_4(X) \quad \text{and} \quad B_3(X) = 2(1 + h^{2,1}(X)).$$

The (topological) Euler characteristic is $\chi(X) = 2(h^{1,1}(X) - h^{2,1}(X))$.

Definition 1.2. A smooth projective Calabi–Yau threefold X over $\mathbb{Q}$ (or over any field) is called *rigid* if $h^{2,1}(X) = h^{1,2}(X) = 0$ so that $B_3(X) = 2$.

Definition 1.3. Let X be a Calabi–Yau threefold defined over $\mathbb{Q}$. Assume that X has a suitable integral model. The L–series of X is defined to be the L–series of the (semi-simplification of the) Galois representation on $H^3_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)$. That is,

$$L(X, s) := L(H^3_{\text{et}}(\bar{X}, \mathbb{Q}_\ell), s).$$

Digression: Let X be a Calabi–Yau threefold defined over $\mathbb{Q}$ with a suitable integral model. Let p be a *good* prime for X, that is, the reduction $X \pmod{p}$ defines a smooth projective variety over the prime field $\mathbb{F}_p$. Let Frob_p denote the (geometric) Frobenius morphism of X at p. We consider the action of Frob_p on the ℓ-adic etale cohomology group $H^3_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)$, and let $t_3(p)$ denote the trace of the Frob_p on $H^3_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)$. By the Lefschetz fixed point formula, $t_3(p)$ can be determined by counting the number of $\mathbb{F}_p$–rational points on X:

$$t_3(p) = 1 + p^3 + (1 + p)t_2(p) - \#X(\mathbb{F}_p).$$

Here $t_2(p) = p\,\text{Trace}(\rho(\text{Frob}_p))$ where ρ is an Artin representation of dimension $h^{1,1}(X)$ so in particular its trace is an integer which is a sum of $h^{1,1}(X)$ roots of unity. This implies that $|t_2(p)| \leq p\, h^{1,1}(X)$, and for a set of primes of positive density, we get $t_2(p) = ph^{1,1}(X)$. Define

$$P_{3,p}(T) := \det(1 - \mathrm{Frob}_p\, T \mid H^3_{\mathrm{et}}(\bar{X}, \mathbb{Q}_\ell)).$$

If X is a rigid Calabi–Yau threefold, then $P_{p,3}(T)$ is an integral polynomial of degree $\deg(P_{3,p}) = 2$; it is of the form:

$$P_{3,p}(T) = 1 - t_3(p)T + p^3T^2 \in 1 + T\mathbb{Z}[T]$$

where $t_3(p)$ is subject to the Riemann Hypothesis: $|t_3(p)| \leq 2p^{3/2}$. The L–series $L(H^3_{\mathrm{et}}(\bar{X}, \mathbb{Q}_\ell), s)$ is then given by

$$L(H^3_{\mathrm{et}}(\bar{X}, \mathbb{Q}_\ell), s) = (*) \prod_p P_{3,p}(p^{-s})^{-1}$$

where p runs over good primes, and $(*)$ is the Euler factor corresponding to bad primes.

2. The modularity conjecture for rigid Calabi–Yau threefolds over $\mathbb{Q}$

Let $k \geq 1$ be an integer. Let Γ be an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. We denote by $S_k(\Gamma)$ the complex vector space of all cusp forms of weight k with respect to Γ. We now formulate the modularity conjecture for rigid Calabi–Yau threefolds defined over $\mathbb{Q}$.

Conjecture 2.1. The modularity conjecture*:* Any rigid Calabi–Yau threefold X defined over $\mathbb{Q}$ is modular in the sense that its L–series of X coincides with the Mellin transform of the L–series of a modular (cusp) form f of weight 4 on $\Gamma_0(N)$. Here N is a positive integer divisible by the primes of bad reduction. More precisely, we have, up to a finite Euler factors,

$$L(X, s) = L(f, s) \quad \text{for} \quad f \in S_4(\Gamma_0(N)).$$

Here are some justifications for formulating the modularity conjecture for rigid Calabi–Yau threefolds over $\mathbb{Q}$.

Remark 2.1. The conjecture of Taniyama–Shimura–Weil that every elliptic curve over $\mathbb{Q}$ is modular, has been established by Wiles and his former students in totality (see Breuil, Conrad, Diamond and Taylor [BCDT]). Noting that elliptic curves are dimension one Calabi–Yau varieties, our modularity conjecture (2.1) may be regarded a dimension three generalization of the Taniyama–Shimura–Weil conjecture to rigid Calabi–Yau threefolds over $\mathbb{Q}$.

Remark 2.2. Livné [Li2] considered a rank 2 motive $\mathcal{M}$ over $\mathbb{Q}$ with Hodge numbers $h^{p,q}(\mathcal{M}) = h^{q,p}(\mathcal{M}) = 1$ where $p > q$, which respects an orthogonal form up to similitudes. Livnè showed how to express the L–series of $\mathcal{M}$ in terms of Hecke characters.

The examples of rank 2 motives $\mathcal{M}$ considered by Livnè arose from an elliptic curve E over $\mathbb{Q}$ or from a singular $K3$ surface X, i.e., $\mathcal{M} = H^1(E) \otimes \mathbb{Q}$ or $H^2(X, \mathbb{Z})/\mathrm{Pic}(X) \otimes \mathbb{Q}$.

In the latter case, Shioda and Inose [SI] determined the L–series of a singular $K3$ surface X (up to a finite numbers of Euler factors) passing to some finite extension of $\mathbb{Q}$:

$$L(X,s) = L(\psi_1^2, s)L(\bar{\psi}_1^2, s)$$

where ψ_1 is the Hecke character associated to an elliptic curve with complex multiplication.

Remark 2.3. Fontaine and Mazur [FM] have conjectured that all irreducible odd 2–dimensional Galois representations "coming from geometry" should be modular, up to a Tate twist. Our modularity conjecture (2.1) may be regarded as a concrete realization of the Fontaine–Mazur conjecture. For a recent progress on a conjecture of Fontaine–Mazur, the reader is referred to a paper of Richard Taylor [T].

Remark 2.4. Let X be a projective variety of odd dimension m over $\mathbb{Q}$ such that the m-th Betti number $B_m = \dim H^m(X \otimes \mathbb{C}, \mathbb{C}) = 2$ and $H^m(X \otimes \mathbb{C}, \mathbb{C})$ has the Hodge decomposition of type $(m, 0) + (0, m)$. Serre [Sr] has formulated a modularity conjecture for the residual mod p 2-dimensional Galois representation attached to X for all primes. In particular, our modularity conjecture 2.1 is a special case of the conjectural Theorem 6 of Serre [Sr] for $m = 3$. Serre has informed us about this in his email dated June 23, 2000. We are thankful to him for pointing this out.

3. A rigid Calabi–Yau threefold arising from the root lattice A_3

This example of a rigid Calabi–Yau threefold is constructed from the root lattice A_3 via a toric construction.

(3.1) Toric construction. For the general backgrounds on toric varieties, the reader is referred to Batyrev [Bat], Fulton [Fu] and Fulton and Harris [FH].

We consider a root system $\mathcal{R}$ of rank r. Let $\mathcal{L}_{\mathcal{R}}$ be the root lattice generated by $\mathcal{R}$, and let $\mathcal{L}_{\mathcal{R}}^*$ be its dual lattice. Let $\Sigma_{\mathcal{R}}$ be the fan in $\mathcal{L}_{\mathcal{R}}^* \otimes \mathbb{Q}$. The fan $\Sigma_{\mathcal{R}}$ gives rise to a toric variety, which is denoted by $X(\Sigma_{\mathcal{R}})$. Let $\Delta_{\mathcal{R}}$ be a polyhedron in $\mathcal{L}_{\mathcal{R}} \otimes \mathbb{Q}$ with vertices in $\mathcal{R}$, and let $L(\Delta_{\mathcal{R}})$ denote the space of Laurent polynomials with support in $\Delta_{\mathcal{R}}$. For each root $r \in \mathcal{R}$, assign a monomial $\mathbf{e}^r$, and define a Laurent polynomial

$$\chi_{\mathcal{R}} := \sum_{r \in \mathcal{R}} \mathbf{e}^r \in L(\Delta_{\mathcal{R}}).$$

Then we have a rational function $\chi_{\mathcal{R}} : X(\Sigma_{\mathcal{R}}) \to \mathbb{P}^1$. For each $\lambda \in \mathbb{P}^1$, define

$$X_\lambda := \overline{\chi_{\mathcal{R}}^{-1}(\lambda)} \subset X(\Sigma_{\mathcal{R}}).$$

There is a base locus $\mathcal{B}$. Let $\mathcal{B} : \mathcal{X}_{\mathcal{R}} \to \mathcal{X}(\Sigma_{\mathcal{R}})$ be the blow up of the base locus. Then we obtain a pencil of varieties

$$\mathcal{X}_{\mathcal{R}} := \{(\lambda, x) \in \mathbb{P}^1 \times X(\Sigma_{\mathcal{R}}) \,|\, \overline{\chi_{\mathcal{R}}(x) = \lambda}\}.$$

We are interested in root lattices $\mathcal{R}$ which give rise to Calabi–Yau threefolds $\mathcal{X}_{\mathcal{R}}$.

(3.2) The root lattice A_3 and the construction of a Calabi–Yau threefold. Verrill[V] constructed a Calabi–Yau threefold associated to the root lattice A_3. We will briefly describe Verrill's construction. Let $\{E_i \mid i = 1, 2, 3, 4\}$ be the standard basis for $\mathbb{R}^4$. The root lattice A_3 is a sublattice of $\mathbb{R}^4$ of rank 3 generated by $v_1 := E_1 - E_2$, $v_2 := E_2 - E_3$, $v_3 := E_3 - E_4$ and the collection of all roots is given by the set

$$\{E_i - E_j \mid 1 \le i, j \le 4,\ i \ne j\}.$$

To a root $E_i - E_j$, we associate a monomial $X_i X_j^{-1}$ by putting $X_i = \mathbf{e}^{E_i}$. Then the character of the adjoint representation is given by

$$\begin{aligned}\chi_{A_3} &= X_1X_2^{-1} + X_1X_3^{-1} + X_1X_4^{-1} + X_2X_3^{-1} + X_2X_4^{-1} + X_3X_4^{-1}\\ &+X_2X_1^{-1} + X_3X_1^{-1} + X_4X_1^{-1} + X_3X_2^{-1} + X_4X_2^{-1} + X_4X_3^{-1}\\ &= (X_1 + X_2 + X_3 + X_4)(X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1}) - 4.\end{aligned}$$

Let $X(\Sigma_{A_3}) \to \mathbb{P}^1$ denote the toric variety defined by $\chi_{A_3} = \lambda \in \mathbb{P}^1$. It is a rational variety. Now we take a double cover of $\mathcal{X}(\Sigma_{A_3})$ by putting $\lambda = \frac{(t-1)^2}{t}$ with $t \in \mathbb{P}^1$. Then a desingularization X of the double cover is a Calabi–Yau threefold defined over $\mathbb{Q}$ whose generic fiber X_λ is a $K3$ surface.

Lemma 3.1. *The numerical characters of X are given as follows:*

$$h^{1,2} = 0,\ h^{3,0} = 1,\ h^{1,1} = 50,\ \chi(X) = 100$$

and hence X is a rigid Calabi–Yau threefold.

We can see from the defining equation of X that 2 and 3 are bad primes. For this rigid Calabi–Yau threefold X over $\mathbb{Q}$, we can establish the modularity conjecture. We will give a proof of the following theorem in the section 6 below. Verrill [V] also proved the theorem with totally different method (à la Livné [Li1]) from ours.

Theorem 3.1. *Let X be the rigid Calabi–Yau threefold constructed by Verrill. Then L–series of X coincides, up to the Euler factors corresponding to primes 2 and 3, with the Mellin transform of a weight 4 newform on $\Gamma_0(6)$. That is,*

$$L(X, s) = L(f, s) \quad \textit{for} \quad f \in S_4(\Gamma_0(6)).$$

The cusp form f has the q–expansion of the form:

$$f(q) = \eta(q)^2\eta(q^2)^2\eta(q^3)^2\eta(q^6)^2.$$

4. **The self-products of elliptic modular surfaces.**

Let Γ be an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with no torsion elements and let $\mathcal{H}$ be the upper half complex plane. Then we can define the modular curve $C_\Gamma^0 := \mathcal{H}/\Gamma$ and its compactification $C_\Gamma := \overline{\mathcal{H}/\Gamma} = \mathcal{H}/\Gamma \cup \{\text{cusps of } \Gamma\}$. It is known (cf. [D]) that the universal family of elliptic curves

$$f^0 : S_\Gamma^0 \longrightarrow C_\Gamma^0 = \mathcal{H}/\Gamma$$

exists, and that its Néron model has the unique smooth minimal compactification

$$f := f_\Gamma : S_\Gamma \longrightarrow C_\Gamma = \overline{\mathcal{H}/\Gamma}.$$

The fibration $f : S_\Gamma \longrightarrow C_\Gamma$ is called the *elliptic modular surface associated to* Γ.

Let us now recall some basic facts about elliptic modular surfaces, which are relevant to our subsequent discussions. Consider the following diagram

$$\begin{array}{ccc} S_\Gamma^0 & \hookrightarrow & S_\Gamma \\ f^0 \downarrow & & f \downarrow \\ C_\Gamma^0 & \overset{j}{\hookrightarrow} & C_\Gamma. \end{array}$$

Making use of the invariant cycle theorem, we obtain an isomorphism of sheaves

$$R^1 f_* \mathbb{Q}_{S_\Gamma} \simeq j_* R^1 f^0_* \mathbb{Q}_{S^0_\Gamma}.$$

In fact, we have the natural map

$$R^1 f_* \mathbb{Q}_{S_\Gamma} \longrightarrow j_* R^1 f^0_* \mathbb{Q}_{S^0_\Gamma} \tag{1}$$

which is an isomorphim over $C^0_\Gamma = C_\Gamma \setminus \Sigma$ where $\Sigma = \{p_1, \cdots, p_r\}$ is the set of cusps of Γ. Looking at the stalks at a cusp p_i, the morphism

$$(R^1 f_* \mathbb{Q}_{S_\Gamma})_{p_i} \longrightarrow (j_* R^1 f^0_* \mathbb{Q}_{S^0_\Gamma})_{p_i} \tag{2}$$

is surjective by local invariant cycle theorem (cf. e.g. [Proposition 15.12, [Z]]). On the other hand, it is easy to see that both of the stalks in (2) are isomorphic to a one-dimensional $\mathbb{Q}$-vector space. Hence the linear map (2) is an isomorphism.

Let V denote the natural representation of $SL_2(\mathbb{Q})$, and let $S^k(V)$ be the k-th symmetric tensor representation of V. Then it is known ([Z], Proposition 12.5) that for each $k \geq 0$ there exists an isomorphism

$$\tilde{H}^1(\Gamma, S^k(V)) \simeq H^1(C_\Gamma, j_*(S^k(R^1 f^0_* \mathbb{Q}_{S^0_\Gamma}))) \quad \text{for each} \quad \mathrm{k} \geq 0. \tag{3}$$

Here $\tilde{H}^1(\Gamma, S^k(V))$ is the parabolic cohomology group associated to the representation of Γ on $S^k(V)$. Let $\omega_{S_\Gamma/C_\Gamma} = K_{S_\Gamma} \otimes f^*(K^{-1}_{C_\Gamma})$ be the relative canonical bundle of f, and we set

$$\omega = f_*(\omega_{S_\Gamma/C_\Gamma}).$$

Then we see that ω is an invertible sheaf on C_Γ.

Now we are ready to describe an isomorphism, the so-called *Shimura isomorphism* and its far-reaching consequences.

Proposition 4.1. ([Sh], [D], [Z].) *There exists a natural isomorphism (the Shimura isomorphism)*

$$H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega^k) \simeq S_{k+2}(\Gamma) \tag{4}$$

and it gives rise to the following commutative diagram:

$$\begin{array}{ccc} \tilde{H}^1(\Gamma, S^k(V)) \otimes_{\mathbb{Q}} \mathbb{C} & \simeq & S_{k+2}(\Gamma) \oplus \overline{S_{k+2}(\Gamma)} \\ \downarrow & & \downarrow \\ H^1(C_\Gamma, j_*(S^k(R^1 f^0_* \mathbb{C}_{S^0_\Gamma}))) & \simeq & H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega^k) \oplus \overline{H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega^k)} \end{array}$$

The first vertical arrow is the isomorphism induced by (3).

With the Shimura isomorphism at our disposal, we may now characterize rational elliptic modular surfaces.

Proposition 4.2. *The elliptic modular surface associated to* Γ, $f : S_\Gamma \longrightarrow C_\Gamma$ *is a rational elliptic surface, if and only if*

$$\dim S_2(\Gamma) = \dim S_3(\Gamma) = 0.$$

Proof. From the formula for the canonical bundle of the elliptic fibration $f : S_\Gamma \longrightarrow C_\Gamma$, we see that

$$K_{S_\Gamma} = f^*(\Omega^1_{C_\Gamma} \otimes \omega). \tag{5}$$

The formula (5) combined with the Shimura isomorphism (4) then give rise to the isomorphisms

$$H^0(C_\Gamma, \Omega^1_{C_\Gamma}) \simeq S_2(\Gamma), \quad H^0(S_\Gamma, K_{S_\Gamma}) \simeq H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega) \simeq S_3(\Gamma). \tag{6}$$

Suppose that S_Γ is a rational surface. Then C_Γ is also a rational curve, so that $\dim S_2(\Gamma) = 0$. We also have $\dim H^0(S_\Gamma, K_{S_\Gamma}) = 0$, and hence $\dim S_3(\Gamma) = 0$.

Conversely, assume that $\dim S_k(\Gamma) = 0$ for $k = 2, 3$. Then again the Shimura isomorphism (4) implies that $H^0(C_\Gamma, \Omega^1_{C_\Gamma}) = 0$. Hence $C_\Gamma \simeq \mathbb{P}^1$. Since f is not isotrivial, the formula for the canonical bundle asserts that we may write ω as $\omega = \mathcal{O}_{\mathbb{P}^1}(a)$ and $K_{S_\Gamma} = f^*(\mathcal{O}_{\mathbb{P}^1}(a-2))$ with a suitable positive integer a. Note that the positivity of a follows from the positivity of the direct image of the relative canonical sheaf $\omega = f_*(\omega_{S_\Gamma/C_\Gamma}) = \omega$.

Under this situation, $\dim S_3(\Gamma) = \dim H^0(S_\Gamma, K_{S_\Gamma}) = \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a-2)) = 0$ implies that $a = 1$. We also have the isomorphisms

$$f_*\mathcal{O}_{S_\Gamma} \simeq \mathcal{O}_{\mathbb{P}^1}, \quad R^1 f_* \mathcal{O}_{S_\Gamma} \simeq \omega^\vee \simeq \mathcal{O}_{\mathbb{P}^1}(-1).$$

Further the Leray spectral sequence shows that $H^1(S_\Gamma, \mathcal{O}_{S_\Gamma}) = 0$. Moreover, for every $k > 0$, $K^k_{S_\Gamma} = f^*(\mathcal{O}_{\mathbb{P}^1}(-k))$ has no section. These facts finally imply that S_Γ is a rational surface. *q.e.d*

Next, let us consider the self-fiber product of the elliptic modular surface associated to Γ

$$f^2 = f \times f : S^2_\Gamma = S_\Gamma \times_{C_\Gamma} S_\Gamma \longrightarrow C_\Gamma$$

In general, S^2_Γ has singularities which arise from the critical points of the map $f : S_\Gamma \longrightarrow C_\Gamma$. However, if f has only semistable fibers (that is, the singular fibers are all reduced and have only nodal singularities), then the fiber product S^2_Γ has only ordinary double points as its singularities.

Proposition 4.3. *Suppose that* $f : S_\Gamma \longrightarrow C_\Gamma$ *has only semistable fibers. Then there exists a small projective resolution* $\pi : \tilde{S}^2_\Gamma \longrightarrow S^2_\Gamma$. *Moreover, if we set* $h = f^2 \circ \pi$, *then the canonical bundle of* $\tilde{S}^2_\Gamma$ *can be written as*

$$K_{\tilde{S}^2_\Gamma} \simeq (h)^*(\Omega^1_{C_\Gamma} \otimes \omega^2). \tag{7}$$

Proof. The first assertion follows from Lemma 3.1 [Sch2]. Since π is a small resolution, $K_{\tilde{S}^2_\Gamma} = \pi^*(K_{S^2_\Gamma})$. On the other hand, since S^2_Γ is a hypersurface in $S_\Gamma \times S_\Gamma$ (i.e., it is the pull-back of the diagonal $\triangle : C_\Gamma \hookrightarrow C_\Gamma \times C_\Gamma$), we can derive from the adjunction formula that

$$\begin{aligned} K_{S^2_\Gamma} &= (K_{S_\Gamma \times S_\Gamma} + f^{-1}(\triangle(C_\Gamma)))_{|f^{-1}(\triangle(C_\Gamma))} \\ &= (f^2)^*((\Omega^1_{C_\Gamma})^{\otimes 2} \otimes \omega^2 \otimes (\Omega^1_{C_\Gamma})^{-1}) \\ &= (f^2)^*(\Omega^1_\Gamma \otimes \omega^2). \end{aligned}$$

This gives rise to the formula (7). *q.e.d.*

Corollary 4.1. (cf. [Sok].) Under the assumption of Proposition 4.3, we have a canonical isomorphism

$$H^0(\tilde{S}^2_\Gamma, K_{\tilde{S}^2_\Gamma}) \simeq H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega^2) \simeq S_4(\Gamma).$$

For the full cohomology group $H^3(\tilde{S}^2_\Gamma, \mathbb{Q})$, the following fact holds (cf. §1, [Sch1]).

Lemma 4.1. (Lemma 1.7, [Sch1].) *The rational Hodge structure of $H^3(\tilde{S}^2_\Gamma, \mathbb{Q})$ is isomorphic to the direct sum of three copies of $H^1(C_\Gamma, \mathbb{Q})[-1]$ and a piece of pure type $(3,0),(0,3)$. In particular, if C_Γ is a rational curve, then $H^3(\tilde{S}^2_\Gamma, \mathbb{Q})$ is of pure type $(3,0),(0,3)$.*

Corollary 4.2. *Assume that $S_\Gamma \longrightarrow C_\Gamma$ is a rational elliptic modular surface associated to Γ with only semistable singular fibers. Then any small resolution $\tilde{S}^2_\Gamma$ of the self-product S^2_Γ is a rigid Calabi–Yau threefold.*

Proof. Since $f : S_\Gamma \longrightarrow C_\Gamma$ is a rational surface, it is immediate that $C_\Gamma \simeq \mathbb{P}^1$. Let $h : \tilde{S}^2_\Gamma \longrightarrow \mathbb{P}^1$ be the natural fibration. We may write ω as $\omega = \mathcal{O}_{\mathbb{P}^1}(1)$ using the formula (7). Then we see that

$$K_{\tilde{S}^2_\Gamma} = h^*(\Omega^1_{\mathbb{P}^1} \otimes \omega^2) = h^*(\mathcal{O}_{\mathbb{P}^1}(-2+2)) = \mathcal{O}_{\tilde{S}^2_\Gamma}.$$

Recall from Lemma (4.1) that the rational Hodge structure $H^3(\tilde{S}^2_\Gamma, \mathbb{Q})$ is of pure type $(3,0),(0,3)$. So it follows that $H^1(\tilde{S}^2_\Gamma, \Omega^2_{\tilde{S}^2_\Gamma}) = H^1(\Theta_{\tilde{S}^2_\Gamma}) = 0$. We also see (cf. [Sch1]) that $h^1(\mathcal{O}_{\tilde{S}^2_\Gamma}) = h^2(\mathcal{O}_{\tilde{S}^2_\Gamma}) = 0$. These facts then imply that $\tilde{S}^2_\Gamma$ is a rigid Calabi–Yau threefold. *q.e.d.*

Remark 4.1. There are in total six arithmetic subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ which give rise to rational elliptic modular surfaces with only semistable fibers. The list of such groups coincides with the list of Beauville [B]. (Beauville classified elliptic surfaces with exactly four semistable fibers (i.e., of type I_b, $b \geq 1$) over $\mathbb{P}^1$.)

In Table 1, groups Γ are listed in the first column, 4 singular fibers of type I_b, $b \geq 1$ in the second column and $h^{1,1}$ (resp. the Euler characteristic χ) of the corresponding Calabi–Yau threefolds $\tilde{S}^2_\Gamma$ in the third (resp. fourth) column. Note that the Euler characteristic $\chi(\tilde{S}^2_\Gamma)$ is exactly equal to $2h^{1,1}$. This is because $\tilde{S}^2_\Gamma$ is rigid. Also note that for any group Γ in Table 1, $\dim S_4(\Gamma) = 1$. This follows from the isomorphism $S_4(\Gamma) \simeq H^{3,0}(\tilde{S}^2_\Gamma) \simeq \mathbb{C}$.

Table 1

Γ					$h^{1,1}(\tilde{S}^2_\Gamma)$	$\chi(\tilde{S}^2_\Gamma)$
$\Gamma(3)$	I_3	I_3	I_3	I_3	36	72
$\Gamma_1(4)\cap\Gamma(2)$	I_4	I_4	I_2	I_2	40	80
$\Gamma_1(5)$	I_5	I_5	I_1	I_1	52	104
$\Gamma_1(6)$	I_6	I_3	I_2	I_1	50	100
$\Gamma_0(8)\cap\Gamma_1(4)$	I_8	I_2	I_1	I_1	70	140
$\Gamma_0(9)\cap\Gamma_1(3)$	I_9	I_1	I_1	I_1	84	168

Now we quote a theorem which is a special case of the fundamental result due to Sato–Kuga–Shimura ([KS]; cf. [D]). We make use the argument of [D], where Deligne worked over the case when $\Gamma=\Gamma(N)$ (full modular case). However his argument works for other cases like $\Gamma=\Gamma_1(N)$ as far as the universal family of generalized elliptic curves exists.

Theorem 4.1. *Let Γ be an arithmetic subgroup of $SL_2(\mathbb{Z})$ listed in Table 1, and let $f: S_\Gamma \longrightarrow \mathbb{P}^1$ the rational elliptic surface associated to Γ. Let $f^2:\tilde{S}^2_\Gamma \longrightarrow \mathbb{P}^1$ be the smooth rigid Calabi–Yau threefold arising from the self-product S^2_Γ. Then $\tilde{S}^2_\Gamma$ is modular, that is, the L-series of $\tilde{S}^2_\Gamma$ coincides with the Mellin transform of a weight 4 newform of Γ up to finite Euler factors coming from the bad primes:*

$$L(\tilde{S}^2_\Gamma, s) = L(f,s) \quad \textit{for} \quad f\in S_4(\Gamma).$$

Remark 4.2. *The bad primes for $\tilde{S}^2_\Gamma$ depends on the level of the discrete group Γ. For example, we can easily see that only 2 and 3 are the bad primes for $\Gamma_1(6)$.*

5. The geometry of Verrill's Calabi–Yau threefold

In this section, we will look into the geometric structure of Verrill's rigid Calabi–Yau threefold X constructed in Section 3. Note that from now on all varieties are defined over $\mathbb{Q}$.

We denote by $[x:y:z:w]$ the homogeneous coordinate of $\mathbb{P}^3$ and by t the inhomogeneous coordinate of $\mathbb{P}^1$. Then Verrill's rigid Calabi–Yau threefold X is a desingularization of the hypersurface in $\mathbb{P}^3\times\mathbb{P}^1$:

$$X' = \{(x+y+z+w)(x^{-1}+y^{-1}+z^{-1}+w^{-1}) - \frac{(t-1)^2}{t} + 4 = 0\} \subset \mathbb{P}^3\times\mathbb{P}^1. \tag{8}$$

It is easily seen that this equation is equivalent to the following equation:

$$\begin{aligned} F(x,y,z,w,t) &= (x+y+z+w)\cdot[(yz+zx+xy)w+xyz]\cdot t \\ &\quad -(t+1)^2\cdot xyzw = 0. \end{aligned} \tag{9}$$

The elliptic modular surface associated to $\Gamma_1(6)$.

Let us consider the hypersurface S in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by

$$H(x,y,z,s) = (s+1)xyz - (x+y+z)(yz+zx+xy) = 0. \tag{10}$$

The hypersurface S has three singular points at

$$[s,[x:y:z]] = [\infty,[1:0:0]],\ [\infty,[0:1:0]],\ [\infty,[0:0:1]].$$

Let

$$\pi : \tilde{S} \longrightarrow S$$

be the blowing up of these three points. Then it is easy to see that $\tilde{S}$ is the minimal resolution of S and $f = p_1 \circ \pi : \tilde{S} \longrightarrow \mathbb{P}^1$ induces the structure of the elliptic surface. Moreover the rational elliptic surface $f : \tilde{S} \longrightarrow \mathbb{P}^1$ has only four semistable singular fibers of types I_1, I_2, I_3, I_6 (see Table 2):

Table 2

s	8	-1	0	∞
singular fiber	I_1	I_2	I_3	I_6

By [B], $f : \tilde{S} \longrightarrow \mathbb{P}^1$ is an integral model of the elliptic modular surface

$$f : S_{\Gamma_1(6)} \longrightarrow \overline{H/\Gamma_1(6)} \simeq \mathbb{P}^1$$

of $\Gamma_1(6)$.

Theorem 5.1. *Verrill's Calabi–Yau threefold X defined in (8) is birationally equivalent over $\mathbb{Q}$ to a crepant resolution $\tilde{S}^2_{\Gamma_1(6)}$ of the self-product of the elliptic modular surface $S_{\Gamma_1(6)}$. More precisely, there exists a birational map defined over $\mathbb{Q}$ between X and $\tilde{S}^2_{\Gamma_1(6)}$.*

Remark 5.1. There are two fibrations, one is the fibration of $X \longrightarrow \mathbb{P}^1$ of K3 surfaces and the other is the fibration $f^2 : \tilde{S}^2_{\Gamma_1(6)} \longrightarrow \mathbb{P}^1$ of abelian surfaces. We will show below that these two fibrations are not equal. (However, we do not know any relation (apart from being non-equal) between these two fibrations.)

Proof of Theorem 5.1: Birational transformations.

To prove Theorem 5.1, we will construct a birational map over $\mathbb{Q}$ between the two varieties explicitly. We define the birational map:

$$\begin{array}{ccc} \pi : \mathbb{P}^2 \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^3 \times \mathbb{P}^1 \\ [x:y:z] \ \times \ [T:W:U] & \mapsto & [x:y:z:w] \times (t) \end{array}$$

by putting

$$x = x,\ y = y,\ z = z,\ t = T/U,\ w = W \cdot (x+y+z)/U.$$

Pulling back the equation $F(x,y,z,w,t)$ in (9) via π, we obtain the equation

$$\begin{aligned}\tilde{F}: \;&=& (x+y+z)^2(yz+xz+xy)TW^2 \\ && -(x+y+z)\cdot[(x+y+z)(yz+xz+xy)-xyz]TWU \\ && +xyz(x+y+z)TU^2-(x+y+z)xyz[T^2+U^2]W=0\end{aligned}$$

Consider the hypersurface $X'' \subset \mathbb{P}^2 \times \mathbb{P}^2$ defined by $\tilde{F}=0$. Of course, X'' is birationally equivalent to the original hypersurface X' in (8). We have the following commutative diagram:

$$\begin{array}{ccc} X'' & \overset{\iota}{\hookrightarrow} & \mathbb{P}^2\times\mathbb{P}^2 \\ f_1\downarrow & \swarrow p_1 & \\ \mathbb{P}^2 & & \end{array}$$

Now recall the hypersurface $S \subset \mathbb{P}^1 \times \mathbb{P}^2$ defined in (10). Considering the birational map $\mu = (p_2)_{|S} : S \longrightarrow \mathbb{P}^2$ and taking the pull-back morphism f_1 by μ, we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2 & \supset & X''' & \overset{\nu}{\longrightarrow} & X'' \\ \downarrow p_{12} & & f_1'\downarrow & & f_1\downarrow \\ \mathbb{P}^1\times\mathbb{P}^2 & \supset & S & \overset{\mu}{\longrightarrow} & \mathbb{P}^2 \end{array}$$

Here we set $X''' := S \times_{\mathbb{P}^2} X''$. Note that the natural induced map $\nu : X''' \longrightarrow X''$ is a birational morphism. Now we see that X''' is a complete intersection defined by the equations

$$\begin{aligned}\tilde{F} &=& 0 \\ H(x,y,z) &=& (s+1)(xyz)-(x+y+z)(yz+zx+xy)=0.\end{aligned}$$

Now using the second equation, $\tilde{F}=0$ can be transformed into the following equation:

$$\begin{aligned}\tilde{F}: \;&=& (x+y+z)\cdot(s+1)xyz\cdot TW^2 \\ && +(x+y+z)\cdot[(s+1)xyz-xyz]TWU \\ && +xyz(x+y+z)TU^2-(x+y+z)[T^2+U^2]xyzW \\ &=& (x+y+z)xyz\times\{(s+1)\cdot TW^2-sTWU+TU^2-T^2W-WU^2\}\end{aligned}$$

Forgetting the factor $(x+y+z)xyz$ from $\tilde{F}$, we obtain the equation

$$G(T,W,U)=(s+1)\cdot TW^2+sTWU+TU^2-T^2W-WU^2=0.$$

Therefore we see that the subvariety X''' is isomorphic to the complete intersection in $\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2$ defined by the equations

$$\begin{aligned}G(T,W,U) &=& (s+1)TW^2+sTWU+TU^2-T^2W-WU^2=0, \\ H(x,y,z) &=& (s+1)xyz-(x+y+z)(yz+zx+xy)=0.\end{aligned}$$

Next, considering the the birational transformation of $\mathbb{P}^2$ given by

$$T = Z \cdot (X+Y+Z), \quad W = -XY, \quad U = Y(X+Y+Z).$$

Then the polynomial $G(T, W, U)$ above can be transformed into the polynomial

$$\tilde{G}(X, Y, Z) = (s+1)XYZ - (X+Y+Z)(YZ+ZX+XY),$$

which is nothing but $H(X, Y, Z)$. Hence the complete intersection $X''' \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ is isomorphic to the fiber product $S \times_{\mathbb{P}^1} S \longrightarrow \mathbb{P}^1$ of the elliptic surface $f : S \longrightarrow \mathbb{P}^1$ defined by the equation $H(x, y, z) = 0$. Hence, X' defined in (8) is birationally equivalent to the self-product $S \times_{\mathbb{P}^1} S$ of S. Since the minimal resolution of S is isomorphic to the elliptic modular surface $S_{\Gamma_1(6)}$, X is birationally equivalent to the self product $\tilde{S}^2_{\Gamma_1(6)}$. Observing that the birational maps discussed above are all defined over $\mathbb{Q}$, this completes the proof of Theorem 5.1. *q.e.d.*

6. Proof for the modularity of X (Theorem 3.1)

Since the birational map between Verrill's example X and $\tilde{S}^2_{\Gamma_1(6)}$ constructed in Section 5 is defined over $\mathbb{Q}$, we obtain the following Lemma.

Lemma 6.1. *The birational map constructed in Section 5 gives rise to the following isomorphism compatible with the Galois action up to bad primes* 2 *and* 3

$$H^3_{et}(\bar{X}, \mathbb{Q}_\ell) \simeq H^3_{et}(\bar{S}^2_{\Gamma_1(6)}, \mathbb{Q}_\ell).$$

Consequently, we have

$$L(X, s) = L(\tilde{S}^2_{\Gamma_1(6)}, s).$$

up to Euler factors at bad primes 2 *and* 3*.*

For a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, its projectivization $\Gamma/\{\pm I_2\}$ will be denoted by $P\Gamma$. By virtue of Theorem 4.1, Theorem 5.1, Remark 4.2 and Lemma 6.1, the proof of Theorem 3.1 (the modularity of Verrill's Calabi–Yau threefold X) is reduced to the proof of the following lemma.

Lemma 6.2. *The projective image of* $\Gamma_1(N)$ *and* $\Gamma_0(N)$ *are the same in* $PSL_2(\mathbb{Z})$ *if and only if* N *is a divisor of* 4 *or a divisor of* 6*. In particular,* $\mathbb{P}\Gamma_1(6) = \mathbb{P}\Gamma_0(6)$ *and so it follows that* $S_4(\mathbb{P}\Gamma_1(6)) \simeq S_4(\mathbb{P}\Gamma_0(6))$*.*

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\Gamma_0(N)$ or $\Gamma_1(N)$. Then $ad \equiv 1 \pmod{N}$. The first assertion follows from the fact that the only positive integers N for which $ad \equiv 1 \pmod{N}$ implies that $a \equiv d \equiv \pm 1 \pmod{N}$ are the divisors of 4 and 6. *q.e.d.*

Remark 6.1. Verrill [V] has an alternative proof of the modularity for X. Her proof is along the lines of Livné's paper [Li1]. It makes use of the Serre criterion [Sr] and Faltings results [F] to prove equality of two L–series. The main point was to show that *finitely many* Euler factors of two L–series coincide.

Remark 6.2. Our geometric proof of the modularity conjecture for X guarantees that the 2–dimensional Galois representations associated to X and that attached to the modular form are indeed isomorphic (i.e., have the same semi-simplification). This provides a confirmation to the conjecture of Fontaine and Mazur [FM].

We close this section by posing an open problem.

Problem 6.1. May one use the method of Wiles to establish the modularity of X? More concretely, find a single good prime ℓ and establish the modularity for the residual mod ℓ Galois representation associated to the rigid Calabi–Yau threefold X in question.

7. The intermediate Jacobians of rigid Calabi–Yau threefolds

In this section, we shall define the intermediate Jacobian for Calabi–Yau threefolds, following the exposition of Bloch [Bl].

Definition 7.1. Let X be a smooth projective Calabi–Yau threefold defined over $\mathbb{C}$. There is a Hodge filtration

$$H^3(X,\mathbb{C}) = F^0 \supset F^1 \supset F^2 \supset F^3 \supset (0)$$

where

$$\begin{aligned} F^1 &= H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \\ F^2 &= H^{3,0} \oplus H^{2,1} \\ F^3 &= H^{3,0}. \end{aligned}$$

This complex vector space also has a $\mathbb{Z}$–structure defined by the image $H^3(X,\mathbb{Z}) \to H^3(X,\mathbb{C})$.

Now we recall some key properties of Hodge filtrations not only for H^3 but for more general cases. We write $\mathcal{F}$ for generic filtrations. We have

$$\mathcal{F}^i \oplus \overline{\mathcal{F}^{2r-i}} \simeq H^{2r-1}(X,\mathbb{C}), \quad \mathcal{F}^i \cap \overline{\mathcal{F}^{2r-i-1}} \simeq H^{i,2r-i-1}$$

where $H^{i,j} \simeq H^j(X,\Omega_X^i)$ and $r \in \{1,2,3\}$. From this, one can compute that

$$\dim_{\mathbb{C}}\mathcal{F}^r = \frac{1}{2}\dim_{\mathbb{C}}H^{2r-1}(X,\mathbb{C})$$

and

$$\mathcal{F}^r \cap \mathrm{Image}(H^{2r-1}(X,\mathbb{Z}) \to H^{2r-1}(X,\mathbb{C})) = (0).$$

Therefore, the quotient

$$J^r(X) = H^{2r-1}(X,\mathbb{C})/(\mathcal{F}^r + H^{2r-1}(X,\mathbb{Z}))$$

is a compact complex torus, called the *intermediate Jacobian*. By the Poincaré duality, we have

$$H^{2r-1}(X,\mathbb{C})/\mathcal{F}^r \simeq \mathcal{F}^{4-r}\, H^{7-2r}(X,\mathbb{C})^*$$

where $*$ denotes $\mathbb{C}$–linear dual. It then follows that

$$J^r(X) \simeq \mathcal{F}^{4-r}\, H^{7-2r}(X,\mathbb{C})^*/H_{7-2r}(X,\mathbb{Z}).$$

Digression. Now assume that X is rigid, and take $r = 2$ and $\mathcal{F} = F$. Then we have

$$F^1 = H^{3,0} = F^2 = F^3$$

and

$$J^2(X) \simeq H^3(X, \mathbb{C})/(F^2 + H^3(X, \mathbb{Z})) \simeq H^{3,0}(X)^*/H_3(X, \mathbb{Z})$$

is a complex torus of dimension one. This means that there is an elliptic curve E such that $E(\mathbb{C}) \simeq J^2(X)$. We will formulate the following question.

Question 7.1. *Is it true that a rigid Calabi–Yau threefold defined over $\mathbb{Q}$ is modular if and only if the intermediate Jacobian is an elliptic curve defined over $\mathbb{Q}$?*

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