MODULI OF STABLE PARABOLIC CONNECTIONS, RIEMANN-HILBERT CORRESPONDENCE AND GEOMETRY OF PAINLEVÉ EQUATION OF TYPE VI, PART I

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Dedicated to Professor Kyoichi Takano on his 60th birthday

In this paper, we will give a complete geometric background for the geometry of Painlevé VI and Garnier equations. By geometric invariant theory, we will construct a smooth fine moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ of stable parabolic connections on $\mathbf{P}^1$ with logarithmic poles at $D(\mathbf{t}) = t_1 + \cdots + t_n$ as well as its natural compactification. Moreover the moduli space $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ of Jordan equivalence classes of $SL_2(\mathbf{C})$-representations of the fundamental group $\pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}), *)$ are defined as the categorical quotient. We define the Riemann-Hilbert correspondence $\mathbf{RH} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ and prove that $\mathbf{RH}$ is a bimeromorphic proper surjective analytic map. Painlevé and Garnier equations can be derived from the isomonodromic flows and Painlevé property of these equations are easily derived from the properties of $\mathbf{RH}$. We also prove that the smooth parts of both moduli spaces have natural symplectic structures and $\mathbf{RH}$ is a symplectic resolution of singularities of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$, from which one can give geometric backgrounds for other interesting phenomena, like Hamiltonian structures, Bäcklund transformations, special solutions of these equations.

1. Introduction

1.1. The purpose.

The purpose of the series of papers is to give a complete geometric background for Painlevé equations of type VI or more generally for the so-called Garnier equations.

As is well-known, these nonlinear differential equations have the Painlevé property which means that *generic* solutions of these equations have no movable singularity except for poles so that solutions have the analytic continuations on whole of the universal covering of the space of time variables.

Besides the Painlevé property, there are several interesting phenomena related to these equations which have been investigated by many authors.

- Each of these equations can be written in a Hamiltonian system by a natural symplectic coordinate system ([Mal], [O3], [Iw1], [Iw2], [K], [ST]).
- These equations have natural parameters $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$. Moreover there exist birational symmetries of these equations, called *Bäcklund transformations of these equations*, which act on both of variables and the parameters and preserve the equations. ([O4]).
- In Painlevé VI case, the group of all Bäcklund transformations is isomorphic to the affine Weyl group $W(D_4^{(1)})$ of the type $D_4^{(1)}$. ([O4], [Sakai], [AL2], [NY], [IIS0]).

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- In Painlevé VI case, if $\boldsymbol{\lambda} \in \mathbf{C}^4$ lies on a reflection hyperplane of a reflection in $W(D_4^{(1)})$, then the corresponding equation has one parameter family of Riccati solutions. ([LY], [FA], [W], [STe], [SU]).
- A natural compactification of each space of initial conditions for P_{VI}, introduced by Okamoto [O1], can be obtained by a series of explicit blowing-ups of $\mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$ or $\mathbf{F}_2$. The compactification is given by a smooth projective rational surface S and it has a unique anti-canonical divisors $-K_S = Y$ such that $S \setminus Y_{red}$ is the space of initial conditions for P_{VI}. The pair (S, Y) becomes an Okamoto-Painlevé pair of type $D_4^{(1)}$ in the sense of [STT]. (See also [Sakai]).

Though these phenomena are discussed and investigated by many authors, the intrinsic mathematical background for these facts remains to be understood. Therefore, for example, it is worthwhile to ask the following fundamental questions:

- What is the geometric meaning of Painlevé property for these equations?
- What is the geometric meaning of the symplectic structure ?
- What is the geometric origin of Bäcklund transformations?
- Why Riccati solutions or some classical solutions appear for the parameters on the reflection hyperplanes of the Bäcklund transformations ?

In the series of the papers, the authors will give answers to these questions in a natural intrinsic framework.

1.2. **Natural Framework.**

It is already known (cf. [F], [Ga], [Sch], [JMU], [O3], [Iw1] and [Iw2]) that these equations can be derived from the isomonodromic deformation of the systems of linear equations of rank 2 with regular singularities over $\mathbf{P}^1$ or equivalently linear connections on vector bundles of rank 2 with logarithmic poles over $\mathbf{P}^1$. Although we will follow this line in this paper, for several essential reasons, we have to introduce a slight generalization of linear connections which will be explained as follows.

Let $n \geq 3$ and let us set $T_n = \{\mathbf{t} = (t_1, \dots, t_n) \in (\mathbf{P}^1_{\mathbf{C}})^n \mid t_i \neq t_j, (i \neq j)\}$, $\Lambda_n = \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n\}$. Fix a data $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$ and set $D(\mathbf{t}) = t_1 + \cdots + t_n$. We also fix a line bundle L on $\mathbf{P}^1_{\mathbf{C}}$ with a logarithmic connection $\nabla_L : L \longrightarrow L \otimes \Omega^1_{\mathbf{P}^1_{\mathbf{C}}}(D(\mathbf{t}))$.

A quadruple $(E, \nabla, \varphi, l = \{l_i\}_{i=1}^n)$ consisting of:

(1) a rank 2 vector bundle E on $\mathbf{P}^1$,
(2) a logarithmic connection $\nabla : E \longrightarrow E \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$
(3) a bundle isomorphism $\varphi : \wedge^2 E \xrightarrow{\simeq} L$ and
(4) one dimensional subspace l_i of the fiber E_{t_i} of E at t_i, $l_i \subset E_{t_i}$, $i = 1, \dots, n$,

is called a $(\mathbf{t}, \boldsymbol{\lambda})$-*parabolic connection* with the determinant (L, ∇_L) if they satisfy the following conditions:

(1) for any local sections s_1, s_2 of E,

$$(\varphi \otimes \mathrm{id})(\nabla s_1 \wedge s_2 + s_1 \wedge \nabla s_2) = \nabla_L(\varphi(s_1 \wedge s_2)),$$

(2) $l_i \subset \mathrm{Ker}(\mathsf{res}_{t_i}(\nabla) - \lambda_i)$, that is, λ_i is an eigenvalue of the residue $\mathsf{res}_{t_i}(\nabla)$ of ∇ at t_i and l_i is a corresponding one-dimensional eigensubspace of $\mathsf{res}_{t_i}(\nabla)$.

We introduce a series of rational numbers $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{2n})$ such that $0 \leq \alpha_1 < \ldots < \alpha_{2n} < 1$, which is called *a weight*. By using a weight $\boldsymbol{\alpha}$, one can define parabolic degrees for $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections (E, ∇, φ, l) and introduce the notion of the parabolic stability. Let $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ be the coarse moduli space of stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections on $\mathbf{P}^1$ with the determinant (L, ∇_L). Considering the relative setting over the parameter space $T_n \times \Lambda_n = \{(\mathbf{t}, \boldsymbol{\lambda})\}$, we can construct a family of moduli spaces

$$\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n \tag{1}$$

such that $\pi_n^{-1}(\mathbf{t}, \boldsymbol{\lambda}) \simeq M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$. Later, we have to extend the family by a finite étale covering $T_n' \longrightarrow T_n$, and for simplicity, we denote it also by $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n' \times \Lambda_n$.

Next, let us fix $\mathbf{t} \in T_n$ and consider a representation $\rho : \pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t}), *) \longrightarrow SL_2(\mathbf{C})$ of the fundamental group $\pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t}), *)$ with a fixed base point $* \in \mathbf{P}^1_{\mathbf{C}}$. Two representations ρ_1 and ρ_2 are said to be equivalent if there exists an element $P \in SL_2(\mathbf{C})$ such that $\rho_2 = P^{-1}\rho_1 P$. To each representation ρ, one can associate a local system $\mathbf{E}_\rho$ of rank 2 on $\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t})$ with an isomorphism $\wedge^2 \mathbf{E}_\rho \simeq \mathbf{C}_{\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t})}$. Moreover, two representations ρ_1 and ρ_2 are equivalent to each other if and only if $\mathbf{E}_{\rho_1}$ and $\mathbf{E}_{\rho_2}$ are isomorphic as local systems. Hence the moduli space of the isomorphism classes of local systems on $\mathbf{P}^1 \setminus D(\mathbf{t})$ with trivial determinants is isomorphic to the moduli space of equivalence classes of the representations.

Since $\pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t}), *)$ is a free group generated by γ_i for $1 \leq i \leq n-1$ where γ_i is a loop around the point t_i, such a representation can be determined by $M_i = \rho(\gamma_i) \in SL_2(\mathbf{C})$ for $1 \leq i \leq n-1$. Therefore the moduli space should be a quotient space of $SL_2(\mathbf{C})^{n-1}$ by a diagonal adjoint action of $SL_2(\mathbf{C})$.

However there is no canonical way to give a scheme structure on the set of equivalence classes of the representations. In this sense, we have to introduce a stronger equivalence relation. Two $SL_2(\mathbf{C})$-representations ρ_1 and ρ_2 of $\pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t}), *)$ are said to be *Jordan equivalent* if their semisimplifications are equivalent. This means that if a local system $\mathbf{E}_\rho$ is an extension of rank one local systems L_1 and L_2 one can not distinguish the extension classes. As is shown by Simpson [Sim2], the set of *the Jordan equivalence classes* of the local systems or representations is equal to the set of closed points of the categorical quotient

$$\mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) = SL_2(\mathbf{C})^{n-1}/Ad(SL_2(\mathbf{C})),$$

of $SL_2(\mathbf{C})^{n-1}$ by the diagonal adjoint action of $SL_2(\mathbf{C})$. The categorical quotients is defined as the affine scheme of the ring of invariant functions on $SL_2(\mathbf{C})^{n-1}$ by the action of $SL_2(\mathbf{C})$. (Cf. §4) .

Fixing the canonical generators γ_i ($1 \leq i \leq n$) of $\pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t}), *)$, to each representation ρ : $\pi_1(\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t}), *) \longrightarrow SL_2(\mathbf{C})$, we can associate n-algebraic functions on $SL_2(\mathbf{C})^{n-1}$

$$\mathrm{Tr}(\rho(\gamma_i)) = a_i, \quad \mathrm{Tr}(\rho((\gamma_1 \cdots \gamma_{n-1})^{-1})) = \mathrm{Tr}(\rho(\gamma_n)) = a_n$$

which are clearly invariant under the adjoint action. Setting $\mathcal{A}_n = \operatorname{Spec} \mathbf{C}[a_1, \ldots, a_n] \simeq \mathbf{C}^n$, we obtain a natural morphism

$$p_n : \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) \longrightarrow \mathcal{A}_n.$$

For a fixed closed point $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}_n$, let us denote by $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = p_n^{-1}(\mathbf{a})$ the closed fiber at $\mathbf{a}$, that is, we set

$$\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = \{\ [\rho] \in \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) \mid \operatorname{Tr}(\rho(\gamma_i)) = a_i,\ 1 \leq i \leq n\}.$$

Moreover, taking a finite étale covering $T'_n \longrightarrow T_n$ we can obtain a family of moduli spaces

$$\phi_n : \mathcal{R}_n \longrightarrow T'_n \times \mathcal{A}_n \tag{2}$$

such that $\phi_n^{-1}(\mathbf{t}, \mathbf{a}) = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ (cf. §4).

Now, we have obtained two kinds of moduli spaces $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ and $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ for fixed $(\mathbf{t}, \boldsymbol{\lambda}) \in T'_n \times \Lambda_n$ and $(\mathbf{t}, \mathbf{a}) \in T'_n \times \mathcal{A}_n$. Moreover we have two families of moduli spaces as in (1) and (2). (Note that we have already pulled back the family in (1) by the finite covering $T'_n \longrightarrow T_n$.)

Next, let us assume that eigenvalues of $\mathsf{res}_{t_i}(\nabla_L)$ are integers for all $1 \leq i \leq n$. Then we can define the *Riemann-Hilbert correspondence* $\mathbf{RH}_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow \mathcal{R}_n$ such that the following diagram commutes:

$$\begin{array}{ccc} M_n^{\boldsymbol{\alpha}}(L) & \xrightarrow{\mathbf{RH}_n} & \mathcal{R}_n \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \phi_n} \\ T'_n \times \Lambda_n & \xrightarrow{(1 \times \mu_n)} & T'_n \times \mathcal{A}_n. \end{array} \tag{3}$$

Here, the map $1 \times \mu_n$ in the bottom row in (3) is given by the map $(1 \times \mu_n)(\mathbf{t}, \boldsymbol{\lambda}) = (\mathbf{t}, \mathbf{a})$ where

$$\boxed{a_i = 2\cos 2\pi\lambda_i} \quad \text{for } 1 \leq i \leq n. \tag{4}$$

Under these relations, $\mathbf{RH}_n$ induces the analytic morphism of the fibers for each $(\mathbf{t}, \boldsymbol{\lambda}) \in T'_n \times \Lambda_n$:

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}. \tag{5}$$

To define the correspondence, take a stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection $(E, \nabla, \varphi, \{l_i\})$. Then restricting the connection ∇ to $\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t})$, define the local system on $\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t})$ by

$$\mathbf{E}(\nabla) := \ker \left(\nabla_{|\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t})}\right)^{an}. \tag{6}$$

(Here $\left(\nabla_{|\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t})}\right)^{an}$ denotes the analytic connection associated to $\nabla_{|\mathbf{P}^1_{\mathbf{C}} \setminus D(\mathbf{t})}$). Then it is easy to see that the map $(E, \nabla, \varphi, \{l_i\}) \mapsto \mathbf{E}(\nabla)$ induces the correspondence in (3) or (5). Basically, our framework for understanding the Painlevé or Garnier equations is the Riemann-Hilbert correspondences in (3) and (5).

There exists one more thing which we should mention here. Let β_1, β_2 be positive integers, $\boldsymbol{\alpha}' = (\alpha'_1, \ldots, \alpha'_{2n})$ a series of rational numbers with $0 \leq \alpha'_1 < \ldots < \alpha'_{2n} < 1$ and set $\boldsymbol{\beta} = (\beta_1, \beta_2)$. Setting $\boldsymbol{\alpha} = \boldsymbol{\alpha}' \frac{\beta_1}{\beta_1 + \beta_2}$, we obtain a weight $\boldsymbol{\alpha}$ for $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections. (Note that since $\alpha_{2n} = \alpha'_{2n} \frac{\beta_1}{\beta_1 + \beta_2} < \frac{\beta_1}{\beta_1 + \beta_2}$, this gives a restriction for the weight $\boldsymbol{\alpha}$). For the weight $\boldsymbol{\alpha}$, we consider the family of moduli spaces $M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T'_n \times \Lambda_n$. On the other hand, we will introduce the notion of $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic ϕ-connection which is a generalization of $\boldsymbol{\alpha}$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections. The moduli space $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L)$ contains the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ as a Zariski open set. Moreover we can construct the family of the moduli spaces such that the following diagram commutes:

$$\begin{array}{ccc} M_n^{\boldsymbol{\alpha}}(L) & \overset{\iota}{\hookrightarrow} & \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \overline{\pi_n}} \\ T_n' \times \Lambda_n & = & T_n' \times \Lambda_n. \end{array} \tag{7}$$

1.3. **Main Results.**

In the framework as above, we can state our main results in this paper as follows.

1.3.1. *Projectivity of the moduli space $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L)$, Smoothness, Irreducibility and the Symplectic Structure of $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$.* We first prove that the moduli space $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L)$ is a projective scheme. Moreover one can show that the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ for each $(\mathbf{t},\boldsymbol{\lambda}) \in T_n \times \Lambda_n$ is smooth and endowed with a natural intrinsic symplectic structure induced by Serre duality of tangent complexes. The irreduciblity of the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ for each $(\mathbf{t},\boldsymbol{\lambda}) \in T_n \times \Lambda_n$ follows from the irreduciblity of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ via the Riemann-Hilbert correspondece (5).

Theorem 1.1. (*Cf. Theorem 2.1, Theorem 5.2, Proposition 6.2 and Proposition 9.1*).

(1) *For a generic weight $(\boldsymbol{\alpha}',\boldsymbol{\beta})$, $\overline{\pi}_n : \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \longrightarrow T_n' \times \Lambda_n$ is a* projective *morphism. In particular, the moduli space $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L)$ is a* projective *algebraic scheme for all $(\mathbf{t},\boldsymbol{\lambda}) \in T_n' \times \Lambda_n$.*

(2) *For a generic weight $\boldsymbol{\alpha}$, $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n' \times \Lambda_n$ is a* smooth morphism *of relative dimension $2n-6$ with irreducible closed fibers. Therefore, the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ is a* smooth, irreducible *algebraic variety of dimension $2n-6$ for all $(\mathbf{t},\boldsymbol{\lambda}) \in T_n' \times \Lambda_n$.*

Theorem 1.2. (*Cf. Proposition 6.1*). *There exists a global relative 2-form*

$$\Omega \in H^0(M_n^{\boldsymbol{\alpha}}(L), \Omega^2_{M_n^{\boldsymbol{\alpha}}(L)/T_n \times \Lambda_n}). \tag{8}$$

which induces a symplectic structure on each fiber of π_n. Consequently, for each $(\mathbf{t},\boldsymbol{\lambda})$, the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ becomes a smooth symplectic algebraic variety.

1.3.2. *Irreducibility, symplectic structure and singularities of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$.*

Let us call a data $\boldsymbol{\lambda} \in \Lambda_n$ a set of local exponents of connections.

Definition 1.1. (1) A set of local exponents $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$ is said to be *special* if

(a) $\boldsymbol{\lambda}$ is *resonant*, that is, for some $1 \leq i \leq n$,

$$2\lambda_i \in \mathbf{Z}, \tag{9}$$

(b) or $\boldsymbol{\lambda}$ is *reducible*, that is, for some $(\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$

$$\sum_{i=1}^n \epsilon_i \lambda_i \in \mathbf{Z} \tag{10}$$

(2) If $\boldsymbol{\lambda} \in \Lambda_n$ is not special, $\boldsymbol{\lambda}$ is said to be *generic*.

(3) The data $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{A}_n$ is said to be *special* if $\mu_n(\boldsymbol{\lambda}) = \mathbf{a}$ for some special $\boldsymbol{\lambda} \in \Lambda_n$.

For a monodromy representation $\rho : \pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}), *) \longrightarrow SL_2(\mathbf{C})$, set $M_i = \rho(\gamma_i) \in SL_2(\mathbf{C})$ for $1 \leq i \leq n$. We consider the following conditions which are invariant under the adjoint action of $SL_2(\mathbf{C})$.

(11) The representation ρ is irreducible.

(12) For all $i, 1 \leq i \leq n$, the local monodromy matrix M_i around t_i is *not* equal to $\pm I_2$.

Theorem 1.3. (*Cf. Proposition 8.1, Proposition 6.3 and Theorem 7.1*). *Assume that* $n \geq 4$.

1. *For any* $\mathbf{a} \in \mathcal{A}_n$*, the moduli space* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ *is an irreducible affine scheme.*
2. *Let* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ *be the Zariski dense open subset of* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ *whose closed points satisfy the conditions (11) and (12). Then* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ *is smooth and there exists a natural symplectic form* Ω_1 *on* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$.
3. *The codimension of the locus* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{sing}_{\mathbf{a}} := \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \setminus \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ *is at least* 2.

1.3.3. *Surjectivity and Properness of the Riemann-Hilbert correspondence.*

Next, the most important result for the Riemann-Hilbert correspondence is the surjectivity and the properness. One can show that the correspondence $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ in (5) gives an analytic isomorphism between two moduli spaces if $\boldsymbol{\lambda} \in \Lambda_n$ is generic (i.e. non-special). However, for a special $\boldsymbol{\lambda} \in \Lambda_n$, one can see that the map (5) contracts some subvarieties of $M^{\boldsymbol{\alpha}}_n(\mathbf{t}, \boldsymbol{\lambda}, L)$ to singular locus of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$. Note that since the correspondence is not an algebraic morphism, one can not directly apply the valuative criterion for the proof of the properness.

Theorem 1.4. (*Cf. Theorem 7.1*). *Under the notation above and assume that* $n \geq 4$ *and* $\boldsymbol{\alpha}$ *is general. For all* $(\mathbf{t}, \boldsymbol{\lambda}) \in T'_n \times \Lambda_n$*, the Riemann-Hilbert correspondence*

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}} : M^{\boldsymbol{\alpha}}_n(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \tag{13}$$

is a bimeromorphic proper surjective morphism.

1.3.4. *The Riemann-Hilbert correspondence as a symplectic resolution of singularities of* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$.

Moreover, we can introduce the natural intrinsic symplectic structure on the smooth part $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ of the moduli spaces $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$. Together with the natural symplectic structure of the moduli space $M^{\boldsymbol{\alpha}}_n(\mathbf{t}, \boldsymbol{\lambda}, L)$, the map $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ gives a symplectic map, which means that the pullback of the symplectic structure on the smooth part of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ coincides with the symplectic structure on $M^{\boldsymbol{\alpha}}_n(\mathbf{t}, \boldsymbol{\lambda}, L)$. This identification will be given by a kind of infinitesimal Riemann-Hilbert correspondence (cf. Lemma 6.6). Together with the surjectivity, the properness of $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ and the fact that $M^{\boldsymbol{\alpha}}_n(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth, we can say that $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ gives an *analytic* symplectic resolution of the singularities of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$. Moreover, we can say that the map $\mathbf{RH}_n$ in (3) gives a *simultaneous resolution of the family* $\phi_n : \mathcal{R}_n \longrightarrow T'_n \times \mathcal{A}_n$ with the base extension $1 \times \mu_n : T'_n \times \Lambda_n \longrightarrow T'_n \times \mathcal{A}_n$. (For definition, see [Definition 4.26, [KM]]).

Theorem 1.5. (*Theorem 7.1, Lemma 6.6*). *Under the assumption of Theorem (1.4), we have the following.*

(1) *For any* $(\mathbf{t}, \boldsymbol{\lambda})$, *let* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ *be as in Theorem 1.3, and set* $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)^{\sharp} = \mathbf{RH}^{-1}_{\mathbf{t},\boldsymbol{\lambda}}(\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}})$. *Then the Riemann-Hilbert correspondence gives an analytic isomorphism*

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda},|M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)^{\sharp}} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)^{\sharp} \xrightarrow{\simeq} \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}. \tag{14}$$

(Note that if $\boldsymbol{\lambda}$ *is not special (cf. Definition 2.3, (36), (37)),* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}} = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$, *hence* $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ *gives an analytic isomorphism between* $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ *and* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$.*)*

(2) *The symplectic structures* Ω *restricted to* $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)^{\sharp}$ *and* Ω_1 *on* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ *can be identified with each other via* $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$, *that is,*

$$\Omega_{|M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)^{\sharp}} = \mathbf{RH}^{*}_{\mathbf{t},\boldsymbol{\lambda}|M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)^{\sharp}}(\Omega_1) \quad \text{on} \quad M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)^{\sharp}. \tag{15}$$

(3) *Putting together all results, the correspondence* $\mathbf{RH}_n$ *in (3) gives an analytic simultaneous symplectic resolution of singularities after the base extension* $1 \times \mu_n : T'_n \times \Lambda_n \to T'_n \times \mathcal{A}_n$.

1.4. **Painlevé and Garnier equations and their Painlevé property.**

In the framework of this paper, we can derive the Painlevé and Garnier equations as follows. Take the universal covering map $\tilde{T}_n \longrightarrow T'_n \longrightarrow T_n$ and pull back the diagram (3) to obtain the following commutative diagram:

$$\begin{array}{ccc} \tilde{M}_n^{\boldsymbol{\alpha}}(L) & \xrightarrow{\mathbf{RH}_n} & \tilde{\mathcal{R}}_n \\ {\scriptstyle \tilde{\pi}_n}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\phi}_n} \\ \tilde{T}_n \times \Lambda_n & \xrightarrow{(1\times\mu_n)} & \tilde{T}_n \times \mathcal{A}_n. \end{array} \tag{16}$$

1.4.1. *The case of generic* $\boldsymbol{\lambda}$.

Now let us fix $\boldsymbol{\lambda} \in \Lambda_n$ and set $\mathbf{a} = \mu_n(\boldsymbol{\lambda})$. First, assume that $\boldsymbol{\lambda}$ is *generic*. We denote by $(\pi_n)_{\boldsymbol{\lambda}} : \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L) \longrightarrow \tilde{T}_n$ and $(\phi_n)_{\mathbf{a}} : (\tilde{\mathcal{R}}_n)_{\mathbf{a}} \longrightarrow \tilde{T}_n$ the families obtained by restricting the families in (16) to $\tilde{T}_n \times \{\boldsymbol{\lambda}\}$ and $\tilde{T}_n \times \{\mathbf{a}\}$. Moreover we denote by $\mathbf{RH}_{\boldsymbol{\lambda}} : \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L) \longrightarrow (\tilde{\mathcal{R}}_n)_{\mathbf{a}}$ the restriction of $\mathbf{RH}_n$ to the restricted families. Since $\boldsymbol{\lambda}$ is generic, $\mathbf{RH}_{\boldsymbol{\lambda}}$ induces an analytic isomorphism between $\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)$ and $(\tilde{\mathcal{R}}_n)_{\mathbf{a}}$. Fix a point $\mathbf{t}_0 \in T'_n$. Since the original fibration $(\phi_n)_{\mathbf{a}} : (\mathcal{R}_n)_{\mathbf{a}} \longrightarrow T'_n \times \{\mathbf{a}\}$ is locally trivial, we can obtain an isomorphism $(\tilde{\mathcal{R}}_n)_{\mathbf{a}} \simeq \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}} \times \tilde{T}_n$ and the following commutative diagram for fixed $\boldsymbol{\lambda}$ and $\mathbf{a}$.

$$\begin{array}{ccc} \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L) & \underset{\simeq}{\xrightarrow{\mathbf{RH}_{\boldsymbol{\lambda}}}} & (\tilde{\mathcal{R}})_{\mathbf{a}} \simeq \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}} \times \tilde{T}_n \\ {\scriptstyle (\tilde{\pi}_n)_{\boldsymbol{\lambda}}}\big\downarrow & & \big\downarrow{\scriptstyle (\tilde{\phi}_n)_{\mathbf{a}}} \\ \tilde{T}_n \times \{\boldsymbol{\lambda}\} & \xrightarrow{=} & \tilde{T}_n \times \{\mathbf{a}\}. \end{array} \tag{17}$$

By using this global trivialization, for each closed point $\mathbf{x} \in \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}}$, we can define the unique constant section $s_{\mathbf{x}} : \tilde{T}_n \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}} \times \tilde{T}_n$ for $(\phi_n)_{\mathbf{a}}$ by the formula $s_{\mathbf{x}}(\mathbf{t}) = (\mathbf{x}, \mathbf{t})$. Pulling back this constant section $s_{\mathbf{x}}$ via $\mathbf{RH}_{\boldsymbol{\lambda}}$ we obtain the global analytic section $\tilde{s}_{\mathbf{x}}$ for the morphism $(\pi_n)_{\boldsymbol{\lambda}}$. Varying the initial points $\mathbf{x}$, we obtain the family of constant sections $\{s_{\mathbf{x}}\}_{\mathbf{x}\in\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}}}$ of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}} \times \tilde{T}_n \longrightarrow \tilde{T}_n$ and also the family of pullback sections $\{\tilde{s}_{\mathbf{x}}\}_{\mathbf{x}\in\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}}}$ for $\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)$.

The family of sections $\{\tilde{s}_{\mathbf{x}}\}_{\mathbf{x}\in\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}}}$ gives the splitting homomorphism

$$\tilde{v}_{\boldsymbol{\lambda}} : (\pi_n)^{*}_{\boldsymbol{\lambda}}(\Theta_{\tilde{T}_n \times \{\boldsymbol{\lambda}\}}) \longrightarrow \Theta_{\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)} \tag{18}$$

for the natural surjective homomorphism $\Theta_{\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)} \longrightarrow (\pi_n)^*_{\boldsymbol{\lambda}}(\Theta_{\tilde{T}_n \times \{\boldsymbol{\lambda}\}})$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L) & \xrightarrow{\tilde{u}} & M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L) \\
{\scriptstyle(\tilde{\pi}_n)_{\boldsymbol{\lambda}}}\downarrow & & \downarrow{\scriptstyle(\pi_n)_{\boldsymbol{\lambda}}} \\
\tilde{T}_n \times \{\boldsymbol{\lambda}\} & \xrightarrow[\text{universal covering}]{u} & T_n \times \{\boldsymbol{\lambda}\}.
\end{array} \tag{19}
$$

We can see that the splitting homomorphism (18) descends to a splitting homomorphism

$$
v_{\boldsymbol{\lambda}} : (\pi_n)^*_{\boldsymbol{\lambda}}(\Theta_{T_n \times \{\boldsymbol{\lambda}\}}) \longrightarrow \Theta_{M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)}. \tag{20}
$$

(One can show that this splitting is an algebraic homomorphism). Therefore, each algebraic vector field θ on $T_n \times \{\boldsymbol{\lambda}\}$ determines an algebraic vector field $v_{\boldsymbol{\lambda}}(\theta)$ on $M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)$. The natural generators of the tangent sheaf of $T_n \times \{\boldsymbol{\lambda}\}$ can be given by

$$
\langle \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n} \rangle.
$$

Defining

$$
v_i(\boldsymbol{\lambda}) = v_{\boldsymbol{\lambda}}(\frac{\partial}{\partial t_i}) \in H^0(M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L), \Theta_{M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)}) \tag{21}
$$

we obtain the differential system

$$
\langle v_1(\boldsymbol{\lambda}), \dots, v_n(\boldsymbol{\lambda}) \rangle \tag{22}
$$

on $M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)$. From the construction, it is obvious that these vector fields $\{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n}$ commute to each other, that is, the differential systems are integrable. Since $\tilde{u} : \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L) \longrightarrow M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)$ in (19) is also a covering map, each section $\tilde{s}_{\mathbf{x}} : \tilde{T}_n \times \{\boldsymbol{\lambda}\} \longrightarrow \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)$ defines a multi-section for $M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L) \longrightarrow T_n \times \{\boldsymbol{\lambda}\}$, which gives an integral submanifold of $M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)$ for the differential system (22) at least locally. Hence the submanifold $\tilde{s}_{\mathbf{x}}(\tilde{T}_n \times \{\boldsymbol{\lambda}\})$ of $\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)$ given by the image of the section $\tilde{s}_{\mathbf{x}}$ can be considered as the integral submanifold (or a solution submanifold) for (22) over the universal covering space $\tilde{T}_n$. (It is natural to call the submanifold $\tilde{s}_{\mathbf{x}}(\tilde{T}_n \times \{\boldsymbol{\lambda}\})$ an *isomonodromic flow*). Since the integral submanifold $\tilde{s}_{\mathbf{x}}(\tilde{T}_n \times \{\boldsymbol{\lambda}\})$ is isomorphic to the parameter space $\tilde{T}_n \times \{\boldsymbol{\lambda}\}$ and the morphism $\tilde{\pi}_n : \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L) \longrightarrow \tilde{T}_n \times \{\boldsymbol{\lambda}\}$ is algebraic, we can conculde that

$$
\boxed{\text{the diffrential system } \{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n} \text{ on } M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L) \text{ has Painlevé property. (See Figure 1).}} \tag{23}
$$

Actually, the dynamical system on $M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)$ detemined by $\{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n}$ has geometric Painlevé property in the sense of [IISA] (cf. [Definition 2.2, [IISA]]). The differential system $\{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n}$ in (22) is called *Painlevé VI system* for $n = 4$ and *Garnier system* for $n \geq 5$. (Moreover we call each vector field $v_i(\boldsymbol{\lambda})$ Painlevé or Garnier vector field).

By using a suitable algebraic local coordinate system for $M_n^{\alpha}(\boldsymbol{\lambda},L)$, one can write down the differential equations associated to $v_i(\boldsymbol{\lambda})$ and see that these differential systems are equivalent to known Painlevé VI systems and Garnier systems. (It is possible to reduce the number of the time variables t_i applying the automorphism of $\mathbf{P}^1_{\mathbf{C}}$ from n to $n-3$). Moreover, one can apply a standard argument to show that the vector fields $v_i(\boldsymbol{\lambda})$ are algebraic vector fields on $M_n^{\alpha}(\boldsymbol{\lambda}, L)$.

1.4.2. *The case of special* $\boldsymbol{\lambda}$.

Next, let us consider the case when $\boldsymbol{\lambda}$ is special. We have the same commutative diagram as (17), however we encounter the following new phenomena.

(1) Although the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is nonsingular, the moduli space $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}}$ has singularities.
(2) The Riemann-Hilbert correspondence $\mathbf{RH}_{\boldsymbol{\lambda}} : \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L) \longrightarrow (\tilde{\mathcal{R}}_n)_{\mathbf{a}}$ (or $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}} : \tilde{M}_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}}$ for a fixed $\mathbf{t}$) is still a bimeromorphic proper surjective map, but it contracts some families of compact subvarieties to singular locus of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}}$.

For example, in case when $n = 4$ (Painlevé VI case) and $\boldsymbol{\lambda}$ is special, $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ contains at least one (-2)-rational curve. For simplicity, assume that there is a unique (-2)-rational curve on $\tilde{M}_4^{\boldsymbol{\alpha}}(\mathbf{t}_0, \boldsymbol{\lambda}, L)$. Since $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})_{\mathbf{a}}$ is an irreducible affine scheme, it cannot contain complete subvarieties of positive dimension, and hence $\mathbf{RH}_{\mathbf{t}_0,\boldsymbol{\lambda}}$ has to contract the (-2)-rational curve onto a singular point of type A_1. (See Figure 2). Let us define the subset $\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)^{\sharp}$ the complement of the subvarieties contracted by $\mathbf{RH}_{\boldsymbol{\lambda}}$ in $\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)$ and set $(\tilde{\mathcal{R}}_n)^{\sharp}_{\mathbf{a}} := \mathbf{RH}_{\boldsymbol{\lambda}}(\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)^{\sharp})$ so that $\mathbf{RH}_{\boldsymbol{\lambda}|\tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)^{\sharp}} : \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)^{\sharp} \longrightarrow (\tilde{\mathcal{R}}_n)^{\sharp}_{\mathbf{a}}$ is an analytic isomorphism. For any $n \geq 4$, we can pull back the constant sections $s_{\mathbf{x}}$ by $\mathbf{RH}_{\boldsymbol{\lambda}}$ for $\mathbf{x} \in (\tilde{\mathcal{R}}_n)^{\sharp}_{\mathbf{a}}$ and obtain analytic sections $\tilde{s}_{\mathbf{x}}$, for $(\pi_n)_{\boldsymbol{\lambda}}$. Now consider the family $(\pi_n)_{\boldsymbol{\lambda}} : M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L) \longrightarrow T_n \times \{\boldsymbol{\lambda}\}$ over $T_n \times \{\boldsymbol{\lambda}\}$ and define $M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)^{\sharp} \subset M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)$ as above. Then we can also obtain mutually commutative Pailevé VI or Garnier vector fields $v_i(\boldsymbol{\lambda})$ for $1 \leq i \leq n$ on $M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)^{\sharp}$, and $\{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n}$ defines an integrable differential system on $M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L)^{\sharp}$. Varying $\boldsymbol{\lambda}$, we obtain the set of algebraic vector fields $\{v_i\}_{1 \leq i \leq n}$ on $M_n^{\boldsymbol{\alpha}}(L)^{\sharp}$ over $T_n \times \Lambda_n$. Since the codimension of $M_n^{\boldsymbol{\alpha}}(L) \setminus M_n^{\boldsymbol{\alpha}}(L)^{\sharp}$ in $M_n^{\boldsymbol{\alpha}}(L)$ is greater than 2, one can extend the algebraic vector field v_i to $M_n^{\boldsymbol{\alpha}}(L)$. Hence $v_i(\boldsymbol{\lambda})$ can also be extended to the total space of the family of the moduli spaces $(\pi_n)_{\boldsymbol{\lambda}} : M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L) \longrightarrow T_n \times \{\boldsymbol{\lambda}\}$. From the properness of the Riemann-Hilbert correspondence $\mathbf{RH}_{\boldsymbol{\lambda}} : \tilde{M}_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}, L) \longrightarrow (\tilde{\mathcal{R}}_n)_{\mathbf{a}}$, we can conclude that the differential system $\{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n}$ also has the geometric Painlvé property (cf. [IISA]).

The extended vector fields should be tangent to the family of contracted subvarieties (see Figure 2). The restriction of Painlevé VI or Garnier vector fields $\{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n}$ to the family of the contracted subvarieties yields integrable differential systems on the subvarieties whose solutions are given by a family of classical solutions like Riccati solutions for Painlevé VI system. For example, in the Painlevé VI case, we can observe the following correspondence (cf. [STe], [IISA]). (See [Iw4] or [IISA] for the meaning of the nonlinear monodromy group for Painlevé VI).

$$
\begin{array}{ccc}
M_4^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L) & & \mathcal{R}(\mathcal{P}_{4,\mathbf{t}})_{\mathbf{a}} \\
\boxed{(-2)\text{ rational curves in } M_4^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)} & \overset{\mathbf{RH}_{\boldsymbol{\lambda}}}{\Longleftrightarrow} & \boxed{\text{Rational double points on } \mathcal{R}(\mathcal{P}_{4,\mathbf{t}})_{\mathbf{a}}} \\
\Updownarrow & & \Updownarrow \\
\boxed{\text{Riccati solutions for } P_{VI}} & \Longleftrightarrow & \boxed{\text{Fixed points of the nonlinear monodromies}}
\end{array} \tag{24}
$$

In Garnier case ($n \geq 5$), when $\boldsymbol{\lambda}$ is reducible (10), one can obtain a special classical solution of the equation integrated by hypergeometric functions F_D of Lauricella (cf. [Proposition 1.7 [K]]). One can see that these classical solutions of Garnier systems $\mathcal{G}_n$ correspond to the subvariety isomorphic to $\mathbf{P}^{n-3}$ which parametrizes reducible stable parabolic connections. Moreover when $\boldsymbol{\lambda}$ is resonant (9), the Garnier system $\mathcal{G}_n$ degenerates into a Riccati system over a Garnier system $\mathcal{G}_{n-1}$. A subvariety which can be contracted by $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ is isomorphic to $\mathbf{P}^1$-bundle over $M_{n-1}^{\boldsymbol{\alpha}}(\mathbf{t}',\boldsymbol{\lambda}',L')$ at a generic point of the contracted subvatiety.

1.4.3. *Painlevé VI or Garnier equations parametrized by $\boldsymbol{\lambda} \in \Lambda_n$.*

In the above formulation, for each fixed local exponent $\boldsymbol{\lambda} \in \Lambda_n$, we obtain the Painlevé or Garnier vector fields $v_i(\boldsymbol{\lambda})$ for $i, 1 \leq i \leq n$ as in (21) such that $\{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n}$ forms an integrable differential system. Moreover the solution manifold for the differential system can be given by the isomonodromic flows. Varying the data $\boldsymbol{\lambda}$, we obtain vector fields

$$
\boxed{v_i \in H^0(M_n^{\boldsymbol{\alpha}}(L), \Theta_{M_n^{\boldsymbol{\alpha}}(L)/\Lambda_n}),\ 1 \leq i \leq n} \tag{25}
$$

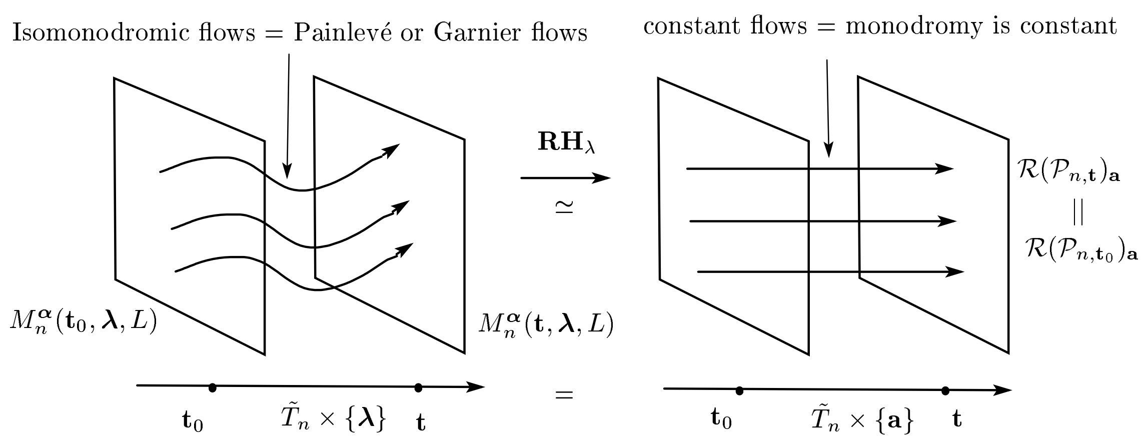
for $M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n$ such that $v_{i|M_n^{\boldsymbol{\alpha}}(\boldsymbol{\lambda},L)} = v_i(\boldsymbol{\lambda})$.

1.5. The Hamiltonian system.

It is well-known that the Painlevé and Garnier equations can be written in the Hamiltonian systems. Now we can explain this as follows. Since the constant flows on $(\phi_n)_{\mathbf{a}} : (\tilde{\mathcal{R}}_n)_{\mathbf{a}} \longrightarrow \tilde{T}_n$ preserve the natural symplectic form Ω_1 on the fiber $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ and the pullback of Ω_1 by Riemann-Hilbert correspondence coincides with the symplectic structure Ω on $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$, Painlevé or Garnier vector fields preserve the symplectic structure Ω. Therefore, we can write the differential equations in the Hamiltonian systems by using suitable cannonical coordinate systems. Then an argument shows that such vector fields are actually regular algebraic, hence the Hamiltonians are given by regular algebraic functions.

1.6. The relation of the space of initial conditions of Okamoto or Okamoto-Painlevé pairs for P_{VI}.

In the case of P_{VI}, Okamoto [O1] constructed the spaces of initial conditions by blowing up the accessible singularities of 4 parameter family of Painlevé VI equations. They are open algebraic surfaces which are complements of the anti-canonical divisors of projective rational surfaces obtained by the 8-point blowing-ups of $\mathbf{P}^1_{\mathbf{C}} \times \mathbf{P}^1_{\mathbf{C}}$ or $\mathbf{F}_2$. In [Sakai], [STT], the notion of the pairs of projective rational surfaces and



Isomonodromic Flows and Painlevé or Garnier Flows

FIGURE 1. Riemann-Hilbert correspondence and isomonodromic flows for generic $\boldsymbol{\lambda}$

its effective anti-canonical divisors with suitable conditions was introduced and its relation to Painlevé equation was revealed. In [STT], such a pair is called an *Okamoto–Painlevé pair.* Okamoto-Painlevé pairs of type $D_4^{(1)}$ correspond to Painlevé VI equations. A semiuniversal family of Okamoto–Painlevé pairs is a family of projective surfaces $\pi : \overline{\mathcal{S}} \longrightarrow T_4' \times \Lambda_4$ with the effective relative anticanonical divisor $\mathcal{Y}$ such that the configuration of the anticanonical divisor $\mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$ is of type $D_4^{(1)}$. Then family of spaces of the initial conditions of Okamoto can be obtained as an open subset $\mathcal{S} := \overline{\mathcal{S}} \setminus \mathcal{Y}$.

In the second part of this paper[IIS2], we will show that the family of Okamoto-Pailevé pairs $\overline{\mathcal{S}} \longrightarrow T_4 \times \Lambda_4$ can be identified with the family of the moduli spaces $\overline{M_4^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathcal{O}_{\mathbf{P}^1}(-1)) \longrightarrow T_4 \times \Lambda_4$, while $\mathcal{S} \longrightarrow T_4 \times \Lambda_4$ can be identified with $M_4^{\boldsymbol{\alpha}}(\mathcal{O}_{\mathbf{P}^1}(-1)) \longrightarrow T_4 \times \Lambda_4$. (In this case, we will take $\beta_1 = \beta_2 = 1$, hence $\boldsymbol{\alpha} = \boldsymbol{\alpha}' \frac{\beta_1}{\beta_1+\beta_2} = \boldsymbol{\alpha}'/2$). So our constructions of the moduli spaces give an intrinsic meaning of Okamoto's explicit hard calculations in [O1].

1.7. **The Bäcklund transformations–Symmetries of the equations.**

In our framework, Bäcklund transformations for the Painlevé equations or Garnier equations can be defined as follows. Consider the Painlevé VI or Garnier system $\{v_i\}_{1 \leq i \leq n}$ defined in (25) and the family of moduli spaces $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n' \times \Lambda_n$.

Definition 1.2. The pair $(\tilde{s}, s)$ of a birational map $\tilde{s} : M_n^{\boldsymbol{\alpha}}(L) \cdots \to M_n^{\boldsymbol{\alpha}}(L)$ (or $\tilde{s} : \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \cdots \to \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$) and an affine transformation $s : \Lambda_n \longrightarrow \Lambda_n$ is said to be a *Bäcklund transformation of the*

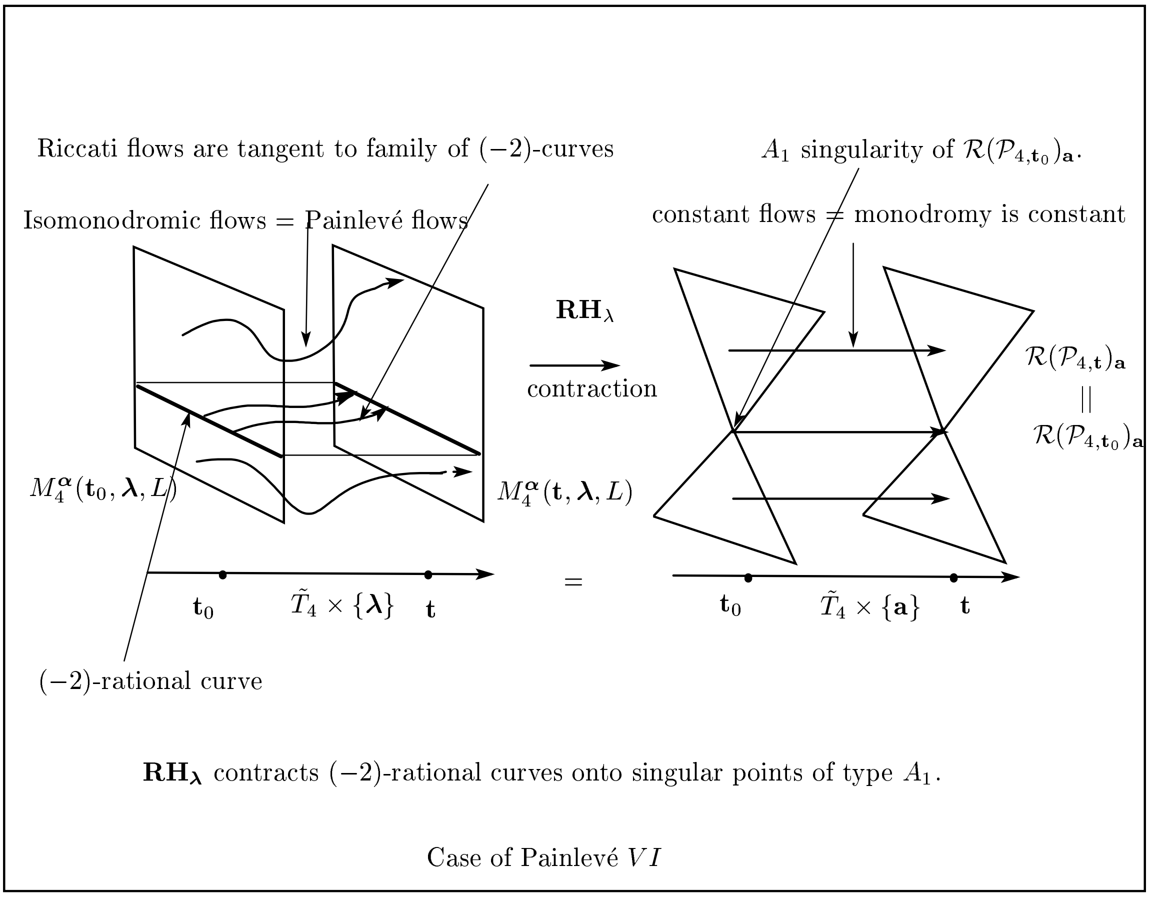


FIGURE 2. Riemann-Hilbert correspondence and isomonodromic flows for special $\boldsymbol{\lambda}$

differential system $\{v_i\}_{1 \leq i \leq n}$ or $\{v_i(\boldsymbol{\lambda})\}_{1 \leq i \leq n, \boldsymbol{\lambda} \in \Lambda_n}$ if they make the following diagram commutative:

$$
\begin{array}{ccc}
M_n^{\boldsymbol{\alpha}}(L) & \overset{\tilde{s}}{\cdots\rightarrow} & M_n^{\boldsymbol{\alpha}}(L) \\
\downarrow \pi_n & & \downarrow \pi_n \\
T_n' \times \Lambda_n & \overset{1 \times s}{\longrightarrow} & T_n' \times \Lambda_n,
\end{array}
\tag{26}
$$

and it satisfies the condition:

$$
\boxed{\tilde{s}_*(v_i) = v_i, \text{ or equivalently } \tilde{s}_*(v_i(\boldsymbol{\lambda})) = v_i(s(\boldsymbol{\lambda}))}\,. \tag{27}
$$

There exists a natural class of Bäcklund transformations of $M_n^{\boldsymbol{\alpha}}(L)$ for any $n \geq 4$ which are induced by *elementary transformations* of stable parabolic connections (cf. §3). Since such transformations induce the identity on the moduli space of the monodromy representations via Riemann-Hilbert correspondence, we can conclude that the transformations preserve the vector field as in (27). (This notion is equivalent to the rational gauge transformation or Schlesinger transformation of connections). In §3, we will list up these kinds of Bäcklund transformations.

As for Painlevé VI equations, the group of the Bäcklund transformation in the above sense is isomorphic to the affine Weyl group $W(D_4^{(1)})$ of type $D_4^{(1)}$, (cf. e.g. [O4], [IIS0]). The affine Weyl group $W(D_4^{(1)})$ is generated by 5 reflections s_i, $i = 0, 1, \dots, 4$ corresponding to the simple roots in the Dynkin diagram of type $D_4^{(1)}$ (see Figure 3). A natural faithful affine action of $W(D_4^{(1)})$ to $\Lambda_4 = \mathbf{C}^4 \ni \boldsymbol{\lambda} = (\lambda_j)$

can be given by

$$
\begin{array}{rcl} s_i(\lambda_j) & = & (-1)^{\delta_{ij}}\lambda_j, \quad i = 1, \dots, 4 \\ s_0(\lambda_i) & = & \lambda_i - \frac{1}{2}\left(\sum_{k=1}^4 \lambda_k\right) + \frac{1}{2}. \end{array} \tag{28}
$$

Recalling the identification of the family $\overline{M_4^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathcal{O}_{\mathbf{P}^1}(-1)) \longrightarrow T_4' \times \Lambda_4$ with the family of Okamoto-Painlevé pairs $\overline{\mathcal{S}} \longrightarrow T_4' \times \Lambda_4$, one can see that the actions of $W(D_4^{(1)})$ in (28) can be lifted to birational actions of the total space of the family $\overline{\pi} : \overline{\mathcal{S}} \longrightarrow T_4' \times \Lambda$, that is, for each $s \in W(D_4^{(1)})$, there exists a commutative diagram

$$
\begin{array}{ccc} \overline{\mathcal{S}} & \overset{\tilde{s}}{\cdots\rightarrow} & \overline{\mathcal{S}} \\ \downarrow \overline{\pi} & & \downarrow \overline{\pi} \\ T_4' \times \Lambda_4 & \overset{1\times s}{\longrightarrow} & T_4' \times \Lambda_4. \end{array} \tag{29}
$$

Moreover it is known [O4] that the actions preserve the Painlevé vector field v_i in (25). That is, for each $s \in W(D_4^{(1)})$, we have

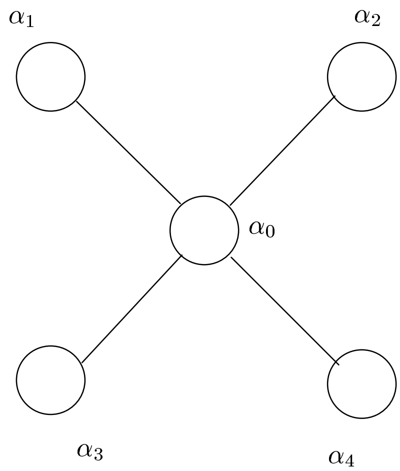
$$
\tilde{s}_*(v_i(\boldsymbol{\lambda})) = v_i(s(\boldsymbol{\lambda})) \quad \text{for} \quad 1 \leq i \leq 4. \tag{30}
$$

In our framework, it is easy to give an intrinsic reason why $\tilde{s_i}$ for $1 \leq i \leq 4$ preserve the vector field. It is simply because these come from elementary transformations. However, the origin of the transformation $\tilde{s_0}$ is still mysterious, and we cannot see any simple reason why $\tilde{s}_0$ preserves the vector field.

As some experts suggested to us, it may be plausible to believe that $\tilde{s_0}$ is induced by Laplace transformations of the stable connection. (The authors were informed by H. Sakai that M. Mazzocco gives some explanations for this fact on this line). For simplicity, let us call the Laplace transform of the original connection the *dual* of the connection. In general, the dual of logarithmic connections of rank 2 becomes a connection of higher rank which may not be logarithmic, so it is not so easy to identify the dual of the connection to the original one. Only in the case of $n = 4$ (Painlevé case), we may miraculously identify the original connection with its dual or a further transformed object, so we have the extra Bäcklund transformation like $\tilde{s}_0$. It may be reasonable to consider the original connection and its dual at once. Then we may include the Laplace transformation as a part of the Bäcklund transformations.

After we have finished the first version of this paper, Philip Boalch informed us that he can obtain $\tilde{s}_0$ using the method of [Boa] as follows. One can embed a rank 2 connection with 4-regular singular points over $\mathbf{P}^1$ into a rank 3 reducible connection. Then there is a simple operation for shifting the eigenvalues of the rank 3 connection. For a special value of shifting, one can obtain a rank 2 subconnection or a rank 1 subconnection in the shifted rank 3 connection, then take the rank 2 connection or the quotient of rank 1 subconnection. This gives a transformation from a rank 2 connection to another rank 2 connection whose transformation on $\boldsymbol{\lambda}$ gives s_0. Note that this transformation only works for the case of $n = 4$. By using this result and another result in [Boa], he also gave a different proof of a result in [IIS0].

Besides these stories, we should mention about the relation of the birational geometry and the Bäcklund transformations. As Saito and Umemura pointed out in [SU], Bäcklund transformations of Painlevé



Root system $D_4^{(1)}$

FIGURE 3. Dynkin diagram $D_4^{(1)}$

equations which are reflections with respect to roots of an affine root system are nothing but *flops* corresponding to (-2)-rational curves in Okamoto spaces of initial conditions.

From the definition of elementary transformations, we can easily see that the locus of indeterminacy of birational transformations correspond to the subvarieties which are contracted by the Riemann-Hilbert correspondence. Since the Riemann-Hilbert correspondence gives a simultaneous symplectic resolution of the singularities of the family $\phi_n : \mathcal{R}_n \longrightarrow T'_n \times \mathcal{A}_n$, it is now obvious that those Bäcklund transformations are flops. (For definition and fundamental facts on flops, see [§6,1 [KM]]).

1.7.1. *Bäcklund transformations and the Riemann-Hilbert correspondence.*

In [IIS0], we have proved that all of the Bäcklund transformations in $W(D_4^{(1)})$ on $\overline{M_4^{\boldsymbol{\alpha}}}(L) \longrightarrow T'_4 \times \Lambda_4$ induce essentially identity on the moduli space $\mathcal{R}_4$ after we take a finite quotient of $\mathcal{A}_n$. (Note that this is nontrivial only for $\tilde{s_0}$). Therefore in this sense, the group of the Bäcklund transformations $W(D_4^{(1)})$ can be considered as the Galois group of the monodromy representations.

1.8. **Related works.**

It is worthwhile to discuss about some works related to this paper and to clarify what are really new in this paper.

The notion of $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ is essentially introduced by Arinkin-Lysenko in [AL1] as a quasiparabolic SL_2-bundle. In [AL1], they also discussed about the moduli problem for quasiparabolic SL_2-bundles and consider the moduli space as an algebraic stack. In the case of $n = 4$, under the assumption that $\boldsymbol{\lambda}$ is generic (cf. Definition 1.1), they give an explicit description of the coarse moduli space. Moreover, in [AL2], by using the explicit descriptions of the moduli spaces, they describe the group of automorphisms of the family of moduli spaces by using an explicit geometry of surfaces. Later, in [A], Arinkin introduced a notion of ϵ-bundle, generalizing Deligne-Simpson's τ-connections in [Sim1]. Again under the assumption that $\boldsymbol{\lambda}$ is generic Arinkin gives a compactification of the moduli space of quasiparabolic SL_2-bundles. Although the basic notions are introduced in their works, from the viewpoint of geometric background for Painlevé or Garnier equations, it is really necessary to construct the moduli spaces even for special $\boldsymbol{\lambda}$. For example, as we pointed out in 1.7 (cf. [SU]), some Bäcklund transformations of these equations are induced by flops in the terminology of the modern birational geometry and the center of flops are lying over the special parameter $\boldsymbol{\lambda}$.

In this sense, the advantage of introducing the notion of the stability for $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection is obvious. In the GIT setting ([Mum]), despite considerable careful computations of stability, we can construct the fine moduli space of stable objects as smooth irreducible schemes even for special $\boldsymbol{\lambda}$. Moreover, we introduce the notion of parabolic ϕ-connections which is a generalization of the notion of ϵ-connections due to Arinkin-Deligne-Simpson and define the stability for them. One can understand the powers of these notions in Theorem 2.1 and Theorem 2.2.

The construction of the family of moduli spaces in (2) of $SL_2(\mathbf{C})$-monodromy representations are essentially due to Simpson [Sim2]. However a systematic treatment of nonlinear monodromies of the braid group is given by Dubrovin and Mazzocco [DM] for a special case of $n = 4$, and by Iwasaki [Iw3], [Iw4] for the general case of $n = 4$, and our construction of the family in this paper is taking care of the action of nonlinear monodromies of the braid groups to the moduli spaces.

Next, we would like to emphasize that only after we establish the natural setting in 1.2 it becomes possible to give a precise formulation of Hilbert's 21th problem for these cases. In our setting, the affirmative answer to the problem is equivalent to the surjectivity of the Riemann-Hilbert correspondences $\mathbf{RH}_n$ and $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ in (3) and (5). As one can imagine easily, only reasonable result which one can apply to proof of the surjectivity is Deligne's theorem in [Del70].

Moreover the properness of $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ is also a new result. In the process of the proof, we need some analysis of the contraction induced by $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ and a technical lemma due to Professor A. Fujiki. The symplectic nature of the moduli spaces is discussed by many authors. (See for example [Go], [Iw1] [Iw2]). Iwasaki gave intrinsic symplectic structures on the moduli spaces of irreducible logarithmic connections on a nonsingular complete curve and show that they are obtained as the pullback of the symplectic structures on the moduli space of the irreducible representations.

Again, in this paper, we extended the symplectic structure to the whole moduli space of the stable parabolic connections and the smooth part of the moduli of the representations. Then one can show that these symplectic structures are identified via $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$. Our proof here is based on some complexes of sheaves whose hypercohomologies describe the tangent spaces to the moduli spaces. Together with the surjectivity and the properness of $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ these results can be understood as $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ gives an analytic symplectic resolution of singularities in the sense of [Bea]. These kinds of viewpoints seem to be new, and this gives a clear explanation that a simple reflection in the group of Bäcklund transformations is nothing but a flop with respect to this resolution.

The derivation of Painlevé equations from the isomonodromic deformation of the linear connections is well-known. (See for example [JMU], [JM] and [O3]). However in most cases, one first takes a normalized linear connection written in certain coordinate systems and then writes up the Painlevé equations as the compatibility conditions for the extended linear connections. For a normalization, one has to assume that the vector bundle E of rank 2 and degree 0 is isomorphic to $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}$, which is not true in general. In fact, the natural subscheme

$$Z_{\mathbf{t}} = \{(E, \nabla, \varphi, l) \in \mathcal{M}_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L); E \not\simeq \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}\}. \tag{31}$$

of $\mathcal{M}_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is a non-empty divisor. We note that the isomonodromic flow starting from some point $p \in \mathcal{M}_n^{\boldsymbol{\alpha}}(\mathbf{t}_0, \boldsymbol{\lambda}, L) \setminus Z_{\mathbf{t}_0}$ does not stay inside the open subset $\bigcup_{\mathbf{t}} (\mathcal{M}_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus Z_{\mathbf{t}})$, that is, the flow intersects with $Z_{\mathbf{t}}$ for some $\mathbf{t}$. Therefore, in order to prove the Painlevé property of Painlevé VI or Garnier equations completely, we have to consider the whole space $\mathcal{M}_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ and the properness of the Riemann-Hilbert correspondence is essential for our proof of Painlevé property. (For a discussion of various definitions of Painlevé property, see [IISA]. Moreover, for some proofs of analytic Painlevé property of isomonodromic deformations, see [Mal] and [Miw]). Moreover most former approaches avoid dealing with the case when $\boldsymbol{\lambda}$ is special, because one has to introduce the notion of the stability of the parabolic connections to obtain a good moduli space which is smooth and Hausdorff.

In our framework, we can also discuss the Painlevé or Garnier equations for special $\boldsymbol{\lambda}$ in a natural framework. Interestingly enough, the classical solutions for these equations can be derived from the family of subvarieties contracted by $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$. Now, the geometric meaning of these facts becomes very clear. (For more detailed treatment in Painlevé VI case, see [W], [STe] and [SU]).

We should mention that Nakajima [N] obtained a smooth moduli space of stable parabolic connections as the moduli space of filtered regular D-modules by the technique of the hyper-Kähler quotients of moment maps. Then he showed that the moduli space is diffeomorphic to the moduli space of parabolic Higgs bundles. Nitsure [Ni] also constructed the moduli space of the stable logarithmic connection without parabolic structures in GIT setting.

Contents

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2. Moduli spaces of stable parabolic connections on $\mathbf{P}^1$ and their compactifications.

2.1. **Parabolic connections on $\mathbf{P}^1$.** Let $n \geq 3$ and set

$$T_n = \{(t_1, \dots, t_n) \in (\mathbf{P}^1)^n \quad | \quad t_i \neq t_j, (i \neq j)\}, \tag{32}$$

$$\Lambda_n = \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n\}. \tag{33}$$

Fixing a data $(\mathbf{t}, \boldsymbol{\lambda}) = (t_1, \dots, t_n, \lambda_1, \dots, \lambda_n) \in T_n \times \Lambda_n$, we define a reduced divisor on $\mathbf{P}^1$ as

$$D(\mathbf{t}) = t_1 + \cdots + t_n. \tag{34}$$

Moreover we fix a line bundle L on $\mathbf{P}^1$ with a logarithmic connection $\nabla_L : L \longrightarrow L \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$.

Definition 2.1. A (rank 2) $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ with the determinant (L, ∇_L) is a quadruplet $(E, \nabla, \varphi, \{l_i\}_{1 \leq i \leq n})$ which consists of

(1) a rank 2 vector bundle E on $\mathbf{P}^1$,
(2) a logarithmic connection $\nabla : E \longrightarrow E \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$

(3) a bundle isomorphism $\varphi : \wedge^2 E \xrightarrow{\simeq} L$

(4) one dimensional subspace l_i of the fiber E_{t_i} of E at t_i, $l_i \subset E_{t_i}$, $i = 1, \dots, n$, such that

(a) for any local sections s_1, s_2 of E,

$$\varphi \otimes id(\nabla s_1 \wedge s_2 + s_1 \wedge \nabla s_2) = \nabla_L(\varphi(s_1 \wedge s_2)),$$

(b) $l_i \subset \mathrm{Ker}(\mathsf{res}_{t_i}(\nabla) - \lambda_i)$, that is, λ_i is aneigenvalue of the residue $\mathsf{res}_{t_i}(\nabla)$ of ∇ at t_i and l_i is a one-dimensional eigensubspace of $\mathsf{res}_{t_i}(\nabla)$.

Definition 2.2. Two $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections $(E_1, \nabla_1, \varphi, \{l_i\}_{1 \le i \le n})$ $(E_2, \nabla_2, \varphi', \{l'_i\}_{1 \le i \le n})$ on $\mathbf{P}^1$ with the determinant (L, ∇_L) are isomorphic to each other if there is an isomorphism $\sigma : E_1 \xrightarrow{\sim} E_2$ and $c \in \mathbf{C}^\times$ such that the diagrams

$$\begin{array}{ccc} E_1 & \xrightarrow{\nabla_1} & E_1 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \\ \sigma \downarrow \cong & & \cong \downarrow \sigma\otimes\mathrm{id} \\ E_2 & \xrightarrow{\nabla_2} & E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \end{array} \qquad \begin{array}{ccc} \bigwedge^2 E_1 & \xrightarrow[\cong]{\varphi} & L \\ \wedge^2\sigma \downarrow \cong & & c \downarrow \cong \\ \bigwedge^2 E_2 & \xrightarrow[\cong]{\varphi'} & L \end{array} \tag{35}$$

commute and $(\sigma)_{t_i}(l_i) = l'_i$ for $i = 1, \dots, n$.

2.2. **The set of local exponents $\boldsymbol{\lambda} \in \Lambda_n$.** Note that a data $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Lambda_n \simeq \mathbf{C}^n$ specifies the set of eigenvalues of the residue matrix of a connection ∇ at $\mathbf{t} = (t_1, \dots, t_n)$, which will be called a set of *local exponents* of ∇.

Definition 2.3. A set of local exponents $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$ is called *special* if

(1) $\boldsymbol{\lambda}$ is *resonant*, that is, for some $1 \le i \le n$,

$$2\lambda_i \in \mathbf{Z}, \tag{36}$$

(2) or $\boldsymbol{\lambda}$ is *reducible*, that is, for some $(\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$

$$\sum_{i=1}^n \epsilon_i \lambda_i \in \mathbf{Z}. \tag{37}$$

If $\boldsymbol{\lambda} \in \Lambda_n$ is not special, $\boldsymbol{\lambda}$ is said to be *generic*.

Lemma 2.1. *Let $(E, \nabla, \varphi, l = \{l_i\})$ be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ with the determinant (L, ∇_L). Assume that eigenvalues of $\mathsf{res}_{t_i}(\nabla_L)$ are integers for $1 \le i \le n$. Suppose that there exists a subline bundle $F \subset E$ such that $\nabla F \subset F \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$. Then $\boldsymbol{\lambda}$ is reducible, that is, $\boldsymbol{\lambda}$ satisfies the condition (37).*

Proof. (Cf. [Proposition 1.1, [AL1]]). Since we have a horizontal bundle isomorphism $\varphi : \wedge^2 E \simeq L$ with respect to the connections, the eigenvalues of the residue matrix $\mathsf{res}_{t_i} \nabla$ at t_i are given by λ_i and $\mathsf{res}_{t_i}(\nabla_L) - \lambda_i$. Since $\nabla F \subset F \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$, the subspace $F_{t_i} \subset E_{t_i}$ is an eigenspace of $\mathsf{res}_{t_i}(\nabla)$. Therefore the eigenvalue of $\mathsf{res}_{t_i}(\nabla_{|F})$ is congruent to $\epsilon_i \lambda_i$ modulo $\mathbf{Z}$ for $\epsilon_i = 1$ or -1. The residue theorem says that

$$\sum_{i=1}^n \mathsf{res}_{t_i}(\nabla_{|F}) \equiv -\deg F \equiv 0 \mod \mathbf{Z}$$

hence we have the lemma. Λ

Remark 2.1. *For $n=4$, the data $\boldsymbol{\lambda} \in \Lambda_4$ is special if and only if $\boldsymbol{\lambda} \in \Lambda_4$ lies on a reflection hyperplane of a reflection $s \in W(D_4^{(1)})$.*

2.3. **Parabolic degrees.** Let us fix a series of positive rational numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{2n})$, which is called *a weight*, such that

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_i < \cdots < \alpha_{2n} < \alpha_{2n+1} = 1. \tag{38}$$

For a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ with the determinant (L, ∇_L), we can define the parabolic degree of (E, ∇, φ, l) with respect to the weight $\boldsymbol{\alpha}$ by

$$\begin{aligned} \operatorname{pardeg}_{\boldsymbol{\alpha}} E = \operatorname{pardeg}_{\boldsymbol{\alpha}}(E, \nabla, \varphi, l) &= \deg E + \sum_{i=1}^{n} (\alpha_{2i-1} \dim E_{t_i}/l_i + \alpha_{2i} \dim l_i) \\ &= \deg L + \sum_{i=1}^{n} (\alpha_{2i-1} + \alpha_{2i}). \end{aligned} \tag{39}$$

Let $F \subset E$ be a rank 1 subbundle of E such that $\nabla F \subset F \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$. We define the parabolic degree of $(F, \nabla_{|F})$ by

$$\operatorname{pardeg}_{\boldsymbol{\alpha}} F = \deg F + \sum_{i=1}^{n} (\alpha_{2i-1} \dim F_{t_i}/l_i \cap F_{t_i} + \alpha_{2i} \dim l_i \cap F_{t_i}) \tag{40}$$

Definition 2.4. Fix a weight $\boldsymbol{\alpha}$. A $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection (E, ∇, φ, l) on $\mathbf{P}^1$ with the determinant (L, ∇_L) is said to be *$\boldsymbol{\alpha}$-stable* (resp. *$\boldsymbol{\alpha}$-semistable*) if for every rank-1 subbundle F with $\nabla(F) \subset F \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$

$$\operatorname{pardeg}_{\boldsymbol{\alpha}} F < \frac{\operatorname{pardeg}_{\boldsymbol{\alpha}} E}{2}, \quad (\text{resp.} \operatorname{pardeg}_{\boldsymbol{\alpha}} F \leq \frac{\operatorname{pardeg}_{\boldsymbol{\alpha}} E}{2}). \tag{41}$$

(For simplicity, "$\boldsymbol{\alpha}$-stable" will be abbreviated to "stable").

We define the coarse moduli space by

$$M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) = \left\{ (E, \nabla, \varphi, l); \begin{array}{l} \text{an } \boldsymbol{\alpha}\text{-stable } (\mathbf{t}, \boldsymbol{\lambda})\text{-parabolic connection} \\ \text{with the determinant } (L, \nabla_L) \end{array} \right\} /\text{isom}. \tag{42}$$

2.4. **Stable parabolic ϕ-connections.** If $n \geq 4$, the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ never becomes projective nor complete. In order to obtain a compactification of the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$, we will introduce the notion of a stable parabolic ϕ-connection, or equivalently, a stable parabolic Λ-triple. Again, let us fix $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n \times \Lambda_n$ and a line bundle L on $\mathbf{P}^1$ with a connection $\nabla_L : L \to L \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$.

Definition 2.5. The data $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^n)$ is said to be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic ϕ-connection of rank 2 with the determinant (L, ∇_L) if E_1, E_2 are rank 2 vector bundles on $\mathbf{P}^1$ with $\deg E_1 = \deg L$, $\phi : E_1 \to E_2$, $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ are morphisms of sheaves, $\varphi : \bigwedge^2 E_2 \xrightarrow{\sim} L$ is an isomorphism and $l_i \subset (E_1)_{t_i}$ are one dimensional subspaces for $i = 1, \dots, n$ such that

1. $\phi(fa) = f\phi(a)$ and $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^1}$, $a \in E_1$,
2. $(\varphi \otimes \mathrm{id})(\nabla(s_1) \wedge \phi(s_2) + \phi(s_1) \wedge \nabla(s_2)) = \nabla_L(\varphi(\phi(s_1) \wedge \phi(s_2)))$ for $s_1, s_2 \in E_1$ and
3. $(\mathsf{res}_{t_i}(\nabla) - \lambda_i \phi_{t_i})|_{l_i} = 0$ for $i = 1, \dots, n$.

Definition 2.6.

Two $(\mathbf{t},\boldsymbol{\lambda})$ parabolic ϕ-connections $(E_1,E_2,\phi,\nabla,\varphi,\{l_i\})$, $(E'_1,E'_2,\phi',\nabla',\varphi',\{l'_i\})$ are said to be isomorphic to each other if there are isomorphisms $\sigma_1 : E_1 \xrightarrow{\sim} E'_1$, $\sigma_2 : E_2 \xrightarrow{\sim} E'_2$ and $c \in \mathbf{C}\setminus\{0\}$ such that the diagrams

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \sigma_1 \downarrow \cong & & \cong \downarrow \sigma_2 \\ E'_1 & \xrightarrow{\phi'} & E'_2 \end{array} \qquad \begin{array}{ccc} E_1 & \xrightarrow{\nabla} & E_2\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \\ \sigma_1 \downarrow \cong & & \cong \downarrow \sigma_2\otimes\mathrm{id} \\ E'_1 & \xrightarrow{\nabla'} & E'_2\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \end{array} \qquad \begin{array}{ccc} \bigwedge^2 E_2 & \underset{\cong}{\xrightarrow{\varphi}} & L \\ \wedge^2\sigma_2 \downarrow \cong & & c \downarrow \cong \\ \bigwedge^2 E'_2 & \underset{\cong}{\xrightarrow{\varphi'}} & L \end{array}$$

commute and $(\sigma_1)_{t_i}(l_i) = l'_i$ for $i=1,\dots,n$.

Remark 2.2. Assume that two vector bundles E_1, E_2 and morphisms $\phi : E_1 \to E_2$, $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ satisfying $\phi(fa) = f\phi(a)$, $\nabla(fa) = \phi(a)\otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^1}$, $a \in E_1$ are given. If ϕ is an isomorphism, then $(\phi\otimes\mathrm{id})^{-1}\circ\nabla : E_1 \to E_1\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ becomes a connection on E_1.

Fix rational numbers $\alpha'_1,\alpha'_2,\dots,\alpha'_{2n},\alpha'_{2n+1}$ satisfying

$$0 \le \alpha'_1 < \alpha'_2 < \cdots < \alpha'_{2n} < \alpha'_{2n+1} = 1$$

and positive integers β_1,β_2. Setting $\boldsymbol{\alpha}' = (\alpha'_1,\dots,\alpha'_{2n}), \boldsymbol{\beta} = (\beta_1,\beta_2)$, we obtain a *weight* $(\boldsymbol{\alpha}',\boldsymbol{\beta})$ for parabolic ϕ-connections.

Definition 2.7. Fix a sufficiently large integer γ. A parabolic ϕ-connection $(E_1,E_2,\phi,\nabla,\varphi,\{l_i\}_{i=1}^n)$ is said to be $(\boldsymbol{\alpha}',\boldsymbol{\beta})$-stable (resp. $(\boldsymbol{\alpha}',\boldsymbol{\beta})$-semistable) if for any subbundles $F_1 \subset E_1$, $F_2 \subset E_2$ satisfying $\phi(F_1)\subset F_2$, $\nabla(F_1)\subset F_2\otimes\Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ and $(F_1,F_2)\neq(E_1,E_2),(0,0)$, the inequality

$$\frac{\beta_1(\deg F_1(-D(\mathbf{t}))) + \beta_2(\deg F_2 - \gamma\,\mathrm{rank}(F_2)) + \sum_{i=1}^n \beta_1(\alpha'_{2i-1}d_{2i-1}(F_1) + \alpha'_{2i}d_{2i}(F_1))}{\beta_1\,\mathrm{rank}(F_1) + \beta_2\,\mathrm{rank}(F_2)}$$
$$\underset{(\text{resp. } \le)}{<} \frac{\beta_1(\deg E_1(-D(\mathbf{t}))) + \beta_2(\deg E_2 - \gamma\,\mathrm{rank}(E_2)) + \sum_{i=1}^n \beta_1(\alpha'_{2i-1}d_{2i-1}(E_1) + \alpha'_{2i}d_{2i}(E_1))}{\beta_1\,\mathrm{rank}(E_1) + \beta_2\,\mathrm{rank}(E_2)}$$

holds, where $d_{2i-1}(F) = \dim((F_1)_{t_i}/l_i\cap(F_1)_{t_i})$, $d_{2i}(F_1) = \dim((F_1)_{t_i}\cap l_i)$, $d_{2i-1}(E_1) = \dim((E_1)_{t_i}/l_i)(= 1)$ and $d_{2i}(E_1) = \dim l_i (= 1)$.

Define the coarse moduli space by

$$\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L) := \left\{ (E_1,E_2,\phi,\nabla,\varphi,\{l_i\});\ \begin{array}{l}\text{a } (\boldsymbol{\alpha}',\boldsymbol{\beta})\text{-stable } (\mathbf{t},\boldsymbol{\lambda})\text{-parabolic } \phi\text{-connection} \\ \text{with the determinant } (L,\nabla_L)\end{array} \right\} /\text{isom.} \tag{43}$$

For a given weight $(\boldsymbol{\alpha}',\boldsymbol{\beta})$ and $1 \le i \le 2n$, define a rational number α_i by

$$\alpha_i = \frac{\beta_1}{\beta_1+\beta_2}\alpha'_i. \tag{44}$$

Then $\boldsymbol{\alpha} = (\alpha_i)$ satisfies the condition

$$0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{2n} < \frac{\beta_1}{(\beta_1+\beta_2)} < 1, \tag{45}$$

hence $\boldsymbol{\alpha}$ defines a weight for parabolic connections. It is easy to see that if we take γ sufficiently large $(E,\nabla,\varphi,\{l_i\})$ is $\boldsymbol{\alpha}$-stable if and only if the associated parabolic ϕ-connection $(E,E,\mathrm{id}_E,\nabla,\varphi,\{l_i\})$ is stable with respect to $(\boldsymbol{\alpha}',\boldsymbol{\beta})$. Therefore we see that the natural map

$$(E,\nabla,\varphi,\{l_i\}) \mapsto (E,E,\mathrm{id}_E,\nabla,\varphi,\{l_i\}) \tag{46}$$

induces an injection

$$M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L) \hookrightarrow \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L). \tag{47}$$

Conversely, assuming that $\boldsymbol{\beta} = (\beta_1, \beta_2)$ are given, for a weight $\boldsymbol{\alpha} = (\alpha_i)$ satisfying the condition (45), we can define $\alpha'_i = \alpha_i \frac{\beta_1+\beta_2}{\beta_1}$ for $1 \leq i \leq 2n$. Since $0 \leq \alpha'_1 < \alpha'_2 < \cdots < \alpha'_{2n} = \alpha_{2n}\frac{\beta_1+\beta_2}{\beta_1} < 1$, $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ give a weight for parabolic ϕ-connections.

Moreover, considering the relative setting over $T_n \times \Lambda_n$, we can define two families of the moduli spaces

$$\overline{\pi_n} : \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \longrightarrow T_n \times \Lambda_n, \quad \pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n \tag{48}$$

such that the following diagram commutes;

$$\begin{array}{ccc} M_n^{\boldsymbol{\alpha}}(L) & \overset{\iota}{\hookrightarrow} & \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \overline{\pi}_n} \\ T_n \times \Lambda_n & = & T_n \times \Lambda_n. \end{array} \tag{49}$$

Here the fibers of π_n and $\overline{\pi}_n$ over $(\mathbf{t},\boldsymbol{\lambda}) \in T_n \times \Lambda_n$ are

$$\pi_n^{-1}(\mathbf{t},\boldsymbol{\lambda}) = M^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L), \quad \overline{\pi}_n^{-1}(\mathbf{t},\boldsymbol{\lambda}) = \overline{M^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L). \tag{50}$$

2.5. The existence of moduli spaces and their properties.

The following theorem is one of our fundamental results in this article which shows that the moduli spaces $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L)$ and $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ exist and they have good properties.

Theorem 2.1. (1) *Fix a weight $\boldsymbol{\beta} = (\beta_1, \beta_2)$. For a generic weight $\boldsymbol{\alpha}'$, $\overline{\pi_n} : \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \longrightarrow T_n \times \Lambda_n$ is a* projective *morphism. In particular, the moduli space $\overline{M^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L)$ is a* projective *algebraic scheme for all $(\mathbf{t},\boldsymbol{\lambda}) \in T_n \times \Lambda_n$.*

(2) *For a generic weight $\boldsymbol{\alpha}$, $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n$ is a* smooth morphism *of relative dimension $2n-6$ with irreducible closed fibers. Therefore, the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ is a* smooth, irreducible *algebraic variety of dimension $2n-6$ for all $(\mathbf{t},\boldsymbol{\lambda}) \in T_n \times \Lambda_n$.*

The proof of Theorem 2.1 can be separated into 3 parts. The construction of the coarse moduli space of the parabolic ϕ-connections over a projective smooth curve will be treated in Section 5. We deal with the relative settings and prove the projectivity of the morphism $\overline{\pi_n} : \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \longrightarrow T_n \times \Lambda_n$. (Cf. Theorem 5.2). Since we have a naturel embedding $M_n^{\boldsymbol{\alpha}}(L) \hookrightarrow \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$, the existence of the moduli space $M_n^{\boldsymbol{\alpha}}(L)$ easily follows from the first assertion. The smoothness of the morphism $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n$ follows from Proposition 6.2. Finally, the irreducibility of the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$ is proved in Section 9, (cf. Proposition 9.1), based on the irreducibility of the moduli space $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ proved in Proposition 8.1.

Remark 2.3. (1) As we mentioned in Introduction, we sometimes extend the base by an étale covering $T'_n \longrightarrow T_n$ in Theorem 2.1, which causes no change in the proof.

(2) The structure of moduli spaces $M_n^{\boldsymbol{\alpha}}(L)$ and $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$ may depend on the weight $\boldsymbol{\alpha}$ and $\deg L$.

(3) The moduli space $M_n^{\boldsymbol{\alpha}}(L)$ is a fine moduli space. In fact, we have the universal family over the moduli space $M_n^{\boldsymbol{\alpha}}(L)$. See §5.

(4) When we describe the explicit algebraic or geometric structure of the moduli spaces $M_n^{\boldsymbol{\alpha}}(L)$ and $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$, it is convenient to fix a determinant line bundle (L, ∇_L). As a typical example of the determinant bundle is

$$(L, \nabla_L) = (\mathcal{O}_{\mathbf{P}^1}(-t_n), d) \tag{51}$$

where the connection is given by

$$\nabla_L(z - t_n) = d(z - t_n) = (z - t_n) \otimes \frac{dz}{z - t_n}. \tag{52}$$

Here z is an inhomogeneous coordinate of $\mathbf{P}^1 = \operatorname{Spec} \mathbf{C}[z] \cup \{\infty\}$. For this $(L, \nabla_L) = (\mathcal{O}_{\mathbf{P}^1}(-t_n), d)$, we set

$$M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1) = M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad (\text{resp. } \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, -1) = \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L) \quad).$$

2.6. **The case of $n = 4$ (Painlevé VI case).**

We will deal with the case of $n = 4$ which corresponds to Painlevé VI equation. Let us fix a sufficiently large intger γ and take a weight $(\boldsymbol{\alpha}', \boldsymbol{\beta})$ for parabolic ϕ-connections where $\boldsymbol{\alpha}' = (\alpha'_1, \dots, \alpha'_8)$, $\boldsymbol{\beta} = (\beta_1, \beta_2), \gamma$ and fix $(\mathbf{t}, \boldsymbol{\lambda}) = (t_1, \dots, t_4, \lambda_1, \dots, \lambda_4) \in T_4 \times \Lambda_4$.

Then the corresponding weight $\alpha = (\alpha_1, \dots, \alpha_8)$ for parabolic connections can be given by

$$\alpha_i = \alpha'_i \frac{\beta_1}{\beta_1 + \beta_2} \quad 1 \leq i \leq 8.$$

Later, for simplicity, we will assume that $\beta_1 = \beta_2$, hence $\boldsymbol{\alpha} = \boldsymbol{\alpha}'/2$. We also assume $(L, \nabla_l) = (\mathcal{O}_{\mathbf{P}^1}(-t_n), d)$ and in this case, we set

$$\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1) = \overline{M_4^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad \overline{M_4^{\boldsymbol{\alpha}'}}(-1) = \overline{M_4^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L).$$

By Theorem 2.1, we can obtain the commutative diagram:

$$\begin{tikzcd} M_4^{\boldsymbol{\alpha}}(-1) \arrow[r, hook, "\iota"] \arrow[d, "\pi_4"'] & \overline{M_4^{\boldsymbol{\alpha}'}}(-1) \arrow[d, "\overline{\pi}_4"] \\ T_4 \times \Lambda_4 \arrow[r, equal] & T_4 \times \Lambda_4, \end{tikzcd} \tag{53}$$

such that $\pi_4^{-1}((\mathbf{t}, \boldsymbol{\lambda})) \simeq M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ and $\overline{\pi}_4^{-1}(\mathbf{t}, \boldsymbol{\lambda}) \simeq \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$. (Note that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'/2$). From Theorem 2.1, we see that for a generic weight $\boldsymbol{\alpha}'$, $\overline{\pi}_4$ is a projective morphism and π_4 is a smooth morphism of relative dimension 2. In Part II, [IIS2], we will give detailed descriptions of the moduli spaces $M_4^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ and $\overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$. The following theorem shows that our family of the moduli space $\overline{M_4^{\boldsymbol{\alpha}'}}(-1) \longrightarrow T_4 \times \Lambda_4$ can be identified with the family of Okamoto-Painlevé pairs constructed by Okamoto [O1]. (See also [Sakai], [STT]). Note also that Arinkin and Lysenko [AL1] give isomorphisms between their moduli spaces and Okamoto spaces for generic $\boldsymbol{\lambda}$.

Theorem 2.2. *(Cf.* [IIS2]*).*

(1) *For a suitable choice of a weight $\boldsymbol{\alpha}'$, the morphism*

$$\overline{\pi}_4 : \overline{M_4^{\boldsymbol{\alpha}'}}(-1) \longrightarrow T_4 \times \Lambda_4$$

is <u>projective and smooth</u> . Moreover for any $(\mathbf{t}, \boldsymbol{\lambda}) \in T_4 \times \Lambda_4$ the fiber $\overline{\pi}_4^{-1}(\mathbf{t}, \boldsymbol{\lambda}) := \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1)$ is irreducible, hence a smooth projective surface.

(2) *Let* $\mathcal{D} = \overline{M_4^{\boldsymbol{\alpha}'}}(-1) \setminus M_4^{\boldsymbol{\alpha}}(-1)$ *be the complement of* $M_4^{\boldsymbol{\alpha}}(-1)$ *in* $\overline{M_4^{\boldsymbol{\alpha}'}}(-1)$. *(Note that* $\boldsymbol{\alpha} = \boldsymbol{\alpha}'/2$*). Then* $\mathcal{D}$ *is a flat reduced divisor over* $T_4 \times \Lambda_4$.

(3) *For each* $(\mathbf{t}, \boldsymbol{\lambda})$, *set*

$$\overline{S}_{\mathbf{t},\boldsymbol{\lambda}} := \overline{\pi}_4^{-1}(\mathbf{t}, \boldsymbol{\lambda}) := \overline{M_4^{\boldsymbol{\alpha}'}}(\mathbf{t}, \boldsymbol{\lambda}, -1).$$

Then $\overline{S}_{\mathbf{t},\boldsymbol{\lambda}}$ *is a smooth projective surface which can be obtained by blowing-ups at* 8 *points of the Hirzebruch surface* $\mathbf{F}_2 = \mathrm{Proj}(\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1})$ *of degree* 2. *The surface has a unique effective anti-canonical divisor* $-K_{S_{\mathbf{t},\boldsymbol{\lambda}}} = \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$ *whose support is* $\mathcal{D}_{\mathbf{t},\boldsymbol{\lambda}}$. *Then the pair*

$$(\overline{S}_{\mathbf{t},\boldsymbol{\lambda}}, \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}) \tag{54}$$

is an Okamoto-Painlevé pair of type $D_4^{(1)}$. *That is, the anti-canonical divisor* $\mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$ *consists of* 5*-nodal rational curves whose configuration is same as Kodaira–Néron degenerate elliptic curves of type* $D_4^{(1)}$ *(=Kodaira type* I_0^**). Moreover we have* $(M_4^{\boldsymbol{\alpha}}(-1))_{\mathbf{t},\boldsymbol{\lambda}} = (\overline{M_4^{\boldsymbol{\alpha}'}}(-1))_{\mathbf{t},\boldsymbol{\lambda}} \setminus \mathcal{Y}_{\mathbf{t},\boldsymbol{\lambda}}$.

3. Elementary transformation of parabolic connections

In this section, we will give basic definitions and some calculations of elementary transformations of stable parabolic connections.

3.1. **Definition.** Let us fix a line bundle L with a connection $\nabla_L : L \longrightarrow L \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ and we set

$$\mu_i = \mathsf{res}_{t_i}(\nabla_L) \quad \text{for } 1 \le i \le n. \tag{55}$$

The residue theorem implies that $\sum_{i=1}^n \mu_i = -\deg L \in \mathbf{Z}$.

For each i, $1 \le i \le n$, we set $L(t_i) = L \otimes \mathcal{O}_{\mathbf{P}^1}(t_i)$, $L(-t_i) = L \otimes \mathcal{O}_{\mathbf{P}^1}(-t_i)$ and so on. We will define two elementary transformations which induce morphisms of moduli spaces.

$$Elm^+_{t_i} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}', L(t_i)) \tag{56}$$

$$Elm^-_{t_i} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}", L(-t_i)) \tag{57}$$

Let $(E, \nabla_E, \varphi, \{l_j\}_{1 \le j \le n})$ be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^1$ with the determinant (L, ∇_L). Note that the eigenvalues of $\mathsf{res}_{t_i}(\nabla)$ are given by the following table.

$$E : \left(\begin{array}{r|ccccc|c} & t_1 & t_2 & \cdots & t_{n-1} & t_n & \wedge^2 E \\ \hline l_j = l_j^+ & \lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n & L \\ E_{t_j}/l_j \simeq l_j^- & \mu_1 - \lambda_1 & \mu_2 - \lambda_2 & \cdots & \mu_{n-1} - \lambda_{n-1} & \mu_n - \lambda_n & \end{array} \right) \tag{58}$$

3.1.1. *Definition of* $Elm^+_{t_i}$. Take a subsheaf F_i as

$$E(-t_i) \subset F_i \subset E \text{ such that } \ l_i = F_i/E(-t_i) \subset E_{t_i} \quad \text{and} \quad l_i(t_i) = F(t_i)/E \tag{59}$$

and define

$$E_i^+ = F_i(t_i) = \mathrm{Ker}\left[E(t_i) \longrightarrow E(t_i)/F(t_i) = E(t_i)_{t_i}/l_i(t_i)\right]. \tag{60}$$

Since l_i is an eigenspace of $\mathsf{res}_{t_i}(\nabla_E)$, it is easy to see that ∇_E induces a connection

$$\nabla_{E_i^+} : E_i^+ \longrightarrow E_i^+ \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \tag{61}$$

and $\varphi : \wedge^2 E \longrightarrow L$ induces a horizontal isomorphism $\varphi' : \wedge^2 E_i^+ \longrightarrow L(t_i)$. Moreover, one can see that the subspace $l'_i = E_{t_i}/l_i \subset (E_i^+)_{t_i}$ defines a new parabolic structure $\{l'_j\}_{j=1}^n$ with $l'_j = l_j$ for $j \neq i$. Now we define

$$Elm^+_{t_i}(E) = (E_i^+, \nabla_{E_i^+}, \varphi', \{l'_i\}), \tag{62}$$

which is called *an upper elementary transformation* of E at t_i. Since $l'_i \simeq E_{t_i}/l_i, (E_i^+)_{t_i}/l'_i \simeq l_i \otimes \mathcal{O}(t_i)$, we see that $(\mathrm{res}_{t_i}(\nabla))_{|l'_i} = \mu_i - \lambda_i,\ (\mathrm{res}_{t_i}(\nabla))_{|(E_i^+)_{t_i}/l'_i} = -1 + \lambda_i$. Therefore the eigenvalues of the residues of $\nabla_{E_i^+}$ on $Elm^+_{t_i}(E) = E_i^+$ and the determinant $\wedge^2 E_i^+$ are given as follows.

$$Elm^+_{t_i}(E) : \left(\begin{array}{c|ccccc|c} t_j & t_1 & \cdots & t_i & \cdots & t_n & \wedge^2 E_i^+ \\ \hline l'_j = l_j^{'+} & \lambda_1 & \cdots & \mu_i - \lambda_i & \cdots & \lambda_n & L(t_i) \\ l_j^{'-} & \mu_1 - \lambda_1 & \cdots & -1 + \lambda_i & \cdots & \mu_n - \lambda_n & \end{array} \right). \tag{63}$$

3.1.2. *Definition of $Elm^-_{t_i}$.* By using (59) subsheaf $F_i \subset E$ we also define a filtration of sheaves

$$E_i^- = F_i \supset E(-t_i) \supset F_i(-t_i) \tag{64}$$

which defines a parabolic connection $(E_i^-, \nabla_{E_i^-}, \varphi', l')$ such that

$$l'_i = E(-t_i)/F_i(-t_i) = (E_{t_i}/l_i) \otimes \mathcal{O}_{\mathbf{P}^1}(-t_i).$$

This is called a *lower elementary transformation* of E at t_i and will be denoted by

$$Elm^-_{t_i}(E) := (E_i^-, \nabla_{E_i^-}, \varphi', l').$$

Note that one has a horizontal isomorphism $\varphi' : \wedge^2 E_i^- \xrightarrow{\simeq} L(-t_i)$ and the eigenvalues of the residues of $\nabla_{E_i^-}$ on $Elm^-_{t_i}(E) = E_i^-$ are given as follows.

$$Elm^-_{t_i}(E) : \left(\begin{array}{c|ccccc|c} & t_1 & \cdots & t_i & \cdots & t_n & \wedge^2 E_i^- \\ \hline l'_j = l_j^{'+} & \lambda_1 & \cdots & 1 + \mu_i - \lambda_i & \cdots & \lambda_n & L(-t_i) \\ l_j^{'-} & \mu_1 - \lambda_1 & \cdots & \lambda_i & \cdots & \mu_n - \lambda_n & \end{array} \right). \tag{65}$$

3.1.3. *Tensoring a line bundle L_1.* Let L_1 be a line bundle with a logarithmic connection ∇_{L_1} and set $\nu_j = \mathrm{res}_{t_j}(\nabla_{L_1})$ for $1 \leq j \leq n$. We can define a transformation $\otimes(L_1, \nabla_{L_1})$ by

$$(E, \nabla_E, \varphi, \{l_j\}) \mapsto (E \otimes L_1, \nabla_{E \otimes L_1}, \varphi', \{l_j \otimes L_1\}) \tag{66}$$

which induces a morphism of moduli spaces

$$\otimes(L_1, \nabla_{L_1}) : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}', L \otimes (L_1)^{\otimes 2}). \tag{67}$$

The set of eigenvalues of new connection can be given as follows.

$$E \otimes L_1 : \left(\begin{array}{c|ccccc|c} & t_1 & \cdots & t_i & \cdots & t_n & \wedge^2(E \otimes L_1) \\ \hline l'_j = l_j^{'+} & \nu_1 + \lambda_1 & \cdots & \nu_i + \lambda_i & \cdots & \nu_n + \lambda_n & L \otimes (L_1)^{\otimes 2} \\ l_j^{'-} & \nu_1 + \mu_1 - \lambda_1 & \cdots & \nu_i + \mu_i - \lambda_i & \cdots & \nu_n + \mu_n - \lambda_n & \end{array} \right). \tag{68}$$

3.1.4. *R_i: Interchanging the eigenspaces.*

Under the assumption

$$\lambda_i \neq \mu_i - \lambda_i, \tag{69}$$

we see that there are unique eigenspaces $l_i^+ = l_i$ and l_i^- of $\mathrm{res}_{t_i}(\nabla_E)$ with the eigenvalues λ_i and $\mu_i - \lambda_i$ respectively. Interchanging the eigenspaces l_i^+ and l_i^- and keeping the other eigenspaces l_j $j \neq i$ unchanged, we obtain a new parabolic connection

$$R_i(E) = (E, \nabla_E, \varphi, \{l_i'\}). \tag{70}$$

If $\lambda_i = \mu_i - \lambda_i$, let us define $R_i(E) = (E, \nabla_E, \varphi, \{l_i\})$, that is, $R_i = Id$.

The set of eigenvalues of new connection can be given as follows.

$$R_i(E) : \left(\begin{array}{r|ccccc|c} & t_1 & \cdots & t_i & \cdots & t_n & \wedge^2 E \\ \hline l_j' = l_j^{'+} & \lambda_1 & \cdots & \mu_i - \lambda_i & \cdots & \lambda_n & L \\ E_{j,t_j}^- / l_j' \simeq l_j^{'-} & \mu_1 - \lambda_1 & \cdots & \lambda_i & \cdots & \mu_n - \lambda_n & \end{array} \right). \tag{71}$$

Now assume that $\mathrm{res}_{t_i}(\nabla_L) \in \mathbf{Z}$ for all $1 \leq i \leq n$.

Lemma 3.1. *Assume that $\boldsymbol{\lambda}$ is not reducible (cf. Definition 2.3). Then R_i induces an isomorphism*

$$R_i : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \xrightarrow{\simeq} M_n^{\alpha}(\mathbf{t}, \boldsymbol{\lambda}', L). \tag{72}$$

Proof. Since $\boldsymbol{\lambda}$ is not reducible, any $(E, \nabla_E, \varphi, \{l_i\}) \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ are irreducible (Lemma 2.1), so is $R_i(E)$. In particular $R_i(E)$ is $\boldsymbol{\alpha}$-stable. Therefore it induces a morphism of moduli spaces. Moreover it is obvious that $R_i^2 = Id$, so it must be an isomorphism. Λ

Later we will extend R_i a birational map of the moduli spaces.

3.2. **Birational transformations arising from elementary transformations.**

Definition 3.1. Assume that $\boldsymbol{\alpha}$ is generic. *An affine birational transformation* of the family of moduli spaces $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n' \times \Lambda_n$ is a pair of maps $(\tilde{s}, s)$ consisting of a birational map $\tilde{s} : M_n^{\boldsymbol{\alpha}}(L) \cdots \longrightarrow M_n^{\boldsymbol{\alpha}}(L)$ and an affine transformation $s : \Lambda_n \longrightarrow \Lambda_n$ such that the following diagram commutes:

$$\begin{array}{ccc} M_n^{\boldsymbol{\alpha}}(L) & \overset{\tilde{s}}{\cdots\rightarrow} & M_n^{\boldsymbol{\alpha}}(L) \\ \downarrow \pi_n & & \downarrow \pi_n \\ T_n' \times \Lambda_n & \xrightarrow{1 \times s} & T_n' \times \Lambda_n. \end{array} \tag{73}$$

3.2.1. *The group BL_n.*

Now we fix a determinant line bundle $(L, \nabla_L) = (\mathcal{O}_{\mathbf{P}^1}(-t_n), d)$ as in Remark 2.3 and consider the family of the moduli spaces $\pi_n : M_n^{\boldsymbol{\alpha}}(\mathcal{O}_{\mathbf{P}^1}(-t_n)) \longrightarrow T_n' \times \Lambda_n$. Let $\mathbf{e}_i \in \Lambda_n$ be the i-th standard base of $\Lambda_n \simeq \mathbf{C}^n$ and set $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$. We define a group BL_n generated by the following affine automorphisms of Λ_n.

$$
(74)\qquad \begin{cases} t_i^+(\boldsymbol{\lambda}) &=& \boldsymbol{\lambda}+\mathbf{e}_i=(\lambda_1,\dots,\lambda_i+1,\dots,\lambda_n) \\ t_{i,j}^+(\boldsymbol{\lambda}) &=& \boldsymbol{\lambda}+\frac{1}{2}(\mathbf{e}_i+\mathbf{e}_j)=(\lambda_1,\dots,\lambda_i+\frac{1}{2},\dots,\lambda_j+\frac{1}{2},\dots,\lambda_n) \\ t_{i,j}^-(\boldsymbol{\lambda}) &=& (\lambda_1,\dots,-\lambda_i+\frac{1}{2},\dots,-\lambda_j+\frac{1}{2},\dots,\lambda_n) \quad (1\le i<j\le n-1) \\ t_{i,n}^-(\boldsymbol{\lambda}) &=& (\lambda_1,\dots,-\lambda_i+\frac{1}{2},\dots,\dots,\dots,-\lambda_n+\frac{3}{2}) \\ r_i(\boldsymbol{\lambda}) &=& (\lambda_1,\dots,-\lambda_i,\dots,\lambda_n) \quad (1\le i\le n-1) \\ r_n(\boldsymbol{\lambda}) &=& (\lambda_1,\dots,\lambda_i,\dots,1-\lambda_n). \end{cases}
$$

We can easily see the following relations.

$$
(75)\qquad t_i^+=(t_{i,j}^-r_i)^2, \quad t_{i,j}^+=t_{i,j}^-r_ir_j.
$$

Therefore we can define the group BL_n as

$$
(76)\qquad BL_n=\langle t_{i,j}^-,(1\le i<j\le n),\ r_k,(1\le k\le n)\rangle.
$$

In [IIS2], we will show the following

Proposition 3.1. *Every element s of the group BL_n of affine transformations of Λ_n can be lifted to a birational transformation*

$$
(77)\qquad \tilde{s}: M_n^{\boldsymbol{\alpha}}(\mathcal{O}_{\mathbf{P}^1}(-t_n))\cdots\longrightarrow M_n^{\boldsymbol{\alpha}}(\mathcal{O}_{\mathbf{P}^1}(-t_n))
$$

such that the pair $(\tilde{s},s)$ becomes an affine birational transformation of the family of moduli spaces.

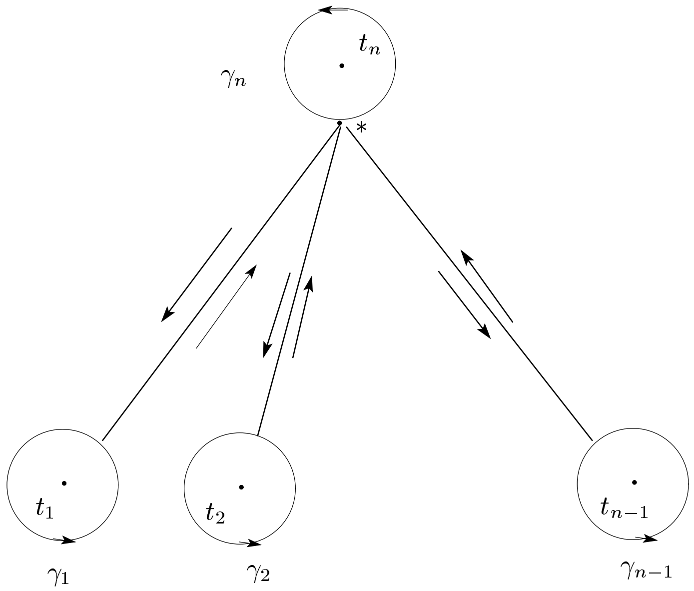

FIGURE 4. Canonical generators of $\pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}), *)$.

4. Moduli of representations of fundamental groups

4.1. **The family of punctured projective lines and their fundamental groups.** For $n \geq 3$, let us consider the space $T_n = \{(t_1, \dots, t_n) \in (\mathbf{P}^1)^n \mid t_i \neq t_j, (i \neq j)\}$ and its open subset

$$W_n = \{(t_1, \dots, t_n) \in \mathbf{C}^n \quad | \quad t_i \neq t_j, (i \neq j)\}. \tag{78}$$

Setting $D(\mathbf{t}) = t_1 + \cdots + t_n$ for each $\mathbf{t} = (t_1, \dots, t_n) \in T_n$, we denote by

$$\Gamma_{n,\mathbf{t}} := \pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}), *), \tag{79}$$

the fundamental group of $\mathbf{P}^1 \setminus D(\mathbf{t})$ with the base point $*$ which we take very near to t_n. It is easy to see that $\Gamma_{n,\mathbf{t}}$ is generated by $\gamma_1, \dots, \gamma_{n-1}, \gamma_n$ in Figure 4 with one relation $\gamma_1\gamma_2 \cdots \gamma_n = 1$. This set of generators $\gamma_1, \dots, \gamma_n$ is called *canonical generators* of $\Gamma_{n,\mathbf{t}}$ with respect to the ordered n-points $\mathbf{t}$.

For each $i, 1 \leq i \leq n$, we define a divisor $\Sigma_{n,i}$ of $\mathbf{P}^1 \times T_n$ as

$$\Sigma_{n,i} = \{(z, (t_1, \dots, t_n)) \in \mathbf{P}^1 \times T_n \mid z = t_i \ \}. \tag{80}$$

Setting $\mathcal{P}_n := \big(\mathbf{P}^1 \times T_n\big) \setminus (\cup_{i=1}^n \Sigma_{n,i}) \simeq T_{n+1}$, we obtain a natural projection map which induces a smooth morphism

$$\tau_n : \mathcal{P}_n \longrightarrow T_n \tag{81}$$

whose fiber $\mathcal{P}_{n,\mathbf{t}}$ over $\mathbf{t} = (t_1, \dots, t_n)$ is $\mathbf{P}^1 \setminus D(\mathbf{t})$. The family $\tau_n : \mathcal{P}_n \longrightarrow T_n$ in (81) is called the *universal family of n-punctured lines.*

By the universal covering map $\tilde{T}_n \longrightarrow T_n$, we can extend the family

$$\begin{array}{ccc} \tilde{\mathcal{P}}_n & \longrightarrow & \mathcal{P}_n \\ \tilde{\tau}_n \downarrow & & \tau_n \downarrow \\ \tilde{T}_n & \longrightarrow & T_n, \end{array} \tag{82}$$

where we set $\tilde{\mathcal{P}}_n = \mathcal{P}_n \times_{T_n} \tilde{T}_n$.

Fix a base point $\mathbf{t}_0 \in T_n$ and consider the fundamental group $\pi_1(T_n, \mathbf{t}_0)$. The natural n-th projection $h_n : T_n \longrightarrow \mathbf{P}^1$ $((t_1, \dots, t_n) \mapsto t_n)$ gives a structure of fiber bundle over $\mathbf{P}^1$ whose fiber at $t_n = \infty$ is isomorphic to W_{n-1}. By using the exact sequence of fundamental groups for fiber bundles, one can see that there exists an isomorphism

$$\pi_1(T_n, \mathbf{t}_0) \simeq \pi_1(W_{n-1}, \mathbf{t}_0). \tag{83}$$

On the other hand, it is well known that the fundamental group $\pi_1(W_{n-1}, \mathbf{t}_0)$ is isomorphic to the pure braid group PB_{n-1} of $n-1$ strings. Therefore the pure braid group PB_{n-1} acts on the universal covering $\tilde{T}_n$ and also the typical fiber $\mathcal{P}_{n,\mathbf{t}_0}$ of $\tilde{\tau}_n$ in (82).

Moreover the fiber bundle $\hat{\pi}_n : \tilde{\mathcal{P}}_n \longrightarrow \tilde{T}_n$ becomes trivial, that is, there exists a diffeomorphism $\tilde{\mathcal{P}}_n \xrightarrow{\simeq} \mathcal{P}_{n,\tilde{\mathbf{t}}^0} \times \tilde{T}_n$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{P}}_n & \xrightarrow{\simeq} & \mathcal{P}_{n,\mathbf{t}_0} \times \tilde{T}_n \\ & \tilde{\tau}_n \searrow \quad \swarrow & \\ & \tilde{T}_n. & \end{array} \tag{84}$$

By using the isomorphism, for every $\tilde{\mathbf{t}} \in \tilde{T}_n$, we can obtain the isomorphism of fundamental groups

$$\pi_1(\tilde{\mathcal{P}}_{n,\tilde{\mathbf{t}}} *) \simeq \pi_1(\mathcal{P}_{n,\mathbf{t}_0}, *) = \Gamma_{n,\mathbf{t}_0} \tag{85}$$

as well as the identification of canonical generators $\gamma_1, \dots, \gamma_n$ in Figure 4 . The action of the pure braid group PB_{n-1} on the fiber bundle $\hat{\pi}_n : \tilde{\mathcal{P}}_n \longrightarrow \tilde{T}_n$ induces an action on canonical generators of $\Gamma_{n,\mathbf{t}_0}$, which can be written in a very explicit way. (For example for the case of $n = 4$, see [Iw3], [Iw4]).

4.2. The moduli space of $SL_2(\mathbf{C})$-representations.

Definition 4.1. An $SL_2(\mathbf{C})$-representation of the fundamental group $\Gamma_{n,\mathbf{t}} = \pi_1(\mathcal{P}_{n,\mathbf{t}}, *)$ of $\mathcal{P}_{n,\mathbf{t}} = \mathbf{P}^1 \setminus D(\mathbf{t})$ is a group homomorphism

$$\rho : \Gamma_{n,\mathbf{t}} = \pi_1(\mathcal{P}_{n,\mathbf{t}}, *) \longrightarrow SL_2(\mathbf{C}). \tag{86}$$

We denote by $\mathrm{Hom}(\Gamma_{n,\mathbf{t}}, SL_2(\mathbf{C}))$ the set of all $SL_2(\mathbf{C})$-representations of $\Gamma_{n,\mathbf{t}}$. If we fix a set of canonical generators $\gamma_1, \dots, \gamma_n$ of $\Gamma_{n,\mathbf{t}}$ as in Figure 4, we have the identification

$$\mathrm{Hom}(\Gamma_{n,\mathbf{t}}, SL_2(\mathbf{C})) = SL_2(\mathbf{C})^{n-1}$$

given by $\rho \mapsto (\rho(\gamma_i))$ for $i = 1, \dots, n-1$.

Definition 4.2. (1) Two $SL_2(\mathbf{C})$-representations ρ_1, ρ_2 are isomorphic to each other, if and only if there exists a matrix $P \in SL_2(\mathbf{C})$ such that

$$\rho_2(\gamma) = P^{-1} \cdot \rho_2(\gamma) \cdot P \quad \text{for all } \gamma \in \pi_1(\tilde{\mathcal{P}}_{n,\mathbf{t}}, *) .$$

(2) A *semisimplification of a representation* ρ is an associated graded of the composition series of ρ.

(3) Two $SL_2(\mathbf{C})$-representation is said to be *Jordan equivalent* if their semisimplifications are isomorphic.

Fixing $\mathbf{t}_0 \in T_n$ and canonical generators $\gamma_1, \ldots, \gamma_n$ of $\Gamma_{n,\mathbf{t}_0}$ and using the isomorphism in (85), for any $\mathbf{t} \in \tilde{T}_n$, we fix an identification

$$\mathrm{Hom}(\Gamma_{n,\mathbf{t}}, SL_2(\mathbf{C})) \xrightarrow{\simeq} SL_2(\mathbf{C})^{n-1} \tag{87}$$

by $\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_{n-1}))$.

Let R_{n-1} denote the affine coordinate ring of $SL_2(\mathbf{C})^{n-1}$ and consider the simultaneous action of $SL_2(\mathbf{C})$ on $SL_2(\mathbf{C})^{n-1}$ as

$$(M_1, \cdots, M_{n-1}) \mapsto (P^{-1}M_1P, \cdots, P^{-1}M_{n-1}P).$$

Hilbert shows that the ring of invariants, denoted by $(R_{n-1})^{Ad(SL_2(\mathbf{C}))}$, is finitely generated. The following lemma is due to Simpson [Sim2],

Lemma 4.1. ([Mum], [Proposition 6.1, [Sim2]]). *For any $\mathbf{t} \in \tilde{T}_n$, under the identification (87), there exists the universal categorical quotient map*

$$\Phi_n : \mathrm{Hom}(\Gamma_{n,\mathbf{t}}, SL_2(\mathbf{C})) \simeq SL_2(\mathbf{C})^{n-1} \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) = SL_2(\mathbf{C})^{n-1}/Ad(SL_2(\mathbf{C}))$$

where

$$\mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) = \mathrm{Spec}[(R_{n-1})^{Ad(SL_2(\mathbf{C}))}]. \tag{88}$$

The closed points of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})$ represent the Jordan equivalence classes of $SL_2(\mathbf{C})$-representations of $\Gamma_{n,\mathbf{t}}$. We say that $\mathcal{RP}_n = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})$ is the moduli space *of $SL_2(\mathbf{C})$-representation of $\pi_1(\mathbf{P}^1 \setminus \Sigma(\mathbf{t}))$.*

Remark 4.1. Lemma 4.1 says that the set $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})$ of Jordan equivalence classes of $SL_2(\mathbf{C})$-representations admits a natural structure of an affine scheme. Moreover, it is easy to see that the moduli stack of isomorphism classes of $SL_2(\mathbf{C})$-representations has no natural scheme structure.

Remark 4.2. It is obvious that the algebraic structure or complex structure of the moduli space $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})$ does not depend on $\mathbf{t} \in \tilde{T}_n$. However in order to define the isomorphism $\mathrm{Hom}(\Gamma_{n,\mathbf{t}}, SL_2(\mathbf{C})) \simeq SL_2(\mathbf{C})^{n-1}$ we have to fix canonical generators of $\Gamma_{n,\mathbf{t}} = \pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}))$. Since the pure braid group $PB_{n-1} := \pi_1(T_n, *)$ acts on the sets of generators of $\Gamma_{n,\mathbf{t}}$ and hence acts on $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})$. This action is called the *topological nonlinear monodromy action* of the pure braid group $PB_{n-1} := \pi_1(T_n, *)$. (Cf. [DM], [Iw3], [Iw4]).

In our case, we can describe the categorical quotient $\mathrm{Spec}[(R_{n-1})^{Ad(SL_2(\mathbf{C}))}]$ more explicitly. Denote the coordinate ring R_{n-1} of $SL_2(\mathbf{C})^{n-1}$ by

$$R_{n-1} = \mathbf{C}[a_i, b_i, c_i, d_i]/(a_id_i - b_ic_i - 1) \quad i = 1, \ldots, n-1 \tag{89}$$

where $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$.

The following Proposition follows from the fundamental theorem for matrix invariants. (See [Theorem 2, Theorem 7, [For]] or [Theorem 1.3, [P]]).

Proposition 4.1.

$$(R_{n-1})^{\mathrm{Ad}(SL_2(\mathbf{C}))} = \mathbf{C}[\mathrm{Tr}(M_{i_1}M_{i_2}\cdots M_{i_k}), 1 \le i_1, \dots, i_k \le n-1]. \tag{90}$$

Moreover, the elements $\mathrm{Tr}(M_{i_1}M_{i_2}\cdots M_{i_k})$ *of degree* $k \le 3$ *generate the invariant ring, that is,*

$$(R_{n-1})^{\mathrm{Ad}(SL_2(\mathbf{C}))} = \mathbf{C}[\mathrm{Tr}(M_i), \mathrm{Tr}(M_iM_j), \mathrm{Tr}(M_iM_jM_k) \mid 1 \le i,j,k \le n-1]. \tag{91}$$

Let us set

$$a_i = \mathrm{Tr}(M_i) \quad \text{for} \quad 1 \le i \le n, \tag{92}$$

which are elements of $(R_{n-1})^{Ad(SL_2(\mathbf{C}))}$ and consider the subring $A_n = \mathbf{C}[a_1, \dots, a_n]$ of $(R_{n-1})^{Ad(SL_2(\mathbf{C})}$. We have a natural morphism

$$p_n : \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) = \mathrm{Spec}\left[(R_{n-1})^{Ad(SL_2(\mathbf{C}))}\right] \longrightarrow \mathcal{A}_n = \mathrm{Spec}\,[A_n]\,. \tag{93}$$

4.3. **Construction of the family of moduli spaces $\phi_n : \mathcal{R}_n \longrightarrow T'_n \times \mathcal{A}_n$.**

Fix $\mathbf{t}_0 \in T_n$ as the base point of fundamental group $\pi_1(T_n, \mathbf{t}_0)$ and fix canonical generators $\gamma_1, \dots, \gamma_n$ of $\Gamma_{n,\mathbf{t}_0}$. Again taking the universal covering map $\tilde{T}_n \longrightarrow T_n$, we can obtain a trivialization (84) and isomorphisms of the fundamental groups (85). By using the isomorphisms, for each $\mathbf{t} \in \tilde{T}_n$, we obtain a canonical isomorphism

$$\mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) \simeq \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0}).$$

Moreover the group $\pi_1(T_n, \mathbf{t}_0) \simeq PB_{n-1}$ acts on the variety $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})$ as the group of nonlinear monodromies and hence defines the action on the product $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0}) \times \tilde{T}_n$. Define the subgroup Γ_{n-1} of $\pi_1(T_n, \mathbf{t}_0)$ as a kernel of the natural homomorphism $\pi_1(T_n, \mathbf{t}_0) \longrightarrow \mathrm{Aut}(\mathbf{C}[a_1, \dots, a_n])$. It is easy to see that Γ_{n-1} is a subgroup of $\pi_1(T_n, \mathbf{t}_0)$ of finite index, so defining as $T'_n = \tilde{T}_n/\Gamma_{n-1}$ we obtain the finite étale covering

$$T'_n := \tilde{T}_n/\Gamma_{n-1} \longrightarrow T_n. \tag{94}$$

Consider the natural action of Γ_{n-1} on the product $\tilde{T}_n \times \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})$. The natural map $1 \times p_n : \tilde{T}_n \times \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0}) \longrightarrow \tilde{T}_n \times \mathcal{A}_n$ is clearly equivariant with respect to the action of Γ_{n-1}, where Γ_{n-1} acts on $\mathcal{A}_n$ as the identity map. Setting

$$\mathcal{R}_n = \tilde{T}_n \times \mathcal{R}(\mathcal{P}_{n,\mathbf{t}_0})/\Gamma_{n-1}, \tag{95}$$

we obtain a morphism

$$\boxed{\phi_n : \mathcal{R}_n \longrightarrow T'_n \times \mathcal{A}_n,} \tag{96}$$

which is said to be the family of the moduli spaces of SL_2-representations of the fundamental group. The fiber of ϕ_n at $(\mathbf{t}, \mathbf{a})$ is given by the affine subscheme of $\mathcal{R}_n$

$$\phi_n^{-1}(\mathbf{t}, \mathbf{a}) = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} := \{[\rho] \in \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) \mid \ \mathrm{Tr}[\rho(\gamma_i)] = a_i, 1 \le i \le n\}. \tag{97}$$

Since a_i determines the eigenvalues of monodromy matrix $\rho(\gamma_i)$, $\mathbf{a}$ may be considered as the set of spectral of local monodromies. Hence the space $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is said to be the moduli space of *isospectral*

SL_2-representations. Note that though the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth for all $(\mathbf{t}, \boldsymbol{\lambda})$ if $\mathbf{a}$ is special in the sense of Definition 1.1 the affine scheme $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ has singularities.

In §8, we will prove the following

Proposition 4.2. *For any $\mathbf{a} \in \mathcal{A}_n$, the scheme $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ in (97) is irreducible.*

4.4. **The case of $n = 4$.** Now we recall the explicit description of the invariant ring for $n = 4$ due to Iwasaki ([Iw3], [Iw4]). We denote by (i, j, k) a cyclic permutation of $(1, 2, 3)$. Then the invariant ring $(R_3)^{Ad(SL_2(\mathbf{C}))}$ is generated by

$$\begin{array}{lcll} x_i & = & \mathrm{Tr}[M_j M_k] & \text{for } i = 1, 2, 3 \\ a_i & = & \mathrm{Tr}[M_i] & \text{for } i = 1, 2, 3 \\ a_4 & = & \mathrm{Tr}[M_1 M_2 M_3] & \end{array} \tag{98}$$

The following proposition is proved in [Iw4].

Proposition 4.3. *The invariant ring $(R_3)^{Ad(SL_2(\mathbf{C}))}$ is generated by seven elements $x_1, x_2, x_3, a_1, a_2, a_3, a_4$ and there exists a relation*

$$f(\mathbf{x}, \mathbf{a}) = x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\mathbf{a}) x_1 - \theta_2(\mathbf{a}) x_2 - \theta_3(\mathbf{a}) x_3 + \theta_4(\mathbf{a}), \tag{99}$$

where we set

$$\theta_i(\mathbf{a}) \;=\; a_i a_4 + a_j a_k, \quad (i, j, k) = \textit{a cyclic permutation of } (1, 2, 3), \tag{100}$$

$$\theta_4(\mathbf{a}) \;=\; a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4. \tag{101}$$

Therefore we have an isomorphism

$$(R_3)^{Ad(SL_2(\mathbf{C}))} \simeq \mathbf{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]/(f(\mathbf{z}, \mathbf{a})). \tag{102}$$

Recall that fixing canonical generators of the fundamental group, for any $\mathbf{t} \in \tilde{T}_4$, the categorical quotient $\mathcal{R}_{4,\mathbf{t}}$ is given by $\mathcal{R}(\mathcal{P}_{4,\mathbf{t}}) := \mathrm{Spec}[(R_3)^{Ad(SL_2(\mathbf{C}))}] \simeq \mathrm{Spec}[\mathbf{C}[\mathbf{x}, \mathbf{a}]/(f(\mathbf{x}, \mathbf{a}))]$. Setting $\mathcal{A}_4 = \mathbf{C}^4 = \mathrm{Spec}[\mathbf{C}[a_1, \ldots, a_4]]$, as in (93) we have a surjective morphism

$$p_4 : \mathcal{R}(\mathcal{P}_{4,\mathbf{t}}) = \mathrm{Spec}[\mathbf{C}[\mathbf{x}, \mathbf{a}]/(f(\mathbf{x}, \mathbf{a}))] \longrightarrow \mathcal{A}$$

whose fiber at $\mathbf{a} \in \mathcal{A}$ is an affine cubic hypersurface in $\mathbf{C}^3$

$$\mathcal{R}(\mathcal{P}_{4,\mathbf{t}})_{\mathbf{a}} \simeq \{(x_1, x_2, x_3) \in \mathbf{C}^3 \mid f(\mathbf{x}, \mathbf{a}) = 0\} \subset \mathbf{C}^3.$$

Therefore, the family in (96) $\phi_4 : \mathcal{R}_4 \longrightarrow T_4' \times \mathcal{A}_4$ is a family of affine cubic hypersurfaces in $\mathbf{C}^3$.

The subgroup Γ_3 of $\pi_1(T_4, \mathbf{t}_0)$ acts both on the space $\mathcal{R}(\mathcal{P}_{4,\mathbf{t}})$ and the space $\mathcal{R}(\mathcal{P}_{4,\mathbf{t}})_{\mathbf{a}}$ as nonlinear monodromies. Iwasaki [Iw3] showed the following

Proposition 4.4. *There exists a one-to-one correspondence between the set of fixed points of the action of Γ_3 on $\mathcal{R}(\mathcal{P}_{4,\mathbf{t}})_{\mathbf{a}}$ and the set of singular points on the affine cubic hypersurface $\mathcal{R}(\mathcal{P}_{4,\mathbf{t}})_{\mathbf{a}}$.*

5. Construction of the moduli space $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(\mathbf{t},\boldsymbol{\lambda},L)$ and Proof of Theorem 2.1, (1)

5.1. **Translation of the definition of parabolic ϕ-connection.** In this section, we will translate the definition of parabolic ϕ-connection, since it is rather convenient to generalize the definition for the construction of the moduli space.

Let X be a smooth projective curve over $\mathbf{C}$ and D be an effective divisor on X.

We define an $\mathcal{O}_X$-bimodule structure on $\Lambda^1_D = \mathcal{O}_X \oplus (\Omega^1_X(D))^\vee$ by

$$(a,v)f := (fa + \langle v, df\rangle, fv) \tag{103}$$
$$f(a,v) := (fa, fv)$$

for $a, f \in \mathcal{O}_X$ and $v \in (\Omega^1_X(D))^\vee$, where $\langle\, ,\, \rangle : (\Omega^1_X(D))^\vee \times \Omega^1_X(D) \to \mathcal{O}_X$ is the canonical pairing.

Definition 5.1. A parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1))$ on X consists of two vector bundles E_1, E_2 on X, a left $\mathcal{O}_X$-homomorphism $\Phi : \Lambda^1_D \otimes_{\mathcal{O}_X} E_1 \to E_2$ and a filtration of coherent subsheaves: $E_1 = F_1(E_1) \supset F_2(E_1) \supset \cdots \supset F_l(E_1) \supset F_{l+1}(E_1) = E_1(-D)$.

Remark 5.1. *Assume that two vector bundles E_1, E_2 on X are given. Then giving morphisms $\phi : E_1 \to E_2$, $\nabla : E_1 \to E_2 \otimes \Omega^1_X(D)$ satisfying $\phi(fa) = f\phi(a)$, $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_X$, $a \in E_1$ is equivalent to giving a left $\mathcal{O}_X$-homomorphism $\Phi : \Lambda^1_D \otimes_{\mathcal{O}_X} E_1 \to E_2$.*

Definition 5.2. A parabolic Λ^1_D-triple $(E'_1, E'_2, \Phi', F_*(E'_1))$ is said to be a parabolic Λ^1_D-subtriple of $(E_1, E_2, \Phi, F_*(E_1))$ if $E'_1 \subset E_1$, $E'_2 \subset E_2$, $\Phi|_{\Lambda^1_D \otimes E'_1} = \Phi'$ and $F_i(E'_1) \subset F_i(E_1)$ for any i.

Fix rational numbers $0 \le \alpha'_1 < \alpha'_2 < \cdots < \alpha'_l < \alpha'_{l+1} = 1$ and positive integers β_1, β_2. We write $\boldsymbol{\alpha}' = (\alpha'_1, \ldots, \alpha'_l)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$. We also fix an ample line bundle $\mathcal{O}_X(1)$ and a rational number γ with $\gamma \gg 0$.

Definition 5.3. For a parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1))$, we put

$$\mu(E_1, E_2, \Phi, F_*(E_1)) := \frac{\beta_1 \deg E_1(-D) + \beta_2 \deg E_2 - \beta_2\gamma \deg \mathcal{O}_X(1) \operatorname{rank} E_2 + \sum_{i=1}^l \beta_1 \alpha'_i \operatorname{length}(F_i(E_1)/F_{i+1}(E_1))}{\beta_1 \operatorname{rank} E_1 + \beta_2 \operatorname{rank} E_2}.$$

Definition 5.4. Assume that γ is sufficiently large. A parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1))$ is $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable (resp. $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-semistable) if for any non-zero proper parabolic Λ^1_D-subtriple $(E'_1, E'_2, \Phi', F_*(E'_1))$ of $(E_1, E_2, \Phi, F_*(E_1))$, the inequality

$$\mu(E'_1, E'_2, \Phi', F_*(E'_1)) \quad < \quad \mu(E_1, E_2, \Phi, F_*(E_1))$$
$$\text{(resp. } \le)$$

holds. (If we fix a weight $(\boldsymbol{\alpha}', \boldsymbol{\beta})$, "$(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable (resp. $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-semistable) " may be abbreviated to "stable (resp. semistable)" for simplicity.)

Let S be a connected noetherian scheme and $\pi_S : \mathcal{X} \to S$ be a smooth projective morphism whose geometric fibers are curves of genus g. Let $\mathcal{D} \subset \mathcal{X}$ be an effective Cartier divisor which is flat over S. A similar formula to (103) enables us to consider the $\mathcal{O}_{\mathcal{X}}$-bimodule structure on $\Lambda^1_{\mathcal{D}/S} := \mathcal{O}_{\mathcal{X}} \oplus (\Omega^1_{\mathcal{X}/S}(\mathcal{D}))^\vee$.

Fix rational numbers $0 \leq \alpha'_1 < \alpha'_2 < \cdots < \alpha'_l < \alpha'_{l+1} = 1$, positive integers $r, d, \{d_i\}_{1 \leq i \leq l}$, β_1, β_2, γ with $\gamma \gg 0$.

Definition 5.5. We define the moduli functor $\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\})$ of the category of locally noetherian schemes over S to the category of sets by

$$\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\})(T) := \{(E_1, E_2, \Phi, F_*(E_1))\}/\sim, \tag{104}$$

where T is a locally noetherian scheme over S and

(1) E_1, E_2 are vector bundles on $\mathcal{X} \times_S T$ such that for any geometric point s of T, $\mathrm{rank}(E_1)_s = \mathrm{rank}(E_2)_s = r$, $\deg(E_1)_s = \deg(E_2)_s = d$,
(2) $\Phi : \Lambda^1_{\mathcal{D}/S} \otimes_{\mathcal{O}_{\mathcal{X}}} E_1 \to E_2$ is a homomorphism of left $\mathcal{O}_{\mathcal{X}\times_S T}$-modules,
(3) $E_1 = F_1(E_1) \supset F_2(E_1) \supset \cdots \supset F_l(E_1) \supset F_{l+1}(E_1) = E_1(-\mathcal{D}_T)$ is a filtration of E_1 by coherent subsheaves such that each $E_1/F_{i+1}(E_1)$ is flat over T and for any geometric point s of T, $\mathrm{length}((E_1/F_{i+1}(E_1))_s) = d_i$,
(4) for any geometric point s of S, the parabolic $\Lambda^1_{\mathcal{D}_s}$-triple $((E_1)_s, (E_2)_s, \Phi_s, F_*(E_1)_s)$ is stable (that is, $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable) .

$(E_1, E_2, \Phi, F_*(E_1)) \sim (E'_1, E'_2, \Phi', F_*(E'_1))$ if there exist a line bundle $\mathcal{L}$ on T and isomorphisms $\sigma_j : E_j \xrightarrow{\sim} E'_j \otimes \mathcal{L}$ for $j = 1, 2$ such that $\sigma_1(F_{i+1}(E_1)) = F_{i+1}(E'_1) \otimes \mathcal{L}$ for any i and the diagram

$$\begin{tikzcd}
\Lambda^1_{\mathcal{D}/S} \otimes_{\mathcal{O}_X} E_1 \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}\otimes\sigma_1"', "\cong"] & E_2 \arrow[d, "\sigma_2"', "\cong"] \\
\Lambda^1_{\mathcal{D}/S} \otimes_{\mathcal{O}_X} E'_1 \otimes_S \mathcal{L} \arrow[r, "\Phi'\otimes\mathrm{id}"] & E'_2 \otimes_S \mathcal{L}
\end{tikzcd}$$

commutes.

We call $(E_1, E_2, \Phi, F_*(E_1))$ a flat family of parabolic $\Lambda^1_{\mathcal{D}_T/T}$-triples on $\mathcal{X}_T \times T$ over T if it satisfies the above conditions $(1), (2)$ and (3).

5.2. **Boundedness and Openness of stability.**

Proposition 5.1. *The family of geometric points of* $\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\})$ *is bounded.*

Proof. Take any geometric point $(E_1, E_2, \Phi, F_*(E_1)) \in \overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\})(K)$. By Serre duality, we have

$$H^1(\mathcal{X}_K, E_1(m-1)) = \mathrm{Hom}(E_1, \omega_{\mathcal{X}_K}(1-m))^\vee.$$

Take any nonzero homomorphism $f : E_1 \to \omega_{\mathcal{X}_K}(1-m)$. Then $(\ker f, E_2, \Phi|_{\ker f}, F_*(E_1) \cap \ker f)$ becomes a parabolic $\Lambda^1_{\mathcal{D}_K}$-subtriple of $(E_1, E_2, \Phi, F_*(E_1))$. Thus we must have the inequality

$$\mu(\ker f, E_2, \Phi|_{\ker f}, F_*(E_1) \cap \ker f) < \mu(E_1, E_2, \Phi, F_*(E_1)).$$

Since $\deg(\ker f) \geq \deg E_1 + m - 2g + 1$, we can find an integer m which depends only on $r, d, d_i, \boldsymbol{\beta}, \boldsymbol{\alpha}', \gamma$, $\mathcal{X}$ and $\mathcal{D}$ such that $\mathrm{Hom}(E_1, \omega_{\mathcal{X}_K}(1-m)) = 0$. Then all E_1 become m-regular.

Similarly we can find an integer m' such that E_2 are all m'-regular. Then the family of (E_1, E_2) is bounded and the boundedness of the family of $(E_1, E_2, \Phi, F_*(E_1))$ can be deduced from it. Λ

We put $\epsilon_i := \alpha'_{i+1} - \alpha'_i$ for $i = 1, \dots, l$. Take an S-ample line bundle $\mathcal{O}_{\mathcal{X}}(1)$ on $\mathcal{X}$.

Proposition 5.2. *There exists an integer m_0 such that for any geometric point $(E_1, E_2, \Phi, F_*(E_1)) \in \overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r, d, \{d_i\})(K)$, the inequality*

$$\frac{\beta_1 \alpha'_1 h^0(E'_1(m)) + \beta_2 h^0(E'_2(m-\gamma)) + \sum_{i=1}^{l} \beta_1 \epsilon_i h^0(F_{i+1}(E'_1)(m))}{\beta_1 \operatorname{rank}(E'_1) + \beta_2 \operatorname{rank}(E'_2)}$$
$$< \frac{\beta_1 \alpha'_1 h^0(E_1(m)) + \beta_2 h^0(E_2(m-\gamma)) + \sum_{i=1}^{l} \beta_1 \epsilon_i h^0(F_{i+1}(E_1)(m))}{\beta_1 \operatorname{rank}(E_1) + \beta_2 \operatorname{rank}(E_2)}$$

holds for any proper non-zero parabolic $\Lambda^1_{\mathcal{D}_K}$-subtriple $(E'_1, E'_2, \Phi', F_*(E'_1))$ of $(E_1, E_2, \Phi, F_*(E_1))$ and any integer $m \geq m_0$.

Proof. By Proposition 5.1, there exists an integer N_1 such that for any geometric point $(E_1, E_2, \Phi, F_*(E_1))$ of $\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r, d, \{d_i\})$, $h^i(F_j(E_1)(m)) = h^i(E_2(m-\gamma)) = 0$ for $i > 0$, $1 \leq j \leq l+1$ and $m \geq N_1$. There also exists an integer e such that for any geometric point $(E_1, E_2, \Phi, F_*(E_1))$ of $\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r, d, \{d_i\})$ and for any coherent subsheaf E' of $E_1^{\oplus\beta_1} \oplus E_2^{\oplus\beta_2}(-\gamma)$, the inequality

$$\deg E' \leq \operatorname{rank} E'(\mu(E_1^{\oplus\beta_1} \oplus E_2(-\gamma)^{\oplus\beta_2}) + e)$$

holds. Note that we write $\mu(E) := \operatorname{rank}(E)^{-1}\deg(E)$ for a vector bundle E. Applying [MY], Lemma 2.6 to the case

$$P(m) = \frac{\beta_1 \alpha'_1 \chi(E_1(m)) + \beta_2 \chi(E_2(m-\gamma)) + \sum_{i=1}^{l} \beta_1 \epsilon_i \chi(F_{i+1}(E_1)(m))}{\beta_1 \operatorname{rank} E_1 + \beta_2 \operatorname{rank} E_2} - 1,$$
$$r = \operatorname{rank}(E_1^{\oplus\beta_1} \oplus E_2^{\oplus\beta_2}), \quad a = \mu(E_1^{\oplus\beta_1} \oplus E_2(-\gamma)^{\oplus\beta_2}) + e,$$

we can take integers L, M such that $M \leq a$ and for any integer $m \geq L$, the inequality

$$h^0(E'(m)) \leq \operatorname{rank}(E')P(m)$$

holds for any vector bundle E' on a fiber of $\mathcal{X}$ over S satisfying $0 < \operatorname{rank}(E') < \beta_1 \operatorname{rank}(E_1) + \beta_2 \operatorname{rank}(E_2)$, $\mu(E') \leq M$ and $\deg \tilde{E}' \leq a \operatorname{rank}(\tilde{E}')$ for any proper nonzero coherent subsheaf $\tilde{E}'$ of E'.

Now we put

$$\mathcal{G} := \left\{ E' \;\middle|\; \begin{array}{l} \text{there exists a geometric point } (E_1, E_2, \Phi, F_*(E_1)) \text{ of } \overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r, d, \{d_i\}) \\ \text{such that } E' \text{ is a subbundle of } E_1^{\oplus\beta_1} \oplus E_2(-\gamma)^{\oplus\beta_2} \text{ and } \mu(E') \geq M \end{array} \right\}.$$

Then $\mathcal{G}$ is bounded. Thus there exists an integer $L' \geq L$ such that for any $E' \in \mathcal{G}$ and any $m \geq L'$, $E'(m-\gamma)$ is generated by its global sections, $h^i(E'(m-\gamma)) = h^i((F_j(E_1) \cap E')(m)) = 0$ for $i > 0$ and $1 \leq j \leq l+1$. If we put

$$\tilde{\mathcal{G}} := \left\{ (E'_1, E'_2) \;\middle|\; \begin{array}{l} E'_1 \subset E_1 \text{ (resp. } E'_2 \subset E_2) \text{ is a subbundle such that} \\ \Phi(\Lambda^1_{\mathcal{D}/S} \otimes E'_1) \subset E'_2 \text{ and } \mu({E'_1}^{\oplus\beta_1} \oplus E'_2(-\gamma)^{\oplus\beta_2}) \geq M \end{array} \right\},$$

then the set of polynomials

$$\left\{ \beta_1 \alpha'_1 \chi(E'_1(m)) + \beta_2 \chi(E'_2(m-\gamma)) + \sum_{i=1}^{l} \beta_1 \epsilon_i \chi((F_{i+1}(E_1) \cap E'_1)(m)) \right\}_{(E'_1, E'_2) \in \tilde{\mathcal{G}}}$$

is finite, because ${E'_1}^{\oplus\beta_1} \oplus E'_2(-\tau)^{\oplus\beta_2} \in \mathcal{G}$ for any $(E'_1, E'_2) \in \tilde{\mathcal{G}}$. Thus there exists an integer $m_0 \geq L'$ such that for any $m \geq m_0$ and for any $(E'_1, E'_2) \in \tilde{\mathcal{G}}$, the inequality

$$\frac{\beta_1\alpha'_1\chi(E'_1(m)) + \beta_2\chi(E'_2(m-\gamma)) + \sum_{i=1}^{l}\beta_1\epsilon_i\chi((F_{i+1}(E_1)\cap E'_1)(m))}{\beta_1\operatorname{rank}(E'_1) + \beta_2\operatorname{rank}(E'_2)} < P(m) + 1$$

holds. We can easily see that this m_0 satisfies the desired condition. Λ

Proposition 5.3. *Let T be a noetherian scheme over S and $(E_1, E_2, \Phi, F_*(E_1))$ be a flat family of parabolic $\Lambda^1_{\mathcal{D}_T/T}$-triples on $\mathcal{X}\times_S T$ over T. Then there is an open subscheme T^s of T such that*

$$T^s(k) = \{t \in T(k) \,|(E_1, E_2, \Phi, F_*(E_1)) \otimes k(t) \text{ is stable}\}$$

for any algebraically closed field k.

Proof. We may assume that T is connected. Put $P_1(m) := \chi((E_1 \otimes k(s))(m))$, $P_2(m) := \chi((E_2 \otimes k(s))(m-\gamma))$ and $P_1^{(i)}(m) := \chi((F_i(E_1)\otimes k(s))(m))$ for a geometric point s of T. Since the family

$$\mathcal{G} = \left\{ E' \;\middle|\; \begin{array}{l} E' \text{ is a subbundle of } (E_1^{\oplus\beta_1}\oplus E_2(-\gamma)^{\oplus\beta_2})\otimes k(s) \text{ for some geometric} \\ \text{point } s \text{ of } T \text{ and } \mu(E') \geq \mu((E_1, E_2, \Phi, F_*(E_1))\otimes k(s)) \end{array} \right\}$$

is bounded, the family

$$\tilde{\mathcal{G}} = \left\{ (E'_1, E'_2, \Phi', F_*(E'_1)) \;\middle|\; \begin{array}{l} (E'_1, E'_2, \Phi', F_*(E'_1)) \text{ is a parabolic } \Lambda^1_{\mathcal{D}_s}\text{-subtriple of} \\ (E_1, E_2, \Phi, F_*(E_1))\otimes k(s) \text{ for some geometric point } s \text{ of } T \\ \text{such that } E'_1 \subset E_1\otimes k(s) \text{ (resp. } E'_2 \subset E_2 \otimes k(s)) \text{ is a subbundle} \\ \text{and } \mu(E'_1, E'_2, \Phi', F_*(E'_1)) \geq \mu((E_1, E_2, \Phi, F_*(E_1))\otimes k(s)) \end{array} \right\}$$

is also bounded. So the set of sequences of polynomials

$$\mathcal{P} := \left\{ (\chi(E'_1(m)), \chi(E'_2(m-\gamma)), (\chi(F_{i+1}(E'_1)(m)))_{1\leq i\leq l}) \;\middle|\; (E'_1, E'_2, \Phi', F_*(E'_1)) \in \tilde{\mathcal{G}} \right\}$$

is finite. For each $\mathbf{P}' := (P'_1, P'_2, ((P'_1)^{(i+1)})) \in \mathcal{P}$, put

$$Q := \mathrm{Quot}^{P_1-P'_1}_{E_1/\mathcal{X}_T/T} \times_T \mathrm{Quot}^{P_2-P'_2}_{E_2/\mathcal{X}_T/T}$$

Let $(E_1)_Q \xrightarrow{\pi_1} G_1$ and $(E_2)_Q \xrightarrow{\pi_2} G_2$ be the universal quotient sheaves. We put

$$Q' := \mathrm{Quot}^{P'_1-(P'_1)^{(2)}}_{\ker\pi_1/\mathcal{X}_Q/Q} \times_Q \cdots \times_Q \mathrm{Quot}^{P'_1-(P'_1)^{(l+1)}}_{\ker\pi_1/\mathcal{X}_Q/Q}.$$

Let $(\ker\pi_1)_{Q'} \xrightarrow{\pi_1^{(i)}} G_1^{(i)}$ $(1 \leq i \leq l)$ be the universal quotient sheaves. We consider the composite homomorphisms

$$\Psi' : \Lambda^1_{\mathcal{D}/S}\otimes(\ker\pi_1)_{Q'} \hookrightarrow \Lambda^1_{\mathcal{D}/S}\otimes(E_1)_{Q'} \xrightarrow{\Phi_{Q'}} (E_2)_{Q'} \xrightarrow{(\pi_2)_{Q'}} (G_2)_{Q'}$$

$$\psi_i : \ker\pi_1^{(i+1)} \hookrightarrow (\ker\pi_1)_{Q'} \xrightarrow{\pi_1^{(i)}} G_1^{(i)} \quad (2 \leq i \leq l)$$

$$\psi_{l+1} : (\ker\pi_1)_{Q'}\otimes\mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \longrightarrow (\ker\pi_1)_{Q'} \xrightarrow{\pi_1^{(l+1)}} G_1^{(l+1)}.$$

Let $\tilde{Q}'_{\mathbf{P}'}$ be the maximal closed subscheme of Q' satisfying $\Psi'_{\tilde{Q}'_{\mathbf{P}'}} = 0$ and $(\psi_i)_{\tilde{Q}'_{\mathbf{P}'}} = 0$ for $2 \leq i \leq l+1$. Since $f_{\mathbf{P}'} : \tilde{Q}'_{\mathbf{P}'} \to T$ is a proper morphism,

$$T^s = T \setminus \bigcup_{\mathbf{P}'\in\mathcal{P}} f_{\mathbf{P}'}(\tilde{Q}'_{\mathbf{P}'})$$

is an open subscheme which satisfies the desired condition. Λ

5.3. **Construction of the moduli space.** Now we construct the moduli scheme of $\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\})$. We define a polynomial $P(m)$ in m by $P(m):=rd_{\mathcal{X}}m+d+r(1-g)$ where $d_{\mathcal{X}}=\deg\mathcal{O}_{\mathcal{X}_s}(1)$ for $s\in S$ and g is the genus of $\mathcal{X}_s$. We take an integer m_0 in Proposition 5.2. By Proposition 5.1, we may assume, by replacing m_0, that for any $m\geq m_0$, $h^j(F_i(E_1)(m))=h^j(E_2(m-\gamma))=0$ for $j>0$, $i=1,\dots,l+1$ and $E_2(m-\gamma)$, $F_i(E_1)(m)$ $(i=1,\dots,l+1)$ are generated by their global sections for any geometric point $(E_1,E_2,\Phi,F_*(E_1))$ of $\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\})$. Put $n_1=P(m_0)$ and $n_2=P(m_0-\gamma)$. Take two free $\mathcal{O}_S$-modules V_1,V_2 such that $\operatorname{rank}V_1=n_1$, $\operatorname{rank}V_2=n_2$. Let Q_1 be the Quot-scheme $\mathrm{Quot}^{P(m)}_{V_1\otimes\mathcal{O}_{\mathcal{X}}(-m_0)/\mathcal{X}/S}$ and $V_1\otimes\mathcal{O}_{\mathcal{X}_{Q_1}}(-m_0)\to\mathcal{E}_1$ be the universal quotient sheaf. Similarly let Q_2 be the Quot-scheme $\mathrm{Quot}^{P(m)}_{V_2\otimes\mathcal{O}_{\mathcal{X}}(-m_0+\gamma)/\mathcal{X}/S}$ and $V_2\otimes\mathcal{O}_{\mathcal{X}_{Q_2}}(-m_0+\gamma)\to\mathcal{E}_2$ be the universal quotient sheaf. We put $Q_1^{(i)}:=\mathrm{Quot}^{d_i}_{\mathcal{E}_1/\mathcal{X}_{Q_1}/Q_1}$. Let $F_{i+1}(\mathcal{E}_1)\subset(\mathcal{E}_1)_{Q_1^{(i)}}$ be the universal subsheaf. We define Q as the maximal closed subscheme of $Q_1^{(1)}\times_{Q_1}\cdots\times_{Q_1}Q_1^{(l)}\times Q_2$ such that there are factorizations

$$(\mathcal{E}_1)_Q\otimes\mathcal{O}_{\mathcal{X}_Q}(-\mathcal{D}_Q)\longrightarrow F_{i+1}(\mathcal{E}_1)_Q\hookrightarrow F_i(\mathcal{E}_1)_Q\subset(\mathcal{E}_1)_Q \tag{105}$$

for $i=1,\dots,l$, where $F_1(\mathcal{E}_1)=\mathcal{E}_1$. Since $(\mathcal{E}_2)_Q$ is flat over Q, there is a coherent sheaf $\mathcal{H}$ on Q such that there is a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{X}_T}(\Lambda^1_{\mathcal{D}/S}\otimes_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}_1)_T,(\mathcal{E}_2)_T\otimes\mathcal{L})\cong\mathrm{Hom}_T(\mathcal{H}\otimes\mathcal{O}_T,\mathcal{L}) \tag{106}$$

for any noetherian scheme T over Q and any quasi-coherent sheaf $\mathcal{L}$ on T.

We denote $\operatorname{Spec}S(\mathcal{H})$ by $\mathbf{V}^*(\mathcal{H})$, where $S(\mathcal{H})$ is the symmetric algebra of $\mathcal{H}$ over $\mathcal{O}_Q$. Let

$$\tilde{\Phi}:\Lambda^1_{\mathcal{D}/S}\otimes_{\mathcal{O}_{\mathcal{X}}}(\mathcal{E}_1)_{\mathbf{V}^*(\mathcal{H})}\longrightarrow(\mathcal{E}_2)_{\mathbf{V}^*(\mathcal{H})}$$

be the universal homomorphism. We define the open subscheme R^s of $\mathbf{V}^*(\mathcal{H})$ by

$$R^s:=\left\{s\in\mathbf{V}^*(\mathcal{H})\;\middle|\;\begin{array}{l}(V_1)_s\to H^0((\mathcal{E}_1)_s(m_0)),\ (V_2)_s\to H^0((\mathcal{E}_2)_s(m_0-\gamma))\text{ are bijective},\\ F_i(\mathcal{E}_1)_s(m_0),(\mathcal{E}_2)_s(m_0-\gamma)\text{ are generated by their global sections},\\ h^j(F_i(\mathcal{E}_1)_s(m_0))=h^j((\mathcal{E}_2)_s(m_0-\gamma))=0\text{ for }j>0,\ 1\leq i\leq l+1\\ \text{and }((\mathcal{E}_1)_s,(\mathcal{E}_2)_s,\tilde{\Phi}_s,F_*(\mathcal{E}_1)_s)\text{ is stable}\end{array}\right\}.$$

For $y\in R^s$ and vector subspaces $V_1'\subset(V_1)_y$, $V_2'\subset(V_2)_y$, let $E(V_1',V_2',y)_1$ be the image of $V_1'\otimes\mathcal{O}_{\mathcal{X}}(-m_0)\to(\mathcal{E}_1)_y$ and $E(V_1',V_2',y)_2$ be that of $V_1'\otimes\Lambda^1_{\mathcal{D}_y}(-m_0)\oplus V_2'\otimes\mathcal{O}_{\mathcal{X}}(-m_0+\gamma)\to(\mathcal{E}_2)_y$. Since the family

$$\mathcal{F}=\{(E(V_1',V_2',y)_1,E(V_1',V_2',y)_2)|y\in R^s,V_1'\subset(V_1)_y,V_2'\subset(V_2)_y\}$$

is bounded, there exists an integer $m_1(\geq m_0)$ such that for all $m\geq m_1$,

$$V_1'\otimes H^0(\mathcal{O}_{\mathcal{X}_y}(m))\to H^0(E(V_1',V_2',y)_1(m)),$$
$$V_1'\otimes H^0(\mathcal{O}_{\mathcal{X}_y}(m_0+m-\gamma)\otimes\Lambda^1_{\mathcal{D}_y}\otimes\mathcal{O}_{\mathcal{X}_y}(-m_0))\oplus V_2'\otimes H^0(\mathcal{O}_{\mathcal{X}_y}(m))\to H^0(E(V_1',V_2',y)_2(m-\gamma))$$

are surjective and $H^i(\mathcal{O}_{\mathcal{X}_y}(m_0+m-\gamma)\otimes\Lambda^1_{\mathcal{D}_y}\otimes\mathcal{O}_{\mathcal{X}_y}(-m_0))=0$, $H^i(\mathcal{O}_{\mathcal{X}_y}(m))=0$ for $i>0$ for all members $(E(V_1',V_2',y)_1,E(V_1',V_2',y)_2)\in\mathcal{F}$ and the inequality

$$
\begin{aligned}
&(\beta_1 \operatorname{rank} E_1' + \beta_2 \operatorname{rank} E_2') d_{\mathcal{X}} \left(\beta_1 h^0((\mathcal{E}_1)_y(m_0)) + \beta_2 h^0((\mathcal{E}_2)_y(m_0-\gamma)) - \sum_{i=1}^{l} \beta_1 \epsilon_i d_i \right) \\
&\quad - (\beta_1+\beta_2) r d_{\mathcal{X}} \Big(\beta_1 \alpha_1' h^0(E_1'(m_0)) + \beta_2 h^0(E_2'(m_0-\gamma)) + \sum_{i=1}^{l} \beta_1 \epsilon_i h^0(F_{i+1}(E_1')(m_0)) \Big) \\
&> m^{-1} \Big(\beta_1 \dim V_1' + \beta_2 \dim V_2' - \beta_1 \chi(E_1'(m_0)) - \beta_2 \chi(E_2'(m_0-\gamma)) \Big) (\beta_1 \dim V_1 + \beta_2 \dim V_2 - \sum_{i=1}^{l} \beta_1 \epsilon_i d_i)
\end{aligned}
$$

holds for $(0,0) \subsetneq (V_1',V_2') \subsetneq ((V_1)_y,(V_2)_y)$, where $E_1' := E(V_1',V_2',y)_1$, $E_2' := E(V_1',V_2',y)_2$ and $F_{i+1}(E_1') := E_1' \cap F_{i+1}(\mathcal{E}_1)_y$ for $i=1,\dots,l$. From now on, we fix such a large integer m_1.

The composite

$$V_1 \otimes \Lambda^1_{\mathcal{D}/S} \otimes \mathcal{O}_{\mathcal{X}}(-m_0)_{R^s} \to \Lambda^1_{\mathcal{D}/S} \otimes (\mathcal{E}_1)_{R^s} \xrightarrow{\tilde{\Phi}} (\mathcal{E}_2)_{R^s}$$

induces a homomorphism

$$V_1 \otimes W_1 \otimes \mathcal{O}_{R^s} \to (\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}),$$

where $W_1 := (\pi_S)_*(\mathcal{O}_{\mathcal{X}}(m_0+m_1-\gamma)\otimes\Lambda^1_{\mathcal{D}/S}\otimes\mathcal{O}_{\mathcal{X}}(-m_0))$ and the quotient $V_2\otimes\mathcal{O}_{\mathcal{X}}(-m_0+\gamma)\to\mathcal{E}_2$ induces a homomorphism

$$V_2 \otimes W_2 \otimes \mathcal{O}_{R^s} \to (\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}),$$

where $W_2 := (\pi_S)_*(\mathcal{O}_{\mathcal{X}}(m_1))$. These homomorphisms induce a quotient bundle

$$(V_1\otimes W_1 \oplus V_2 \otimes W_2)\otimes\mathcal{O}_{R^s} \longrightarrow (\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}).$$

This quotient and the canonical quotient bundles

$$
\begin{aligned}
&V_1\otimes W_2\otimes\mathcal{O}_{R^s} = V_1\otimes(\pi_S)_*(\mathcal{O}_{\mathcal{X}}(m_1))\otimes\mathcal{O}_{R^s} \to (\pi_{R^s})_*(\mathcal{E}_1(m_0+m_1)_{R^s}),\\
&V_1\otimes\mathcal{O}_{R^s} \to (\pi_{R^s})_*(\mathcal{E}_1/F_{i+1}(\mathcal{E}_1)(m_0)_{R^s}) \quad (i=1,\dots,l)
\end{aligned}
$$

determine a morphism

$$\iota : R^s \to \mathrm{Grass}_{r_2}(V_1\otimes W_1\oplus V_2\otimes W_2)\times\mathrm{Grass}_{r_1}(V_1\otimes W_2)\times\prod_{i=1}^{l}\mathrm{Grass}_{d_i}(V_1),$$

where $r_1 = h^0(\mathcal{E}_1(m_0+m_1)_s)$, $r_2 := h^0(\mathcal{E}_2(m_0+m_1-\gamma)_s)$ for any point $s\in R^s$ and $\mathrm{Grass}_r(V)$ is the Grassmannian parametrizing r-dimensional quotient vector spaces of V. We can check that ι is an immersion.

We set $G := (GL(V_1)\times GL(V_2))/(\mathbf{G}_m\times S)$, where $\mathbf{G}_m\times S$ is contained in $GL(V_1)\times GL(V_2)$ as scalar matrices. Then G acts canonically on R^s and on $\mathrm{Grass}_{r_2}(V_1\otimes W_1\oplus V_2\otimes W_2)\times\mathrm{Grass}_{r_1}(V_1\otimes W_2)\times\prod_{i=1}^{l}\mathrm{Grass}_{d_i}(V_1)$. We can see that ι is a G-equivariant immersion. There is an S-ample line bundle $\mathcal{O}_{\mathrm{Grass}_{r_2}(V_1\otimes W_1\oplus V_2\otimes W_2)}(1)$ on $\mathrm{Grass}_{r_2}(V_1\otimes W_1\oplus V_2\otimes W_2)$ induced by Plücker embedding. Similarly there are canonical S-ample line bundles $\mathcal{O}_{\mathrm{Grass}_{r_1}(V_1\otimes W_2)}(1)$, $\mathcal{O}_{\mathrm{Grass}_{d_i}(V_1)}(1)$, on $\mathrm{Grass}_{r_1}(V_1\otimes W_2)$, $\mathrm{Grass}_{d_i}(V_1)$, respectively. We define positive rational numbers ν_1, ν_2, $\nu_1^{(i)}$ $(1 \leq i \leq l)$ by

$$\begin{aligned}
\nu_1 &= \beta_1(\beta_1 P(m_0) + \beta_2 P(m_0-\gamma) - \sum_{i=1}^{l} \beta_1 \epsilon_i d_i), \\
\nu_2 &= \beta_2(\beta_1 P(m_0) + \beta_2 P(m_0-\gamma) - \sum_{i=1}^{l} \beta_1 \epsilon_i d_i), \\
\nu_1^{(i)} &= (\beta_1+\beta_2)\beta_1 r d_{\mathcal{X}} m_1 \epsilon_i.
\end{aligned}$$

Let us consider the $\mathbf{Q}$-line bundle

$$L := \iota^* \left(\mathcal{O}_{\mathrm{Grass}_{r_2}(V_1\otimes W_1 \oplus V_2 \otimes W_2)}(\nu_1) \otimes \mathcal{O}_{\mathrm{Grass}_{r_1}(V_1 \otimes W_2)}(\nu_2) \otimes \bigotimes_{i=1}^{l} \mathcal{O}_{\mathrm{Grass}_{d_i}(V_1)}(\nu_1^{(i)}) \right)$$

on R^s. Then for some positive integer N, $L^{\otimes N}$ becomes a G-linearized S-ample line bundle on R^s.

Proposition 5.4. *All points of R^s are properly stable with respect to the action of G and the G-linearized S-ample line bundle $L^{\otimes N}$.*

Proof. Take any geometric point x of R^s. Let y be the induced geometric point of S. We must show that x is a properly stable point of the fiber R^s_y with respect to the action of G_y and the polarization $L_y^{\otimes N}$. So we may assume that $S = \operatorname{Spec} K$ with K an algebraically closed field. We put

$$(E_1, E_2, \Phi, F_*(E_1)) := ((\mathcal{E}_1)_x, (\mathcal{E}_2)_x, \tilde{\Phi}_x, F_*(\mathcal{E}_1)_x).$$

Let

$$\pi_2 : V_1 \otimes W_1 \oplus V_2 \otimes W_2 \to N_2, \quad \pi_1 : V_1 \otimes W_2 \to N_1, \quad \pi_1^{(i)} : V_1 \to N_1^{(i)} \quad (i = 1, \dots, l)$$

be the quotient vector spaces corresponding to $\iota(x)$. We will show that $\iota(x)$ is a properly stable point with respect to the action of G and the linearization of $L^{\otimes N}$. Consider the character

$$\chi : GL(V_1) \times GL(V_2) \longrightarrow \mathbf{G}_m; \quad (g_1, g_2) \mapsto \det(g_1)^{\beta_1} \det(g_2)^{\beta_2}.$$

Then there is an isogeny $\ker\chi \longrightarrow G$ and we may prove the stability with respect to the action of $\ker\chi$ instead of G. Take any one parameter subgroup λ of $\ker\chi$. For a suitable basis $e_1^{(1)}, \dots, e_{n_1}^{(1)}$ (resp. $e_1^{(2)}, \dots, e_{n_2}^{(2)}$) of V_1 (resp. V_2), the action of λ on V_1 (resp. V_2) is represented by

$$e_i^{(1)} \mapsto t^{u_i^{(1)}} e_i^{(1)} \ (\text{resp.}\ e_i^{(2)} \mapsto t^{u_i^{(2)}} e_i^{(2)}) \quad (t \in \mathbf{G}_m),$$

where $u_1^{(1)} \leq \cdots \leq u_{n_1}^{(1)}$ (resp. $u_1^{(2)} \leq \cdots \leq u_{n_2}^{(2)}$) and $\sum_{i=1}^{n_1} \beta_1 u_i^{(1)} + \sum_{i=1}^{n_2} \beta_2 u_i^{(2)} = 0$. Take a basis $f_1^{(k)}, \dots, f_{b_k}^{(k)}$ of W_k for $k = 1, 2$.

We define functions $a_1(p)$ and $a_2(p)$ in $p \in \{0, 1, \dots, \beta_1 n_1 + \beta_2 n_2\}$ as follows. First we put $(a_1(0), a_2(0)) := (0, 0)$. We put

$$(a_1(1), a_2(1)) := \begin{cases} (1,0) & \text{if } \beta_1 u_1^{(1)} \leq \beta_2 u_1^{(2)} \\ (0,1) & \text{if } \beta_1 u_1^{(1)} > \beta_2 u_2^{(2)}. \end{cases}$$

Inductively we define

$$\begin{cases} (a_1(p+1), a_2(p+1)) := (a_1(p), a_2(p)) & \text{if } p < \beta_1 a_1(p) + \beta_2 a_2(p) \\ (a_1(p+1), a_2(p+1)) := (a_1(p)+1, a_2(p)) & \text{if } p = \beta_1 a_1(p) + \beta_2 a_2(p),\ \beta_1 u^{(1)}_{a_1(p)+1} \leq \beta_2 u^{(2)}_{a_2(p)+1} \\ & \text{and } a_1(p) < n_1 \\ (a_1(p+1), a_2(p+1)) := (a_1(p), a_2(p)+1) & \text{if } p = \beta_1 a_1(p) + \beta_2 a_2(p),\ \beta_1 u^{(1)}_{a_1(p)+1} > \beta_2 u^{(2)}_{a_2(p)+1} \\ & \text{and } a_2(p) < n_2. \\ (a_1(p+1), a_2(p+1)) := (a_1(p)+1, a_2(p)) & \text{if } p = \beta_1 a_1(p) + \beta_2 a_2(p) \text{ and } a_2(p) = n_2 \\ (a_1(p+1), a_2(p+1)) := (a_1(p), a_2(p)+1) & \text{if } p = \beta_1 a_1(p) + \beta_2 a_2(p) \text{ and } a_1(p) = n_1 \end{cases}$$

Then $a_1(p)$ and $a_2(p)$ are integers with $0 \leq a_1(p) \leq n_1$, $0 \leq a_2(p) \leq n_2$, $a_1(p) \leq a_1(p+1)$ and $a_2(p) \leq a_2(p+1)$. We define $v_1, \ldots, v_{\beta_1 n_1 + \beta_2 n_2}$ and $e'_1, \ldots, e'_{\beta_1 n_1 + \beta_2 n_2}$ by

$$\begin{cases} v_p := \beta_1 u^{(1)}_{a_1(p)},\ e'_p := e^{(1)}_{a_1(p)} & \text{if } a_1(p-1) < a_1(p) \\ v_p := \beta_2 u^{(2)}_{a_2(p)},\ e'_p := e^{(2)}_{a_2(p)} & \text{if } a_2(p-1) < a_2(p) \\ v_p := v_{p-1},\ e'_p := e'_{p-1} & \text{if } a_1(p-1) = a_1(p) \text{ and } a_2(p-1) = a_2(p). \end{cases}$$

We put $\delta_p := (v_{p+1} - v_p)(\beta_1 n_1 + \beta_2 n_2)^{-1}$ for $p = 1, \ldots, \beta_1 n_1 + \beta_2 n_2 - 1$. Then δ_p are non-negative rational numbers and for each $1 \leq i \leq n_1$

$$\beta_1 u^{(1)}_i = \sum_{\substack{1 \leq p \leq \beta_1 n_1 + \beta_2 n_2 - 1 \\ a_1(p) < i}} p\delta_p + \sum_{\substack{1 \leq p \leq \beta_1 n_1 + \beta_2 n_2 - 1 \\ a_1(p) \geq i}} (p - \beta_1 n_1 - \beta_2 n_2)\delta_p$$

and for each $1 \leq i \leq n_2$

$$\beta_2 u^{(2)}_i = \sum_{\substack{1 \leq p \leq \beta_1 n_1 + \beta_2 n_2 - 1 \\ a_2(p) < i}} p\delta_p + \sum_{\substack{1 \leq p \leq \beta_1 n_1 + \beta_2 n_2 - 1 \\ a_2(p) \geq i}} (p - \beta_1 n_1 - \beta_2 n_2)\delta_p.$$

For $\mu = 1, \ldots, \beta_1 n_1 b_1 + \beta_2 n_2 b_2$, we can find unique integers $p_0, p_1 \in \{0, 1, \ldots, \beta_1 n_1 + \beta_2 n_2\}$ such that

$$(a_1(p_1), a_2(p_1)) = (a_1(p_0+1), a_2(p_0+1)) = (a_1(p_0)+1, a_2(p_0)), \text{ or}$$
$$(a_1(p_1), a_2(p_1)) = (a_1(p_0+1), a_2(p_0+1) = (a_1(p_0), a_2(p_0)+1)$$

and

$$\mu = \begin{cases} (a_1(p_0)\beta_1 + (p_1 - p_0 - 1))b_1 + a_2(p_0)\beta_2 b_2 + j & \text{for some } 1 \leq j \leq b_1 \\ & \text{if } (a_1(p_1), a_2(p_1)) = (a_1(p_0)+1, a_2(p_0)) \\ a_1(p_0)\beta_1 b_1 + (a_2(p_0)\beta_2 + (p_1 - p_0 - 1))b_2 + j & \text{for some } 1 \leq j \leq b_2 \\ & \text{if } (a_1(p_1), a_2(p_1)) = (a_1(p_0), a_2(p_0)+1). \end{cases}$$

For such μ, we put $s^{(2)}_\mu := v_{p_1}$ and

$$h'_\mu := \begin{cases} e'_{p_1} \otimes f^{(1)}_j & \text{if } (a_1(p_1), a_2(p_1)) = (a_1(p_0)+1, a_2(p_0)) \\ e'_{p_1} \otimes f^{(2)}_j & \text{if } (a_1(p_1), a_2(p_1)) = (a_1(p_0), a_2(p_0)+1). \end{cases}$$

Let $U^{(2)}_\mu$ be the vector subspace of $V_1 \otimes W_1 \oplus V_2 \otimes W_2$ generated by $h'_1, \ldots, h'_\mu$. We put $U^{(2)}_0 = 0$. For $q = 1, \ldots, r_2$, we can find an integer $\mu^{(2)}_q \in \{1, \ldots, \beta_1 n_1 b_1 + \beta_2 n_2 b_2\}$ such that $\dim \pi_2(U^{(2)}_{\mu^{(2)}_q}) = q$ and

$\dim \pi_2(U^{(2)}_{\mu^{(2)}_q-1}) = q-1$. Then

$$\begin{aligned}
\sum_{q=1}^{r_2} s^{(2)}_{\mu^{(2)}_q} &= \sum_{q=1}^{r_2} s^{(2)}_{\mu^{(2)}_q}\left(\dim \pi_2(U^{(2)}_{\mu^{(2)}_q}) - \dim \pi_2(U^{(2)}_{\mu^{(2)}_q-1})\right) \\
&= \sum_{\mu=1}^{\beta_1 n_1 b_1+\beta_2 n_2 b_2} s^{(2)}_{\mu}\left(\dim \pi_2(U^{(2)}_{\mu}) - \dim \pi_2(U^{(2)}_{\mu-1})\right) \\
&= r_2 s^{(2)}_{\beta_1 n_1 b_1+\beta_2 n_2 b_2} - \sum_{\mu=1}^{\beta_1 n_1 b_1+\beta_2 n_2 b_2-1} (s^{(2)}_{\mu+1}-s^{(2)}_{\mu}) \dim \pi_2(U^{(2)}_{\mu}) \\
&= r_2 v_{\beta_1 n_1+\beta_2 n_2} - \sum_{p=1}^{\beta_1 n_1+\beta_2 n_2-1} (v_{p+1}-v_p) \dim \pi_2(U^{(2)}_{\beta_1 a_1(p) b_1+\beta_2 a_2(p) b_2}) \\
&= \sum_{p=1}^{\beta_1 n_1+\beta_2 n_2-1} \left(r_2 p - (\beta_1 n_1+\beta_2 n_2)\dim \pi_2(U^{(2)}_{\beta_1 a_1(p) b_1+\beta_2 a_2(p) b_2})\right)\delta_p.
\end{aligned}$$

For $\mu = (i-1)b_2 + j$, put $h^{(1)}_{\mu} := e^{(1)}_i \otimes f^{(2)}_j$ for $i = 1, \dots, n_1$, $j = 1, \dots, b_2$. We define integers $s^{(1)}_1, \dots, s^{(1)}_{b_2 n_1}$ by putting $s^{(1)}_{\mu} := \beta_1 u^{(1)}_i$ for $\mu = (i-1)b_2 + j$ with $1 \le j \le b_2$. Let $U^{(1)}_{\mu}$ be the vector subspace of $V_1 \otimes W_2$ generated by $h^{(1)}_1, \dots, h^{(1)}_{\mu}$ for $\mu = 1, \dots, b_2 n_1$. We put $U^{(1)}_0 = 0$. For $q = 1, \dots, r_1$, let $\mu^{(1)}_q$ be the integer such that $\dim \pi_1(U^{(1)}_{\mu^{(1)}_q}) = q$ and $\dim \pi_1(U^{(1)}_{\mu^{(1)}_q-1}) = q-1$. Then

$$\begin{aligned}
\sum_{q=1}^{r_1} s^{(1)}_{\mu^{(1)}_q} &= \sum_{\mu=1}^{b_2 n_1} s^{(1)}_{\mu}\left(\dim \pi_1(U^{(1)}_{\mu}) - \dim \pi_1(U^{(1)}_{\mu-1})\right) \\
&= r_1 s^{(1)}_{b_2 n_1} - \sum_{\mu=1}^{b_2 n_1-1} (s^{(1)}_{\mu+1}-s^{(1)}_{\mu}) \dim \pi_1(U^{(1)}_{\mu}) \\
&= r_1 \beta_1 u^{(1)}_{n_1} - \sum_{i=1}^{n_1-1} (u^{(1)}_{i+1}-u^{(1)}_{i})\beta_1 \dim \pi_1(U^{(1)}_{i b_2}) \\
&= r_1 \beta_1 u^{(1)}_{n_1} + \sum_{a_1(p)<n_1} (v_{p+1}-v_p) \dim \pi_1(U^{(1)}_{a_1(p) b_2}) \\
&= r_1 \left(\sum_{\substack{1\le p\le \beta_1 n_1+\beta_2 n_2-1 \\ a_1(p)<n_1}} p\delta_p + \sum_{\substack{1\le p\le \beta_1 n_1+\beta_2 n_2-1 \\ a_1(p)\ge n_1}} (p-\beta_1 n_1-\beta_2 n_2)\delta_p \right) \\
&\quad - \sum_{\substack{1\le p\le \beta_1 n_1+\beta_2 n_2-1 \\ a_1(p)<n_1}} (\beta_1 n_1+\beta_2 n_2)\delta_p \dim \pi_1(U^{(1)}_{a_1(p) b_2}) \\
&= \sum_{p=1}^{\beta_1 n_1+\beta_2 n_2-1} \left(r_1 p - (\beta_1 n_1+\beta_2 n_2)\dim \pi_1(U^{(1)}_{a_1(p) b_2})\right)\delta_p
\end{aligned}$$

Let $V^{(1)}_p$ be the vector subspace of V_1 generated by $e^{(1)}_1, \dots, e^{(1)}_p$. We put $V^{(1)}_0 = 0$. For $i = 1, \dots, l$ and for $q = 1, \dots, d_i$, let μ^i_q be the integer such that $\dim \pi^{(i)}_1(V^{(1)}_{\mu^i_q}) = q$ and $\dim \pi^{(i)}_1(V^{(1)}_{\mu^i_q-1}) = q-1$.

Then

$$\begin{aligned}
\sum_{q=1}^{d_i}\beta_1 u^{(1)}_{\mu^i_q} &= \sum_{q=1}^{d_i}\beta_1 u^{(1)}_{\mu^i_q}\left(\dim\pi_1^{(i)}(V^{(1)}_{\mu^i_q})-\dim\pi_1^{(i)}(V^{(1)}_{\mu^i_q-1})\right)\\
&= \sum_{p=1}^{n_1}\beta_1 u^{(1)}_{p}\left(\dim\pi_1^{(i)}(V^{(1)}_{p})-\dim\pi_1^{(i)}(V^{(1)}_{p-1})\right)\\
&= d_i\beta_1 u^{(1)}_{n_1}-\sum_{p=1}^{n_1-1}\beta_1(u^{(1)}_{p+1}-u^{(1)}_p)\dim\pi_1^{(i)}(V^{(1)}_p)\\
&= d_i\beta_1 u^{(1)}_{n_1}-\sum_{a_1(p)<n_1}(v_{p+1}-v_p)\dim\pi_1^{(i)}(V^{(1)}_{a_1(p)})\\
&= d_i\left(\sum_{\substack{1\le p\le\beta_1n_1+\beta_2n_2-1\\ a_1(p)<n_1}}p\delta_p+\sum_{\substack{1\le p\le\beta_1n_1+\beta_2n_2-1\\ a_1(p)\ge n_1}}(p-\beta_1n_1-\beta_2n_2)\delta_p\right)\\
&\quad-\sum_{\substack{1\le p\le\beta_1n_1+\beta_2n_2-1\\ a_1(p)<n_1}}(\beta_1n_1+\beta_2n_2)\delta_p\dim\pi_1^{(i)}(V^{(1)}_{a_1(p)})\\
&= \sum_{p=1}^{\beta_1n_1+\beta_2n_2-1}\left(d_ip-(\beta_1n_1+\beta_2n_2)\dim\pi_1^{(i)}(V^{(1)}_{a_1(p)})\right)\delta_p.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mu^{L^{\otimes N}}(x,\lambda) &= -\left(\sum_{k=1}^{2}\nu_k\sum_{q=1}^{r_k}s^{(k)}_{\mu^{(k)}_q}+\sum_{i=1}^{l}\nu_1^{(i)}\sum_{q=1}^{d_i}\beta_1u^{(1)}_{\mu^i_q}\right)N\\
&= \sum_{p=1}^{\beta_1n_1+\beta_2n_2-1}N\delta_p\Bigg\{\left(-p\sum_{i=1}^{l}\nu_1^{(i)}d_i+(\beta_1n_1+\beta_2n_2)\sum_{i=1}^{l}\nu_1^{(i)}\dim\pi_1^{(i)}(V^{(1)}_{a_1(p)})\right)\\
&\quad-(\nu_1r_1+\nu_2r_2)p+(\beta_1n_1+\beta_2n_2)\left(\nu_1\dim\pi_1(U^{(1)}_{a_1(p)b_2})+\nu_2\dim\pi_2(U^{(2)}_{\beta_1a_1(p)b_1+\beta_2a_2(p)b_2})\right)\Bigg\}.
\end{aligned}$$

See [Mum], Definition 2.2 for the definition of $\mu^{L^{\otimes N}}(x,\lambda)$. By [Mum], Theorem 2.1, x is a properly stable point if

$$\begin{aligned}
&-p(\nu_1r_1+\nu_2r_2)+(\beta_1n_1+\beta_2n_2)(\nu_1\dim\pi_1(U^{(1)}_{a_1(p)b_2})+\nu_2\dim\pi_2(U^{(2)}_{\beta_1a_1(p)b_1+\beta_2a_2(p)b_2}))\\
&-p\sum_{i=1}^{l}\nu_1^{(i)}d_i+(\beta_1n_1+\beta_2n_2)\sum_{i=1}^{l}\nu_1^{(i)}\dim\pi_1^{(i)}(V^{(1)}_{a_1(p)})>0
\end{aligned}$$

for all $p=1,\dots,\beta_1n_1+\beta_2n_2-1$.

For each p $(1\le p\le\beta_1n_1+\beta_2n_2-1)$, let V'_k be the vector subspace of V_k generated by $e^{(k)}_1,\dots,e^{(k)}_{a_k(p)}$ for $k=1,2$. Then $U^{(1)}_{a_1(p)b_2}=V'_1\otimes W_2$ and $U^{(2)}_{\beta_1a_1(p)b_1+\beta_2a_2(p)b_2}=V'_1\otimes W_1\oplus V'_2\otimes W_2$. Put

$$\begin{aligned}
&E'_1:=\mathrm{Im}(V'_1\otimes\mathcal{O}_{\mathcal{X}}(-m_0)\to E_1),\quad F_{i+1}(E'_1):=F_{i+1}(E_1)\cap E'_1,\quad (i=1,\dots,l),\\
&E'_2:=\mathrm{Im}(V'_1\otimes\Lambda^1_{\mathcal{D}/S}(-m_0)\oplus V'_2\otimes\mathcal{O}_{\mathcal{X}}(-m_0+\gamma)\to E_2),\quad \Phi':=\Phi|_{\Lambda^1_{\mathcal{D}/S}\otimes E'_1}.
\end{aligned}$$

Then $(E'_1,E'_2,\Phi',F_*(E'_1))$ is a parabolic $\Lambda^1_{\mathcal{D}_K}$-subtriple of $(E_1,E_2,\Phi,F_*(E_1))$. By the choice of m_1, we have $\pi_2(U^{(2)}_{\beta_1a_1(p)b_1+\beta_2a_2(p)b_2})=H^0(E'_2(m_0+m_1-\gamma))$ and $\pi_1(U^{(1)}_{a_1(p)b_2})=H^0(E'_1(m_0+m_1))$. Put

$r'_1 := \operatorname{rank} E'_1$, $r'_2 := \operatorname{rank} E'_2$. Let ${V'}_1^{(i)}$ be the kernel of the composite $V'_1 \hookrightarrow V_1 \xrightarrow{\pi_1^{(i)}} N_1^{(i)}$. Then we have

$$
\begin{aligned}
&-p(\nu_1 r_1+\nu_2 r_2)+(\beta_1 n_1+\beta_2 n_2)(\nu_1 \dim \pi_1(U^{(1)}_{a_1(p)b_2})+\nu_2 \dim \pi_2(U^{(2)}_{\beta_1 a_1(p)b_1+\beta_2 a_2(p)b_2}))\\
&\quad -p\sum_{i=1}^{l}\nu_1^{(i)}d_i+(\beta_1 n_1+\beta_2 n_2)\sum_{i=1}^{l}\nu_1^{(i)}\dim \pi_1^{(i)}(V^{(1)}_{a_1(p)})\\
&\geq (\beta_1 \dim V_1+\beta_2 \dim V_2-\sum_{i=1}^{l}\beta_1\epsilon_i d_i)\times\\
&\quad \Big\{-(\beta_1 \dim V'_1+\beta_2 \dim V'_2)(\beta_1 h^0(E_1(m_0+m_1))+\beta_2 h^0(E_2(m_0+m_1-\gamma)))\\
&\qquad\qquad +(\beta_1 \dim V_1+\beta_2 \dim V_2)(\beta_1 h^0(E'_1(m_0+m_1))+\beta_2 h^0(E'_2(m_0+m_1-\gamma)))\Big\}\\
&\quad -(\beta_1 \dim V'_1+\beta_2 \dim V'_2)\sum_{i=1}^{l}\nu_1^{(i)}d_i+(\beta_1 \dim V_1+\beta_2 \dim V_2)\sum_{i=1}^{l}\nu_1^{(i)}(\dim V'_1-\dim {V'}_1^{(i)})\\
&=(\beta_1 \dim V_1+\beta_2 \dim V_2-\sum_{i=1}^{l}\beta_1\epsilon_i d_i)\times\\
&\quad \Big\{-(\beta_1 \dim V'_1+\beta_2 \dim V'_2)\,(rd_{\mathcal{X}}(\beta_1+\beta_2)m_1+\beta_1 \dim V_1+\beta_2 \dim V_2)\\
&\quad +(\beta_1 \dim V_1+\beta_2 \dim V_2)\,((\beta_1 r'_1+\beta_2 r'_2)d_{\mathcal{X}}m_1+\beta_1\chi(E'_1(m_0))+\beta_2\chi(E'_2(m_0-\gamma)))\Big\}\\
&\quad -(\beta_1 \dim V'_1+\beta_2 \dim V'_2)\sum_{i=1}^{l}\nu_1^{(i)}d_i+(\beta_1 \dim V_1+\beta_2 \dim V_2)\sum_{i=1}^{l}\nu_1^{(i)}(\dim V'_1-\dim {V'}_1^{(i)})\\
&=(\beta_1 \dim V_1+\beta_2 \dim V_2-\sum_{i=1}^{l}\beta_1\epsilon_i d_i)\times\\
&\quad \Big\{-rd_{\mathcal{X}}(\beta_1+\beta_2)m_1(\beta_1 \dim V'_1+\beta_2 \dim V'_2)+(\beta_1 r'_1+\beta_2 r'_2)d_{\mathcal{X}}m_1(\beta_1 \dim V_1+\beta_2 \dim V_2)\Big\}\\
&\quad +(\beta_1 \dim V_1+\beta_2 \dim V_2-\sum_{i=1}^{l}\beta_1\epsilon_i d_i)(\beta_1 \dim V_1+\beta_2 \dim V_2)\times\\
&\quad \Big\{-(\beta_1 \dim V'_1+\beta_2 \dim V'_2)+(\beta_1\chi(E'_1(m_0))+\beta_2\chi(E'_2(m_0-\gamma)))\Big\}\\
&\quad -(\beta_1 \dim V'_1+\beta_2 \dim V'_2)\sum_{i=1}^{l}(\beta_1+\beta_2)\beta_1 rd_{\mathcal{X}}m_1\epsilon_i d_i\\
&\quad +(\beta_1 \dim V_1+\beta_2 \dim V_2)\sum_{i=1}^{l}(\beta_1+\beta_2)\beta_1 rd_{\mathcal{X}}m_1\epsilon_i(\dim V'_1-\dim {V'}_1^{(i)})\\
&=-(\beta_1 \dim V_1+\beta_2 \dim V_2)(\beta_1+\beta_2)rd_{\mathcal{X}}m_1\Big(\beta_1 \dim V'_1+\beta_2 \dim V'_2-\sum_{i=1}^{l}\beta_1\epsilon_i(\dim V'_1-\dim {V'}_1^{(i)})\Big)\\
&\quad +(\beta_1 \dim V_1+\beta_2 \dim V_2)(\beta_1 r'_1+\beta_2 r'_2)d_{\mathcal{X}}m_1\Big(\beta_1 \dim V_1+\beta_2 \dim V_2-\sum_{i=1}^{l}\beta_1\epsilon_i d_i\Big)\\
&\quad +\Big(\beta_1 \dim V_1+\beta_2 \dim V_2-\sum_{i=1}^{l}\beta_1\epsilon_i d_i\Big)(\beta_1 \dim V_1+\beta_2 \dim V_2)\times\\
&\quad \Big(-(\beta_1 \dim V'_1+\beta_2 \dim V'_2)+(\beta_1\chi(E'_1(m_0))+\beta_2\chi(E'_2(m_0-\gamma)))\Big)
\end{aligned}
$$

$$\geq (\beta_1 \dim V_1 + \beta_2 \dim V_2)\Big\{(\beta_1 r'_1 + \beta_2 r'_2) d_{\mathcal{X}} m_1 \Big(\beta_1 h^0(E_1(m_0)) + \beta_2 h^0(E_2(m_0-\gamma)) - \sum_{i=1}^{l} \beta_1 \epsilon_i d_i\Big)$$
$$- (\beta_1+\beta_2) r d_{\mathcal{X}} m_1 \Big(\beta_1 h^0(E'_1(m_0)) + \beta_2 h^0(E'_2(m_0-\gamma)) - \sum_{i=1}^{l} \beta_1 \epsilon_i (h^0(E'_1(m_0)) - h^0(F_{i+1}(E'_1)(m_0)))\Big)\Big\}$$
$$+ \Big(\beta_1 \dim V_1 + \beta_2 \dim V_2 - \sum_{i=1}^{l} \beta_1 \epsilon_i d_i\Big)(\beta_1 \dim V_1 + \beta_2 \dim V_2) \times$$
$$\Big(- (\beta_1 \dim V'_1 + \beta_2 \dim V'_2) + (\beta_1 \chi(E'_1(m_0)) + \beta_2 \chi(E'_2(m_0-\gamma)))\Big)$$
$$> 0.$$

Note that the last inequality holds by the choice of m_1. Hence x is a properly stable point. Λ

By Proposition 5.4, there exists a geometric quotient R^s/G. The following proposition follows from a standard argument.

Theorem 5.1. *$\overline{M^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\}) := R^s/G$ is a coarse moduli scheme of $\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\})$.*

Remark 5.2. The quotient map $R^s \to \overline{M^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r,d,\{d_i\})$ is a principal G-bundle, which we can see by the following lemma and the same argument as [M], Proposition 6.4.

Lemma 5.1. *Take any geometric point $(E_1,E_2,\Phi,F_*(E_1)) \in \overline{M^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}(r,d,\{d_i\})}(K)$. Then for any endomorphisms $f_1 : E_1 \to E_1$, $f_2 : E_2 \to E_2$ satisfying $\Phi\circ(1\otimes f_1) = f_2\circ\Phi$ and $f_1(F_{i+1}(E_1)) \subset F_{i+1}(E_1)$ for $1 \leq i \leq l$, there exists $c \in K$ such that $(f_1,f_2) = (c\cdot \mathrm{id}_{E_1}, c\cdot\mathrm{id}_{E_2})$.*

Proof. Take such (f_1,f_2). Let $c \in K$ be an eigenvalue of $f_1 \otimes k(x)$ for some $x \in \mathcal{X}_K(K)$. Then $f_1 - c\cdot\mathrm{id}_{E_1}$ becomes an endomorphism of E_1 which is not an isomorphism. Put $E'_1 := \mathrm{Im}(f_1 - c\cdot\mathrm{id}_{E_1})$, $E'_2 := \mathrm{Im}(f_2 - c\cdot\mathrm{id}_{E_2})$, $\Phi' := \Phi|_{\Lambda^1_{\mathcal{D}_K}\otimes E'_1}$ and $F_{i+1}(E'_1) := (f_1 - c\cdot\mathrm{id}_{E_1})(F_{i+1}(E_1))$ for $i = 1,\dots,l$. Then $(E'_1,E'_2,\Phi',F_*(E'_1))$ becomes a parabolic $\Lambda^1_{\mathcal{D}_K}$-subtriple of $(E_1,E_2,\Phi,F_*(E_1))$. If we put $G_1 := \ker(E_1 \to E'_1)$, $G_2 := \ker(E_2 \to E'_2)$, $\Phi_G := \Phi|_{\Lambda^1_{\mathcal{D}_K}\otimes G_1}$ and $F_{i+1}(G_1) := F_{i+1}(E_1)\cap G_1$ for $i = 1,\dots,l$, then $(G_1,G_2,\Phi_G,F_*(G_1))$ becomes a parabolic $\Lambda^1_{\mathcal{D}_K}$-subtriple of $(E_1,E_2,\Phi,F_*(E_1))$. If $(E'_1,E'_2) \neq (0,0)$, then, by the stability of $(E_1,E_2,\Phi,F_*(E_1))$, we must have the inequalities

$$\frac{\beta_1\alpha'_1\chi(E_1(m)) + \beta_2\chi(E_2(m-\gamma)) + \sum_{i=1}^{l}\beta_1\epsilon_i\chi(F_{i+1}(E_1)(m))}{\beta_1\,\mathrm{rank}(E_1) + \beta_2\,\mathrm{rank}(E_2)}$$
$$> \frac{\beta_1\alpha'_1\chi(E'_1(m)) + \beta_2\chi(E'_2(m-\gamma)) + \sum_{i=1}^{l}\beta_1\epsilon_i\chi(F_{i+1}(E'_1)(m))}{\beta_1\,\mathrm{rank}(E'_1) + \beta_2\,\mathrm{rank}(E'_2)}$$
$$> \frac{\beta_1\alpha'_1\chi(E_1(m)) + \beta_2\chi(E_2(m-\gamma)) + \sum_{i=1}^{l}\beta_1\epsilon_i\chi(F_{i+1}(E_1)(m))}{\beta_1\,\mathrm{rank}(E_1) + \beta_2\,\mathrm{rank}(E_2)}$$

for $m \gg 0$, which is a contradiction. Therefore we have $(E'_1,E'_2) = (0,0)$, which means that $(f_1,f_2) = (c\cdot\mathrm{id}_{E_1}, c\cdot\mathrm{id}_{E_2})$. Λ

5.4. **Projectivity of the moduli space.**

Proposition 5.5. *Let R be a discrete valuation ring over S with residue field $k = R/m$ and quotient field K. Let $(E_1,E_2,\Phi,F_*(E_1))$ be a semistable parabolic $\Lambda^1_{\mathcal{D}_K}$-triple on $\mathcal{X}_K$. Then there exists a* *flat family* $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1))$ *of parabolic* $\Lambda^1_{\mathcal{D}_R}$*-triples on* $\mathcal{X}_R$ *over* R *such that* $(E_1, E_2, \Phi, F_*(E_1)) \cong (\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1)) \otimes_R K$ *and that* $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1)) \otimes_R k$ *is semistable.*

Proof. Two surjections

$$V_1 \otimes \mathcal{O}_{\mathcal{X}_K}(-m_0) \cong H^0(E_1(m_0)) \otimes \mathcal{O}_{\mathcal{X}_K}(-m_0) \to E_1,$$
$$V_2 \otimes \mathcal{O}_{\mathcal{X}_K}(-m_0 + \gamma) \cong H^0(E_2(m_0 - \gamma)) \otimes \mathcal{O}_{\mathcal{X}_K}(-m_0 + \gamma) \to E_2$$

and the quotients $E_1 \to E_1/F_{i+1}(E_1)$ $(i = 1, \dots, l)$ give a morphism $f : \operatorname{Spec} K \to Q$, where Q is defined by the property (105) in subsection 5.3. Since Q is proper over S, f extends to a morphism $\tilde{f} : \operatorname{Spec} R \to Q$. Thus there are coherent sheaves $E_1^{(0)}$, $E_2^{(0)}$ on $\mathcal{X}_R$ flat over R and a flat family of filtrations $F_*(E_1^{(0)})$ of $E_1^{(0)}$ such that $E_1^{(0)} \otimes K \cong E_1$, $E_2^{(0)} \otimes K \cong E_2$ and $F_*(E_1^{(0)}) \otimes_R K = F_*(E_1)$. The pullback of $\mathcal{H}$ by the morphism $\tilde{f} : \operatorname{Spec} R \to Q$ is denoted by $\mathcal{H}_R$. Recall that $\mathcal{H}$ is defined by (106) in subsection 5.3. The homomorphism $\Phi : \Lambda^1_{\mathcal{D}/S} \otimes E_1 \to E_2$ corresponds to a homomorphism $\psi : \mathcal{H}_R \otimes_R K \to K$. There is a non-zero element $t \in K \setminus \{0\}$ and a homomorphism $\tilde{\psi} : \mathcal{H}_R \to R$ such that $t\psi = \tilde{\psi} \otimes_R K$. Let $\Phi^{(0)} : \Lambda^1_{\mathcal{D}/S} \otimes E_1^{(0)} \to E_2^{(0)}$ be the homomorphism corresponding to $\tilde{\psi}$. Then we have $(E_1, E_2, \Phi, F_*(E_1)) \cong (E_1^{(0)}, E_2^{(0)}, \Phi^{(0)}, F_*(E_1^{(0)})) \otimes_R K$, since $(E_1, E_2, \Phi, F_*(E_1)) \cong (E_1, E_2, t\Phi, F_*(E_1))$. Our proposition follows from the following claim:

Claim There is a flat family $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1))$ of parabolic $\Lambda^1_{\mathcal{D}_R}$-triples on $\mathcal{X}_R$ over R such that $\tilde{E}_j \subset E_j^{(0)}$ for $j = 1, 2$, $F_{i+1}(\tilde{E}_1) \subset F_{i+1}(E_1^{(0)})$ for $i = 1, \dots, l$, $\tilde{\Phi} = \Phi^{(0)}|_{\tilde{E}_1 \otimes \Lambda^1_{\mathcal{D}/S}}$, $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1)) \otimes_R K \cong (E_1, E_2, \Phi, F_*(E_1))$ and $(\tilde{E}_1, \tilde{E}_2, \tilde{\Phi}, F_*(\tilde{E}_1)) \otimes_R k$ is semistable.

Assume that $E_1^{(0)} \otimes k$ or $E_2^{(0)} \otimes k$ have torsions. In this case let $B_1^{(0)}$ and $B_2^{(0)}$ be the torsion parts of $E_1^{(0)} \otimes k$ and $E_2^{(0)} \otimes k$, respectively. Then there are exact sequences

$$0 \to B_1^{(0)} \longrightarrow E_1^{(0)} \otimes k \longrightarrow G_1^{(0)} \to 0$$
$$0 \to B_2^{(0)} \longrightarrow E_2^{(0)} \otimes k \longrightarrow G_2^{(0)} \to 0,$$

where $G_1^{(0)}$ and $G_2^{(0)}$ are vector bundles on $\mathcal{X}_k$. Put $E_1^{(1)} := \ker(E_1^{(0)} \to ((E_1^{(0)} \otimes k)/B_1^{(0)}))$, $E_2^{(1)} := \ker(E_2^{(0)} \to ((E_2^{(0)} \otimes k)/B_2^{(0)}))$, $\Phi^{(1)} := \Phi^{(0)}|_{\Lambda^1_{\mathcal{D}_R} \otimes E_1^{(1)}}$ and $F_{i+1}(E_1^{(1)}) := F_{i+1}(E_1^{(0)}) \cap E_1^{(1)}$ for $i = 1, \dots, l$. Then there are exact sequences

$$0 \to G_1^{(0)} \longrightarrow E_1^{(1)} \otimes k \longrightarrow B_1^{(0)} \to 0$$
$$0 \to G_2^{(0)} \longrightarrow E_2^{(1)} \otimes k \longrightarrow B_2^{(0)} \to 0.$$

Again let $B_1^{(1)}$ and $B_2^{(1)}$ be the torsion parts of $E_1^{(1)} \otimes k$ and $E_2^{(1)} \otimes k$, respectively. Repeating these operations, we obtain sequences $(E_1^{(n)}, E_2^{(n)}, \Phi^{(n)}, F_*(E_1^{(n)}))_{n \geq 0}$, $(B_1^{(n)}, B_2^{(n)})_{n \geq 0}$ and $(G_1^{(n)}, G_2^{(n)})_{n \geq 0}$. Then the injections $B_1^{(n+1)} \hookrightarrow B_1^{(n)}$, $B_2^{(n+1)} \hookrightarrow B_2^{(n)}$ are induced by the homomorphisms $E_1^{(n+1)} \otimes k \to E_1^{(n)} \otimes k$, $E_2^{(n+1)} \otimes k \to E_2^{(n)} \otimes k$. Since $(\operatorname{length} B_1^{(n)}, \operatorname{length} B_2^{(n)})_{n \geq 0}$ is stationary, we may assume that it is constant. Then we have isomorphisms $B_1^{(n+1)} \xrightarrow{\sim} B_1^{(n)}$, $B_2^{(n+1)} \xrightarrow{\sim} B_2^{(n)}$, $G_1^{(n)} \xrightarrow{\sim} G_1^{(n+1)}$, $G_2^{(n)} \xrightarrow{\sim} G_2^{(n+1)}$ for all n. Assume that $(B_1^{(n)}, B_2^{(n)}) \neq (0, 0)$. There is an exact sequence

$$E_j^{(n)}/m^n E_j^{(0)} \xrightarrow{u} E_j^{(0)}/m^n E_j^{(0)} \longrightarrow E_j^{(0)}/E_j^{(n)} \to 0$$

for $n \geq 1$ and $j = 1, 2$. We can see that $(E_j^{(n)}/m^n E_j^{(0)}) \otimes k \cong B_j^{(n-1)}$ and that

$$u \otimes k : (E_j^{(n)}/m^n E_j^{(0)}) \otimes k \cong B_j^{(n-1)} \to E_j^{(0)} \otimes k$$

is injective. Thus $E_j^{(0)}/E_j^{(n)}$ is flat over R/m^n and the quotient $E_j^{(0)}/m^n E_j^{(0)} \to E_j^{(0)}/E_j^{(n)}$ determines a morphism $f_n : \operatorname{Spec} R/m^n \to \operatorname{Quot}_{E_j^{(0)}/\mathcal{X}_R/R}$ for $n \geq 1$. So we obtain a morphism $f : \operatorname{Spec} \hat{R} \to \operatorname{Quot}_{E_j^{(0)}/\mathcal{X}_R/R}$, where $\hat{R}$ is the completion of R. f corresponds to a quotient sheaf $E_j^{(0)} \otimes \hat{R} \xrightarrow{\pi} G$. Since $(\ker \pi) \otimes R/m \cong B_j^{(0)}$, $\ker \pi \otimes \hat{K}$ is a torsion submodule of $E_j^{(0)}$, which is nonzero either for $j = 1$ or $j = 2$, where $\hat{K}$ is the quotient field of $\hat{R}$. However, it is a contradiction, because $E_1^{(0)} \otimes \hat{K}$, $E_2^{(0)} \otimes \hat{K}$ are vector bundles. Hence we must have $(B_1^{(n)}, B_2^{(n)}) = (0, 0)$ for some n. So we may assume without loss of generality that $E_1^{(0)} \otimes k$ and $E_2^{(0)} \otimes k$ are locally free.

Now assume that the claim does not hold. Then we can define a descending sequence of flat families of parabolic $\Lambda^1_{\mathcal{D}_R}$-triples

$$(E_1^{(0)}, E_2^{(0)}, \Phi^{(0)}, F_*(E_1^{(0)})) \supset (E_1^{(1)}, E_2^{(1)}, \Phi^{(1)}, F_*(E_1^{(1)})) \supset (E_1^{(2)}, E_2^{(2)}, \Phi^{(2)}, F_*(E_1^{(2)})) \supset \cdots$$

as follows: Suppose $(E_1^{(n)}, E_2^{(n)}, \Phi^{(n)}, F_*(E_1^{(n)}))$ has already been defined. There exists a maximal destabilizer $(B_1^{(n)}, B_2^{(n)}, \Phi_{B^{(n)}}, F_*(B_1^{(n)}))$ of $(E_1^{(n)}, E_2^{(n)}, \Phi^{(n)}, F_*(E_1^{(n)})) \otimes k$ as in the usual case of semistability of coherent sheaves. We can see that $B_j^{(n)}$ is a subbundle of $E_j^{(n)} \otimes k$ for $j = 1, 2$ and $F_{i+1}(B_1^{(n)}) = B_1^{(n)} \cap (F_{i+1}(E_1^{(n)}) \otimes k)$ for $i = 1, \ldots, l$. We put $G_j^{(n)} := (E_j^{(n)} \otimes k)/B_j^{(n)}$ for $j = 1, 2$. Then $G_1^{(n)}$ has an induced quotient parabolic structure $F_*(G_1^{(n)})$. A homomorphism $\Phi_{G^{(n)}} : \Lambda^1_{\mathcal{D}/S} \otimes G_1^{(n)} \to G_2^{(n)}$ is induced by $\Phi^{(n)}$ and $(G_1^{(n)}, G_2^{(n)}, \Phi_{G^{(n)}}, F_*(G_1^{(n)}))$ becomes a parabolic $\Lambda^1_{\mathcal{D}_k}$-triple. Put

$$E_j^{(n+1)} = \ker(E_j^{(n)} \to G_j^{(n)}), \quad \Phi^{(n+1)} := \Phi^{(n)}|_{\Lambda^1_{\mathcal{D}/S} \otimes E_1^{(n+1)}},$$
$$F_{i+1}(E_1^{(n+1)}) = \ker(F_{i+1}(E_1^{(n)}) \to F_{i+1}(G_1^{(n)}))$$

Then $(E_1^{(n+1)}, E_2^{(n+1)}, \Phi^{(n+1)}, F_*(E_1^{(n+1)}))$ becomes a flat family of parabolic $\Lambda^1_{\mathcal{D}_R}$-triples on $\mathcal{X}_R$ over R. There are exact sequences

$$0 \to B_j^{(n)} \to E_j^{(n)} \otimes k \to G_j^{(n)} \to 0 \quad \text{and} \quad 0 \to G_j^{(n)} \to E_j^{(n+1)} \otimes k \to B_j^{(n)} \to 0 \tag{107}$$

for $j = 1, 2$. Then we can see that $(G_1^{(n)}, G_2^{(n)}, \Phi_{G^{(n)}}, F_*(G_1^{(n)}))$ becomes a parabolic $\Lambda^1_{\mathcal{D}_k}$-subtriple of $(E_1^{(n+1)}, E_2^{(n+1)}, \Phi^{(n+1)}, F_*(E_1^{(n+1)})) \otimes k$. We can check that $F_{i+1}(G_1^{(n)}) = G_1^{(n)} \cap (F_{i+1}(E_1^{(n+1)}) \otimes k)$ for $i = 1, \ldots, l$. Put

$$C_j^{(n)} := G_j^{(n)} \cap B_j^{(n+1)}, \quad \Phi_{C^{(n)}} := (\Phi^{(n+1)} \otimes k)|_{\Lambda^1_{\mathcal{D}_k} \otimes C_1^{(n)}},$$
$$F_{i+1}(C_1^{(n)}) := F_{i+1}(G_1^{(n)}) \cap F_{i+1}(B_1^{(n+1)}) \quad (i = 1, \ldots, l).$$

Then $(C_1^{(n)}, C_2^{(n)}, \Phi_{C^{(n)}}, F_*(C_1^{(n)}))$ becomes a parabolic $\Lambda^1_{\mathcal{D}_k}$-triple and $F_{i+1}(C_1^{(n)}) = C_1^{(n)} \cap (F_{i+1}(E_1^{(n+1)}) \otimes k)$ for $i = 1, \ldots, l$. A quotient parabolic structure $F_*(B_1^{(n+1)}/C_1^{(n)})$ is induced on $B_1^{(n+1)}/C_1^{(n)}$ and a homomorphism $\Phi_{B^{(n+1)}/C^{(n)}} : \Lambda^1_{\mathcal{D}_k} \otimes B_1^{(n+1)}/C_1^{(n)} \to B_2^{(n+1)}/C_2^{(n)}$ is induced by $\Phi^{(n+1)}$. Then

$$(B_1^{(n+1)}/C_1^{(n)}, B_2^{(n+1)}/C_2^{(n)}, \Phi_{B^{(n+1)}/C^{(n)}}, F_*(B_1^{(n+1)}/C_1^{(n)}))$$

becomes a parabolic $\Lambda^1_{\mathcal{D}_k}$-triple. If $(C_1^{(n)}, C_2^{(n)}) \neq (0,0)$, then we have

$$\begin{aligned}\mu((C_1^{(n)}, C_2^{(n)}, \Phi_{C^{(n)}}, F_*(C_1^{(n)}))) &\leq \mu_{\max}((G_1^{(n)}, G_2^{(n)}, \Phi_{G^{(n)}}, F_*(G_*^{(n)}))) \\ < \mu_{\max}((E_1^{(n)}, E_2^{(n)}, \Phi^{(n)}, F_*(E_1^{(n)})) \otimes k) &= \mu((B_1^{(n)}, B_2^{(n)}, \Phi_{B^{(n)}}, F_*(B_1^{(n)}))),\end{aligned}$$

where $\mu_{\max}$ means the value of μ at the maximal destabilizer. Thus, in any case, we have the inequality

$$\begin{aligned}&\mu((B_1^{(n+1)}, B_2^{(n+1)}, \Phi_{B^{(n+1)}}, F_*(B_1^{(n+1)}))) \\ \leq &\mu((B_1^{(n+1)}/C_1^{(n)}, B_2^{(n+1)}/C_2^{(n)}, \Phi_{B^{(n+1)}/C^{(n)}}, F_*(B_1^{(n+1)}/C_1^{(n)}))) \\ \leq &\mu((B_1^{(n)}, B_2^{(n)}, \Phi_{B^{(n)}}, F_*(B_1^{(n)})))\end{aligned}$$

with equality if and only if $(C_1^{(n)}, C_2^{(n)}) = (0,0)$.

The descending sequence

$$\{\mu((B_1^{(n)}, B_2^{(n)}, \Phi_{B^{(n)}}, F_*(B_1^{(n)})))\}_{n \in \mathbf{N}}$$

must become stationary since it is bounded below. We may assume without loss of generality that

$$\mu((B_1^{(n)}, B_2^{(n)}, \Phi_{B^{(n)}}, F_*(B_1^{(n)})))$$

is constant for all n. In this case we must have $(C_1^{(n)}, C_2^{(n)}) = (0,0)$ and

$$(B_1^{(n+1)}, B_2^{(n+1)}, \Phi_{B^{(n+1)}}, F_*(B_1^{(n+1)}))$$

becomes a parabolic $\Lambda^1_{\mathcal{D}_k}$-subtriple of

$$(B_1^{(n)}, B_2^{(n)}, \Phi_{B^{(n)}}, F_*(B_1^{(n)}))$$

for all n. Since the descending sequence $\{\operatorname{rank} B_1^{(n)} + \operatorname{rank} B_2^{(n)}\}_{n \in \mathbf{N}}$ must be stationary, we may assume without loss of generality that $\operatorname{rank} B_1^{(n)} + \operatorname{rank} B_2^{(n)}$ is constant for all n. Then we must have

$$(B_1^{(n)}, B_2^{(n)}, \Phi_{B^{(n)}}, F_*(B_1^{(n)})) = (B_1^{(n+1)}, B_2^{(n+1)}, \Phi_{B^{(n+1)}}, F_*(B_1^{(n+1)}))$$

for all n. Thus the sequences (107) split and

$$(E_1^{(n)}, E_2^{(n)}, \Phi^{(n)}, F_*(E_1^{(n)})) \otimes k \cong (B_1^{(n)}, B_2^{(n)}, \Phi_{B^{(n)}}, F_*(B_1^{(n)})) \oplus (G_1^{(n)}, G_2^{(n)}, \Phi_{G^{(n)}}, F_*(G_1^{(n)})).$$

Then all the maps $G_j^{(n)} \to G_j^{(n+1)}$ are isomorphisms. Since $B_j^{(n+1)} \to B_j^{(n)}$ are all isomorphic, every image of $E_j^{(n)} \otimes k \to E_j^{(0)} \otimes k$ is $B_j^{(0)}$ for $j = 1, 2$. So we have an isomorphism $(E_j^{(0)}/E_j^{(n)}) \otimes k \cong G_j^{(0)}$ for any n. On the other hand, every image of $m^n/m^{n+1} \otimes E_j^{(0)} \to E_j^{(n)} \otimes k$ is $G_j^{(n-1)}$. So we have an isomorphism $(E_j^{(n)}/m^n E_j^{(0)}) \otimes k \cong B_j^{(n-1)}$. Consider the exact sequence

$$0 \longrightarrow E_j^{(n)}/m^n E_j^{(0)} \xrightarrow{u} E_j^{(0)}/m^n E_j^{(0)} \longrightarrow E_j^{(0)}/E_j^{(n)} \longrightarrow 0.$$

Then $u \otimes k : (E_j^{(n)}/m^n E_j^{(0)}) \otimes k \cong B_j^{(n-1)} \to E_j^{(0)} \otimes k$ is injective. Thus u is injective and $E_j^{(0)}/E_j^{(n)}$ is flat over $R/m^n R$. Then quotients $E_j^{(0)} \otimes R/m^n \to E_j^{(0)}/E_j^{(n)}$ define a system of morphisms $\operatorname{Spec} R/m^n \to Q'_j := \operatorname{Quot}^{\chi(G_j^{(0)}(n))}_{E_j^{(0)}/\mathcal{X}_R/R}$ which induces a morphism $f_j : \operatorname{Spec} \hat{R} \to Q'_j$, where $\hat{R}$ is the completion of R. If $\tilde{G}_j$ is the quotient sheaf of $E_j^{(0)} \otimes \hat{R}$ corresponding to f_j, then we have $\tilde{G}_j \otimes R/m^n R \cong E_j^{(0)}/E_j^{(n)}$. Similarly we can lift the parabolic structure $F_*(G_1^{(0)})$ to a flat family $F_*(\tilde{G}_1)$ of parabolic structure on $\tilde{G}_1$ over $\hat{R}$. We can also lift $\Phi_{G^{(0)}}$ to $\Phi_{\tilde{G}} : \Lambda^1_{\mathcal{D}/S} \otimes \tilde{G}_1 \to \tilde{G}_2$ and $(\tilde{G}_1, \tilde{G}_2, \Phi_{\tilde{G}}, F_*(\tilde{G}_1))$ becomes a flat family of parabolic $\Lambda^1_{\mathcal{D}_R}$-triples which is a quotient of $(E_1^{(0)}, E_2^{(0)}, \Phi^{(0)}, F_*(E_*^{(0)})) \otimes \hat{R}$. If $\hat{K}$ is the quotient field of $\hat{R}$, then $(\tilde{G}_1, \tilde{G}_2, \Phi_{\tilde{G}}, F_*(\tilde{G}_1)) \otimes \hat{K}$ becomes a destabilizing quotient parabolic $\Lambda^1_{\mathcal{D}_{\hat{K}}}$-triple of $(E_1, E_2, \Phi, F_*(E_1)) \otimes \hat{K}$, which contradicts the semistability of $(E_1, E_2, \Phi, F_*(E_1))$. Λ

As a corollary of Proposition 5.5, we obtain the following proposition:

Proposition 5.6. *Assume that $\alpha'_1, \dots, \alpha'_l$ are sufficiently general so that all the semistable parabolic $\Lambda^1_{\mathcal{D}/S}$-triples are stable. Then the moduli scheme $\overline{M^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}}(r, d, \{d_i\})$ is projective over S.*

There is another corollary of Proposition 5.5 which is used in the proof of the surjectivity of the Riemann-Hilbert morphism in Lemma 7.1. For a parabolic connection $(E, \nabla, \varphi, \{l_i\})$, let $(0, 0) = (F_0, \nabla_0) \subset (F_1, \nabla_1) \subset \cdots \subset (F_l, \nabla_l) = (E, \nabla)$ be a Jordan-Hölder filtration of (E, ∇), that is, each $(F_i/F_{i+1}, \overline{\nabla_i})$ is irreducible, where $\overline{\nabla_i} : F_i/F_{i+1} \to F_i/F_{i+1} \otimes \Omega^1_X(D(\mathbf{t}))$ is the connection induced by ∇_i. Then we put

$$gr(E, \nabla) := \bigoplus_{i=1}^{l} (F_i/F_{i+1}, \overline{\nabla_i}).$$

Corollary 5.1. *Let R be a discrete valuation ring with quotient field K and residue field k. Let $(E, \nabla, \varphi, \{l_i\})$ be a flat family of connections with parabolic structures on $X \times \operatorname{Spec} R$ over R such that the generic fiber $(E, \nabla, \varphi, \{l_i\}) \otimes_R K$ is $\boldsymbol{\alpha}$-semistable. Then there exists a flat family $(\tilde{E}, \tilde{\nabla}, \tilde{\varphi}, \{\tilde{l}_i\})$ of $\boldsymbol{\alpha}$-semistable parabolic connections such that $(\tilde{E}, \tilde{\nabla}, \tilde{\varphi}, \{\tilde{l}_i\}) \otimes K \cong (E, \nabla, \varphi, \{l_i\}) \otimes K$ and $gr((\tilde{E}, \tilde{\nabla}) \otimes k) \cong gr((E, \nabla) \otimes k)$.*

5.5. **Proof of Theorem 2.1 (1).** Now we prove the assertion (1) of Theorem 2.1.

We take S for $T_n \times \Lambda_n$ and $\mathcal{X}$ for $\mathbf{P}^1 \times T_n \times \Lambda_n$.

Let $\mathcal{D}_i \subset \mathbf{P}^1 \times T_n \times \Lambda_n$ be the effective divisor determined by the section

$$T_n \times \Lambda_n \hookrightarrow \mathbf{P}^1 \times T_n \times \Lambda_n; \quad ((t_j)_{1 \le j \le n}, (\lambda_k)_{1 \le k \le n}) \mapsto (t_i, (t_j)_{1 \le j \le n}, (\lambda_k)_{1 \le k \le n})$$

for $i = 1, \dots, n$ and put $\mathcal{D} := \sum_{i=1}^n \mathcal{D}_i$. Then $\mathcal{D}$ becomes an effective Cartier divisor on $\mathbf{P}^1 \times T_n \times \Lambda_n$ which is flat over $T_n \times \Lambda_n$.

We fix a line bundle L on $\mathbf{P}^1 \times T_n \times \Lambda_n$ with a relative connection

$$\nabla_L : L \to L \otimes \Omega^1_{\mathbf{P}^1 \times T_n \times \Lambda_n / T_n \times \Lambda_n}(\mathcal{D})$$

over $T_n \times \Lambda_n$. Let $\boldsymbol{\alpha}' = (\alpha'_1, \dots, \alpha'_{2n})$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$, and $\gamma \gg 0$ be as in Theorem 2.1.

We define a moduli functor $\overline{\mathcal{M}^{\boldsymbol{\alpha}'\boldsymbol{\beta}}_n}(L)$ of the category of locally noetherian schemes over $T_n \times \Lambda_n$ to the category of sets by

$$\overline{\mathcal{M}^{\boldsymbol{\alpha}'\boldsymbol{\beta}}_n}(L)(S) := \{(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}_{i=1}^n)\}/\sim,$$

where S is a locally noetherian scheme over $T_n \times \Lambda_n$ corresponding to $(\mathbf{t}, \boldsymbol{\lambda}) = (t_1, \dots, t_n, \lambda_1, \dots, \lambda_n) \in T_n(S) \times \Lambda_n(S)$ and

(1) E_1, E_2 are rank 2 vector bundles on $\mathbf{P}^1 \times S$,

(2) $\phi : E_1 \to E_2$ is an $\mathcal{O}_{\mathbf{P}^1\times S}$-homomorphism, $\nabla : E_1 \to E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t}))$ is a morphism such that $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^1\times S}$, $a \in E_1$,

(3) $l_i \subset E_1|_{t_i}$ are rank 1 subbundles such that $(\mathrm{res}_{t_i}(\nabla) - \lambda_i\phi|_{t_i})|_{l_i} = 0$ for $i = 1, \dots, n$,

(4) $\varphi : \bigwedge^2 E_2 \xrightarrow{\sim} L \otimes \mathcal{L}_\varphi$ is an isomorphism such that $(\varphi \otimes 1)(\nabla(s_1) \wedge \phi(s_2) + \phi(s_1) \wedge \nabla(s_2)) = (\nabla_L \otimes \mathrm{id}_{\mathcal{L}_\varphi})(\varphi(\phi(s_1) \wedge \phi(s_2)))$ for $s_1, s_2 \in E_1$ where $\mathcal{L}_\varphi$ is a line bundle on S,

(5) for any geometric point s of S, the fiber $((E_1)_s, (E_2)_s, \phi_s, \nabla_s, \varphi_s, \{l_i|_{t_i\otimes k(s)}\}_{i=1}^n)$ is $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-stable and $\deg(E_1)_s = \deg L_s$.

Here $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \sim (E_1', E_2', \phi', \nabla', \varphi', \{l_i'\})$ if there exist a line bundle $\mathcal{L}$ on S and isomorphisms $\sigma_j : E_j \xrightarrow{\sim} E_j' \otimes \mathcal{L}$ for $j = 1, 2$ such that $\sigma_1|_{t_i\times S}(l_i) = l_i'$ for any i, the diagrams

$$\begin{tikzcd} E_1 \arrow[r, "\phi"] \arrow[d, "\sigma_1"', "\cong"] & E_2 \arrow[d, "\cong"', "\sigma_2"] \\ E_1' \otimes \mathcal{L} \arrow[r, "\phi'"] & E_2' \otimes \mathcal{L} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E_1 \arrow[r, "\nabla"] \arrow[d, "\sigma_1"', "\cong"] & E_2 \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \arrow[d, "\cong"', "\sigma_2\otimes\mathrm{id}"] \\ E_1' \otimes \mathcal{L} \arrow[r, "\nabla'"] & E_2' \otimes \Omega^1_{\mathbf{P}^1}(D(\mathbf{t})) \otimes \mathcal{L} \end{tikzcd}$$

commute and there is an isomorphism $\sigma : \mathcal{L}_\varphi \xrightarrow{\sim} \mathcal{L}_{\varphi'} \otimes \mathcal{L}^{\otimes 2}$ such that the diagram

$$\begin{tikzcd} \bigwedge^2 E_2 \arrow[r, "\varphi", "\sim"'] \arrow[d, "\wedge\sigma_2"', "\cong"] & L \otimes \mathcal{L}_\varphi \arrow[d, "\cong"', "\mathrm{id}\otimes\sigma"] \\ \bigwedge^2 E_2' \otimes \mathcal{L} \arrow[r, "\varphi'\otimes\mathrm{id}_{\mathcal{L}}", "\sim"'] & L \otimes \mathcal{L}_{\varphi'} \otimes \mathcal{L}^{\otimes 2} \end{tikzcd}$$

commutes.

We can define another weight $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{2n})$ with $0 \le \alpha_1 < \dots < \alpha_{2n} < \frac{\beta_1}{\beta_1+\beta_2} < 1$ by

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}' \frac{\beta_1}{\beta_1 + \beta_2}.$$

Theorem 2.1, (1) follows from the following theorem:

Theorem 5.2. *There exists a coarse moduli scheme $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$ of $\overline{\mathcal{M}_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$, which is projective over $T_n \times \Lambda_n$ if $\boldsymbol{\alpha}'$ is generic. If we put*

$$M_n^{\boldsymbol{\alpha}}(L) := \left\{ (E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \in \overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \,\middle|\, \phi : E_1 \to E_2 \text{ is an isomorphism} \right\},$$

then $M_n^{\boldsymbol{\alpha}}(L)$ is a Zariski open subset of $\overline{M_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)$, which is a fine moduli scheme of $\boldsymbol{\alpha}$-stable parabolic connections.

Proof. We put $r = 2$, $d = \deg L_s$ for $s \in T_n \times \Lambda_n$, $l = 2n$ and $d_i = i$ for $i = 1, \dots, 2n$ and consider the moduli scheme $\overline{M^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathbf{P}^1\times T_n\times\Lambda_n/T_n\times\Lambda_n}}(2, d, \{d_i\})$. For each $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \in \overline{\mathcal{M}_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L)(S)$, let $\Phi : \Lambda^1_{\mathcal{D}_S} \otimes E_1 \to E_2$ be the left $\mathcal{O}_{\mathbf{P}^1\times S}$-homomorphism corresponding to (ϕ, ∇) and put $F_{2i+1}(E_1) := E_1(-\sum_{j=1}^i t_j)$ for $i = 0, \dots, l$ and $F_{2i}(E_1) := \ker(F_{2i-1}(E_1) \to (E_1|_{t_i}/l_i))$ for $i = 1, \dots, l$, where $(t_1, \dots, t_n, \lambda_1, \dots, \lambda_n) \in T_n(S) \times \Lambda_n(S)$ corresponds to the structure morphism $S \to T_n \times \Lambda_n$. Then the correspondence $(E_1, E_2, \phi, \nabla, \varphi, \{l_i\}) \mapsto (E_1, E_2, \Phi, F_*(E_1))$ determines a morphism of functors

$$\iota : \overline{\mathcal{M}_n^{\boldsymbol{\alpha}'\boldsymbol{\beta}}}(L) \to \overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathbf{P}^1\times T_n\times\Lambda_n/T_n\times\Lambda_n}}(2, d, \{d_i\}).$$

We can easily see that ι is represented by a closed immersion. Recall that $R^s \to \overline{M^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathbf{P}^1\times T_n\times\Lambda_n/T_n\times\Lambda_n}}(2,d,\{d_i\})$ is a principal G-bundle. Then there exists a closed subscheme $Z\subset R^s$ such that

$$h_Z = h_{R^s}\times_{\overline{\mathcal{M}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathbf{P}^1\times T_n\times\Lambda_n/T_n\times\Lambda_n}}(2,d,\{d_i\})}\overline{\mathcal{M}^{\boldsymbol{\alpha}'\boldsymbol{\beta}}_n}(L).$$

Z descends to a closed subscheme of $\overline{M^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathbf{P}^1\times T_n\times\Lambda_n/T_n\times\Lambda_n}}(2,d,\{d_i\})$ which is just the coarse moduli scheme of $\overline{\mathcal{M}^{\boldsymbol{\alpha}'\boldsymbol{\beta}}_n}(L)$.

If we take γ sufficiently large, we can check that a parabolic connection $(E,\nabla_E,\varphi,\{l_i\})$ is $\boldsymbol{\alpha}$-stable if and only if the associated parabolic ϕ-connection $(E,E,\mathrm{id}_E,\nabla_E,\varphi,\{l_i\})$ is $(\boldsymbol{\alpha}',\boldsymbol{\beta})$-stable. Thus the open subscheme

$$M^{\boldsymbol{\alpha}}_n(L) := \left\{ (E_1,E_2,(\phi,\nabla),F_*(E_1))\in\overline{M^{\boldsymbol{\alpha}'\boldsymbol{\beta}}_n}(L) \,\middle|\, \phi: E_1\to E_2 \text{ is an isomorphism}\right\}$$

of $\overline{M^{\boldsymbol{\alpha}'\boldsymbol{\beta}}_n}(L)$ is just the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections with the determinant L.

If $\deg L$ is odd, we can see by the same argument as [[M], Theorem 6.11] or [[HL], Theorem 4.6.5] that $M^{\boldsymbol{\alpha}}_n(L)$ is in fact a fine moduli scheme. If $\deg L$ is even, then we can obtain, by an elementary transform, an isomorphism

$$\sigma:\mathcal{M}_n(L)\to\mathcal{M}_n(L')$$

of moduli stacks of parabolic connections without stability condition, where $\deg L'$ is odd. Then we can see by the same argument that $\sigma(M^{\boldsymbol{\alpha}}_n(L))$ becomes a fine moduli scheme, and so $M^{\boldsymbol{\alpha}}_n(L)$ is also fine. Λ

6. Tangent spaces of the moduli spaces and Canonical symplectic structure.

In this section, we will work over the finite étale covering $T'_n\longrightarrow T_n$ defined in (94). Fix $(\mathbf{t},\boldsymbol{\lambda})\in T'_n\times\Lambda_n$ and set $a_i = 2\cos 2\pi\lambda_i$ and $\mathbf{a}=(a_1,\dots,a_n)$. Moreover fix a determinant line bundle $L=(L,\nabla_L)$ on $\mathbf{P}^1$ such that $\mathsf{res}_{t_i}(\nabla_L)\in\mathbf{Z}$. We have defined two moduli spaces $M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L),\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ where $M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)$ is the moduli space of stable $(\mathbf{t},\boldsymbol{\lambda})$-parabolic connections with the determinant L and $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is the moduli of Jordan equivalence classes of the $SL_2(\mathbf{C})$-representations of $\pi_1(\mathbf{P}^1\setminus D(\mathbf{t}),*)$ with fixed local exponents $\mathbf{a}=(a_1,\dots,a_n)$. As we show in Theorem 2.1, for a suitable (or generic) weight $\boldsymbol{\alpha}$, the moduli space $M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)$ is a non-singular complex scheme. In this section, we will describe the tangent space to $M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)$ and a non-degenerate holomorphic 2-form on the moduli space $M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)$.

Although the moduli space $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ may be singular, we can define a Zariski dense open set $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ such that $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ is a non-singular variety. (Note that for generic $\mathbf{a}\in\mathcal{A}_n$, $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}} = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$). Moreover on $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ we can also define a canonical symplectic structure Ω_1. In §7 we define the Riemann-Hilbert correspondence $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}: M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)\longrightarrow\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$. We show that $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ is bimeromorphic proper surjective morphism and gives an analytic isomorphism between $M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)^{\sharp} := \mathbf{RH}^{-1}_{\mathbf{t},\boldsymbol{\lambda}}(\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}})$ and $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$. (Again, for a generic $\boldsymbol{\lambda}$, $M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)^{\sharp} = M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)$). Note that $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ is not an algebraic morphism, and hence the algebraic structures of $M^{\boldsymbol{\alpha}}_n(\mathbf{t},\boldsymbol{\lambda},L)^{\sharp}$ and $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ are completely different). The canonical symplectic structures on both moduli spaces can be identified via $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}|M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)^\sharp}$, that is, $\left(\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}|M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)^\sharp}\right)^*(\Omega_1) = \Omega$.

6.1. **Tangent space to $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$.** Consider the base extension of the family of moduli spaces in (48) by the étale covering $T_n' \longrightarrow T_n$:

$$\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n' \times \Lambda_n, \tag{108}$$

such that for every $(\mathbf{t},\boldsymbol{\lambda}) \in T_n' \times \Lambda_n$, we have $\pi_n^{-1}((\mathbf{t},\lambda)) \simeq M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$. For simplicity, we will omit L from now on, so we write as $M_n^{\boldsymbol{\alpha}} = M_n^{\boldsymbol{\alpha}}(L)$, $M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda}) = M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)$. We assume that $\boldsymbol{\alpha}$ is generic so that π_n is a smooth morphism (cf. Theorem 2.1).

Let us consider natural projection maps

$$\begin{array}{ccccc} & & M_n^{\boldsymbol{\alpha}} & & \\ & & \downarrow \pi_n & & \\ & & T_n' \times \Lambda_n & & \\ & \overset{p_1}{\swarrow} & & \overset{p_2}{\searrow} & \\ T_n' & & & & \Lambda_n \end{array}$$

and set $\varphi_i = p_i \circ \pi_n$. Since $\varphi_1 : M_n^{\boldsymbol{\alpha}} \longrightarrow T_n'$ is smooth, we have the following exact sequence of tangent sheaves on $M_n^{\boldsymbol{\alpha}}$

$$0 \longrightarrow \Theta_{M_n^{\boldsymbol{\alpha}}/T_n'\times\Lambda_n} \longrightarrow \Theta_{M_n^{\boldsymbol{\alpha}}/T_n'} \longrightarrow \pi_n^*(\Theta_{T_n'\times\Lambda_n/T_n'}) \longrightarrow 0. \tag{109}$$

We will describe this exact sequence in terms of the infinitesimal deformation of the stable parabolic connections. Let us consider the natural projection map $q_2 : \mathbf{P}^1 \times T_n' \longrightarrow T_n'$ and defines a divisor $\mathcal{D} \subset \mathbf{P}^1 \times T_n'$ such that $q_2^{-1}(\mathbf{t}) \cap \mathcal{D} = D(\mathbf{t}) = t_1 + \cdots + t_n \subset \mathbf{P}^1$.

Let $(\tilde{E}, \tilde{\nabla}, \tilde{\varphi}, \{\tilde{l}_i\})$ be a universal family on $\mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}}$. Consider the following commutative diagram:

$$\begin{array}{ccc} & & \mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}} \\ & \overset{\tilde{\varphi}_1}{\swarrow} & \downarrow \tilde{\pi}_n \\ & & \mathbf{P}^1 \times T_n' \times \Lambda_n \\ & \swarrow & \\ \mathbf{P}^1 \times T_n' & & \\ q_2 \downarrow & & \\ T_n' & & \end{array} \tag{110}$$

For a coherent sheaf $\mathcal{G}$ on $\mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}}$ and a closed point $\mathbf{x} \in M_n^{\boldsymbol{\alpha}}$, we set $\mathcal{G}_{\mathbf{x}} := \mathcal{G}_{|\mathbf{P}^1\times\mathbf{x}}$.

We define coherent sheaves on $\mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}}$ as follows.

$$\mathcal{F}^0 := \left\{ s \in \mathcal{E}nd(\tilde{E}) |\, \mathrm{Tr}(s) = 0, ; \; (s|_{t_i \times M_n^{\boldsymbol{\alpha}}})(\tilde{l}_i) \subset \tilde{l}_i \right\} \tag{111}$$

$$\mathcal{F}^1 := \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \tilde{\varphi}_1^*(\Omega^1_{\mathbf{P}^1\times T_n'/T_n'}(\mathcal{D})) \;|\; \mathrm{Tr}(s) = 0, \;\; (s|_{t_i \times M_n^{\boldsymbol{\alpha}}})(\tilde{l}_i) = 0 \right\} \tag{112}$$

$$\mathcal{F}^{1,+} := \left\{ s \in \mathcal{E}nd(\tilde{E}) \otimes \tilde{\varphi}_1^*(\Omega^1_{\mathbf{P}^1\times T_n'/T_n'}(\mathcal{D})) |\, \mathrm{Tr}(s) = 0, \;\; \mathrm{res}_{(t_i \times M_n^{\boldsymbol{\alpha}})}(s)(\tilde{l}_i) \subset \tilde{l}_i \right\} \tag{113}$$

For a local section s of $\mathcal{F}^0$, define $\nabla_1(s) := \tilde{\nabla}s - s\tilde{\nabla}$. Then it is easy to see that $\nabla_1(s)$ is a local section of $\mathcal{F}^1$. Since we have a natural inclusion of sheaves $\iota : \mathcal{F}^1 \hookrightarrow \mathcal{F}^{1,+}$, we can define two complexes of sheaves on $\mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}}$:

$$\mathcal{F}^\bullet := \left[\nabla_1 : \mathcal{F}^0 \longrightarrow \mathcal{F}^1\right], \tag{114}$$

$$\mathcal{F}^{\bullet,+} := \left[\nabla_1^+ : \mathcal{F}^0 \longrightarrow \mathcal{F}^{1,+}\right]. \tag{115}$$

Let $\mathbf{x} \in M_n^{\boldsymbol{\alpha}}$ be a closed point and set $\pi(\mathbf{x}) = (\mathbf{t}, \boldsymbol{\lambda})$. Setting $\mathcal{T}_1 = \mathcal{F}^{1,+}/\mathcal{F}^1$, we have the following exact sequences of the complexes on $\mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}}$ and $\mathbf{P}^1 \times \{\mathbf{x}\}$.

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ \mathcal{F}^0 & \xrightarrow{\nabla_1} & \mathcal{F}^1 \\ \| & & \downarrow \\ \mathcal{F}^0 & \xrightarrow{\nabla_1^+} & \mathcal{F}^{1,+} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{T}_1 \\ & & \downarrow \\ & & 0 \end{array} \quad \text{on } \mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}}, \qquad \begin{array}{ccc} & & 0 \\ & & \downarrow \\ \mathcal{F}^0_{\mathbf{x}} & \xrightarrow{\nabla_{1,\mathbf{x}}} & \mathcal{F}^1_{\mathbf{x}} \\ \| & & \downarrow \\ \mathcal{F}^0_{\mathbf{x}} & \xrightarrow{\nabla_{1,\mathbf{x}}^+} & \mathcal{F}^{1,+}_{\mathbf{x}} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{T}_{1,\mathbf{x}} \\ & & \downarrow \\ & & 0 \end{array} \quad \text{on } \mathbf{P}^1 \times \{\mathbf{x}\}. \tag{116}$$

Note that at each point t_i, $1 \le i \le n$, the stalk $(\mathcal{T}_{1,\mathbf{x}})_{t_i}$ is isomorphic to $\mathbf{C}((t_i, \mathbf{x}))$, hence $H^0(\mathcal{T}_{1,\mathbf{x}}) \simeq \oplus_{i=1}^n \mathbf{C}((t_i, \mathbf{x})) \simeq \mathbf{C}^n$.

Lemma 6.1. *At each closed point $\mathbf{x} \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}) \subset M_n^{\boldsymbol{\alpha}}$ the tangent spaces can be given as follows.*

$$(\Theta_{M_n^{\boldsymbol{\alpha}}/T_n'})_{\mathbf{x}} \simeq \mathbf{H}^1(\mathbf{P}^1, [\mathcal{F}^0_{\mathbf{x}} \xrightarrow{\nabla_{1,\mathbf{x}}^+} \mathcal{F}^{1,+}_{\mathbf{x}}]), \tag{117}$$

$$(\Theta_{M_n^{\boldsymbol{\alpha}}/T_n' \times \Lambda_n})_{\mathbf{x}} \simeq \mathbf{H}^1(\mathbf{P}^1, [\mathcal{F}^0_{\mathbf{x}} \xrightarrow{\nabla_{1,\mathbf{x}}} \mathcal{F}^1_{\mathbf{x}}]), \tag{118}$$

$$(\Theta_{T_n' \times \Lambda_n / T_n'})_{\pi(\mathbf{x})} \simeq H^0(\mathcal{T}_{1,\mathbf{x}}) \simeq \mathbf{C}^n. \tag{119}$$

Under these isomorphisms, we have the following identification of the natural exact sequences of the tangent spaces with the exact sequences of the hypercohomologies:

$$\begin{array}{ccccccccc} 0 \longrightarrow & (\Theta_{M_n^{\boldsymbol{\alpha}}/T_n' \times \Lambda_n})_{\mathbf{x}} & \longrightarrow & (\Theta_{M_n^{\boldsymbol{\alpha}}/T_n'})_{\mathbf{x}} & \longrightarrow & (\Theta_{T_n' \times \Lambda_n / T_n'})_{\pi(\mathbf{x})} & \longrightarrow 0 \\ & \| & & \| & & \| & \\ 0 \longrightarrow & \mathbf{H}^1([\mathcal{F}^0_{\mathbf{x}} \xrightarrow{\nabla_{1,\mathbf{x}}} \mathcal{F}^1_{\mathbf{x}}]) & \longrightarrow & \mathbf{H}^1([\mathcal{F}^0_{\mathbf{x}} \xrightarrow{\nabla_{1,\mathbf{x}}^+} \mathcal{F}^{1,+}_{\mathbf{x}}]) & \longrightarrow & H^0(\mathcal{T}_{1,\mathbf{x}}) & \longrightarrow 0. \end{array} \tag{120}$$

Proof. The smoothness of the natural map $\pi_n : M_n^{\boldsymbol{\alpha}} \longrightarrow T_n' \times \Lambda_n$ follows from Theorem 2.1. (Actually, one can show that $\mathbf{H}^2([\mathcal{F}^0_{\mathbf{x}} \xrightarrow{\nabla_{1,\mathbf{x}}} \mathcal{F}^1_{\mathbf{x}}]) = \{0\}$ (cf. Lemma 6.3)). The space of the infinitesimal deformations of logarithmic parabolic connection with fixing the eigenvalues of the residue matrix of $\tilde{\nabla}_{\mathbf{x},t_i}$ at t_i is given by the hypercohomology

$$\mathbf{H}^1(\mathbf{P}^1, [\mathcal{F}^0_{\mathbf{x}} \xrightarrow{\nabla_{1,\mathbf{x}}} \mathcal{F}^1_{\mathbf{x}}]).$$

(Cf. Arinkin [A]). Moreover it is easy to see that $\mathbf{H}^1(\mathbf{P}^1, [\mathcal{F}^0_{\mathbf{x}} \xrightarrow{\nabla_{1,\mathbf{x}}^+} \mathcal{F}^{1,+}_{\mathbf{x}}])$ is the set of infinitesimal deformations of logarithmic parabolic connections without fixing the eigenvalues of the residues of $\tilde{\nabla}_{\mathbf{x}}$.

Since $\mathcal{T}_{1,\mathbf{x}}$ is a skyscraper sheaf supported on $D(\mathbf{t}) \subset \mathbf{P}^1 \times \{\mathbf{x}\}$, we see that $\mathbf{H}^0(0 \to \mathcal{T}_{1,\mathbf{x}}) = \{0\}$, $\mathbf{H}^1(0 \to \mathcal{T}_{1,\mathbf{x}}) = H^0(\mathcal{T}_{1,\mathbf{x}}) \simeq \mathbf{C}^n$. Local calculations of the maps $\nabla_1, \nabla_{1,+}$ in the commutative diagram in (116) show that the natural map

$$d\pi_{n,\mathbf{x}} : \mathbf{H}^1([\nabla_{1,\mathbf{x}}^+ : \mathcal{F}^0_{\mathbf{x}} \longrightarrow \mathcal{F}^{1,+}]) \longrightarrow H^0(T_1)$$

gives the differential of the map $\pi_n : M_n^{\boldsymbol{\alpha}} \longrightarrow \Lambda_n$ at $\mathbf{x}$. Since $\mathbf{H}^2([\mathcal{F}^0_{\mathbf{x}} \xrightarrow{\nabla_{1,\mathbf{x}}} \mathcal{F}^1_{\mathbf{x}}]) = \{0\}$ or equivalently π_n is smooth at $\mathbf{x}$, the map $d\pi_{\mathbf{x}}$ is surjective. Λ

6.2. **The relative symplectic form Ω for $\pi_n : M_n^{\boldsymbol{\alpha}} \longrightarrow T_n' \times \Lambda_n$.** Let us consider the smooth family of moduli spaces of stable parabolic connections:

$$\pi_n : M_n^{\boldsymbol{\alpha}} \longrightarrow T_n' \times \Lambda_n. \tag{121}$$

Now we will show that each closed fiber $\pi_n^{-1}(\mathbf{t}, \boldsymbol{\lambda}) = M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ admits a canonical *symplectic structure* Ω, which induces a non-degenerate skew symmetric bilinear form on its tangent sheaf:

$$\Omega_{|M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)} : \Theta_{M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)} \otimes \Theta_{M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)} \longrightarrow \mathcal{O}_{M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)}. \tag{122}$$

First, a local calculation shows the following

Lemma 6.2. *For each point $\mathbf{x} \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) = \pi_n^{-1}(\mathbf{t}, \boldsymbol{\lambda}) \subset M_n^{\boldsymbol{\alpha}}$, set $\mathcal{F}^i_{\mathbf{x}} = \mathcal{F}^i_{|\mathbf{P}^1 \times \mathbf{x}}$ for $i = 0, 1$. Then we have isomorphisms*

$$\mathcal{F}^1_{\mathbf{x}} \simeq \mathcal{F}^{0\vee}_{\mathbf{x}} \otimes \Omega^1_{\mathbf{P}^1}, \quad \mathcal{F}^0_{\mathbf{x}} \simeq \mathcal{F}^{1\vee}_{\mathbf{x}} \otimes \Omega^1_{\mathbf{P}^1}. \tag{123}$$

where $\mathcal{F}^{i\vee}_{\mathbf{x}} = \mathcal{H}om(\mathcal{F}^i_{\mathbf{x}}, \mathcal{O}_{\mathbf{P}^1})$.

The following lemma is a key of proof of the smoothness of the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$. The stability assumption on the objects in $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is essential in this lemma.

Lemma 6.3. *Under the notation as above, we have*

$$\mathbf{H}^2(\mathbf{P}^1, \mathcal{F}^{\bullet}_{\mathbf{x}}) = \{0\}. \tag{124}$$

Proof. Consider the dual complex $(\nabla^1)^\vee : (\mathcal{F}^1)^\vee \otimes \Omega^1_{\mathbf{P}^1} \longrightarrow (\mathcal{F}^0)^\vee \otimes \Omega^1_{\mathbf{P}^1}$ which can be identified with the original complex ∇^1 by Killing form (cf. Lemma 6.2). Therefore

$$\begin{aligned}
\mathbf{H}^2(\mathcal{F}^{\bullet}) &\simeq \operatorname{coker}\left[H^1(\mathcal{F}^0_{\mathbf{x}}) \xrightarrow{\nabla^1} H^1(\mathcal{F}^1_{\mathbf{x}})\right] \\
&\simeq \ker\left[H^1(\mathcal{F}^1_{\mathbf{x}})^\vee \xrightarrow{(\nabla^1)^\vee} H^1(\mathcal{F}^0_{\mathbf{x}})^\vee\right]^\vee \\
&\simeq \ker\left[H^0(\mathcal{F}^0_{\mathbf{x}}) \xrightarrow{\nabla^1} H^0(\mathcal{F}^1_{\mathbf{x}})\right]^\vee.
\end{aligned}$$

Since $\mathcal{F}^0_{\mathbf{x}}$ is in the trace free part of the endomorphisms, it suffices to show that any $s \in H^0(\mathcal{F}^0_{\mathbf{x}})$ such that $s\nabla = \nabla s$ is a scalar. For any $\lambda \in \mathbf{C}$, let us set $E^0_\lambda = \ker(s - \lambda)$ and $E^1_\lambda = \mathrm{Im}(s - \lambda)$. Then both E^0_λ and E^1_λ are subsheaves of E stable under ∇. If E^0_λ is locally free of rank 1, one can see that either E^0_λ or E^1_λ violates the stability of E. Hence E^0_λ is zero or coincides with E. Therefore s is scalar. Λ

Proposition 6.1. *There exists a global relative 2-form*

$$\Omega \in H^0(M_n^{\boldsymbol{\alpha}}, \Omega^2_{M_n^{\boldsymbol{\alpha}}/T_n' \times \Lambda_n}). \tag{125}$$

which induces a symplectic structure on each fiber of π_n.

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}} & \xrightarrow{p_2} & M_n^{\boldsymbol{\alpha}} \\ \downarrow & & \downarrow \\ \mathbf{P}^1 \times T'_n \times \Lambda_n & \xrightarrow{q_2} & T'_n \times \Lambda_n \end{array} \tag{126}$$

Let $\mathcal{F}^\bullet := \left[\nabla_1 : \mathcal{F}^0 \longrightarrow \mathcal{F}^1\right]$ be the complex of sheaves defined in (114). From Lemma 6.1, we have a natural isomorphism of sheaves:

$$\mathbf{R}^1 p_{2,*}(\mathcal{F}^\bullet) \xrightarrow{\simeq} \Theta_{M_n^{\boldsymbol{\alpha}}/T'_n \times \Lambda_n}. \tag{127}$$

By this isomorphism, it suffices to define a non-degenerate skew-symmetric form

$$\Omega : \mathbf{R}^1 p_{2,*}(\mathcal{F}^\bullet) \otimes \mathbf{R}^1 p_{2,*}(\mathcal{F}^\bullet) \longrightarrow \mathbf{R}^2 p_{2,*}(\Omega^\bullet_{\mathbf{P}^1 \times M_n^{\boldsymbol{\alpha}}/M_n^{\boldsymbol{\alpha}}}) \cong \mathcal{O}_{M_n^{\boldsymbol{\alpha}}}. \tag{128}$$

Let us fix a point $\mathbf{x} \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) = \pi_n^{-1}(\mathbf{t}, \boldsymbol{\lambda}) \subset M_n^{\boldsymbol{\alpha}}$ and define the restriction as $\mathcal{F}^\bullet_{\mathbf{x}} = \mathcal{F}^\bullet_{|\mathbf{P}^1 \times \{\mathbf{x}\}}$. From the following definition of $\Omega(\mathbf{x})$ at the stalk level of (128), it is obvious the definition of the global relative 2-form Ω in (128), and the non-degeneracy of Ω will be checked at the stalk of each closed point $\mathbf{x}$.

Take an affine open covering $\{U_\alpha\}$ of $\mathbf{P}^1$ and consider the following pairing

(129)

$$\begin{array}{rccc} \Omega(\mathbf{x}) : & \mathbf{H}^1(\mathbf{P}^1, \mathcal{F}^\bullet_{\mathbf{x}}) \otimes \mathbf{H}^1(\mathbf{P}^1, \mathcal{F}^\bullet_{\mathbf{x}}) & \longrightarrow & \mathbf{H}^2(\Omega^\bullet_{\mathbf{P}^1}) \simeq \mathbf{C}(\mathbf{x}) \\ & ([\{v_{\alpha\beta}\}, \{u_\alpha\}], [\{v'_{\alpha\beta}\}, \{u'_\alpha\}]) & \mapsto & [\{\mathrm{Tr}(v_{\alpha\beta} \circ u'_\beta) - \mathrm{Tr}(u_\alpha \circ v'_{\alpha\beta})\}] - [\{\mathrm{Tr}(v_{\alpha\beta} \circ v'_{\beta\gamma})\}] \end{array}$$

where we consider in Čech cohomology and $\{v_{\alpha\beta}\} \in C^1(\mathcal{F}^0_{\mathbf{x}})$, $\{u_\alpha\} \in C^0(\mathcal{F}^1_{\mathbf{x}})$, $\{\tilde{\nabla}_{\mathbf{x}} v_{\alpha\beta} - v_{\alpha\beta}\tilde{\nabla}_{\mathbf{x}}\} = \{u_\beta - u_\alpha\}$ and so on. We can check that $\Omega(\mathbf{x})$ is a skew symmetric pairing. Let us show that $\Omega(\mathbf{x})$ is non-degenerate for any point $\mathbf{x} \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$. From Lemma 6.3, one can show that $\mathbf{H}^2(\mathcal{F}^\bullet_{\mathbf{x}}) = 0$ for any $\mathbf{x} \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$. $\Omega(\mathbf{x})$ induces a homomorphism

$$\mathbf{H}^1(\mathcal{F}^\bullet_{\mathbf{x}}) \xrightarrow{\xi} \mathbf{H}^1(\mathcal{F}^\bullet_{\mathbf{x}})^\vee.$$

From the spectral sequence $H^p\left(H^q(\mathcal{F}^0_{\mathbf{x}}) \to H^q(\mathcal{F}^1_{\mathbf{x}})\right) \Rightarrow \mathbf{H}^{p+q}(\mathcal{F}^\bullet_{\mathbf{x}})$, we obtain the following exact sequence

$$0 \longrightarrow H^0(\mathcal{F}^0_{\mathbf{x}}) \longrightarrow H^0(\mathcal{F}^1_{\mathbf{x}}) \longrightarrow \mathbf{H}^1(\mathcal{F}^\bullet_{\mathbf{x}}) \longrightarrow H^1(\mathcal{F}^0_{\mathbf{x}}) \longrightarrow H^1(\mathcal{F}^1_{\mathbf{x}}) \longrightarrow 0. \tag{130}$$

Then we obtain the exact commutative diagram

$$\begin{array}{ccccccccc} H^0(\mathcal{F}^0_{\mathbf{x}}) & \longrightarrow & H^0(\mathcal{F}^1_{\mathbf{x}}) & \longrightarrow & \mathbf{H}^1(\mathcal{F}^\bullet_{\mathbf{x}}) & \longrightarrow & H^1(\mathcal{F}^0_{\mathbf{x}}) & \longrightarrow & H^1(\mathcal{F}^1_{\mathbf{x}}) \\ {\scriptstyle b_1}\downarrow & & {\scriptstyle b_2}\downarrow & & {\scriptstyle \xi}\downarrow & & {\scriptstyle b_3}\downarrow & & {\scriptstyle b_4}\downarrow \\ H^1(\mathcal{F}^1_{\mathbf{x}})^\vee & \longrightarrow & H^1(\mathcal{F}^0_{\mathbf{x}})^\vee & \longrightarrow & \mathbf{H}^1(\mathcal{F}^\bullet_{\mathbf{x}})^\vee & \longrightarrow & H^0(\mathcal{F}^1_{\mathbf{x}})^\vee & \longrightarrow & H^0(\mathcal{F}^0_{\mathbf{x}})^\vee, \end{array}$$

where $b_1, \ldots, b_4$ are isomorphisms induced by the isomorphisms $\mathcal{F}^0_{\mathbf{x}} \cong (\mathcal{F}^1)^\vee_{\mathbf{x}} \otimes \Omega^1_{\mathbf{P}^1}$, $\mathcal{F}^1_{\mathbf{x}} \cong (\mathcal{F}^0)^\vee_{\mathbf{x}} \otimes \Omega^1_{\mathbf{P}^1}$ and Serre-duality. Thus ξ becomes an isomorphism by five lemma. Λ

6.3. **Smoothness of $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ and its dimension.** In this subsection, we prove that the morphism $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n$ is smooth of equidimension $2n - 6$, which is stated in Theorem 2.1, (2).

Proposition 6.2. (1) *The morphism $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n \times \Lambda_n$ is smooth.*

(2) *For any closed point* $\mathbf{x} \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$, *we have*

$$\dim_{\mathbf{C}} \mathbf{H}^1(\mathcal{F}_{\mathbf{x}}^{\bullet}) = 2n - 6. \tag{131}$$

In particular, the moduli space $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ *is smooth of equidimension* $2n-6$.

Proof. (1): By a standard argument as in [Lemma 4, [A]], the smoothness of π_n at $\mathbf{x}$ follows from Lemma 6.3. (2): First, by (123), we have $\mathcal{F}^1 \simeq (\mathcal{F}^0)^{\vee} \otimes \Omega^1_{\mathbf{P}^1}$, and hence Serre duality implies that $\chi(\mathbf{P}^1, \mathcal{F}^1_{\mathbf{x}}) = \chi((\mathcal{F}^0_{\mathbf{x}})^{\vee} \otimes \Omega^1_{\mathbf{P}^1}) = -\chi(\mathbf{P}^1, \mathcal{F}^0)$. Together with the exact sequence (130), we obtain

$$\dim \mathbf{H}^1(\mathcal{F}^{\bullet}_{\mathbf{x}}) = -\chi(\mathbf{P}^1, \mathcal{F}^0_{\mathbf{x}}) + \chi(\mathbf{P}^1, \mathcal{F}^1_{\mathbf{x}}) = -2\chi(\mathbf{P}^1, \mathcal{F}^0_{\mathbf{x}}). \tag{132}$$

Setting $\mathcal{E}nd^0(\mathbf{E_x}) = \{s \in \mathcal{E}nd(\tilde{E}_{\mathbf{x}}) \mid \operatorname{Tr}(s) = 0\}$, by definition of $\mathcal{F}^0_{\mathbf{x}}$ (111), we obtain the following exact sequence

$$0 \longrightarrow \mathcal{F}^0_{\mathbf{x}} \longrightarrow \mathcal{E}nd^0(\tilde{E}_{\mathbf{x}}) \longrightarrow \otimes_{i=1}^n \mathbf{C}((t_i, \mathbf{x})) \longrightarrow 0. \tag{133}$$

Since $\mathcal{E}nd^0(\mathbf{E_x})$ is a self-dual locally free sheaf of rank 3 on $\mathbf{P}^1$, Riemann-Roch theorem implies that $\chi(\mathbf{P}^1, \mathcal{E}nd^0(\mathbf{E_x})) = 3 + \deg \mathcal{E}nd^0(\mathbf{E_x}) = 3$. Then the exact sequence (133) together with (132) shows that

$$\chi(\mathbf{P}^1, \mathcal{F}^0_{\mathbf{x}}) = \chi(\mathcal{E}nd^0(\tilde{E}_{\mathbf{x}})) - n = 3 - n,$$

which implies the assertion (131) . Λ

Remark 6.1. *One can also show that*

$$\mathbf{H}^2(\mathbf{P}^1, \mathcal{F}^{\bullet +}_{\mathbf{x}}) = \{0\}, \tag{134}$$

which implies that the morphism $M_n^{\boldsymbol{\alpha}} \longrightarrow T_n$ *is smooth.*

6.4. **Tangent space to $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$.** Let $(E, \nabla, \varphi, \{l_i\})$ be a stable parabolic connection on $\mathbf{P}^1$ corresponding to a point $\mathbf{x}$ in $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$. Let us consider the inclusion $j : \mathbf{P}^1 \setminus D(\mathbf{t}) \hookrightarrow \mathbf{P}^1$ and define

$$\mathbf{E} = \ker \left[\nabla^{an} : E \longrightarrow (E \otimes \Omega^1_{\mathbf{P}^1})\right]_{\mathbf{P}^1 \setminus D(\mathbf{t})}. \tag{135}$$

Then $\mathbf{E}$ becomes a locally constant sheaf on $\mathbf{P}^1 \setminus D(\mathbf{t})$. The correspondence $(E, \nabla, \varphi, \{l_i\}) \mapsto \mathbf{E}$ induces an analytic morphism

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \tag{136}$$

which is called the *Riemann-Hilbert correspondence.* (Here we set $\mathbf{a} = (a_i), a_i = 2\cos 2\pi\lambda_i$). For the precise definition, see Definition 7.1 in §7.

The morphism $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ will be studied in detail in the next section.

Define another locally constant sheaf on $\mathbf{P}^1 \setminus D(\mathbf{t})$ by

$$\mathbf{V} := \{s \in Hom(\mathbf{E}, \mathbf{E}) \mid \operatorname{Tr}(s) = 0\}. \tag{137}$$

Note that for each point $u \in \mathbf{P}^1 \setminus D(\mathbf{t})$ the fiber of $\mathbf{V}_u$ is isomorphic to the Lie algebra $sl_2(\mathbf{C})$. Therefore $\mathbf{V}$ admits the natural non-degenerate pairing $q : \mathbf{V} \otimes \mathbf{V} \longrightarrow \mathbf{C}_{\mathbf{P}^1 \setminus D(\mathbf{t})}$ induced by the Killing form on each fiber $\mathbf{V}_u, u \in \mathbf{P}^1 \setminus D(\mathbf{t})$. Now consider the constructible sheaf $j_*(\mathbf{V})$ and the following exact sequence induced by the Leray spectral sequence for the inclusion $j : \mathbf{P}^1 \setminus D(\mathbf{t}) \hookrightarrow \mathbf{P}^1$:

(138)
$$0 \to H^1(\mathbf{P}^1, j_*\mathbf{V}) \to H^1(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V}) \to H^0(\mathbf{P}^1, R^1 j_*(\mathbf{V})) \to H^2(\mathbf{P}^1, j_*(\mathbf{V})) \to H^2(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V}).$$

Recall that in §4 we have obtained the morphism

$$\phi_n : \mathcal{R}_n \longrightarrow T'_n \times \mathcal{A}_n \tag{139}$$

such that $\phi_n^{-1}((\mathbf{t}, \mathbf{a})) = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$. Fixing $\mathbf{t} \in T'_n$, we can also define

$$\phi_{n,\mathbf{t}} : \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) \longrightarrow \mathbf{t} \times \mathcal{A}_n. \tag{140}$$

Lemma 6.4. *Let $(E, \nabla, \varphi, \{l_i\}) \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ be a stable parabolic connection, and $\mathbf{E} := \ker \nabla_{|\mathbf{P}^1 - D(\mathbf{t})}$ the corresponding local system. Moreover let $\mathbf{V}$ be the trace free part of $\mathrm{End}(\mathbf{E})$. Let us fix a monodromy representation $\rho_{\mathbf{E}} : \pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}), *) \longrightarrow SL_2(\mathbf{C})$ associated to the local system $\mathbf{E}$. Fix canonical generators $\gamma_i, 1 \leq i \leq n$ of $\pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}), *)$ and set $M_i = \rho_{\mathbf{E}}(\gamma_i) \in SL_2(\mathbf{C})$ for $1 \leq i \leq n$. Consider the following conditions.*

$$\text{The representation } \rho_{\mathbf{E}} \text{ is irreducible.} \tag{141}$$

$$\text{For each } i, 1 \leq i \leq n, \text{ the local monodromy matrix } M_i \text{ around } t_i \text{ is not equal to } \pm I_2. \tag{142}$$

(1) *Under the condition* (141), *we have*

$$H^2(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V}) = \{0\}. \tag{143}$$

(2) *Under the conditions* (141) *and* (142), *we have a sheaf isomorphism*

$$R^1 j_*(\mathbf{V}) \simeq \oplus_{i=1}^n \mathbf{C}(t_i), \tag{144}$$

and the exact sequence of cohomology groups.

$$0 \longrightarrow H^1(\mathbf{P}^1, j_*\mathbf{V}) \longrightarrow H^1(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V}) \longrightarrow H^0(\mathbf{P}^1, R^1 j_*(\mathbf{V})) \longrightarrow 0. \tag{145}$$

Proof. Since we have a canonical non-degenerate pairing

$$j_*(\mathbf{V}) \otimes j_*(\mathbf{V}) \stackrel{\text{Killing}}{\longrightarrow} \mathbf{C}_{\mathbf{P}^1},$$

we have a self-duality $(j_*\mathbf{V})^\vee \simeq j_*\mathbf{V}$ and hence a duality isomorphism

$$H^2(\mathbf{P}^1, j_*(\mathbf{V})) \simeq H^0(\mathbf{P}^1, j_*\mathbf{V})^\vee \simeq H^0(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V})^\vee.$$

Since by (141) the monodromy representation $\rho_{\mathbf{E}}$ is irreducible, $H^0(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathrm{End}(\mathbf{E})) \simeq \mathbf{C} \cdot Id_{\mathbf{E}}$ by Schur's lemma and hence its trace free part $H^0(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V})$ is $\{0\}$, thus

$$H^2(\mathbf{P}^1, j_*(\mathbf{V})) = \{0\}. \tag{146}$$

Moreover $H^1(\mathbf{P}^1, R^1 j_*\mathbf{V}) = \{0\}$, for the sheaf $R^1 j_*\mathbf{V}$ is supported only on $D(\mathbf{t}) = t_1 + \cdots + t_n$. Then the assertion (143) now easily follows from the Leray spectral sequence for $j : \mathbf{P}^1 \setminus D(\mathbf{t}) \hookrightarrow \mathbf{P}^1$.

From (138), we obtain the exact sequence (145) because of (146).

For the assertion (144), we first remark that the sheaf $R^1 j_* \mathbf{V}$ is supported on $D(\mathbf{t}) = t_1 + \cdots + t_n$. We will determine the stalk $R^1 j_* \mathbf{V}_{t_i}$ at each t_i. Let us take a small neighborhood U_i of t_i and $u_i \in U_i - \{t_i\}$. Then one can identify the fiber $\mathbf{V}_{u_i}$ with the symmetric tensor $Sym^2(\mathbf{E}_{u_i})$. Consider the $\mathbf{V}_{u_i} \simeq Sym^2(\mathbf{E}_{u_i})$ as the vector space with the action of M_i. Then define the invariant part as

$$\mathbf{V}_{u_i}^{<M_i>} = Sym^2(\mathbf{E}_{u_i})^{<M_i>} := \ker(Sym^2(M_i) - Id : Sym^2(\mathbf{E}_{u_i}) \to Sym^2(\mathbf{E}_{u_i})).$$

Then it is easy to see that

$$R^1 j_* \mathbf{V}_{t_i} \simeq \left(\mathbf{V}_{u_i}^{<M_i>}\right)^{\vee}.$$

Choose a suitable basis of $\mathbf{E}_{u_i}$ and write M_i as $M_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. Then the action of M_i on $Sym^2(\mathbf{E}_{u_i})$ has the following matrix representation.

$$Sym^2(M_i) := \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \tag{147}$$

Then it is easy to check that the eigenvalues of $Sym^2(M_i)$ are given by the roots of

$$(x - 1)(x^2 - ((a + d)^2 - 2)x + 1) = 0.$$

If neither of the roots of $x^2 - ((a + d)^2 - 2)x + 1 = 0$ is 1, then $\dim\ker(Sym^2(M_i) - Id) = 1$. If one of the roots of $x^2 - ((a+d)^2 - 2)x + 1 = 0$ is one, then we have $(a + d)^2 = 4$, which implies that $a + d = \pm 2$. For those cases, the eigenvalues of M_i are 1 or -1 respectively. We may assume that $M_i \neq \pm I_2$. Then we can assume that $M_i = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ or $M_i = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ with $b \neq 0$. For these cases, we can write

$$Sym^2(M_i) = \begin{pmatrix} 1 & b & b^2 \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix}.$$

Now it is easy to check that $\dim\ker(Sym^2(M_i) - Id) = 1$. Λ

Lemma 6.5. *Let us fix* $\mathbf{t} \in T'_n$. *The notation being as in Lemma 6.4, let us take a point* $\mathbf{y} := [\mathbf{E}] \in \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \subset \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})$.

(1) *Assume that the condition (141) holds for* $\mathbf{E}$. *Then the total space* $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})$ *is smooth at* $\mathbf{y} = [\mathbf{E}]$ *and we have the isomorphism*

$$\Theta_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}}),\mathbf{y}} \simeq H^1(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V}).$$

(2) *Assume that the conditions (141) and (142) hold for* $\mathbf{E}$. *Then, the map* $\phi_{n,\mathbf{t}} : \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) \longrightarrow \mathbf{t} \times \mathcal{A}_n$ *is also smooth at* $\mathbf{y} = [\mathbf{E}]$. *Hence the fiber* $\phi_{n,\mathbf{t}}^{-1}(\mathbf{a}) = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ *is smooth at* $\mathbf{y}$ *where* $\mathbf{a} = \phi_{n,\mathbf{t}}(\mathbf{y})$. *Moreover we have the following linear isomorphisms:*

$$(\Theta_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}})_{\mathbf{y}} \simeq H^1(\mathbf{P}^1, j_* \mathbf{V})$$

$$(\Theta_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})})_{\mathbf{y}} \simeq H^1(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V})$$

$$(\Theta_{\mathbf{t} \times \mathcal{A}_n})_{\phi_{n,\mathbf{t}}(\mathbf{y})} \simeq H^0(\mathbf{P}^1, R^1 j_*(\mathbf{V}))$$

Under the isomorphisms above, we have the following identification of the natural exact sequences of the tangent spaces with the sequence (145)

$$
\begin{array}{ccccccccc}
0 \longrightarrow & (\Theta_{\mathcal{R}(\mathcal{P}(\mathcal{P}_{n,\mathbf{t}}))_{\mathbf{a}}})_{\mathbf{y}} & \longrightarrow & (\Theta_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})})_{\mathbf{y}} & \stackrel{d\phi_{n,\mathbf{t}*}}{\longrightarrow} & (\Theta_{\mathbf{t}\times\mathcal{A}_n})_{\phi_{n,\mathbf{t}}(\mathbf{y})} & \longrightarrow 0. \\
& || & & || & & || & \\
0 \longrightarrow & H^1(\mathbf{P}^1, j_*\mathbf{V}) & \longrightarrow & H^1(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V}) & \longrightarrow & H^0(\mathbf{P}^1, R^1 j_*(\mathbf{V})) & \longrightarrow 0.
\end{array} \tag{148}
$$

Proof. 1. Since $\mathbf{E}$ is irreducible, it is easy to see that the Zariski tangent space $\Theta_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}}),\mathbf{y}}$ of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})$ at $\mathbf{y} = [\mathbf{E}]$ is given by $H^1(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V})$ and the obstructions to deformations lie in $H^2(\mathbf{P} \setminus D(\mathbf{t}), \mathbf{V})$. Since we assume that $\rho_{\mathbf{E}}$ is irreducible, we have $H^2(\mathbf{P} \setminus D(\mathbf{t}), \mathbf{V}) = \{0\}$ (cf. (143)) , from which the assertion follows.

2. From Lemma 6.4, under the assumptions, we can see that the differential $(\Theta_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})})_{\mathbf{y}} \stackrel{d\phi_{n,\mathbf{t}*}}{\longrightarrow} (\Theta_{\mathbf{t}\times\mathcal{A}_n})_{\phi_{n,\mathbf{t}}(\mathbf{y})}$ can be identified with the linear map

$$H^1(\mathbf{P}^1 \setminus D(\mathbf{t}), \mathbf{V}) \longrightarrow H^0(\mathbf{P}^1, R^1 j_*(\mathbf{V})) \simeq \mathbf{C}^n,$$

which is surjective because of $H^2(\mathbf{P}^1, j_*\mathbf{V}) = \{0\}$. Therefore the map $\phi_{n,\mathbf{t}}$ is smooth at $\mathbf{y} = [\mathbf{E}]$ and the fiber $\phi_{n,\mathbf{t}}^{-1}(\mathbf{a}) = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is smooth at $\mathbf{y}$. Other assertions now easily follow from the exact sequence (145). Λ

Lemma 6.6. *Under the conditions (141) and (142) for* $\mathbf{E}$*, we have an isomorphism of locally constant sheaves*

$$j_*\mathbf{V} \simeq \ker \nabla_1 \simeq \left[\nabla_1 : \mathcal{F}^0 \longrightarrow \mathcal{F}^1\right], \tag{149}$$

which induces a canonical isomorphism

$$H^1(\mathbf{P}^1, j_*\mathbf{V}) \stackrel{\simeq}{\longrightarrow} \mathbf{H}^1(\left[\nabla_1 : \mathcal{F}^0 \longrightarrow \mathcal{F}^1\right]). \tag{150}$$

Moreover we have the canonical non-degenerate pairing

$$H^1(\mathbf{P}^1, j_*\mathbf{V}) \otimes H^1(\mathbf{P}^1, j_*\mathbf{V}) \longrightarrow H^2(\mathbf{P}^1, \mathbf{C}_{\mathbf{P}^1}) \simeq \mathbf{C}, \tag{151}$$

which induces the non-degenerate pairing $\Omega_1(\mathbf{y})$ *at* $\mathbf{y} = \mathbf{E}$

$$\Omega_1(\mathbf{y}) : (\Theta_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}})_{\mathbf{y}} \otimes (\Theta_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}})_{\mathbf{y}} \longrightarrow (\mathcal{O}_{\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}})_{\mathbf{y}}. \tag{152}$$

This pairing can be identified with (129) via the isomorphism (150).

Proof. The assertion (149) is trivial at the point $u \in \mathbf{P}^1 \setminus D(\mathbf{t})$. At each point t_i $i = 1, \ldots, n$, we will describe the connection ∇ and ∇_1 locally around t_i. Let us set $n = n_i = \mathrm{res}_{t_i}(\nabla_L) \in \mathbf{Z}$. We separate the proof into two cases.

i) Let $\lambda, n - \lambda$ be the eigenvalues of $\mathrm{res}_{t_i}(\nabla)$. First assume that $2\lambda \notin \mathbf{Z}$. Then $\lambda \neq n - \lambda$. By a standard reduction theory of connection near a regular singularity, we can choose a suitable local coordinate z around $t = t_i$ and write down the connection matrix of ∇ by

$$\nabla = \frac{dz}{z - t} \begin{pmatrix} \lambda & 0 \\ 0 & n - \lambda \end{pmatrix}.$$

Then for a local section $s = \begin{pmatrix} s_1 & s_2 \\ s_3 & -s_1 \end{pmatrix} \in \mathrm{End}(E)$, the connection $\nabla_1 s = \nabla_E s - s\nabla_E$ is given by

$$\begin{pmatrix} s_1 & s_2 \\ s_3 & -s_1 \end{pmatrix} \mapsto \begin{pmatrix} ds_1 & ds_2 + (2\lambda - n)s_2(z-t)^{-1}dz \\ ds_3 + (n-2\lambda)s_3(z-t)^{-1}dz & -ds_1 \end{pmatrix}.$$

Solving $\nabla_1 = 0$ locally near $z = t$, we obtain the local solutions for $z \neq t$ as follows.

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & (z-t)^{n-2\lambda} \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ (z-t)^{2\lambda - n} & 0 \end{pmatrix}.$$

Here, $c_1, c_2, c_3 \in \mathbf{C}$ are constants. These solutions have to be single-valued well-defined section around $z = t$, hence $\ker \nabla_1$ is generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (Note that this local section lies in $\mathcal{F}^0$.) On the other hand, the stalk $(j_*\mathbf{V})_t$ is the space of monodromy invariant trace-free endomorphisms of $\mathbf{E}_u$, which is also generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence we have an isomorphism $(j_*\mathbf{V})_t \simeq \ker(\nabla_1)_t$.

ii) Again, let $\lambda, n - \lambda$ be the eigenvalues of $\mathsf{res}_{t_i}(\nabla)$ and assume that $2\lambda \in \mathbf{Z}$. Since we assume that the local monodromy M_i is not $\pm I_2$, by a reduction theory of a connection near a regular singularity, we can choose a suitable local coordinate z around $t = t_i$ and write down the connection matrix of ∇ by

$$\nabla = \frac{dz}{z-t} \begin{pmatrix} m_1 & (z-t)^{m_2 - m_1} \\ 0 & m_2 \end{pmatrix},$$

where $2m_1, 2m_2 \in \mathbf{Z}$, $m_2 - m_1 \in \mathbf{Z}$ and $m_1 \leq m_2$. For local section $\begin{pmatrix} s_1 & s_2 \\ s_3 & -s_1 \end{pmatrix} \in \mathrm{End}(E)$, the connection $\nabla_1 s$ can be given by

$$\begin{pmatrix} s_1 & s_2 \\ s_3 & -s_1 \end{pmatrix} \mapsto \begin{pmatrix} ds_1 + s_3(z-t)^{m_2-m_1-1}dz & ds_2 - 2s_1(z-t)^{m_2-m_1-1}dz + s_2(m_1 - m_2)(z-t)^{-1}dz \\ ds_3 + s_3(m_2 - m_1)(z-t)^{-1}dz & -ds_1 - s_3(z-t)^{m_2-m_1-1}dz \end{pmatrix}.$$

Solving $\nabla_1 s = 0$ locally for $z \neq t$, we obtain the solutions

$$\begin{aligned} s = c_1 \begin{pmatrix} 0 & (z-t)^{m_2-m_1} \\ 0 & 0 \end{pmatrix} & + c_2 \begin{pmatrix} 1 & 2(z-t)^{m_2-m_1}\log(z-t) \\ 0 & -1 \end{pmatrix} \\ & + c_3 \begin{pmatrix} \log(z-t) & (z-t)^{m_2-m_1}(\log(z-t))2 \\ -(z-t)^{m_1-m_2} & -\log(z-t) \end{pmatrix} \end{aligned}$$

where $c_1, c_2, c_3 \in \mathbf{C}$ are the constants. Then we can see that all single valued solutions for $\ker \nabla_1$ are

$$c_1 \begin{pmatrix} 0 & (z-t)^{m_2-m_1} \\ 0 & 0 \end{pmatrix},$$

which are also sections of $(j_*\mathbf{V})_t$. Therefore we have an isomorphism $(\ker \nabla_1)_t \simeq (j_*\mathbf{V})_t$. Hence we have proved the assertion (149) which shows also (150).

It is easy to see that the pairing of sheaves $j_*\mathbf{V} \otimes j_*\mathbf{V} \longrightarrow j_*\mathbf{C}_{\mathbf{P}^1 \setminus D(\mathbf{t})} \simeq \mathbf{C}_{\mathbf{P}^1}$ is non-degenerate at each point of $\mathbf{P}^1$. Therefore, the pairing (151) is also non-degenerate.

Λ

Summarizing all results in this section, we have the following

Proposition 6.3. *Let $\phi_n : \mathcal{R}_n \longrightarrow T'_n \times \mathcal{A}_n$ be a family of moduli spaces of representations of the fundamental group $\pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}), *)$ as in (139). Let $\mathcal{R}^\sharp_n$ be the subset of $\mathcal{R}_n$ whose closed points satisfy the conditions (141) and (142).*

Then $\mathcal{R}^\sharp_n$ is a non-singular variety and the restricted morphism

(153) $$\phi_n : \mathcal{R}^\sharp_n \longrightarrow T'_n \times \mathcal{A}_n$$

is smooth, so that all fibers $\mathcal{R}^{\sharp}_{n,(\mathbf{t},\mathbf{a})} = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ are non-singular varieties. On $\mathcal{R}^{\sharp}_n$, there exists a relative symplectic form

$$\Omega_1 \in \Gamma(\mathcal{R}^{\sharp}_n, \Omega^2_{\mathcal{R}^{\sharp}_n/T'_n\times\mathcal{A}_n}) \tag{154}$$

induced by (152).

Remark 6.2. (1) Since $p_2 \circ \phi_n : \mathcal{R}^{\sharp}_n \longrightarrow T'_n \times \mathcal{A}_n \longrightarrow T'_n$ is locally trivial, one can lift $\Omega_1 \in \Gamma(\mathcal{R}^{\sharp}_n, \Omega^2_{\mathcal{R}^{\sharp}_n/T'_n\times\mathcal{A}_n})$ to a relative regular 2-form

$$\tilde{\Omega}_1 \in \Gamma(\mathcal{R}^{\sharp}_n, \Omega^2_{\mathcal{R}'^{\sharp}_n/\mathcal{A}_n}). \tag{155}$$

In §7, we can define the Riemann-Hilbert correspondence $\mathbf{RH}_n : M^{\boldsymbol{\alpha}}_n \longrightarrow \mathcal{R}_n$ which is a surjective holomorphic map. Set $(M^{\boldsymbol{\alpha}}_n)^{\sharp} = \mathbf{RH}_n^{-1}(\mathcal{R}^{\sharp}_n)$. From Lemma 6.6, one can see that $\mathbf{RH}^*_{n|\mathcal{R}^{\sharp}_n}(\Omega_1)$ coincides with the two form $\Omega_{|(M^{\boldsymbol{\alpha}}_n)^{\sharp}} \in \Gamma((M^{\boldsymbol{\alpha}}_n)^{\sharp}, \Omega^2_{(M^{\boldsymbol{\alpha}}_n)^{\sharp}/T'_n\times\Lambda_n})$ defined in (129). Pulling back $\tilde{\Omega}_1$ via $\mathbf{RH}_{n|\mathcal{R}^{\sharp}_n}$, we obtain

$$\tilde{\Omega} \in \Gamma((M^{\boldsymbol{\alpha}}_n)^{\sharp}, \Omega^2_{(M^{\boldsymbol{\alpha}}_n)^{\sharp}/\Lambda_n}) \tag{156}$$

which is a lift of $\Omega_{|(M^{\boldsymbol{\alpha}}_n)^{\sharp}}$ via the canonical morphism $\Gamma((M^{\boldsymbol{\alpha}}_n)^{\sharp}, \Omega^2_{(M^{\boldsymbol{\alpha}}_n)^{\sharp}/\Lambda_n}) \longrightarrow \Gamma((M^{\boldsymbol{\alpha}}_n)^{\sharp}, \Omega^2_{(M^{\boldsymbol{\alpha}}_n)^{\sharp}/T'_n\times\Lambda_n})$. (Note that a lift $\tilde{\Omega}$ can be induced from the splitting homomorphism (20) and the splitting homomorphism can be defined algebraically). Since the codimension of $M^{\boldsymbol{\alpha}}_n \setminus (M^{\boldsymbol{\alpha}}_n)^{\sharp}$ in $M^{\boldsymbol{\alpha}}_n$ is at least two, the two form $\tilde{\Omega}$ can be extended to a regular relative two form on $M^{\boldsymbol{\alpha}}_n$ which will be denoted also by $\tilde{\Omega}$. This extended two form $\tilde{\Omega} \in \Gamma(M^{\boldsymbol{\alpha}}_n, \Omega^2_{M^{\boldsymbol{\alpha}}_n/\Lambda_n})$ is a lift of $\Omega \in \Gamma(M^{\boldsymbol{\alpha}}_n, \Omega^2_{M^{\boldsymbol{\alpha}}_n/T'_n\times\Lambda_n})$ in (129) on the whole total space $M^{\boldsymbol{\alpha}}_n$.

(2) The closedness of $\tilde{\Omega}$, $(d_{M^{\boldsymbol{\alpha}}_n/\Lambda_n}(\tilde{\Omega}) = 0)$, can be proved as follows. It is easy to see that the two form $\tilde{\Omega}$ here coincides with the symplectic two form introduced in [Iw1] and [Iw2] on a Zariski dense open subset $(M^{\boldsymbol{\alpha}}_n)'$ of M^{α}_n. As proved in [Iw1], [Iw2], there exists a suitable affine open covering $\{U_i\}_i$ of $(M^{\boldsymbol{\alpha}}_n)'$ with local coordinates (for U_i)

$$(q^i_1, \dots, q^i_r, p^i_1, \dots, p^i_r, t_1, \dots, t_n, \lambda_1, \dots, \lambda_n)$$

such that $\tilde{\Omega}_{|U_i}$ can be written as

$$\tilde{\Omega}_{|U_i} = \sum_{k=1}^{r} dq^i_k \wedge dp^i_k - \sum_{l=1}^{n} dt_l \wedge dH^i_l(\mathbf{p}, \mathbf{q}, \mathbf{t}, \boldsymbol{\lambda}). \tag{157}$$

where $r = n-3$ (= the half of the relative dimension of π_n) and $H^i_l(\mathbf{p}, \mathbf{q}, \mathbf{t}, \boldsymbol{\lambda})$ are regular algebraic functions on U_i. The closedness $d_{M^{\boldsymbol{\alpha}}_n/\Lambda_n}(\tilde{\Omega}) = 0$ on U_i easily follows from the expression (157), hence by analytic continuation we see that $d_{M^{\boldsymbol{\alpha}}_n/\Lambda_n}(\tilde{\Omega}) = 0$ on the total space $M^{\boldsymbol{\alpha}}_n$.

(3) The regular functions $H^i_l(\mathbf{p}, \mathbf{q}, \mathbf{t}, \boldsymbol{\lambda})$ on U_i in (157) are called *Hamiltonians* for Painlevé or Garnier systems with respect to the time variable t_l. Actually on an affine open set U_i one can obtain the Hamiltonian systems (Cf. [Iw1], [Iw2]).

$$\frac{\partial q^i_k}{\partial t_l} = \frac{\partial H^i_l}{\partial p^i_k}, \quad \frac{\partial p^i_k}{\partial t_l} = -\frac{\partial H^i_l}{\partial q^i_k} \quad (1 \le k \le n-3, 1 \le l \le n). \tag{158}$$

Although these Hamiltonian systems are defined on a Zariski open subset $(M_n^{\boldsymbol{\alpha}})'$ of $M_n^{\boldsymbol{\alpha}}$, these Hamitonian systems can be extended to Hamitonian systems on the total space $M_n^{\boldsymbol{\alpha}}$. This is because global vector fields on $M_n^{\boldsymbol{\alpha}}$ induced from the isomonodromic flows coincide with these Hamitonian systems on the Zariski open set $(M_n^{\boldsymbol{\alpha}})'$ and the global vector fields on $M_n^{\boldsymbol{\alpha}}$ also preserves the symplectic form $\tilde{\Omega}$.

7. The Riemann-Hilbert correspondence

In this section, we also work over T_n' (cf. (94)). Fix $(\mathbf{t}, \boldsymbol{\lambda}) \in T_n' \times \Lambda_n$ and set $D(\mathbf{t}) = t_1 + \cdots + t_n \subset \mathbf{P}^1$, $a_i = 2\cos 2\pi\lambda_i$ and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}_n$. Moreover fix a determinant line bundle $L = (L, \nabla_L)$ on $\mathbf{P}^1$ such that $\mathsf{res}_{t_i}(\nabla_L) \in \mathbf{Z}$ for every $1 \leq i \leq n$. We have defined two moduli spaces $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ in (42) and $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ in (97). In this section, we define the *Riemann-Hilbert correspondence* $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}} : M^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$, and show our main results for the Riemann-Hilbert correspondence (Theorem 7.1).

7.1. **Definition of $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$.** As in (136), take $E = (E, \nabla, \varphi, l) \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ and define the local system on $\mathbf{P}^1 \setminus D(\mathbf{t})$ as $\mathbf{E} = \ker\left(\nabla_{|\mathbf{P}^1 \setminus D(\mathbf{t})}\right)^{an}$. (Here we denote by $\left(\nabla_{|\mathbf{P}^1 \setminus D(\mathbf{t})}\right)^{an}$ the analytic connection associated to $\left(\nabla_{|\mathbf{P}^1 \setminus D(\mathbf{t})}\right)$.) Choosing a suitable flat basis for the fiber $\mathbf{E}_*$ at the base point $* \in \mathbf{P}^1 \setminus D(\mathbf{t})$, one can define a monodromy representation $\rho_{\mathbf{E}} : \pi^1(\mathbf{P}^1 \setminus D(\mathbf{t}), *) \longrightarrow SL_2(\mathbf{C})$. The difference of choices of flat basis can be given by the adjoint action of $SL_2(\mathbf{C})$, and hence one has a correspondence

$$E = (E, \nabla, \varphi, l) \mapsto [\rho_{\mathbf{E}}]. \tag{159}$$

Here$[\rho_{\mathbf{E}}]$ denotes the Jordan equivalence class of $\rho_{\mathbf{E}}$.

Fix canonical generators $\gamma_i, 1 \leq i \leq n$ of $\pi^1(\mathbf{P}^1 \setminus D(\mathbf{t}), *)$. For a monodromy representation $\rho_{\mathbf{E}}$ of (E, ∇, φ, l), set $M_i = \rho_{\mathbf{E}}(\gamma_i)$ as in §4. Since eigenvalues of $\mathsf{res}_{t_i}(\nabla)$ can be given by $\lambda_i, \mathsf{res}_{t_i}(\nabla_L) - \lambda_i$ and $\mathsf{res}_{t_i}(\nabla_L) \in \mathbf{Z}$, we see that the eigenvalues of M_i are given by $\exp(\mp 2\pi\sqrt{-1}\lambda_i)$. Therefore, we have local exponents for $\rho_{\mathbf{E}}$

$$a_i := \mathrm{Tr}[M_i] = \exp(-2\pi\sqrt{-1}\lambda_i) + \exp(2\pi\sqrt{-1}\lambda_i) = 2\cos(2\pi\lambda_i), \tag{160}$$

which are invariant under the adjoint action.

Definition 7.1. Under the relation (160), the correspondence (159) gives an analytic morphism

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}, \tag{161}$$

which is called the *Riemann-Hilbert correspondence.*

7.2. **Fundamental properties of Riemann-Hilbert correspondence.** Let us assume that $n \geq 4$. In §4, (96), we have defined the family of moduli spaces of representations of fundamental group $\phi_n : \mathcal{R}_n \longrightarrow T_n' \times \mathcal{A}_n$ and we also have a smooth family $\pi_n : M_n^{\boldsymbol{\alpha}}(L) \longrightarrow T_n' \times \Lambda_n$ whose geometric fibers are

$M^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ (cf. Theorem 2.1). From Definition 7.1 we obtain the following commutative diagram:

$$\begin{array}{ccc} M_n^{\boldsymbol{\alpha}}(L) & \xrightarrow{\mathbf{RH}_n} & \mathcal{R}_n \\ {\scriptstyle \pi_n}\downarrow & & {\scriptstyle \phi_n}\downarrow \\ T'_n \times \Lambda_n & \xrightarrow{id \times \mu_n} & T'_n \times \mathcal{A}_n. \end{array} \tag{162}$$

Here $\mu_n : \Lambda_n \longrightarrow \mathcal{A}_n$ is given by

$$\mu_n(\lambda_1, \ldots, \lambda_n) = (a_1, \ldots, a_n) = (2\cos(2\pi\lambda_1), \ldots, 2\cos(2\pi\lambda_n)). \tag{163}$$

Of course, for each $(\mathbf{t}, \boldsymbol{\lambda}) \in T'_n \times \Lambda_n$, the morphism $\mathbf{RH}_{n|M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)}$ is equal to $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ in (161).

Theorem 7.1. *Under the notation and the assumption as above, we have the following assertions.*

1. *For all $(\mathbf{t}, \boldsymbol{\lambda}) \in T'_n \times \Lambda_n$, the Riemann-Hilbert correspondence $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ in (161) is a* <u>*bimeromorphic proper surjective*</u> *morphism.*
2. *For any $(\mathbf{t}, \boldsymbol{\lambda})$, let $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ be the Zariski open subset of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ whose closed points satisfy the conditions (141) and (142) in §6, and set $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)^{\sharp} = \mathbf{RH}^{-1}_{\mathbf{t},\boldsymbol{\lambda}}(\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}})$. Then the Riemann-Hilbert correspondence gives an analytic isomorphism*

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda},|M_n^{\boldsymbol{\alpha}}(\mathbf{t},\boldsymbol{\lambda},L)^{\sharp}} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)^{\sharp} \xrightarrow{\simeq} \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}. \tag{164}$$

   (Note that if $\boldsymbol{\lambda}$ is generic (cf. Definition 2.3, (36), (37)), $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}} = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$, hence $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ gives an analytic isomorphism between $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ and $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$.)
3. *Let us set $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{sing}_{\mathbf{a}} = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \setminus \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$. Then the codimension of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{sing}_{\mathbf{a}}$ in $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is at least 2.*
4. *The symplectic structures Ω on $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ and Ω_1 on $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}$ can be identified with each other via $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$, that is,*

$$\Omega = \left(\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda},|\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^{\sharp}_{\mathbf{a}}}\right)^{*}(\Omega_1) \quad on \quad M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)^{\sharp}. \tag{165}$$

Remark 7.1. (1) The moduli spaces $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ and $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ are irreducible. (See §8 and §9).
(2) The statement (165) is originally shown by Iwasaki in [Iw1], [Iw2].

Let us denote $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ in (161) simply by $\mathbf{RH}$. We first show the following

Lemma 7.1. *Assume that $n \geq 4$ and that α_i ($i = 1, \ldots, n$) are so general that all the semistable parabolic connections are stable. Then the morphism $\mathbf{RH} : M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is a bimeromorphic surjective morphism.*

Proof. Let $\mathcal{R}^{\mathrm{irr}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ be the open subscheme of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ whose points correspond to the irreducible representations. First we will show that $\mathcal{R}^{\mathrm{irr}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is contained in the image of $\mathbf{RH}$.

Let $M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ be the open subscheme of $M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ consisting of the points corresponding to the irreducible connections. Note that if $(E, \nabla_E, \varphi, \{l_i\})$ is a parabolic connection such that (E, ∇_E) is an irreducible connection, we have $(E, \nabla_E, \varphi, \{l_i\}) \in M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$. We consider the isomorphism of the moduli spaces

$$Elm^{-}_{t_i} : M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}, L) \xrightarrow{\sim} M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}', L(-t_i)); \quad (E, \nabla_E, \varphi, \{l_i\}) \mapsto (E', \nabla_{E'}, \varphi', \{l'_i\}),$$

where $E' = \ker(E \to E_{t_i}/l_i)$, $\nabla_{E'}$ is a connection on E' induced by ∇_E, $l'_i = \ker(E'_{t_i} \to E_{t_i})$, $l'_j = l_j$ for $j \neq i$, $\lambda'_i = 1 + \mathsf{res}_{t_i}(\nabla_L) - \lambda_i$, $\lambda'_j = \lambda_j$ for $j \neq i$ and $\varphi' : \bigwedge^2 E' \xrightarrow{\sim} L(-t_i)$ is the horizontal isomorphism induced by φ. We also consider the isomorphisms of the moduli spaces

$$\otimes\mathcal{O}(t_i) : M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}', L \otimes \mathcal{O}(2t_i));$$
$$(E, \nabla_E, \varphi, \{l_i\}) \mapsto (E \otimes \mathcal{O}(t_i), \nabla_{E\otimes\mathcal{O}(t_i)}, \varphi \otimes 1, \{l'_i \otimes \mathcal{O}(t_i)|_{t_i}\}),$$

where $\lambda'_i = \lambda_i - 1$ and $\lambda'_j = \lambda_j$ for $j \neq i$ and we consider for $\lambda_i \neq \mathsf{res}_{t_i}(\nabla_L) - \lambda_i$, an isomorphism

$$s_i : M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}, L) \xrightarrow{\sim} M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}', L); \quad (E, \nabla_E, \varphi, \{l_i\}) \mapsto (E, \nabla_E, \varphi, \{l'_i\}),$$

where $\lambda'_i = \mathsf{res}_{t_i}(\nabla_L) - \lambda_i$, $\lambda'_j = \lambda_j$ for $j \neq i$, $l'_i = \ker(\mathsf{res}_{t_i}(\nabla_E) - \lambda'_i)$ and $l'_j = l_j$ for $j \neq i$. Note that these isomorphisms all commute with the Riemann-Hilbert morphism $\mathbf{RH}$.

Now we fix $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ and put $\lambda_i^+ := \lambda_i$, $\lambda_i^- := \mathsf{res}_{t_i}(\nabla_L) - \lambda_i$ for $i = 1, \ldots, n$. Applying a certain composition of $Elm^-_{t_i}$, $\otimes\mathcal{O}_{\mathbf{P}^1}(t_i)$ and s_i for $i = 1, \ldots, n$, we obtain an isomorphism

$$\tau : M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}, L) \xrightarrow{\sim} M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}', L'),$$

where $\lambda'_i := \lambda_i + m_i^+$ and $\mathsf{res}_{t_i}(\nabla_{L'}) = \mathsf{res}_{t_i}(\nabla_L) + m_i^+ + m_i^-$ for some integers $m_i^+, m_i^- \in \mathbf{Z}$ such that $0 \leq \mathrm{Re}(\lambda_i^+ + m_i^+) < 1$, $0 \leq \mathrm{Re}(\lambda_i^- + m_i^-) < 1$ for $1 \leq i \leq n$.

Let $N_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}', L')$ be the moduli space of rank 2 irreducible connections (E, ∇_E) with a horizontal isomorphism $\bigwedge^2 E \xrightarrow{\sim} L'$ such that $\det(\mathsf{res}_{t_i}(\nabla_E) - \lambda'_i) = 0$ for $i = 1, \ldots, n$. By [[Del70], Proposition 5.4], we obtain an isomorphism

$$\mathrm{rh} : N_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}', L') \xrightarrow{\sim} \mathcal{R}^{\mathrm{irr}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}. \tag{166}$$

There is a canonical surjective morphism

$$M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}', L') \to N_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}', L') \tag{167}$$

which is obtained by forgetting parabolic structures. Composing τ, (167) and rh, we obtain a surjective morphism

$$\mathbf{RH} : M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}, L) \longrightarrow \mathcal{R}^{\mathrm{irr}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}. \tag{168}$$

Note that the morphism (167) is isomorphic except on the locus where the parabolic structures are not uniquely determined by (E, ∇_E), namely,

$$M_n^{\mathrm{app}}(\mathbf{t}, \boldsymbol{\lambda}', L') = \left\{ (E, \nabla_E, \varphi, \{l_j\}) \in M_n^{irr}(\mathbf{t}, \boldsymbol{\lambda}', L') \,\middle|\, \mathrm{Res}_{t_i}(\nabla_E) = O \text{ or } \frac{1}{2}\mathrm{id}_{E_{t_i}} \text{ for some } i \right\}$$

whose image in $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is

$$\mathcal{R}^{\mathrm{app}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = \left\{ \rho \in \mathcal{R}^{irr}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \,\middle|\, \rho(\gamma_i) = \pm\mathrm{id} \text{ for some } i \right\}.$$

Thus the restriction of $\mathbf{RH}$

$$M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)^\sharp = M_n^{\mathrm{irr}}(\mathbf{t}, \boldsymbol{\lambda}, L) \setminus \tau^{-1}(M_n^{\mathrm{app}}(\mathbf{t}, \boldsymbol{\lambda}', L')) \xrightarrow{\mathrm{RH}} \mathcal{R}^{\mathrm{irr}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \setminus \mathcal{R}^{\mathrm{app}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})^\sharp_{\mathbf{a}}$$

is an isomorphism. Since $\dim \mathcal{R}^{\mathrm{app}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} < \dim \mathcal{R}^{\mathrm{irr}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ for $n \geq 4$, $\mathbf{RH}$ is a bimeromorphic morphism.

Next we will show that $M_n^{\mathrm{red}}(\mathbf{t},\boldsymbol{\lambda},L) \to \mathcal{R}^{\mathrm{red}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is surjective. Take any point $[\rho] \in \mathcal{R}^{\mathrm{red}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$. Then the representation ρ is Jordan equivalent to the representation $\rho_1 \oplus \rho_2$ for some one dimensional representations ρ_1, ρ_2 of $\pi_1(\mathbf{P}^1 \setminus D(\mathbf{t}), *)$. Put

$$U_{n,\mathbf{a}} := \left\{ (M_1, \dots, M_{n-1}) \in SL_2(\mathbf{C})^{n-1} \,\middle|\, \mathrm{Tr}(M_i) = a_i \ (1 \leq i \leq n-1), \quad \mathrm{Tr}((M_1 M_2 \cdots M_{n-1})^{-1}) = a_n \right\}.$$

Then $U_{n,\mathbf{a}}$ is irreducible by Proposition 8.2. Let $\Phi_n : U_{n,\mathbf{a}} \to \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ be the quotient map. Then there exists a point $p_0 \in U_{n,\mathbf{a}}$ such that $\Phi_n(p_0) = [\rho]$. Since $U_{n,\mathbf{a}}$ is irreducible, there exists a smooth irreducible curve C, a point p of C and a morphism $f : C \to U_{n,\mathbf{a}}$ such that $f(p) = p_0$ and that $\Phi_n(f(C)) \cap \mathcal{R}^{\mathrm{irr}}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \neq \emptyset$. From [[Del70], Proposition 5.4], there exists an analytic flat family $(\tilde{E}, \nabla_{\tilde{E}}, \tilde{\varphi})$ of connections such that $\ker \nabla_{\tilde{E}}|_{(\mathbf{P}^1 \setminus D(\mathbf{t})) \times C}$ is equivalent to the flat family of local systems on $(\mathbf{P}^1 \setminus D(\mathbf{t})) \times C$ over C induced by the morphism f. Applying certain elementary transformations and tensoring line bundles to $(\tilde{E}, \nabla_{\tilde{E}}, \tilde{\varphi})$, we may assume that the eigenvalues of $\mathsf{res}_{t_i}(\tilde{E})$ are λ_i and $\mathsf{res}_{t_i}(\nabla_L) - \lambda_i$ for $i = 1, \dots, n$. We can construct a flat family of parabolic structures $\{\tilde{l}_i\}$ and $(\tilde{E}, \nabla_{\tilde{E}}, \tilde{\varphi}, \{\tilde{l}_i\})$ becomes a flat family of parabolic connections. Taking the completion at p, we obtain a flat family of parabolic connections $(E, \nabla_E, \varphi, \{l_i\})$ on $\mathbf{P}^1_{\mathbf{C}[[x]]}$ over $\mathbf{C}[[x]]$. By Corollary 5.1, there exists a flat family $(E', \nabla_{E'}, \varphi', \{l'\})$ of $\boldsymbol{\alpha}$-semistable parabolic connections such that $(E, \nabla_E, \varphi, \{l_i\}) \otimes \mathbf{C}((x)) \cong (E', \nabla_{E'}, \varphi', \{l'\}) \otimes \mathbf{C}((x))$ and $gr((E', \nabla_{E'}) \otimes \mathbf{C}[[x]]/(x)) \cong gr((E, \nabla_E) \otimes \mathbf{C}[[x]]/(x))$. Then $(E', \nabla_{E'}, \varphi', \{l'\})$ determines a morphism $\mathrm{Spec}\, \mathbf{C}[[x]] \to M_n^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$. If q is the image of the closed point by this morphism, then we have $\mathbf{RH}(q) = [\rho]$. Λ

7.2.1. *Proof of Theorem 7.1 except for the properness of* $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$.

Proof. Here we prove the assertions in Theorem 7.1 except for the properness of $\mathbf{RH}_{\mathbf{t},\boldsymbol{\lambda}}$ which will be proved in Proposition 10.1. The first assertion except for the properness follows from Lemma 7.1 and the second assertion is proved in the proof of Lemma 7.1. The last assertion follows from these assertions and Lemma 6.6. For the third assertion recall the definition of $\mathcal{R}^{irr}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ and $\mathcal{R}^{app}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ in the proof of Lemma 7.1. Let us set $\mathcal{R}^{red}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \setminus \mathcal{R}^{irr}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$. Then we see that

$$\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}^{sing} = \mathcal{R}^{red}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} \cup \mathcal{R}^{app}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}.$$

If $[\rho] \in \mathcal{R}^{red}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$, then ρ is a reducible representation. Then the semisimplification of ρ is a direct sum of one dimensional representation ρ_1, ρ_2. Since $\wedge^2 \rho$ is trivial, $\rho_2 \simeq \rho_1^{-1}$. Moreover since $\mathrm{Tr}[\rho(\gamma_i)] = a_i$ are fixed for all $1 \leq i \leq n$, we see that Jordan equivalence class of ρ, which is equal to the Jordan equivalence class of $\rho_1 \oplus \rho_1^{-1}$, has finitely many possiblity. Hence $\mathcal{R}^{red}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is a zero dimensional subscheme. Moreover for a closed point $[\rho] \in \mathcal{R}^{app}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$, ρ is irreducible and $\rho(\gamma_i) = \pm id$ for some i by definition. This means that ρ is determined by $\rho(\gamma_j)$ for $j \neq i$ and so $\dim \mathcal{R}^{app}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = \dim \mathcal{R}(\mathcal{P}_{n-1,\mathbf{t}})_{\mathbf{a}'}$. Noting that $\dim \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = 2n-6$ for $n \geq 3$, we have $\dim \mathcal{R}^{app}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = 2n-8$ for $n \geq 4$. In both cases, if $n \geq 4$, the codimensions of the subschemes are at least 2. Λ

7.3. **The case of** $n = 4$**.** In the case of $n = 4$, let us recall the isomorphism

$$T_4'/PGL_2 \simeq B = \mathbf{P}^1 - \{0, 1, \infty\},$$

where B is one-dimensional space of time variables as usual. Here the group PGL_2 acts on the base space $\mathbf{P}^1$ by linear fractional transformations. Therefore the family and the morphism can be descended and one obtains the following commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{S} & \overset{u}{\simeq} & M_4^{\boldsymbol{\alpha}} & \xrightarrow{\mathbf{RH}_4} & \mathcal{R}_4 \\
\pi \downarrow & & \pi_4 \downarrow & & \phi_4 \downarrow \\
B \times \Lambda_4 & = & B \times \Lambda_4 & \xrightarrow{id \times \mu} & B \times \mathcal{A}_4 .
\end{array}
\tag{169}
$$

Here the family $\pi : \mathcal{S} \longrightarrow B \times \Lambda_4$ is the family of Okamoto space of initial conditions. The isomorphism u will be constructed in [IIS2].

8. Irreducibility of $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$

As in Lemma 4.1, we have the natural quotient morphism

$$
\begin{array}{rcl}
\Phi_n : SL_2(\mathbf{C})^{n-1} & \longrightarrow & \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) \simeq \operatorname{Spec}[(R_{n-1})^{Ad(SL_2(\mathbf{C}))}] \\
(M_1, M_2, \dots, M_{n-1}) & \mapsto & [M_1, M_2, \dots, M_{n-1}]
\end{array}
$$

where R_{n-1} denotes the affine coordinate ring of $SL_2(\mathbf{C})^{n-1}$. Under this quotient morphism, for $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{A}_n = \mathbf{C}^n$, the subscheme $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ in (97) is isomorphic to

$$
\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}} = \{[M_1, \dots, M_{n-1}] \in \mathcal{R}(\mathcal{P}_{n,\mathbf{t}}) \mid \operatorname{Tr}(M_i) = a_i, (1 \leq i \leq n-1), \operatorname{Tr}(M_1 M_2 \cdots M_{n-1})^{-1} = a_n\}.
$$

Proposition 8.1. *Assume that $n \geq 4$. The affine scheme $\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ is irreducible.*

Set $U_{n,\mathbf{a}} := \Phi_n^{-1}(\mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}})$ so that we have a surjective morphism $U_{n,\mathbf{a}} \longrightarrow \mathcal{R}(\mathcal{P}_{n,\mathbf{t}})_{\mathbf{a}}$ of schemes. Because $\operatorname{Tr}(M_1 M_2 \cdots M_{n-1})^{-1} = \operatorname{Tr}(M_1 M_2 \cdots M_{n-1})$ for $M_i \in SL_2(\mathbf{C})$, we have

$$
U_{n,\mathbf{a}} = \{(M_1, \dots, M_{n-1}) \in SL_2(\mathbf{C})^{n-1} \mid \operatorname{Tr}(M_i) = a_i, 1 \leq i \leq n-1, \operatorname{Tr}(M_1 M_2 \cdots M_{n-1}) = a_n\}. \tag{170}
$$

Then it suffices to show the following

Proposition 8.2. *The scheme $U_{n,\mathbf{a}}$ is irreducible.*

Let us prove some easy lemma which we will use later. The proof of the following lemma is easy and we omit it.

Lemma 8.1. *Fix $a \in \mathbf{C}$ and define*

$$
V_a = \{A = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in SL_2(\mathbf{C}) \mid \operatorname{Tr}(A) = a\}.
$$

(1) *Then V_a is an irreducible affine subscheme of $\mathbf{C}^3$.*

(2) *Let us define a quadratic hypersurface in $\mathbf{P}^3_{\mathbf{C}}$ as:*

$$
\overline{V}_a := \{[x : y : z : w] \in \mathbf{P}^3_{\mathbf{C}} \mid x^2 - axw + w^2 + yz = 0\}. \tag{171}
$$

Then we have an isomorphism $V_a \simeq \overline{V}_a \setminus \{w = 0\}$, that is, $\overline{V}_a$ is a compactification of V_a. If $a \neq \pm 2$, $\overline{V}_a$ is a smooth quadric hypersurface, and if $a = \pm 2$, $\overline{V}_a$ is a cone over a conic and have a unique singular point at $p_a = [x : y : z : w] = [a/2 : 0 : 0 : 1]$.

Fix $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{A}_n$ and set $\mathbf{a}' = (a_1, \dots, a_{n-1})$. Using the notation in Lemma 8.1, we set

$$V_{\mathbf{a}'} := V_{a_1} \times V_{a_2} \times \cdots \times V_{a_{n-1}} \subset \overline{V}_{\mathbf{a}'} := \overline{V}_{a_1} \times \overline{V}_{a_2} \cdots \times \overline{V}_{a_{n-1}} \tag{172}$$

It is obvious that $U_{n,\mathbf{a}}$ is a Cartier divisor of the scheme

$$V_{\mathbf{a}'} = \{(M_1, \dots, M_{n-1}) \in SL_2(\mathbf{C}) \mid \mathrm{Tr}(M_i) = a_i, 1 \leq i \leq n-1\}$$

defined by the equation

$$\mathrm{Tr}(M_1 M_2 \cdots M_{n-1}) = a_n. \tag{173}$$

Again from Lemma 8.1, we can introduce a homogeneous coordinates $[x_i : y_i : z_i : w_i] \in \mathbf{P}^3_{\mathbf{C}}$ such that

$$\overline{V}_{a_i} = \{[x_i : y_i : z_i : w_i] \in \mathbf{P}^3_{\mathbf{C}} \mid F_{a_i} = x_i^2 - a_i x_i w_i + w_i^2 + y_i z_i = 0\}$$

Let us denote by $\overline{U}_{n,\mathbf{a}}$ the closure of $U_{n,\mathbf{a}} \subset V_{\mathbf{a}'}$ in $\overline{V}_{\mathbf{a}'} \subset (\mathbf{P}^3_{\mathbf{C}})^{n-1}$. It is easy to see that $\overline{U}_{n,\mathbf{a}}$ is also a Cartier divisor in $\overline{V}_{\mathbf{a}'}$.

For $1 \leq i \leq n-1$, set $T_{n-2,i} = V_{a_1} \times \cdots \times \widehat{V_{a_i}} \times \cdots \times V_{a_{n-1}}$ and $\overline{T}_{n-2,i} = \overline{V}_{a_2} \times \cdots \times \widehat{\overline{V}_{a_i}} \times \cdots \times \overline{V}_{a_{n-1}}$ (omitting i-th factors) and consider the i-th projections

$$\begin{array}{ccc} U_{n,\mathbf{a}} & \hookrightarrow & V_{a_1} \times V_{a_2} \times \cdots \times V_{a_{n-1}} \\ \pi_i \downarrow & & \pi'_i \downarrow \\ T_{n-2,i} & = & T_{n-2,i} \end{array} \qquad \begin{array}{ccc} \overline{U}_{n,\mathbf{a}} & \hookrightarrow & \overline{V}_{a_1} \times \overline{V}_{a_2} \times \cdots \times \overline{V}_{a_{n-1}} \\ \pi_i \downarrow & & \pi'_i \downarrow \\ \overline{T}_{n-2,i} & = & \overline{T}_{n-2,i} \end{array} \tag{174}$$

Lemma 8.2. *For each $1 \leq i \leq n-1$, the family $\pi_i : \overline{U}_{n,\mathbf{a}} \longrightarrow \overline{T}_{n-2,i}$ can be considered as a family of hyperplane sections of $\overline{V}_{a_i} \subset \mathbf{P}^3_{\mathbf{C}}$ parametrized by $\overline{T}_{n-2,i}$. Therefore $\overline{U}_{n,\mathbf{a}} \subset \overline{V}_{\mathbf{a}'}$ is a hypersurface defined by a multi-homogeneous polynomial*

$$H_{\mathbf{a}} = H_{\mathbf{a}}(x_1, y_1, z_1, w_1, \dots, x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \tag{175}$$

in the homogeneous coordinate ring of $(\mathbf{P}^3_{\mathbf{C}})^{n-1}$ of multi-degree $(1, \dots, 1)$.

Proof. First we prove the assertion for $i = 1$. For simplicity, we set $T = T_{n-2,1}$ and $\overline{T} = \overline{T}_{n-2,1}$ and we write as $\pi_1 : U_{n,\mathbf{a}} \longrightarrow T$, $\overline{\pi}_1 : \overline{U}_{n,\mathbf{a}} \longrightarrow \overline{T}$. Take an element $(M_2, M_3, \dots, M_{n-1}) \in T$ and set

$$M_2 M_3 \cdots M_{n-1} = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$$

and $M_1 = \begin{pmatrix} s & t \\ u & a_1 - s \end{pmatrix} \in V_{a_1}$ with $s(a_1 - s) - tu = 1$. Then we can write as

$$M_1(M_2 \cdots M_{n-1}) = \begin{pmatrix} s & t \\ u & a_1 - s \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} = \begin{pmatrix} sf_1 + tf_3 & sf_2 + tf_4 \\ f_3(a_1 - s) + uf_1 & (a_1 - s)f_4 + uf_2 \end{pmatrix}.$$

Hence the Cartier divisor $U_{n,\mathbf{a}} \subset V_{\mathbf{a}'}$ is defined by the polynomial

$$\begin{aligned} \mathrm{Tr}(M_1 M_2 \cdots M_{n-1}) - a_n &= sf_1 + tf_3 + (a_1 - s)f_4 + uf_2 - a_n \\ &= (f_1 - f_4)s + f_3 t + f_2 u + (a_1 f_4 - a_n). \end{aligned} \tag{176}$$

First, we will show that any irreducible component of $U_{n,\mathbf{a}}$ is not a pullback divisor via π_1. Consider the subscheme Z of T defined by the ideal generated by the following elements:

$$f_1 - f_4, \ f_3, \ f_2, \ a_1 f_4 - a_n.$$

Then, it suffices to show that the codimension of Z in T is at least 2. Recall that T is a product of V_{a_i}'s for $i = 2, \dots, n-1$.

If $n \geq 4$, let us consider the natural projection $\varphi : Z \longrightarrow V_{a_2} \times \cdots \times V_{a_{n-2}}$. We will show that every closed fiber of φ consists of a finite number of points or becomes empty, which means that codimension of Z in T is at least 2. For this purpose, let us set

$$M_2 \cdots M_{n-2} = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$$

and

$$M_{n-1} = \begin{pmatrix} s & t \\ u & b-s \end{pmatrix}.$$

(Note that $g_1 g_4 - g_2 g_3 = 1, s(b-s) - tu = 1$, $s = s_{n-1}, t = t_{n-1}, u = t_{n-1}, b = a_{n-1}$). Since

$$\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} = M_2 \cdots M_{n-2} M_{n-1} = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} s & t \\ u & b-s \end{pmatrix} = \begin{pmatrix} g_1 s + g_2 u & g_1 t + g_2(b-s) \\ g_3 s + g_4 u & g_3 t + g_4(b-s) \end{pmatrix},$$

the ideal of Z contains the following element

$$\begin{array}{lll} f_1 - f_4 & = & (g_1 + g_4)s + g_2 u - g_3 t - g_4 b \\ f_2 & = & g_1 t + g_2(b-s) \\ f_3 & = & g_3 s + g_4 u. \end{array}$$

Using the relations $g_1 g_4 - g_2 g_3 = 1, s(b-s) - tu = 1$, from these elements we can obtain the following elements of the ideal of Z

$$s^2 - g_4^2, \quad t^2 - g_2^2, \quad u^2 - g_3^2.$$

This means that every closed fiber of the projection $\varphi : Z \longrightarrow V_{a_2} \times \cdots \times V_{a_{n-2}}$ consists of finitely many points or becomes the empty set as desired. Let us recall the natural projection $\pi_1 : U_{n,\mathbf{a}} \longrightarrow T$. The assertion implies that the Cartier divisor $U_{n,\mathbf{a}}$ defined by the polynomial (176) has no irreducible component which is a pullback Cartier divisor by π_1. Then, from the expression in (176), we conclude that the polynomial (176) is of degree 1 with respect to s, t, u and hence the fibers of the compactifications $\overline{\pi}_1 : \overline{U}_{n,\mathbf{a}} \longrightarrow \overline{T}$ of morphism π_1 are hyperplane sections of the quadric hypersurface $\overline{V}_{a_1} \subset \mathbf{P}^3_{\mathbf{C}}$. This proves the assertion for $i = 1$. Since

$$\mathrm{Tr}(M_i M_{i+1} \cdots M_{n-1} M_1 \cdots M_{i-1}) = \mathrm{Tr}(M_1 M_2 \cdots M_{n-1}), \tag{177}$$

the same is true for the i-th factor. Now we can conclude that $\overline{U}_{n,\mathbf{a}}$ is defined by the multi-homogeneous polynomial of $H_{\mathbf{a}}$ of multi-degree $(1, \ldots, 1)$. Λ

Now we prove

Lemma 8.3. *For any $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{A}_n$ and $i, 1 \leq i \leq n-1$, the general fiber of $\pi_i : \overline{U}_{n,\mathbf{a}} \longrightarrow \overline{T}_{n-2,i}$ is irreducible and reduced.*

Proof. By (177), we only have to prove the assertion for $i = 1$. From now on, we set $T_{n-2} := T_{n-2,1}, \overline{T}_{n-2} = \overline{T}_{n-2,1}$ and $\pi = \pi_1$, etc.

For $(M_1, M_2, M_3, \ldots, M_{n-1}) \in U_{n,\mathbf{a}}$, write

$$M_1 = \begin{pmatrix} s_1 & t_1 \\ u_1 & a_1 - s_1 \end{pmatrix}, \quad M_2 M_3 \cdots M_{n-1} = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}.$$

Then for a fixed $(M_2, M_3, \ldots, M_{n-1}) \in T_{n-2}$, the fiber of $\pi_{|U_{n,\mathbf{a}}} : U_{n,\mathbf{a}} \longrightarrow T_{n-2}$ is defined by the equations

$$s_1 f_1 + t_1 f_3 + (a_1 - s_1) f_4 + u_1 f_2 - a_n = 0, \quad s_1^2 - a_1 s_1 + 1 + t_1 u_1 = 0.$$