DEFORMATION OF OKAMOTO–PAINLEVÉ PAIRS AND PAINLEVÉ EQUATIONS

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0. Abstract

In this note, we will explain how Painlevé equations can be understood from algebro-geometric viewpoint of deformation of Okamoto–Painlevé pairs.

1. Painlevé equations

In this section, we review briefly how the Painlevé equations are discovered and why it is important. Main reference will be [3.1, Ch. 3, [IKSY]].

Let us consider the following algebraic ordinary equation

$$F(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \cdots, \frac{d^nx}{dt}) = 0 \qquad (1)$$

where

$$F(t, x_0, x_1, x_2, \cdots, x_n) \in \mathbf{C}(t)[x_0, x_1, \cdots, x_n]$$

is a polynomial in $\mathbf{x} = (x_0, x_1, x_2, \cdots, x_n)$ with coefficients rational in t.

Take $(t_0, \mathbf{c}_0) = (t_0, c_0, c_1, \cdots, c_n) \in \{(t_0, \mathbf{c}_0) \in \mathbf{C}^{n+2} | F(t_0, c_0, c_1, c_2, \cdots, c_n) = 0\}$ and consider the Cauchy problem for (1) to find a solution $x(t) = \varphi(t)$ such that

$$\frac{d^i\varphi}{dt^i}(t_0) = c_i, \quad (i = 0, \ldots, n). \qquad (2)$$

The function obtained by an analytic continuation of the local solution $x = \varphi(t)$ is also denoted by $\varphi(t)$.

If an ODE (1) is a linear ordinary differential equation, that is, if $F(t, x_0, x_1, \cdots, x_n)$ is linear with respect to the variables x_i ($i = 0, \cdots, n$), then the sigularities of the solution $\varphi(t)$ can be detected from the differential equation and do not depend on the initial values $(t_0, \mathbf{c}_0)$. These kinds of singularities of the solutions are called <u>*non-movable singularities*</u> of the solutions of (1). On the other hand, the solutions of non-linear algebraic ODE may have <u>*movable singulairities*</u>. For example, let us consider the following famous ODE:

$$(x')^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in \mathbf{C}, \ g_2^3 - 27g_3^2 \neq 0. \qquad (3)$$

(Here we denote by $'$ the derivation with respect to t.)

Let $\tau \in \mathcal{H} = \{z \in \mathbf{C} | \operatorname{Im} z > 0\}$ be the normalized period of the elliptic curve $y^2 = 4x^3 - g_2x - g_3$ so that $1, \tau$ is the fundamental periods of the elliptic curve and let

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\in \mathbf{Z}^2-(0,0)} \left(\frac{1}{(z - m - n\tau)^2} - \frac{1}{(m + n\tau)^2} \right)$$

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be the Weierstrass $\wp$-function with the periods $\mathbf{Z} + \mathbf{Z}\tau$.

If we consider the solution of (3) with the initial values $(t, x, x') = (0, a, b) \in \mathbf{C}^3$ such $b^2 = 4a^3 - g_2 a - g_3$, the solution can be written as

$$x = \wp(t + c)$$

with $a = \wp(c)$ and $b = \wp'(c)$. It is well-known that $\wp(t)$ has poles of order 2 at $t \equiv 0 \mod \mathbf{Z} + \mathbf{Z}\tau$. Therefore the solution $x(t) = \wp(t + c)$ has poles of order 2 at $t \equiv -c \mod \mathbf{Z} + \mathbf{Z}\tau$. The singularities of the solution $x(t)$ do depend on the initial values and this is an example of movable singularities of a non-linear ODE.

There are many examples of non-linear algebraic ODE whose solutions have movable essential singularities or movable branch points and one can easily realized that it is rather rare that the movable singularities of ODE are only poles.

Definition 1.1. An algebraic ODE (1) has *Painlevé property* if the generic solution of (1) has only poles as its movable singularities.

Theorem 1.1. *(L. Fuchs, H. Poincaré). For $n = 1$, an algebraic ODE (1) has Painlevé property if and only if (1) can be transformed, by a holomorphic change of the variable t and by a linear fractinal change of the unknown x with coefficients in holomorphic functions of t, into one of the following equations:*

(1) **The equation of the Weierstrass $\wp$ function** .

$$(x')^2 = 4x^3 - g_2 x - g_3 \tag{4}$$

(2) **Riccati equation**

$$x' = a(t)x^2 + b(t)x + c(t). \tag{5}$$

Remark 1.1. *By the change of unknown*

$$x = -\frac{1}{a(t)}\frac{d}{dt}\log(u) = -\frac{1}{a(t)}\frac{u'}{u}, \tag{6}$$

the Riccati equation (5) is transformed into the linear equation

$$u'' - [\frac{a'(t)}{a(t)} + b(t)]u' + a(t)c(t)u = 0. \tag{7}$$

Hence the solutions $u(t)$ of (7) has only nonmovable singularities and only movable singularities of $x(t)$ is the zero of $u(t)$. Since the zero of $u(t)$ has a finite order, then the movable singularities $x(t)$ are only poles.

E. Picard took a pessimistic view of finding ODEs with with Painlevé property in case of $n \geq 2$. However it is Paul Painlevé (1863–1933) who attacked te classification problem for $n = 2$.

Definition 1.2. *Painlevé equation is a second order algebraic ODE of rational type, that is,*

$$x'' = R(x, x', t), \quad R(x, y, t) \in \mathbf{C}(x, y, t) \tag{8}$$

satisfying Painlevé property.

Painlevé and his student B.O. Gambier showed that Painlevé equation reduces, by an approptiate transformation of the variables, to an equation which can be integrated by quadrature, or to a linear equation, or to P_J, $J = I, II, III, IV, V, VI$. (See Table 1). Here α, β, γ and δ are complex constants.

$$
\begin{aligned}
P_I: \quad \frac{d^2x}{dt^2} &= 6x^2+t,\\
P_{II}: \quad \frac{d^2x}{dt^2} &= 2x^3+tx+\alpha,\\
P_{III}: \quad \frac{d^2x}{dt^2} &= \frac{1}{x}\left(\frac{dx}{dt}\right)^2-\frac{1}{t}\frac{dx}{dt}+\frac{1}{t}(\alpha x^2+\beta)+\gamma x^3+\frac{\delta}{x},\\
P_{IV}: \quad \frac{d^2x}{dt^2} &= \frac{1}{2x}\left(\frac{dx}{dt}\right)^2+\frac{3}{2}x^3+4tx^2+2(t^2-\alpha)x+\frac{\beta}{x},\\
P_V: \quad \frac{d^2x}{dt^2} &= \left(\frac{1}{2x}+\frac{1}{x-1}\right)\left(\frac{dx}{dt}\right)^2-\frac{1}{t}\frac{dx}{dt}+\frac{(x-1)^2}{t^2}\left(\alpha x+\frac{\beta}{x}\right)\\
&\qquad +\gamma\frac{x}{t}+\delta\frac{x(x+1)}{x-1},\\
P_{VI}: \quad \frac{d^2x}{dt^2} &= \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{x-t}\right)\left(\frac{dx}{dt}\right),\\
&\quad +\frac{x(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha-\beta\frac{t}{x^2}+\gamma\frac{t-1}{(x-1)^2}+\left(\frac{1}{2}-\delta\right)\frac{t(t-1)}{(x-t)^2}\right].
\end{aligned}
$$

TABLE 1

It is known that Painlevé equations appear in many fields of mathematics and physics. For example, Painlevé equations can be derived from isomonodromic deformations of linear ordinary equations over $\mathbf{P}^1$. ([IKSY]). Besides the important relation to correlation functions of Ising models, the later is deeply related to the notion of Frobenius structure or Frobenius manifolds due to Dubrovin [Du]. Frobenius structure is a kind of generalization of WDVV equations, which is equivalent to the fact that related quantum cohomology rings are associative. (Note that the Frobenius structure is essentially same as the flat structure, originally introduced by Kyoji Saito in case of versal deformation of isolated singularities.)

2. Okamoto's spaces of initial conditions for Painlevé equations

In this section, we recall a series of works of Okamoto which shows the importance of study of the space of initial conditions of Painlevé equation. Later we will introduce the notion of Okamoto–Painlevé pairs, which is a generalization of Okamoto's space of intial conditions.

First, let us recall that each P_J is equivalent to a Hamiltonian system H_J (cf. [Mal], [O1], [IKSY], [MMT]):

$$
(H_J): \begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H_J}{\partial y}, \\[2mm] \dfrac{dy}{dt} = -\dfrac{\partial H_J}{\partial x}, \end{cases} \tag{9}
$$

where the Hamiltonians $H_J = H_J(x, y, t)$ are given in TABLE 2.

$$
\begin{aligned}
H_I(x,y,t) &= \frac{1}{2}y^2 - 2x^3 - tx, \\
H_{II}(x,y,t) &= \frac{1}{2}y^2 - \left(x^2 + \frac{t}{2}\right) y - \left(\alpha + \frac{1}{2}\right) x, \\
H_{III}(x,y,t) &= \frac{1}{t}\Big[2x^2y^2 - \Big\{2\eta_\infty t x^2 + (2\kappa_0 + 1)x - 2\eta_0 t\Big\} y + \eta_\infty (\kappa_0 + \kappa_\infty) tx\Big], \\
H_{IV}(x,y,t) &= 2xy^2 - \Big\{x^2 + 2tx + 2\kappa_0\Big\} y + \kappa_\infty x, \\
H_V(x,y,t) &= \frac{1}{t}\Big[x(x-1)^2y^2 - \Big\{\kappa_0(x-1)^2 + \kappa_t x(x-1) - \eta t x\Big\} y + \kappa_\infty (x-1)\Big], \\
&\quad \left(\kappa := \frac{1}{4}\Big\{(\kappa_0 + \kappa_t)^2 - \kappa_\infty^2\Big\}\right), \\
H_{VI}(x,y,t) &= \frac{1}{t(t-1)}\Big[x(x-1)(x-t)y^2 - \{\kappa_0(x-1)(x-t) \\
&\quad + \kappa_1 x(x-t) + (\kappa_t - 1)x(x-1)\} y + \kappa(x-t)\Big] \\
&\quad \left(\kappa := \frac{1}{4}\Big\{(\kappa_0 + \kappa_1 + \kappa_t - 1)^2 - \kappa_\infty^2\Big\}\right).
\end{aligned}
$$

TABLE 2

Let us consider the Hamiltonian system H_J from now on. Let $\Sigma_J = \{a_1, \dots, a_l\} \subset \mathbf{C}$ be the set of non-movable singularities for (H_J) and set

$$
B_J = \mathbf{C} - \Sigma_J = \operatorname{Spec} \mathbf{C}[t, \frac{1}{t - a_1}, \dots, \frac{1}{t - a_l}].
$$

Note that the non-movable singularities can be easily detected from the differential equation itself.

Consider the product space $\mathbf{C}^2 \times B_J \ni (x, y, t)$ and the projection map $\pi : \mathbf{C}^2 \times B_J \longrightarrow B_J$. On the product space $\mathbf{C}^2 \times B_J \ni (x, y, t)$, we can consider the Hamiltonaian system (H_J). Let us take a relative compactification of π:

$$
\begin{array}{ccc}
\mathbf{C}^2 \times B_J & \hookrightarrow & \mathbf{P}^2 \times B_J \\
\downarrow & & \downarrow \\
B_J & = & B_J
\end{array} \tag{10}
$$

If (H_J) satisfies Painlevé property, all movable singularities of the solutions of (H_J) are just poles. The solution curve $(x(t), y(t), t)$ of (H_J) starting from an initial vaule $(x_0, y_0, t_0) \in \mathbf{C}^2 \times B_J$ may have singularities at some point at $t_1 \in B_J$ depending on the initial vaule (x_0, y_0, t_0). However Painlevé property ensures that the solutions $x(t), y(t)$ have only poles as it singularities, hence they have limits in $\mathbf{P}^2 \times B_J$ thanks to the properness of $\mathbf{P}^2$. Setting $L = \mathbf{P}^2 - \mathbf{C}^2$, the solution curve $(x(t), y(t), t)$ starting from $(x_0, y_0, t_0) \in \mathbf{C}^2 \times B_J$ passes through a point at the boundary $L \times \{t_1\}$. Such a singulairity of solution curve is called an *accesible singularity.* For a classical Painlevé equation P_J, or equivalently Hamiltonian system (H_J), for each fixed $t = t_1$,

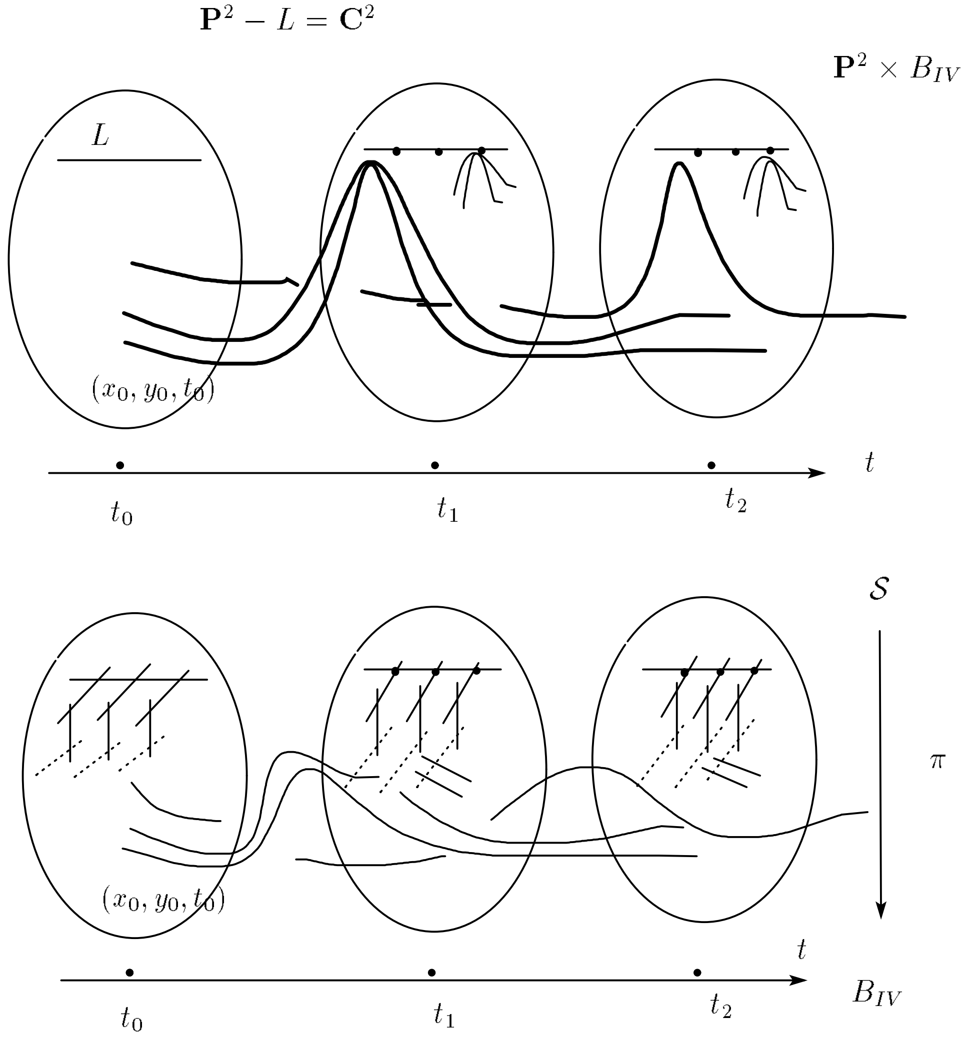

FIGURE 1

there are only finitely many accesible singular points at the boundary $L \times t_1$. At those points, infinitely many solution curves come together.

Okamoto analyses these kinds of singularities in details [O1]. He blowed up at the accesible singular points including infinitely near points of boundaries, so that all of solution curves of (H_J) can be separated (see FIGURE 1). More precisely, he constructed a smooth morphism $\pi : \mathcal{S} \longrightarrow B_J$ by blowing up $\mathbf{P}^2 \times B_J$.

$$\begin{array}{ccc} \mathbf{P}^2 \times B_J & \stackrel{\tau}{\leftarrow} & \mathcal{S} \\ \downarrow & \swarrow \pi & \\ B_J & & \end{array} \tag{11}$$

From the construction of Okamoto, one can obtain the following theorem (cf. [O1], see also [Sakai], [STa], [STT]).

Proposition 2.1. *For a general $t \in B_J$, a fiber $S = \mathcal{S}_t$ of $\pi : \mathcal{S} \longrightarrow B_J$ over t is obtained by blowing ups of $\mathbf{P}^2$ at 9 points of boundary L (including infinitely near points). The anti-canonical divisor $-K_S$ of $S = \mathcal{S}_t$ is an effective divisor $Y = \sum_{i=1}^r m_i Y_i$. Moreover the configuration of an effective divisior Y is same as in the list of Kodaira–Néron's minimal model of singular elliptic curves of additive types (Figure 2).*

We remark that the cases $\tilde{D}_7, \tilde{D}_8$ did not appear in Okamoto's paper [O1]. These cases really appear when parameters in Painlevé III equations take special values. Moreover we can show that for the case $\tilde{D}_8$ the $S - Y_{red}$ does not contains $\mathbf{C}^2$ but S can be obtained by blowing up of $\mathbf{P}^2$.

Note that in Figure 2, the real line shows that a smooth rational curve $C \simeq \mathbf{P}^1$ with $C^2 = -2$ and the number near the each rational curve denotes the multiplicity in $Y = -K_S$.

Let us set

$$-K_S = Y = \sum_{i=1}^{r} m_i Y_i.$$

Under the assumption that S is a rational surface, one can show that Y has the same configuration as Kodaira–Néron's minimal model of singular elliptic curves if and only if the following conditions are satisfied (cf. [Kod], [STa]).

$$\boxed{\deg -(K_S)_{|Y_i} = -K_S \cdot Y_i = Y \cdot Y_i = 0 \quad \text{for all } i, \quad 1 \leq i \leq m}. \tag{12}$$

Here we should remark that rational surfaces S appeared above are not elliptic surface. Actually, one can show that the dimension of the linear system $|-K_S| = |Y|$ is zero, while for an rational elliptic surface S' one should have $|-K_{S'}| = |Y| \simeq \mathbf{P}^1$.

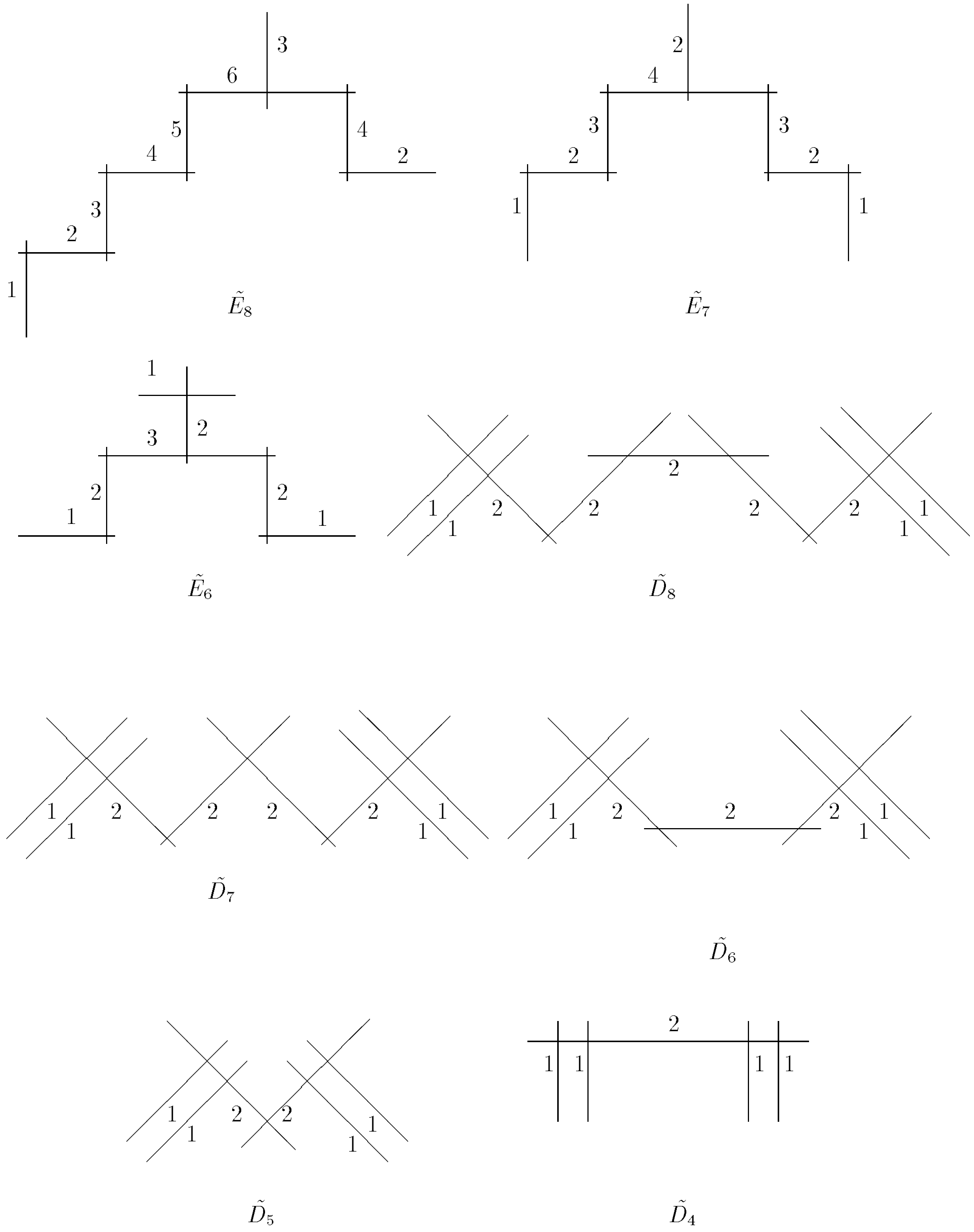
$\tilde{E}_8$
$\tilde{E}_7$
$\tilde{E}_6$
$\tilde{D}_8$
$\tilde{D}_7$
$\tilde{D}_6$
$\tilde{D}_5$
$\tilde{D}_4$

FIGURE 2

3. Okamoto–Painlevé pairs

From the construction of Okamoto, one can understand that the following definition is natural.

Definition 3.1. Let (S, Y) be a pair of a complex projective surface S and an anti-canonical divisor $Y \in |-K_S|$ of S. Let $Y = \sum_{i=1}^{r} m_i Y_i$ be the irreducible decomposition of Y. We call a pair (S, Y) a *(generalized) Okamoto–Painlevé Pair* if for all $i, 1 \leq i \leq r$,

$$Y \cdot Y_i = \deg Y_{|Y_i} = 0. \tag{13}$$

An Okamoto–Painlevé pair (S, Y) is called *rational*, if S is a rational surface. We will consider only rational Okamoto–Painlevé pairs from now on.

The first assertion of the following propositon is proved in [Sakai] and the rest is proved by Riemann–Roch theorem and standard arguments.

Proposition 3.1. *Let (S, Y) be a rational Okamoto–Painlevé pair. Then S can be obtained by 9 points blowings-up of $\mathbf{P}^2$. Moreover*

(1) $\dim |-nK_S| = \dim |nY| \leq 1$ *for all* $n \geq 1$.
(2) *If* $\dim |-nK_S| = \dim |nY| = 1$ *for some* $n \geq 1$, *there exists an elliptic fibraiton* $f : S \longrightarrow \mathbf{P}^1$ *with* $f^*(\infty) = nY$.

Definition 3.2. A rational Okamoto–Painlevé pair (S, Y) is called *of fibered–type* if there exists an elliptic fibration $f : S \longrightarrow \mathbf{P}^1$ such that $f^*(\infty) = nY$ for some positive integer $n \geq 1$. If (S, Y) is not of fibered type, we call (S, Y) "*of non-fibered type* ".

Note that a rational Okamoto–Painlevé pair (S, Y) is of fibered–type if and only if $\dim |nY| \geq 1$ for some positive integer $n \geq 1$. Moreover, if (S, Y) is of fibered type and $\varphi : S \longrightarrow \mathbf{P}^1$ is an elliptic fibration with $\varphi^*(\infty) = nY$ with $n > 1$, $\varphi^*(\infty)$ is called a multiple fiber. This happens only when Y is of elliptic type or multiplicative type in the notation below.

Let $Y = \sum_{i=1}^{r} m_i Y_i$ be the irreducible decomposition of Y. Denote by $M(Y)$ the sublattice of $\mathrm{Pic}(S) \simeq H^2(S, \mathbf{Z})$ generated by the irreducible components $\{Y_i\}_{i=1}^{r}$. Here the bilinear form on $\mathrm{Pic}(S)$ is (-1) times the intersection form on $\mathrm{Pic}(S)$. Then $\{Y_i\}_{i=1}^{r}$ forms a root basis of $M(Y)$ and we denote by $R(Y)$ the type of the root system. (The root system $R(Y)$ is of affine type.)

According to the type of Y, $R(Y)$ can be classified into three classes: *elliptic type* when Y is a smooth elliptic curve, *multiplicative type* when Y is a cycle of rational curves, *additive type* when the configuration of Y is tree. These types also correspond to the types of generalized Jacobians $\mathrm{Pic}^0(Y)$ of Y.

For the classification of ratinal Okaomoto–Painlevé pairs (S, Y) with normal crossing divisor Y_{red}, we can easily show the following theorem.

Proposition 3.2. *Let (S, Y) be a rational Okamoto–Painlevé pair such that Y_{red} is a divisor with only normal crossings. Then the type of Y is same as one in the list of Table 3.*

The following proposition is technically important (cf. [STT]).

Proposition 3.3. *Let (S, Y) be a rational Okamoto–Painlevé pair. The following conditions are equivalent to each other.*

(1) (S, Y) *is of non-fibered type.*
(2) $H^0(S - Y, \mathcal{O}^{alg}) \simeq \mathbf{C}$, *that is, all regular algebraic functions of* $S - Y$ *are constant functions.*

Y *or* $R(Y)$	$\tilde{E}_8$	$\tilde{D}_8$	$\tilde{E}_7$	$\tilde{D}_7$	$\tilde{D}_6$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{D}_4$	$\tilde{A}_{r-1}$ $1 \leq r \leq 9$	$\tilde{A}_0^{\ *}$ $r=1$
Kodaira's notation	II^*	I_4^*	III^*	I_3^*	I_2^*	IV^*	I_1^*	I_0^*	I_r	I_0
Painlevé equation	P_I	$P_{III}^{\tilde{D}_8}$	P_{II}	$P_{III}^{\tilde{D}_7}$	P_{III}	P_{IV}	P_V	P_{VI}	*none*	*none*
r	*9*	*9*	*8*	*8*	*7*	*7*	*6*	*5*	r	*1*

Table 3

4. Deformation of rational Okamoto–Painlevé pairs

Let (S, Y) be a generalized rational Okamoto–Painlevé pair. Recall that $Y = \sum_{i=1}^{r} m_i Y_i$ is the anti-canonical divisor $-K_S$. Moreover we set $D_i = Y_{red} = \sum_{i=1}^{r} Y_i$.

We will consider the deformation of smooth pairs $(S, D) = (S, Y_{red})$ [Kaw], [SSU] and [STT]. By a generalization of Kodaira–Spencer theory due to Kawamata [Kaw], the set of infinitesimal deformations of (S, D) is isomorphic to the cohomology group

$$H^1(S, \Theta_S(-\log D)) \tag{14}$$

and the set of obstructions to lift the infinitesimal deformations to higher order ones are contained in

$$H^2(S, \Theta_S(-\log D)). \tag{15}$$

The following results are proved in [STT]. (See also [AL], [SU]).

Proposition 4.1. *For a rational Okamoto–Painlevé pair (S, Y), we have the following.*

(1) $H^1(S - D, \mathbf{C}) = 0$.
(2) $H^0(S, \Omega_S^1(\log D)) = 0$.
(3) $H^2(S, \Theta_S(-\log D)) = 0$.
(4) $H^2(S, \Theta_S) = 0$.
(5) *If furthermore S is of non-fibered type, $H^0(S, \Theta_S(-\log D)(H)) = 0$ for any effective divisor H supported on D.*

From the above results, one can obtain the following proposition [STT].

Proposition 4.2. *Let (S, Y) be a generalized rational Okamoto–Painlevé pair such that $D = Y_{red}$ is a simple normal crossing divisor and $Y \neq \tilde{A}_0$-type. Then we have*

$$c_2(S) = \textit{topological Euler characteristic} = 12, \tag{16}$$

$$b_2(S) = \operatorname{rank} H^2(S, \mathbf{Z}) = 10, \tag{17}$$

$$\dim H^1(S, \Theta_S) = 10, \tag{18}$$

and

$$\dim H^1(S, \Theta_S(-\log D)) = 10 - r \tag{19}$$

where r is the number of irreducible components of Y. Moreover, the Kuranishi space of the local deformation of the pair (S, D) is smooth and of dimension $10 - r$.

5. Local cohomology sequences and Time variables

Let (S, Y) be a generalized rational Okamoto–Painlevé pair and set $D = Y_{red}$. Moreover, in this section, we assume that

(1) (S, Y) is of non-fibered type and
(2) Y_{red} is a normal crossing divisor with at least two irreducible components, i.e. $(r \geq 2)$ so that all irreducible components of Y_{red} are smooth rational curves.

In what follows, $\mathcal{O}_S$ and $\mathcal{O}_{S-D}$ denote the sheaves of germs of algebraic regular functions on S and $S - D$ respectively. Moreover all sheaves of $\mathcal{O}_S$-modules are considered in algebraic category unless otherwise stated. Let us consider the following exact sequence of local cohomology groups ([Corollary 1.9, [Gr]])

$$H^0(S, \Theta_S(-\log D)) \to H^0(S - D, \Theta_S(-\log D)) \to H^1_D(\Theta_S(-\log D)) \to \tag{20}$$
$$H^1(S, \Theta_S(-\log D)) \overset{\mathrm{res}}{\to} H^1(S - D, \Theta_S(-\log D)) \quad . \tag{21}$$

Since (S, Y) is of non-fibered type, from (2), Proposition 4.1, we see that

$$H^0(S - D, \Theta_S(-\log D)) = H^0(S - D, \Theta_S) = \{0\}.$$

Hence, we have the following

Proposition 5.1. *For a generalized rational Okamoto–Painlevé pair of non-fibered type, we have the following exact sequence:*

$$0 \to H^1_D(\Theta_S(-\log D)) \to H^1(S, \Theta_S(-\log D)) \overset{\mathrm{res}}{\to} H^1(S - D, \Theta_S(-\log D)) \tag{22}$$

The following theorem is proved in [T].

Theorem 5.1. *Let (S, Y) be a generalized rational Okamoto-Painlevé pair (S, Y) with the condition above. Moreover $D = Y_{red}$ is of additive type. Then we have*

$$\dim H^0(D, \Theta_S(-\log D) \otimes N_D) = 1. \tag{23}$$

Here we put $N_D = \mathcal{O}_S(D)/\mathcal{O}_S$.

Since we have a natural inclusion

$$H^0(D, \Theta_S(-\log D) \otimes N_D) \hookrightarrow H^1_D(\Theta_S(-\log D)),$$

we obtain

$$\dim H^1_D(\Theta_S(-\log D)) \geq 1. \tag{24}$$

This theorem plays an important role to understand the Painlevé equation related to (S, Y). Note that for a generic rational Okamoto–Painlevé pair (S, Y) in Theorem 5.1, one can show that

$$H^1_D(\Theta_S(-\log D)) \simeq H^0(D, \Theta_S(-\log D) \otimes N_D) \simeq \mathbf{C}. \tag{25}$$

From the exact sequence (22), we see that the subspace $H^1_D(S, \Theta_S(-\log Y))$ of $H^1(S, \Theta_S(-\log Y))$ coincides with the kernel of μ. This implies that:

$$H^1_D(S, \Theta_S(-\log D)) \simeq \left\{ \begin{array}{l} \text{Infinitesimal deformations of } (S, D) \text{ whose restriction} \\ \text{to } S - D \text{ induces the trivial deformation} \end{array} \right\}.$$

We can construct semi-universal families of Okamoto–Painlevé pairs with nice coordinate systems ([Sakai], [STT], [ShT]).

Proposition 5.2. *Let $R = R(Y)$ be one of types of the root systems appeared in Proposition 3.2 which is additive type, so that*

$$\dim H^0(D, \Theta_S(-\log D) \otimes N_D) = 1$$

for corresponding generalized rational Okamoto–Painlevé pair (S, Y) (cf. Theorem 5.1). Moreover denote by r the number of irreducible components of $D = Y_{red}$.

Let $\mathcal{M}_R$ be an affine open subscheme in $\mathbf{C}^s = \operatorname{Spec} \mathbf{C}[\alpha_1, \cdots, \alpha_s]$ of dimension $s = s(R) = 9 - r$ and $\mathcal{B}_R$ be an affine open subscheme of $\mathbf{C} = \operatorname{Spec} \mathbf{C}[t]$. Then there exists the following commutative diagram satisfying the conditions below.

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{M}_R \times \mathcal{B}_R & & . \end{array} \tag{26}$$

(1) *The above diagram is a deformation of non-singular pair of projective surfaces and normal crossing divisors.*
(2) *There exists a rational relative 2-form*
$$\omega_{\mathcal{S}} \in \Gamma(\mathcal{S}, \Omega^2_{\mathcal{S}/\mathcal{M}_R \times \mathcal{B}_R}(\mathcal{Y}))$$
whose pole divisor is $\mathcal{Y}$ with $\mathcal{Y}_{red} = \mathcal{D}$.
(3) *If we denote by $\mathcal{Y}$ the pole divisor of $\omega_{\mathcal{S}}$, then for each point $(\boldsymbol{\alpha}, t) \in \mathcal{M}_R \times \mathcal{B}_R$, $(\mathcal{S}_{\boldsymbol{\alpha},t}, \mathcal{Y}_{\boldsymbol{\alpha},t})$ is a generalized Okamoto–Painlevé pair of non fibered type with type $R = R(Y)$. Moreover $\mathcal{Y}_{red} = \mathcal{D}$.*
(4) *The family is semiuniversal at a general point $(\boldsymbol{\alpha}, t) \in \mathcal{M}_R \times \mathcal{B}_R$, that is, the Kodaira–Spencer map*
$$\rho : T_{\boldsymbol{\alpha},t}(\mathcal{M}_R \times \mathcal{B}_R) \longrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \tag{27}$$
is an isomorphism for a general point $(\boldsymbol{\alpha}, t)$.

The Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ of t-direction ($= \mathcal{B}_R$) lies in the image of the natural map :
$$\delta : \mathbf{C} \simeq H^0(\mathcal{D}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}) \otimes N_{\mathcal{D}_{\boldsymbol{\alpha},t}}) \hookrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})). \tag{28}$$
(5) *Let M_R and B_R denote the affine coordinate rings of $\mathcal{M}_R$ and $\mathcal{B}_R$ respectively so that $\mathcal{M}_R = \operatorname{Spec} M_R$ and $\mathcal{B}_R = \operatorname{Spec} B_R$. (Note that M_R and B_R is obtained by some localization's of $\mathbf{C}[\alpha_1, \cdots, \alpha_s]$ and $\mathbf{C}[t]$ respectively.) There exists a finite affine covering $\{\tilde{U}_i\}_{i=1}^{l+k}$ of $\mathcal{S}$ relative to $\mathcal{M}_R \times \mathcal{B}_R$ such that there exists an isomorphism for each i*
$$\tilde{U}_i \simeq \operatorname{Spec}(M_R \otimes B_R)[x_i, y_i, \frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)}] \subset \operatorname{Spec} \mathbf{C}[\boldsymbol{\alpha}, t, x_i, y_i] \simeq \mathbf{C}^{s+3} \simeq \mathbf{C}^{12-r} \tag{29}$$
Here $f_i(x_i, y_i, \boldsymbol{\alpha}, t)$ is a polynomial in $(M_R \otimes B_R)[x_i, y_i]$. Moreover we may assume that $\mathcal{S} - \mathcal{D}$ can be covered by $\{\tilde{U}_i\}_{i=1}^{l}$. Moreover for each i the restriction of the rational two form $\omega_{\mathcal{S}}$ can be written as
$$\omega_{\mathcal{S}|\tilde{U}_i} = \frac{dx_i \wedge dy_i}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)^{m_i}} \tag{30}$$
(6) *For each pair i, j such that $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$, the coordinate transformation functions*
$$x_i = f_{ij}(x_j, y_j, \boldsymbol{\alpha}, t), \quad y_i = g_{ij}(x_j, y_j, \boldsymbol{\alpha}, t) \tag{31}$$

are rational functions in variables $x_j, y_j, \boldsymbol{\alpha}, t$.

6. From Global Deformations to Hamiltonian systems

In this section, we will explain how one can derive differential equations from the one dimensional global deformation of rational Okamoto-Painlevé pairs of additive type which arises from the image of linear map

$$\delta : \mathbf{C} \simeq H^0(D, \Theta_S(-\log D) \otimes N_D) \longrightarrow H^1(S, \Theta_S(-\log D)).$$

We will be able to obtain a global rational vector field $\tilde{v}$ on $\mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ whose poles are only supported on $\mathcal{D}$, which is a lift of $\frac{\partial}{\partial t}$. Hence the restriction of $\tilde{v}$ to $\mathcal{S} - \mathcal{D}$ gives a global regular algebraic vector field.

Moreover we will show that such a lift $\tilde{v}$ of $\frac{\partial}{\partial t}$ is unique. The restriction of $\tilde{v}$ to the open affine subset $\tilde{U}_i$ of $\mathcal{S} - \mathcal{D}$ gives an explicit system of algebraic ODEs of 1st order, which is equivalent to Painlevé equation.

Let $R = R(Y)$ be one of types of additive affine root systems appeared in Proposition 3.2 and let

$$\begin{array}{ccc} \mathcal{S} & \hookleftarrow & \mathcal{D} \\ \pi \downarrow & \swarrow & \varphi \\ \mathcal{M}_R \times \mathcal{B}_R & & \end{array} \tag{32}$$

be a global deformation of generalized Okamoto–Painlevé pairs of type R as in Proposition 5.2. The total space $\mathcal{S}$ has a finite affine covering $\{\tilde{U}_i\}_{i=1}^{l+k}$ such that

$$\tilde{U}_i \simeq \mathrm{Spec}(M_R \otimes B_R)[x_i, y_i, \frac{1}{f_i(x_i, y_i, \boldsymbol{\alpha}, t)}] \subset \mathrm{Spec}\, \mathbf{C}[\boldsymbol{\alpha}, t, x_i, y_i] \tag{33}$$

as in (29). Moreover, we may assume that $\mathcal{S} - \mathcal{D}$ can be covered by $\{\tilde{U}_i\}_{i=1}^{l}$, that is,

$$\mathcal{S} - \mathcal{D} = \cup_{i=1}^{l} \tilde{U}_i.$$

Let us recall that the coordinate transformations in (31) for $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$ are given by the rational functions

$$x_i = f_{ij}(x_j, y_j, \boldsymbol{\alpha}, t), \quad y_i = g_{ij}(x_j, y_j, \boldsymbol{\alpha}, t) \tag{34}$$

The Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ can be represented by the Čech 1-cocycles

$$\rho(\frac{\partial}{\partial t}) = \{\ \theta_{ij} = \frac{\partial f_{ij}}{\partial t}\frac{\partial}{\partial x_i} + \frac{\partial g_{ij}}{\partial t}\frac{\partial}{\partial y_i} \in \Gamma(\tilde{U}_i \cap \tilde{U}_j, \Theta_{\mathcal{S}/\mathcal{M}_R \times \mathcal{B}_R}(-\log \mathcal{D}))\ \} \tag{35}$$

Recall that (cf. Proposition 5.2, 3) on $\mathcal{M}_R \times \mathcal{B}_R$ the corresponding Okamoto–Painlevé pairs $(\mathcal{S}_{\boldsymbol{\alpha},t}, \mathcal{Y}_{\boldsymbol{\alpha},t})$ are *of non-fibered type.*

From (28) of Proposition 5.2, we may assume that $\rho(\frac{\partial}{\partial t})$ is non-zero element of the image of δ:

$$\begin{array}{ccc} \mathbf{C} \simeq H^0(D, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}) \otimes N_D) & \subset & H^1_{\mathcal{D}_{\boldsymbol{\alpha},t}}(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})). \\ & \delta \searrow & \downarrow \\ & & H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \end{array} \tag{36}$$

Since the local cohomology group is the kernel of the natural restriction map (cf. Proposition 5.1)

$$\mathsf{res} : H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})) \longrightarrow H^1(\mathcal{S}_{\boldsymbol{\alpha},t} - \mathcal{D}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t})), \tag{37}$$

the Kodaira–Spencer class $\rho(\frac{\partial}{\partial t})$ is cohomologus to zero in $H^1(\mathcal{S}_{\boldsymbol{\alpha},t}-\mathcal{D}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}))$. More precisely, we see that the 1-cocycle $\rho(\frac{\partial}{\partial t})$ is cohomologus to zero in

$$H^1(\mathcal{S}_{\boldsymbol{\alpha},t}, \Theta_{\mathcal{S}_{\boldsymbol{\alpha},t}}(-\log \mathcal{D}_{\boldsymbol{\alpha},t}) \otimes \mathcal{O}_S(\mathcal{D}_{\boldsymbol{\alpha},t})). \tag{38}$$

Since dimensions of these cohomology groups are constant as functions of $(\boldsymbol{\alpha}, t)$, by an argument using the base change theorem, we can see that for $1 \leq i \leq l+k$ there exist rational vector fields

$$\theta_i(x_i, y_i, \boldsymbol{\alpha}, t) = \eta_i(x_i, y_i, \boldsymbol{\alpha}, t)\frac{\partial}{\partial x_i} + \zeta_i(x_i, y_i, \boldsymbol{\alpha}, t)\frac{\partial}{\partial y_i} \ \in \ \Gamma(\tilde{U}_i, \Theta_{\tilde{U}_i}(-\log \mathcal{D}) \otimes \mathcal{O}(\mathcal{D})) \tag{39}$$

such that

$$\theta_{ij}(x_i, y_i, \boldsymbol{\alpha}, t) = \theta_j(x_j, y_j, \boldsymbol{\alpha}, t) - \theta_i(x_i, y_i, \boldsymbol{\alpha}, t). \tag{40}$$

Note that for $1 \leq i \leq l$ one has $\tilde{U}_i \cap \mathcal{D} = \emptyset$. Hence for $1 \leq i \leq l$, $\theta_i(x_i, y_i, \boldsymbol{\alpha}, t)$ is a regular algebraic vector field, that is, $\eta_i(x_i, y_i, \boldsymbol{\alpha}, t)$ and $\zeta(x_i, y_i, \boldsymbol{\alpha}, t)$ are regular algebraic functions on $\tilde{U}_i$.

For any pair i, j such that $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$, we can obtain the identity on $\tilde{U}_i \cap \tilde{U}_j$

$$\left(\frac{\partial}{\partial t}\right)_j = \left(\frac{\partial}{\partial t}\right)_i + \theta_{ij}(\boldsymbol{\alpha}, t), \tag{41}$$

and hence we have

$$\left(\frac{\partial}{\partial t}\right)_j = \left(\frac{\partial}{\partial t}\right)_i + (\theta_j(x_j, y_j, \boldsymbol{\alpha}, t) - \theta_i(x_i, y_i, \boldsymbol{\alpha}, t)), \tag{42}$$

or

$$\left(\frac{\partial}{\partial t}\right)_j - \theta_j(x_j, y_j, \boldsymbol{\alpha}, t) = \left(\frac{\partial}{\partial t}\right)_i - \theta_i(x_i, y_i, \boldsymbol{\alpha}, t). \tag{43}$$

This means that the vector fields

$$\{\left(\frac{\partial}{\partial t}\right)_i - \theta_i(x_i, y_i, \boldsymbol{\alpha}, t)\}_{1 \leq i \leq l+k} \tag{44}$$

can be patched together and define a global rational vector field

$$\tilde{v} \in \Gamma(\mathcal{S}, \Theta_{\mathcal{S}}(-\log \mathcal{D}) \otimes \mathcal{O}(\mathcal{D})).$$

Note that this global rational vector field $\tilde{v}$ is a lift of $\frac{\partial}{\partial t}$ via $\pi : \mathcal{S} \longrightarrow \mathcal{M}_R \times \mathcal{B}_R$ and

$$\tilde{v}_{|\mathcal{S}-\mathcal{D}} \tag{45}$$

is a regular algebraic vector field. Now we obtain the following

Theorem 6.1. *Let $R = R(Y)$, $\mathcal{S}, \mathcal{D}, \mathcal{M}_R \times \mathcal{B}_R$ be as above. Then there exists a unique global rational vector field*

$$\tilde{v} \in \Gamma(\mathcal{S}, \Theta(-\log \mathcal{D}) \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{D}))$$

on $\mathcal{S}$ which is a lift of $\frac{\partial}{\partial t}$. The restriction $\tilde{v}_{\mathcal{S}-\mathcal{D}}$ to $\mathcal{S}-\mathcal{D}$ is a regular algebraic vector field. Moreover the restriction $\tilde{v}$ to each open covering $\tilde{U}_i$ for $1 \leq i \leq l$ (corresponding to the open coverings of $\mathcal{S}-\mathcal{D}$) can be written as

$$\tilde{v}_{|\tilde{U}_i} \ = \ \frac{\partial}{\partial t} \ - \ \theta_i \ = \ \frac{\partial}{\partial t} \ - \ \eta_i\frac{\partial}{\partial x_i} - \zeta_i\frac{\partial}{\partial y_i} \tag{46}$$

and defines a system of differential equations

$$\begin{cases} \dfrac{dx_i}{dt} & = & -\eta_i(x_i, y_i, \boldsymbol{\alpha}, t) \\ \dfrac{dy_i}{dt} & = & -\zeta_i(x_i, y_i.\boldsymbol{\alpha}, t) \end{cases} . \tag{47}$$

Here the functions η_i, ζ_i are regular algebraic functions on $\tilde{U}_i$.

Moreover we will be able to explain the reason why the systems obtained as above are Hamiltonian systems as in (9). Our proof is geometrically clear. In fact we will be able to show that global vector field $\tilde{v}$ preserves the relative 2 forms $\omega_{\mathcal{S}}$ in (30) . This fact can be expressed as (cf. [STT])

$$\boxed{d_{\mathcal{S}/\mathcal{M}}(\tilde{v} \cdot (\omega_{\mathcal{S}} \wedge dt)) = 0,} \tag{48}$$

where $d_{\mathcal{S}/\mathcal{M}}$ is the relative differential with respect to the morphism $\mathcal{S} \longrightarrow \mathcal{M}$ and $\tilde{v} \cdot (\omega_{\mathcal{S}} \wedge dt)$ denotes the contraction of vector fields and 3-forms.

On an affine open subset $U \times M_R \times B_R$ with $U \simeq \mathbf{C}^2$ and coordinate systems (x, y) such that $\omega_{\mathcal{S}|\mathbf{C}^2} = dx \wedge dy$, the equation (48) implies that the differential equation obtained as above becomes a Hamiltonian system with a polynomial Hamiltonian.

We note that in the case of $R = \tilde{D}_7$, $\tilde{D}_8$, $\tilde{E}_7$ $S - D$ can not be covered by affine open sets all of which are isomorphic to $\mathbf{C}^2$. On such affine open subsets, the differential equation (47) can not be written in Hamiltonian systems globaly.

Furthermore, one can obtain explicit description of differential equations 47 by using the explicit affine coordinates (cf. [STT], [STe]).

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