A construction of flat structures from tt* geometry

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Motivation

Want to construct

$$\{\ ?\ \}$$

$\{$ Variation of Hodge structures (VHS's) $\}$ $\quad$ $\{$ K. Saito's flat structures $\}$

$\{\text{VHS}\} \supset$ Classical deformation theory

$\mathcal{X} \to S \iff H^1(X, TX)$

$\cap$ $\qquad\qquad$ $\cap$

"A_∞-deformation theory"

"$\bigcup_{s \in S} D_s \to S$" $\iff \bigoplus_{p,q} H^p(X, \Lambda^q TX)$

$\{\ ?\ \} \supset$ deformation theory for families of triangulated categories

(Barannikov, Kontsevich)

$\{$Flat structures$\}$ (K. Saito)

$\supset$ $\{$Frobenius structures$\}$ (Dubrobin)

$\cup$

$\{$Gromov-Witten theory at $g=0\}$

$\supset$ period mapping of primitive forms

$\cup$

elliptic integrals (A_2-singularity)

A candidate for {?}

- tt^* geometry (Cecotti - Vafa)

  the geometry of moduli spaces of $N=2$ supersymmetric field theories in 2-dim

But tt^* geometry is based on physical intuitions and path-integrals.

$\Rightarrow$ Axioms of the mathematical structure

<u>Cecotti - Vafa structure</u>

A mathematical formulation of tt^* geometry

Dubrovin, Hertling (VTERP str.)

Our results

- CV structure of Calabi-Yau type
  a genelarization of VHS for Calabi-Yau manifolds.

- CV structure of CY type
  \+ a choice of a "base point"
  $\Rightarrow$ K. Saito's flat structure
  $\Rightarrow$ Frobenius structure
  (via a real analytic gauge transform)

- Weil - Peterson metric on a parameter space

- A characterization of primitive forms and flat coordinates

0. Some notations

S : a complex manifold.

A_S : the sheaf of real analytic complex valued functions on S

$\mathcal{O}_S$: the sheaf of holomophic functions on S

T_S : the tangent bundle

$\mathcal{T}_S$: the sheaf of holomorphic vector fields on S

$\mathcal{T}_S^{1,0} := \mathcal{T}_S \otimes A_S$

$A_S^{p,q}$: the sheaf of real analytic complex valued (p,q)-forms on S

$A_S^r := \bigoplus_{p+q=r} A_S^{p,q}$ $\qquad A_S^0 = A_S$

Ω_S^p : the sheaf of holomorphic p-forms on S

$\vdots$

1. CV structure

Let $H \to S$: real analytic cpx. vect. bdle of rank μ.

$D : A_S(H) \to A^1_S(H)$ a connection

$C : A_S(H) \to A^{1,0}_S(H)$

$\overline{C} : A_S(H) \to A^{0,1}_S(H)$

$Q : A_S(H) \to A_S(H)$

} A_S- linear map

$E \in \Gamma(S, T_S)$ (called Euler vector field)

$$\begin{aligned} \pi : \mathbb{P}^1_z \times S &\to S \\ (z, s) &\mapsto s \end{aligned}$$ projection

<u>Def</u>. CV connection ∇

Define ∇ on $\pi^* H|_{\mathbb{C}^\times \times S}$ by

$$\nabla := D + zC + z^{-1}\overline{C} + \left(zC_E - Q - \frac{w}{2}\mathrm{id.} - z^{-1}\overline{C}_{\overline{E}}\right)\frac{dz}{z}$$

where $w \in \mathbb{Z}$, $C, \overline{C}, Q$ are canonical extensions,

D : extended to $\pi^* H|_{\mathbb{C}^\times \times S}$ by

$$D_{\frac{d}{dz}} \pi^{-1} A_S(H) = D_{\frac{d}{d\bar{z}}} \pi^{-1} A_S(H) = 0$$

Let $J : \mathcal{A}_S(H) \times \mathcal{A}_S(H) \to \mathcal{A}_S$

a non degenerate $\mathcal{A}_S$-bilinear form.

<u>Def.</u> Higher residue

Set $\mathcal{H}_{CV}^{(0)} := \pi^{-1}\mathcal{A}_S(H)|_{\mathbb{P}^1_z \setminus \{0\} \times S} \otimes \mathcal{O}_{\mathbb{P}^1_z \setminus \{0\}} \cdot \mathcal{A}_S$

$\left(\mathcal{O}_{\mathbb{P}^1 \setminus \{0\}} \cdot \mathcal{A}_S := \{ f \in \mathcal{A}_{\mathbb{P}^1_z \setminus \{0\} \times S} \mid \bar{\partial}_z f = 0 \} \right)$

$$\widehat{\pi_* \mathcal{H}_{CV}^{(0)}} := \varprojlim_k \pi_* \mathcal{H}_{CV}^{(0)} / \pi_* \mathcal{H}_{CV}^{(-k)}$$

$$\pi_* \mathcal{H}_{CV}^{(-k)} := \{ \zeta \in \pi_* \mathcal{H}_{CV}^{(0)} \mid z^k \zeta \in \pi_* \mathcal{H}_{CV}^{(0)} \}$$

Therefore $\zeta, \zeta' \in \widehat{\pi_* \mathcal{H}_{CV}^{(0)}}$ are formal series of z^{-1}

$$\zeta = \sum_{k \geq 0} \zeta_k z^{-k}, \quad \zeta' = \sum_{l \geq 0} \zeta'_l z^{-l}$$

$\zeta_k, \zeta'_l \in \mathcal{A}_S(H)$.

Define $\mathcal{A}_S$-bilinear, z-sesqui linear form

$$K_{CV} : \widehat{\pi_* \mathcal{H}_{CV}^{(0)}} \times \widehat{\pi_* \mathcal{H}_{CV}^{(0)}} \to z^{-w} \mathcal{A}_S[[z^{-1}]]$$

by $K_{CV}(\zeta, \zeta') := z^{-w} \sum_{k,l \geq 0} (-1)^l z^{-k-l} J(\zeta_k, \zeta'_l)$

Def CV structure

A tuple $(H, D, C, \overline{C}, Q, E, J, \chi, w)$

is called a CV structure if

(CV 1) $\nabla^2 = 0$

(CV 2) $\nabla K_{CV} = 0$

(CV 3) $\nabla \chi = 0$

where $\chi : A_S^{p,q}(H) \to A^{p,q}(H)$

is an A_S-anti linear involution (real str.)

extended to $\pi^* H|_{\mathbb{C}^\times \times S}$ by

$$\chi(z^k \zeta) = z^{-k}\chi(\zeta) \quad , \zeta \in A_S(H)$$

(CV4) J takes real values on

$$H_{\mathbb{R}} := \mathrm{Ker}\,(\chi - id.) \subset H$$

i.e.

$$J : H_{\mathbb{R},s} \times H_{\mathbb{R},s} \to \mathbb{R} \quad \text{for } {}^\forall s \in S$$

Remark

Let $(s^1, \cdots, s^n)$ local coordinates of S

Set $D'_i := D'_{\frac{\partial}{\partial s^i}}$, $D''_{\bar{j}} := D''_{\frac{\partial}{\partial \bar{s}^j}}$

where $D = D' + D''$ (1.0), (0.1)-parts,

$C_i := C_{\frac{\partial}{\partial s^i}}$, $\bar{C}_{\bar{j}} := \bar{C}_{\frac{\partial}{\partial \bar{s}^j}}$

Then CV structure gives

$$[D'_i, D'_j] = 0 = [D''_{\bar{i}}, D''_{\bar{j}}]$$

$$[D'_i, C_j] = [D'_j, C_i]$$

$$[D''_{\bar{i}}, C_j] = 0$$

$$[D'_i, D''_{\bar{j}}] + [C_i, \bar{C}_{\bar{j}}] = 0$$

$$[D'_i, C_E] + [Q, C_i] - C_i = 0$$

$$[D'_i, Q] + [C_i, \bar{C}_{\bar{E}}] = 0$$

$$DJ = 0, \quad J(C_i \mathfrak{z}, \mathfrak{z}') = J(\mathfrak{z}, C_i \mathfrak{z}')$$

$$J(Q\mathfrak{z}, \mathfrak{z}') = -J(\mathfrak{z}, Q\mathfrak{z}'), \quad \mathfrak{z}, \mathfrak{z}' \in A_S(H)$$

$$DK = 0, \quad \bar{C} = KCK, \quad Q = -KQK.$$

In particular,

$$[D''_{\bar{i}}, D''_{\bar{j}}] = 0, \ [D''_{\bar{i}}, C_j] = 0, \ DJ = 0$$

$\Rightarrow$ $\exists$ holomorphic structure on H.

C, J : holomorphic

$$g(\zeta, \zeta') := J(\zeta, \kappa\zeta'), \quad \zeta, \zeta' \in A_S(H)$$

is Hermitian.

Since $g(Q\zeta, \zeta') = g(\zeta, Q\zeta')$,

if g is positive definite,

then Q is semi-simple with real eigenvalues.

$\Rightarrow$ We have Hodge decomposition

$$H = \bigoplus_q H^q$$

q: real analytically varying eigenvalues of Q

Example of CV structure.

<u>Def.</u> (variation of Hodge structure)

A variation of polarized structure of weight $w \in \mathbb{Z}$ is a tuple $(H, \nabla, I, K, F^{\cdot})$ where

$H \to S$: a hol. vect. bdle.

$\nabla : \mathcal{O}_S(H) \to \Omega^1_S(H)$ a flat hol. conn.

$K : A^{p,q}_S(H) \to A^{q,p}_S(H)$ a flat A_S-anti linear involution

$F^{\cdot}$: a dicreasing filtration by hol. subbdles

$$F^p \subset H \quad \text{satisfying} \quad \bigcup_{p \in \mathbb{Z}} F^p = H, \ \bigcap_{p \in \mathbb{Z}} F^p = \{0\}$$

$$H_s = F^p_s \oplus \overline{F^{w+1-p}_s} \quad \text{for } {}^\forall s \in S \quad \overline{F^q} := K(F^q)$$

$$\nabla(\mathcal{O}_S(F^p)) \subset \Omega^1_S(F^{p-1}) \quad \text{Griffith transversality}$$

$I : \mathcal{O}_S(H) \otimes \mathcal{O}_S(H) \to \mathcal{O}_S$ nondeg. $(-1)^w$-symmetric ∇-flat $\mathcal{O}_S$-bilinear form satisfying

$$I(F^p_s, F^q_s) = 0 \text{ for } p+q > w \ ; {}^\forall s \in S$$

$$(\sqrt{-1})^{2p-w} I(\xi, K\xi) > 0 \text{ for } \xi \in F^p_s \cap \overline{F^{w-p}_s}, \ {}^\forall s \in S$$

Lemma

A variation of polarized Hodge structure of wt. w $(H, \nabla, I, \kappa, F^\cdot)$ gives a CV structure of wt. w.

$\because$)

$$\nabla : A_S(H^{p,w-p}) \to A_S^{1,0}(H^{p,w-p}) \oplus A_S^{0,1}(H^{p,w-p}) \oplus A_S^{1,0}(H^{p-1,w-p+1}) \oplus A_S^{0,1}(H^{p+1,w-p-1})$$

(Griffith transversality + real structure)

$$\nabla \zeta = D'\zeta + D''\zeta + C\zeta + \bar{C}\zeta$$

$$D = D' + D'' \quad : \text{a connection}$$

$$J(\zeta, \zeta') := (\sqrt{-1})^{2p-w} I(\zeta, \zeta') \quad \text{for } \zeta, \zeta' \in A_S(H^{p,w-p})$$

$$E := 0$$

$$Q|_{H^{p,w-p}} := \left(\frac{w}{2} - p\right) \mathrm{id}.$$

$\Rightarrow$ $(H, D, C, \bar{C}, Q, E(=0), J, \kappa, w)$

CV structure.

<u>Def</u> CV structure of Calabi-Yau type.

A tuple $(H, D, C, \bar{C}, Q, E, J, \varkappa, w, L)$ is called a CV structure of Calabi-Yau type if $(H, D, C, \bar{C}, Q, E, J, \varkappa, w)$ is a CV structure with positive definite Hermitian form $g(\cdot, \cdot) := J(\cdot, \varkappa \cdot)$ and L is a real analytic subbundle $L \subset H$ of rank 1 such that

(CY1) $$T_S^{1,0} \otimes L \xrightarrow{\sim} H, \quad (\delta, \zeta) \mapsto C_\delta \zeta$$

and $$D(A_S(L)) \subset A_S^1(L)$$

(CY2) $$Q(A_S(L)) \subset A_S(L)$$

2. Flat structure

S : a cpx mfd. of $\dim_{\mathbb{C}} S = \mu$.

$H \to S$: a holomorphic vector bundle of rank μ.

$\nabla : \mathcal{O}_S(H) \to \Omega^1_S(H)$: a hol. connection.

$$\left.\begin{array}{l} C : \mathcal{O}_S(H) \to \Omega^1_S(H) \\ N : \mathcal{O}_S(H) \to \mathcal{O}_S(H) \end{array}\right\} \ \mathcal{O}_S\text{-linear}$$

$E \in \Gamma(S, T_S)$

$J : \mathcal{O}_S(H) \otimes \mathcal{O}_S(H) \to \mathcal{O}_S$ non-deg. $\mathcal{O}_S$-bilinear

$w \in \mathbb{Z}$. $\pi : \mathbb{P}^1_z \times S \to S$: projection.

Def. First structure connection.

Define a hol. connection $\hat{\nabla}$ on $\pi^* H|_{\mathbb{C}^* \times S}$ by

$$\hat{\nabla} := \nabla + zC + \left(z C_E - N - \frac{w}{2}\mathrm{id.}\right)\frac{dz}{z}$$

Def. Higher residue. $\left(\mathcal{H}^{(0)} := \pi^*\mathcal{O}_S(H)|_{\mathbb{P}^1_z \setminus \{0\} \times S}\right)$

$$K : \pi_*\hat{\mathcal{H}}^{(0)} \times \pi_*\hat{\mathcal{H}}^{(0)} \to z^{-w}\mathcal{O}_S[[z^{-1}]]$$

$$K(\zeta, \zeta') := z^{-w} \sum_{k,l \geq 0} (-1)^l z^{-k-l} J(\zeta_k, \zeta_l)$$

$$\zeta = \sum_{k \geq 0} \zeta_k z^{-k}, \quad \zeta' = \sum_{l \geq 0} \zeta'_l z^{-l}, \quad \zeta_k, \zeta'_l \in \mathcal{O}_S(H)$$

Def. Primitive form.

$\zeta \in \Gamma(S, \mathcal{O}_S(H))$ is called a primitive form for a tuple $(H, \nabla, C, N, E, J, w)$ if

(P1) $$\begin{array}{ccc} \mathcal{T}_S & \xrightarrow{\sim} & \mathcal{O}_S(H) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \delta & \mapsto & C_\delta \zeta \end{array} \qquad \mathcal{O}_S\text{-iso}$$

(P2) $$\nabla \zeta = 0$$

(P3) $\exists d \in \mathbb{R}$ (called the dimension)
$$N\zeta = -\frac{d}{2}\zeta$$

Def. Flat structure. (K. Saito)

A tuple $(H, \nabla, C, N, E, J, w, \zeta)$ is called a flat structure of weight w if

(F1) $$\hat{\nabla}^2 = 0$$

(F2) $$\hat{\nabla} K = 0$$

(F3) ζ is a primitive form for $(H, \nabla, C, N, E, J, w)$

Def. Frobenius structure. (Dubrobin)

Frobenius structure on S

= flat structure on T_S with $\zeta = e$ (identity)
$(T_S, \nabla, C, N, E, \omega, e)$

In particular,

- $\exists$ associative, commutative multiplication $\circ$ on T_S

$$T_S \otimes T_S \to T_S \quad , \quad (\delta, \delta') \mapsto \delta \circ \delta' := C_\delta \delta'$$
$$J(\delta \circ \delta', \delta'') = J(\delta, \delta' \circ \delta'') . \quad \delta, \delta', \delta'' \in T_S$$
$$e \circ \delta = \delta \circ e = \delta$$

★ ∇ is flat, torsion free and metric

$$[\nabla_\delta , \nabla_{\delta'}] = \nabla_{[\delta, \delta']}$$
$$\nabla_\delta \delta' - \nabla_{\delta'} \delta = [\delta, \delta']$$
$$\delta J(\delta', \delta'') = J(\nabla_\delta \delta', \delta'') + J(\delta', \nabla_\delta \delta'')$$

and satisfies $\nabla C = 0$. (potentiality)

- E satisfies

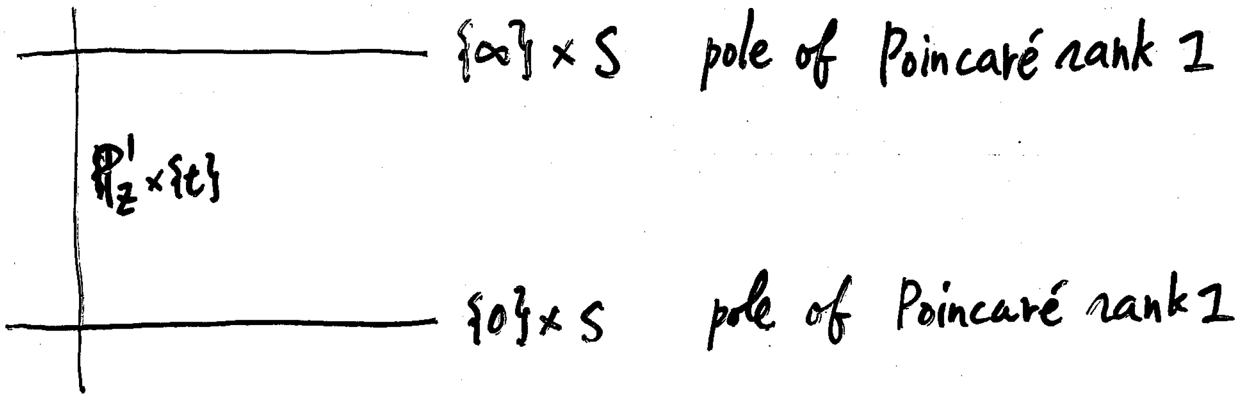
$$\mathrm{Lie}_E(J) = (2-d) J , \quad \mathrm{Lie}_E(\circ) = 0$$

Rem.

★ $\Rightarrow$ $\exists$ flat coodinates on S and $\exists$ Frobenius potential.

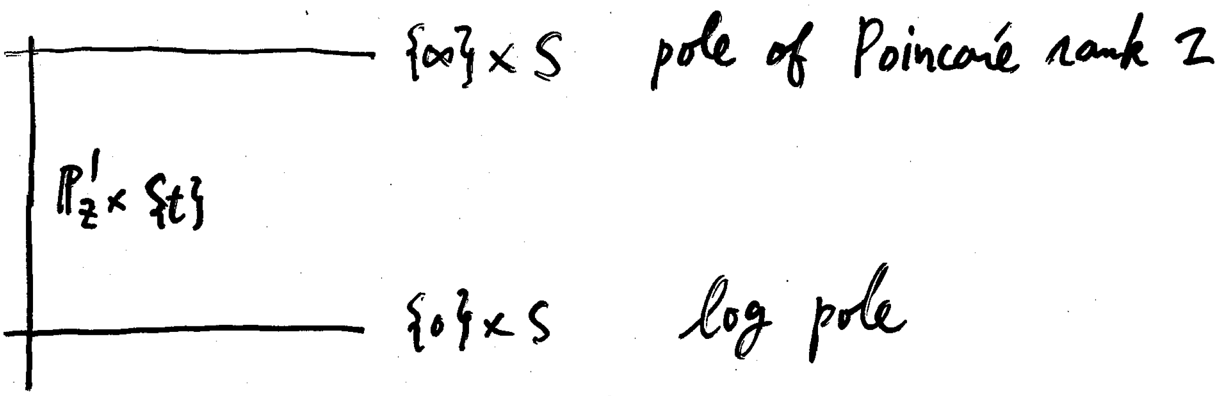
3. CV structure ⇒ flat structure.

- CV connection $\mathbb{D} = D + zC + z^{-1}\overline{C} + \left(zC_E - Q - \frac{w}{2}\mathrm{id} - z^{-1}\overline{C}_E\right)\frac{dz}{z}$



H, C, J, E : holomorphic w.r.t D''

- First structure connection $\widehat{\nabla} = \nabla + zC + \left(zC_E - N - \frac{w}{2}\mathrm{id}\right)\frac{dz}{z}$



everything are defined over $\mathcal{O}_S$

Very similar! (if H, C, E, J, w are same)

The difference may be only the extensions to $\{0\} \times S$.

(may be isomorphic on $\mathbb{P}^1_z \setminus \{0\} \times S$)

Consider the following problem.

Problem Let $(H, D, C, \bar{C}, Q, E, J, \varkappa, w, L)$ be a CV structure of CY type.

Find a gauge transformation ϕ of $\pi^* H|_{\mathbb{P}^1_z \setminus \{0\} \times S}$ such that $\phi^* \nabla$ defines the first structure connection $\hat{\nabla}$ of a flat structure on S with the same H, C, E, J, w.

Remark $\exists$ real analytic gauge transformation ϕ such that

$$\phi^* \nabla = \nabla + zC + \sum_{k \geq 1} z^{-k} C^{(k)} + \left(z C_E - N - \frac{w}{2} \mathrm{id} + \sum_{k \geq 1} z^{-k} N^{(k)} \right) \frac{dz}{z}$$

for a hol. conn. ∇ on H, a hol. end. N of H, and $\mathcal{O}_S$-linear maps

$$\left. \begin{aligned} C^{(k)} &: \mathcal{O}_S(H) \to \Omega^1_S(H) \\ N^{(k)} &: \mathcal{O}_S(H) \to \mathcal{O}_S(H) \end{aligned} \right\} k \geq 1$$

Therefore we need to find ϕ such that

$$C^{(k)} = N^{(k)} = 0 \quad \text{for } \forall k \geq 1.$$

Thm. Let $(H, D, C, \bar{C}, Q, E, J, \kappa, w, L)$ be a CV structure of CY type.

Then a choice of $p \in S$ such that $E|_p = 0$ gives a gauge transformation ϕ of $\pi^* H|_{\mathbb{P}^1_z \setminus \{0\} \times S}$ such that $C^{(k)} = N^{(k)} = 0$ for $\forall k \geq 1$, and gives a flat structure.

Remark

Choice of point $\bar{C}_E|_p = 0$ $p \in \bar{S}$ also gives a gauge transformation with $C^{(k)} = N^{(k)} = 0$ for $\forall k \geq 1$

$\bar{C}_E|_p = 0$ $p \in \bar{S} \Rightarrow$ 1) $p \in S$ and $E|_p = 0$

2) $p \in \bar{S} \setminus S$ and $g|_{L_s} \to +\infty$ $s \in S$, $(s \to p)$

1) $\Leftrightarrow$ pure Hodge structure

2) $\Leftrightarrow$ mixed Hodge structure (we need to know the behavior of g.)

Choice of base point $p \Leftrightarrow$ choice of the canonical opposite filtration (given by Deligne's $I^{p,q}$)

<u>Key lemma</u> Fix a local hol. coordinates on S.

$\exists$ unique A_S-endmorphisms $\{U_a\}_{a\geq 0}$ of H

such that

$$\bar{\partial}_{\bar{i}} U_a = - U_{a-1} \bar{C}_{\bar{i}} \qquad a \geq 1$$

$$U_0 = id.$$

$$\partial_{i_1} \cdots \partial_{i_n} U_a |_p = 0$$

for $i_1, \cdots, i_n \in \{1, \cdots, \mu\}$

$\forall n \geq 1$.

$\because$). Induction.

Use $[\bar{C}_{\bar{i}}, \bar{C}_{\bar{j}}] = 0$ and

$[\bar{\partial}_{\bar{i}}, \bar{C}_{\bar{j}}] = [\bar{\partial}_{\bar{j}}, \bar{C}_{\bar{i}}]$

$\Rightarrow$ $\sum_{a\geq 0} U_a z^{-a}$ gives the gauge transfomation.

<u>Thm</u>. $\exists\, \zeta \in \Gamma(S, \mathcal{O}_S(L))$ unique up to a cnst. factor

such that $\partial_{i_1} \cdots \partial_{i_n} \log g(\zeta, \zeta) |_p = 0$

for $i_1, \cdots, i_n \in \{1, \cdots, \mu\}$ and $\forall\, n \geq 1$.

$\Rightarrow$ primitive form.

Thm. Let G be the Hermitian metric on T_S canonically defined by the isomorphism

$$T_S \otimes \mathcal{O}_S(L) \xrightarrow{\sim} \mathcal{O}_S(H)$$

Then G is Kähler.

Proof. Tensor analysis.

Use facts $\quad D(A_S(L)) \subset A'_S(L)$

$$[D'_i, \partial_{\bar{j}}] + [C_i, \bar{C}_{\bar{j}}] = 0, \qquad (\zeta \in \Gamma(S, \mathcal{O}_S(L)))$$

$$[D'_i, C_j] = [D'_j, C_i] \quad \text{and} \quad C_{\frac{\partial}{\partial s^0}}\zeta = \zeta$$

($\frac{\partial}{\partial s^0}$: the hol. vector field corresponding to L)

$$G_{i\bar{j}} = g_{i\bar{j}} / g_{0\bar{0}} := g(C_{\frac{\partial}{\partial s^i}}\zeta, C_{\frac{\partial}{\partial s^j}}\zeta) / g(\zeta, \zeta)$$

Thm. (Characterization of flat coordinates)

$\exists$ local hol. coordinates $(t^0, \cdots, t^{\mu-1})$ of S unique up to a linear transformation such that

$$\partial_{i_1} \cdots\cdots \partial_{i_n} G_{i\bar{j}} \Big|_p = 0$$

for $i, j, i_1, \cdots\cdots, i_n \in \{0, \cdots\cdots, \mu-1\}$ and $\forall n \geq 1$.

4. Example

$$S = \mathbb{C} \underset{(t^0,\ \tau)}{\times} \mathbb{H}. \qquad \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$$

$$\nabla_{\partial_0} = \partial_0 + z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nabla_{\frac{1}{i}\partial_\tau} = \frac{1}{i}\partial_\tau + \frac{-i}{\tau - \bar\tau}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\nabla_{\bar\partial_0} = \bar\partial_0 + z^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\nabla_{-\frac{1}{i}\bar\partial_\tau} = -\frac{1}{i}\bar\partial_\tau + z^{-1}\frac{1}{(\tau-\bar\tau)^2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\nabla_{z\partial_z} = z\partial_z + z\begin{pmatrix} t^0 & 0 \\ a & t^0 \end{pmatrix} - q\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} - z^{-1}\begin{pmatrix} \bar{t^0} & \frac{-a}{(\tau-\bar\tau)^2} \\ 0 & \bar{t^0} \end{pmatrix}$$

where

$$E = t^0\partial_{t^0} + a\partial_\tau \quad , a \in \mathbb{R}$$

$$q = -\frac{1}{2} + \frac{ia}{\tau - \bar\tau}$$

$$g_{i\bar j} = \begin{pmatrix} 2\,\mathrm{Im}\,\tau & 0 \\ 0 & (2\,\mathrm{Im}\,\tau)^{-1} \end{pmatrix}$$

$a = 0 \iff$ Extended moduli for elliptic curve

$$T_tS \xrightarrow{\sim} H^0(E, \mathcal{O}_E) \oplus H^1(E, TE)$$

$$G_{i\bar{j}} := g_{i\bar{j}} / g_{0\bar{0}} = \begin{pmatrix} 1 & 0 \\ 0 & (2\,\mathrm{Im}\,\tau)^{-2} \end{pmatrix}$$

$(2\,\mathrm{Im}\,\tau)^{-2}$: Weil-Peterson metric on $\mathbb{H}$

- $a = 0 \Rightarrow$ { base points }
  $= \mathbb{H} \cup \{\infty\}$ large cpx str. limit.

  flat coordinates

  $t = \dfrac{\tau - a}{\tau - \bar{a}}$, $a \in \mathbb{H}$

  τ , $a = \infty$
  $(g_{0\bar{0}} = 2\,\mathrm{Im}\,\tau \to +\infty)$

- Observation

  R : the real analytic subvar. of S of codim 1
  such that on $S \setminus R$ $\exists$ (Vtx TERP)-str

  $R = \{ g_{0\bar{0}} = 0 \}$.