A_∞-deformation of representations of groups

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(i) Take a complex torus $X = \mathbb{C}^g / \Lambda$.

Λ : lattice $\curvearrowright \underset{\text{contractible.}}{\mathbb{C}^g}$ free, discontinuous

In this situation, we have

$$\underset{\text{group theoretic obj.}}{H^*(\Lambda, \mathbb{C})} \cong \underset{\text{tangent space of the "}A_\infty\text{-deformation space" of } X.}{H^*(X, \mathbb{C})}$$

(ii) Γ : Fuchsian group of the 1-st kind.

$A_*(\Gamma)$: space of automorphic forms

$\cong$

$\{$sections of a line bundle over $X = \mathbb{H}^*/\Gamma\}$

$\mathbb{H}^* = \mathbb{H} \cup \{\text{cusps}\}$

A^∞-deformations of X would induce "higher products" on $A_*(\Gamma)$.

However, X is an orbifold in general.

On the other hand, $A_*(\Gamma)$ has some group theoretic meaning in terms of Γ.

§ A_∞ - algebra , A_∞ - category

Def. (V, m_k) : A_∞-alg. $/K$ (K : a field)

$\Leftrightarrow$ V : $\mathbb{Z}$-graded K-vect. sp.

$m_k : V^{\otimes k} \to V$ ($k \geq 1$)

multi-linear maps of deg. $2-k$.

such that

$$(*) \quad \sum_{k+\ell=n+1} \sum_{j=0}^{k-1} (-1)^{\ell \sum_{i=1}^{j} |a_i| + j(\ell-1) + (k-1)\ell} m_k(a_1, \cdots, a_j, m_\ell(a_{j+1}, \cdots, a_{j+\ell}), \cdots, a_n) = 0$$

Rem. (1) In more general definition, m_0 is also considered. Here we are assuming $m_0 = 0$.

(2) $W := \Pi V^*$ $\quad \Pi$: parity change (degree shift by 1.)

$T(W) := \bigoplus_{n=0}^{\infty} W^{\otimes n}$ tensor alg.

Consider the (super) derivation

$\hat{d} : T(W) \to T(W)$

$\hat{d}(a \otimes b) = \hat{d}(a) \otimes b + (-1)^{|a||\hat{d}|} a \otimes \hat{d}(b)$.

$d := \hat{d}|_W$ determines $\hat{d}$ uniquely.

$d = d_1 + d_2 + \cdots \qquad d_k : W \to W^{\otimes k}$

$\hat{d} = \sum d_{\ell,k} \qquad d_{\ell,k} : W^{\otimes k} \to W^{\otimes \ell}$

Then, $\hat{d}^2 = 0 \Leftrightarrow \sum_{k+\ell=n} d_{k,\ell}\, d_\ell = 0 \quad \forall n > 1$

This is dual to the condition $(*)$

- A_∞ - module structure on a graded vector space M

$\Leftrightarrow$ D : integrable connection on $M^* \otimes T(W)$

$[D, D] = 0$ with some condition on degree.

We can see that the deformations of A_∞-structure are obtained by deforming d or D.

- $\mathcal{C}$: A_∞ - category

$\Leftrightarrow$ $Hom(E, E')$: a graded vect. sp. $\forall E, E' \in Ob\, \mathcal{C}$.

$m_{k-1} : Hom(E_1, E_2) \otimes \cdots \otimes Hom(E_{k-1}, E_k) \to Hom(E_1, E_k)$.

satisfies $(*)$.

§ A_∞ - deformation of the category of representations of groups

G : a group

$Mod(G)$: category of $\mathbb{C}\langle G \rangle$ - bimodules

$D\,Mod(G)$: derived category

$Hom(M, N) = Ext^*_G(M, N)$ $\quad M, N \in DMod(G)$

For $M \in Mod(G)$, we define the space of cochains by

$$C^k(G, M) = \{ \mu : \mathbb{C}\langle G \rangle^{\otimes k} \to M, \ \mathbb{C}\text{-linear} \}.$$

$$(d\mu)(g_1, \cdots, g_{k+1}) := g_1 \mu(g_2, \cdots, g_{k+1}) - \mu(g_1 g_2, g_3, \cdots, g_{k+1}) + \cdots + (-1)^k \mu(g_1, \cdots, g_k) g_{k+1}.$$

$C^\bullet = (C^\bullet(G, \mathbb{C}\langle G\rangle), \wedge, d)$ has a structure of diff. graded algebra.

This is an analogue of the space of diff. forms.

$C^\bullet$ has a canonical bracket $[\ ,\ ]$.

$$[\mu, \nu] := \mu \circ \nu - (-1)^{(|\mu|-1)(|\nu|-1)} \nu \circ \mu$$

$$\mu \circ \nu (g_1, g_2, \cdots, g_n) = \sum_i \pm \mu(g_1, \cdots, g_i, \nu(g_{i+1}, \cdots), \cdots)$$

- Maurer - Cartan equation

$t = (t_g)_{g \in G}$ parameters

$$R = \mathbb{C}[[t]] = \varprojlim_n \mathbb{C}[t_g | g \in G]/m^n$$

m : ideal generated by t_g's.

$\gamma(t) \in C^\bullet \otimes R$

$$Q(\gamma(t)) := d\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)]$$

Maurer - Cartan equation

$$Q(\gamma(t)) = 0$$

Let $\gamma(t)$ be a formal solution of the Maurer - Cartan equation

Then, $\gamma(t) + Q_{\gamma(t)} \gamma'(t)$ also satisfies MC equation $(\gamma'(t) \in C^\bullet \otimes R,\ Q_{\gamma(t)}(\gamma'(t)) = d\gamma'(t) + [\gamma(t), \gamma'(t)]$

$\gamma(t) \mapsto Q_{\gamma(t)} \gamma'(t)$ is an analogue of the Gauge transformation.

We introduce the moduli space of the formal solutions of Maurer-Cartan equations

$$\mathcal{M}_G = \{\text{formal solutions of MC eq.}\} / \text{Gauge equivalence}$$

Then, we have the following.

Th. 1. (1) $\mathcal{M}_G$ is a formal A_∞-deformation space of the category $D\,Mod(G)$

(2) (tangent space at $t=0$) $\cong H^*(G, \mathbb{C}\langle G\rangle)$

(3) The exponential map $H^*(G, \mathbb{C}\langle G\rangle) \to \mathcal{M}_G$ is surjective.

Surjectivity of the exponential map follows from Tamarkin and Tsygan's formality theorem.

- X : complex manifold, contractible
  $G \curvearrowright X$: hol. action, free, discontinuous
  $X \xrightarrow{\pi} Y = X/G$

  Assume that Y : compact.

Consider $\mathcal{M}_Y$: Barannikov and Kontsevich's extended moduli space of Y.

$\mathcal{M}_Y$ parametrizes formal A_∞-deformations of $D\,Coh(Y)$.

Th. 2. Each section $s \in H = H^*(X, \pi^*(\bigoplus_k \Lambda^k T_Y))$ defines a morphism

$$\alpha_s : \mathcal{M}_G \to \mathcal{M}_Y.$$

The induced map α_{s*} on the tangent space coincides with

$$H^*(G, \mathbb{C}\langle G\rangle) \to H^*(G, \mathbb{C}\langle G\rangle)\otimes H \to H^*(Y, \wedge^* T_Y)$$
$$\mu \mapsto \mu \otimes s \quad .$$

§ Examples

$A = \mathbb{C}^g/\Lambda$ complex torus

In this case, there exists a standard morphism $\alpha_0 : \mathcal{M}_\Lambda \to \mathcal{M}_A$

s.t.

$$\begin{array}{ccccc} \alpha_{0*} : T_0\mathcal{M}_\Lambda & & \longrightarrow & & T_0\mathcal{M}_A \\ \| & & & & \| \\ H^*(\Lambda, \mathbb{C}[\Lambda]) & \to & H^*(\Lambda, \mathbb{C}[\Lambda])\otimes\mathbb{C} & \to & H^*(A, \mathbb{C}) \\ \mu & \mapsto & \mu\otimes 1 & & \end{array}$$

Group ring $\mathbb{C}[\Lambda]$ can be considered as a polynomial ring $\mathbb{C}[Q_1, \cdots, Q_{2g}]$.

By assigning some $q_i \in \mathbb{C}^\times$ to Q_{2g},

we have a morphism $j : \mathbb{C}[\Lambda] \to \Gamma(\mathbb{C}^g, \mathcal{O}^\times_{\mathbb{C}^g})$, $Q^n \mapsto q^{2\pi i\langle n, \vec{\alpha}\rangle}$

where q_i are determined by $\Lambda \hookrightarrow \mathbb{C}^g$

Differentiating j, we have $H^1(\Lambda, \mathbb{C}[\Lambda]) \to H^1(A, \mathcal{O}^{\oplus g})$,

$H^1(A, \mathcal{O}^{\oplus g})$ is the tangent space of classical deformations of A.

$C^\cdot(\Lambda, \mathbb{C}) \cong \mathbb{C}\langle Z^n \mid n\in\Lambda\rangle$

noncommutative polynomial ring freely generated by the symbols Z^n.

$H^2(\Lambda, \mathbb{C}) \cong \wedge^2 \mathrm{Hom}(\Lambda, \mathbb{C})$

tangent space of noncommutative deformations.

Another viewpoint

$$\mathbb{C}[\Lambda] \xrightarrow[\substack{\text{Fourier}\\ \text{transf.}}]{\varphi} C(\Lambda^\vee) \quad \text{ring hom.} \quad (\Lambda^\vee : \text{space of irred. rep.s of } \Lambda)$$

$C(\Lambda^\vee)$ can be considered as a space of functions on $Pic^\circ(A)$

φ induces $\mathcal{M}_\Lambda \to \mathcal{M}(Pic^\circ(A))$

This can be generalized for the cases G: noncommutative

Prop. Fourier transformation $\mathbb{C}\langle G\rangle \to C(G^\vee)$ induces $\mathcal{M}_G \to \mathcal{M}(Pic^\circ(Y))$

$(Y = X/G)$

Case $Y = \mathbb{H}/\Gamma$, $\Gamma \curvearrowright \mathbb{H}$ free
(Γ: Fuchsian grp. of 1-st kind, $\mathbb{H}$: upper half plane)

$\Rightarrow$ $\mathcal{M}_G \to \mathcal{M}(Pic^\circ(Y))$ surjective