2003年06月23日　13時35分

$$\text{“}1\times 2\times 3\times\cdots\text{”} = \sqrt{2\pi} \leftarrow \underline{\text{Riemann}}$$

$$\text{“}1+2+3+\cdots\text{”} = -\frac{1}{12}$$

発散をうまくくり込んでいる。

“解析接続”　正規化する

2003年06月23日　13時47分

$a_1, \dots, a_N$: 有限正項数列

$$\prod_{j=1}^{N} a_j$$

$\Downarrow$

$a = \{a_n\}_{n=1,2,\dots}$ の $\underbrace{\text{正規化積}}_{}$ (ゼータ)

C. Deninger '91 invent. math.

$$\prod_{n=1}^{\infty} a_n = \exp\left(-\zeta_a'(0)\right)$$

但し, $\zeta_a(s) \overset{def}{=} \sum_{n=1}^{\infty} a_n^{-s}$: a のゼータ (zeta)

Dirichlet series.

$\Rightarrow$ $\zeta_a(s) = \sum_{n=1}^{N} a_n^{-s}$　　$a_n^{-s} = e^{-s\log a_n}$

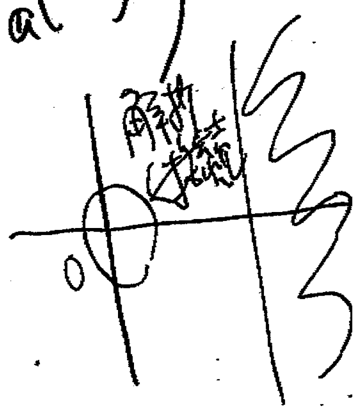
$\zeta_a'(s) = -\sum_{n=1}^{N} (\log a_n) \cdot a_n^{-s}$

$\zeta_a'(0) = -\sum_{n=1}^{N} \log a_n = -\log \prod_{n=1}^{N} a_n$

$e^{-\zeta_a'(0)} = \prod_{n=1}^{N} a_n$

絶対収束

$\downarrow$

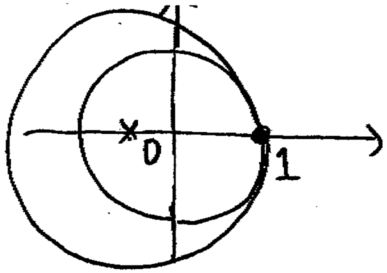


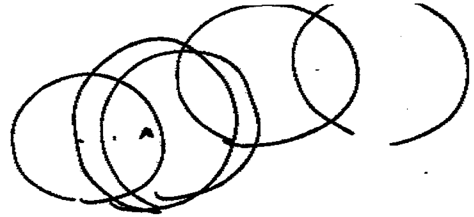
2003年06月23日　13時54分

解析接続

$|z|<1$

$$1+z+z^2+\cdots\cdots = \frac{1}{1-z}$$





$$= \frac{1}{1+\frac{1}{2}-(z+\frac{1}{2})} = \frac{1}{\frac{3}{2}\left(1-\frac{2}{3}(z+\frac{1}{2})\right)} = \frac{2}{3}\sum_{n=0}^{\infty}\left\{\frac{2}{3}\left(z+\frac{1}{2}\right)\right\}^n$$

$$\left|\frac{2}{3}\left(z+\frac{1}{2}\right)\right| < 1$$

$$\left|z+\frac{1}{2}\right| < \frac{3}{2}$$

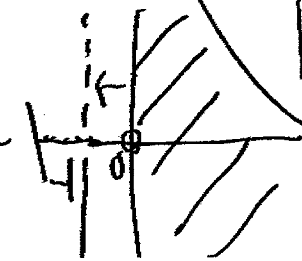
・ガンマ関数

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt \underset{\text{部分積分}}{\Longrightarrow} \Gamma(s+1) = s\Gamma(s)$$

$\mathrm{Re}(s)>0$

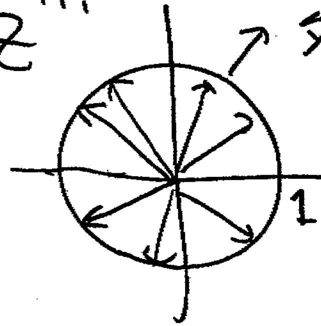
$\mathrm{Re}(s+1)>0$

$\mathrm{Re}(s)>-1$



$$\varphi(z)=\sum_{n=0}^{\infty} z^{2^n},\quad \sum z^{n!}$$

$$\boxed{|z|<1}$$



外に接続できない

自然境界

$$\varphi(1)=\infty$$

$$\zeta=e^{\frac{2\pi i m}{2^N}}\quad (m\in\mathbb{Z},\ N\in\mathbb{Z}_{>0})$$

$$\zeta^{2^N}=1$$

$$\varphi(\zeta z)=\zeta z+\cdots+(\zeta z)^{2^{N-1}}+\sum_{n=N}^{\infty} z^{2^n}=(\zeta-1)z+\cdots+(\zeta^{2^{N-1}}-1)z^{2^{N-1}}+\varphi(z)$$

$$\infty ! = \sqrt{2\pi}\qquad (x=1)$$

$$\frac{1}{\Gamma(x)}=\frac{1}{\sqrt{2\pi}}\prod_{n=0}^{\infty}(n+x)$$

$$\|$$

Euler 定数

$$e^{\gamma x}\cdot x\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}$$

零点情報

Weierstrass canonical product

$$\Gamma(x+1)=x\Gamma(x) \longleftarrow ◎ \prod_{n\in I\sqcup J} a_n = \prod_{n\in I} a_n \prod_{n\in J} a_n$$

$$\prod_{n\in I} a_n = \exp\left(-\frac{d}{ds}\sum_{n\in I} a_n^{-s}\right)\Big|_{s=0}$$

$$\therefore \sum_{n\in I\sqcup J} a_n^{-s} = \sum_{n\in I} a_n^{-s} + \sum_{n\in J} a_n^{-s}$$

☆を認めると

$$\frac{\sqrt{2\pi}}{\Gamma(x+1)} = \prod_{n=0}^{\infty}(n+x+1)$$

$$\frac{\sqrt{2\pi}}{\Gamma(x)} = \prod_{n=0}^{\infty}(n+x) = x\prod_{n=1}^{\infty}(n+x)$$

$\{0\}\sqcup\mathbb{N}$

Γの duplication formula

$$\frac{\sqrt{2\pi}}{\Gamma(2x)} = \prod_{n=0}^{\infty}(n+2x) = \prod_{m=0}^{\infty}(2m+2x)\prod_{m=0}^{\infty}(2m+1+2x)$$

$$= \prod_{m=0}^{\infty} 2(m+x)\prod_{m=0}^{\infty} 2\left(m+x+\frac{1}{2}\right)$$

2003年06月23日　14時26分

$\lambda > 0$

$$\prod_{n \in I} \lambda a_n \quad -\pi \leq \arg a_n < \pi \quad = \lambda^{\zeta_a(0)} \prod_{n \in I} a_n$$

$a = \{a_n\}$

$\lambda a = \{\lambda a_n\}$

$$\zeta_{\lambda a}(s) = \sum_{n \in I} (\lambda a_n)^{-s} = \lambda^{-s} \sum_{n \in I} a_n^{-s}$$

$$\zeta'_{\lambda a}(s) = -(\log \lambda)\lambda^{-s}\zeta_a(s) + \lambda^{-s}\zeta'_a(s)$$

$$\therefore \zeta'_{\lambda a}(0) = -\zeta_a(0)\log\lambda + \zeta'_a(0)$$

$$e^{-\zeta'_{\lambda a}(0)} = \lambda^{\zeta_a(0)} \cdot e^{-\zeta'_a(0)}$$

Hurwitz zeta

$$\zeta(s, x) = \sum_{n=0}^{\infty} (n+x)^{-s}$$

$\mathrm{Re}\, s > 1$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$$

$\Gamma(Nx)$ Gauss-Legendre

$$\zeta(0, x) = \frac{1}{2} - x$$ を使う

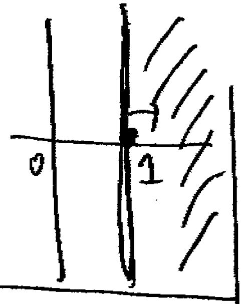
$$\frac{\sqrt{2\pi}}{\Gamma(2x)} = 2^{\zeta(0,x)} \prod_{m=0}^{\infty}(m+x) \cdot 2^{\zeta(0,x+\frac{1}{2})} \prod_{m=0}^{\infty}\left(m+x+\frac{1}{2}\right)$$

$$= 2^{\frac{1}{2}-x+\frac{1}{2}-(x+\frac{1}{2})} \frac{\sqrt{2\pi}}{\Gamma(x)} \cdot \frac{\sqrt{2\pi}}{\Gamma(x+\frac{1}{2})}$$

$2^{\frac{1}{2}-2x}$

@ 零点情報と、変換則が見れることが期待される。

@ "ゼータ"の行列式表示。



$\boxed{例}$ $t\frac{d}{dt} : \mathbb{C}[t] \to \mathbb{C}[t]$

t^n : 固有値 n の固有ベクトル。

$$\frac{\sqrt{2\pi}}{\Gamma(x)} = \prod_{n=0}^{\infty}(n+x) \Rightarrow \det_{\infty}\left(t\frac{d}{dt}+x\right)$$

$$\pi(x) = \sum_{p \le x} 1 = \#\{x以下の素数\} \sim \frac{x}{\log x} \quad (x \to \infty)$$ $\boxed{\mathrm{PNT}}$

Γ-factors

Γ_1, Γ_2

$\Gamma \subset SL_2(\mathbb{R})$

$\frac{1}{y^2}(dx^2+dy^2)$

$\Delta = -y^2(\partial_x^2+\partial_y^2)$

R : リーマン面の

$\det(\Delta + s(1-s))$

Selberg zeta ft.

$$Z_\Gamma(s) = \prod_{n=0}^{\infty}\prod_{P}\left(1-N(P)^{-s-n}\right)$$

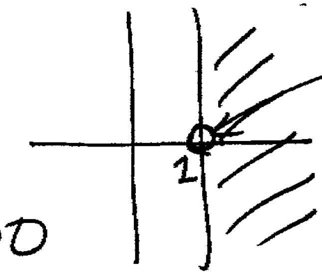
P : 素元

$\boxed{\mathrm{PGT}}$ $P \sim \begin{pmatrix}\alpha_P & 0 \\ 0 & \beta_P\end{pmatrix}$

$N(P) = \max(|\alpha_P|, |\beta_P|)$

2003年06月23日　14時52分

$$\boxed{\infty! = \sqrt{2\pi}} \qquad e^{-\zeta'(0)}$$



$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \mathrm{Re}(s)>0$$

$$= \prod_{p:\text{prime}} (1-p^{-s})^{-1}$$

$$(\zeta(s,1) = \zeta(s))$$

meromorphic

$\mathbb{C}$.

$\boxed{\text{ft. eq}}$

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\underline{\cos\left(\frac{\pi s}{2}\right)\zeta(s)}$$

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n \quad (\text{Laurent expansion})$$

$$\cos\left(\frac{\pi s}{2}\right) = -\frac{\pi}{2}(s-1) + O((s-1)^3)$$

$$\left(\cos\left(\frac{\pi s}{2}\right)\right)' = -\frac{\pi}{2}\sin\left(\frac{\pi s}{2}\right)$$

$$\cos\left(\frac{\pi s}{2}\right)\zeta(s) = \underline{-\frac{\pi}{2} - \frac{\pi\gamma}{2}(s-1)} + O((s-1)^2)$$

$$2(2\pi)^{-s} = \underline{\frac{2}{2\pi}\left(1 - \log(2\pi)(s-1) + \cdots\right)}$$

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$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1} \left(1+\frac{s}{n}\right) e^{-\frac{s}{n}}$$

$$-\log\Gamma(s) = \gamma s + \log s + \sum_{n=1}^{\infty}\left\{\log\left(1+\frac{s}{n}\right) - \frac{s}{n}\right\}$$

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \gamma + \frac{1}{s} + \sum_{n=1}^{\infty}\left\{\underbrace{\frac{\frac{1}{n}}{1+\frac{s}{n}} - \frac{1}{n}}_{\frac{1}{n+s}-\frac{1}{n}}\right\}$$

$$-\frac{\Gamma'(1)}{\Gamma(1)} = \gamma + 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n}\right) = \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n+1} - \frac{1}{n}\right) = \frac{1}{N+1} - 1$$

$$= \gamma + 1 - 1 = \gamma \quad \therefore \Gamma'(1) = -\gamma$$

$$\Gamma(s) = 1 - \gamma(s-1) + \cdots$$

$$\zeta'(0) = -\frac{d}{ds}\zeta(1-s)\Big|_{s=0} = -\zeta(1-s)\text{の}\underset{(s-1)}{\text{1次の項}}$$

$$= \left\{-\frac{\pi}{2} - \frac{\pi\gamma}{2}\underline{(s-1)} + \cdots\right\}\left\{\frac{1}{\pi}\left(1 - \log(2\pi)\underline{(s-1)} + \cdots\right)\right\}$$

$$\times\left\{1 - \gamma\underline{(s-1)} + \cdots\right\} = \left(-\frac{\pi\gamma}{2}\right)\frac{1}{\pi} + \frac{\pi}{2}\frac{1}{\pi}\gamma - \frac{\pi}{2}\frac{1}{\pi}\log(2\pi)$$

の1次の~~項~~

$$= -\frac{1}{2}\log(2\pi)$$

$$\infty! = \prod^{\infty} n = e^{-\zeta'(0)} = \sqrt{2\pi}$$

2003年06月23日　15時11分

$$\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s}$$

$$\sum n^{-s} \cdots \int_1^{\infty} x^{-s}dx \qquad \underline{\mathrm{Re}\, s>1}$$

$$e^{-\zeta'(0,x)}=\frac{\sqrt{2\pi}}{\Gamma(x)}$$ Lerch

$$\frac{\partial}{\partial x}\zeta(s,x)=-s\sum(n+x)^{-s-1}$$

$$\frac{\partial^2}{\partial x^2}\zeta(s,x)=s(s+1)\sum(n+x)^{-s-2}$$

$$=s(s+1)\zeta(s+2,x)$$

$$\frac{\partial^2}{\partial x^2}\zeta'(s,x)=(2s+1)\zeta(s+2,x)+s(s+1)\underline{\zeta'(s+2,x)}$$

$s=0$

$$+\frac{\partial^2}{\partial x^2}\log\Gamma(x)=+\sum_{n=0}^{\infty}\frac{1}{(n+x)^2}$$

$$\|$$

$$\frac{\partial^2}{\partial x^2}\zeta'(0,x)=\zeta(2,x)$$

$$\log\Gamma(x)-\zeta'(0,x)=Ax+B$$

$\boxed{x=1}$

$$-\zeta'(0,1)=B$$

$$\|$$

$$-\zeta'(0)=-\log\sqrt{2\pi}$$

$x \to x+1$ で成立

$\therefore A=0$

$$\zeta(s,x+1)=\sum_{n=0}^{\infty}(n+x+1)^{-s}=\sum_{n=0}^{\infty}(n+x)^{-s}-x^{-s}$$

$$=\zeta(s,x)-x^{-s}$$

$$\boxed{\zeta'(0,x+1)=\zeta'(0,x)+\log x}$$

$$\log\Gamma(x+1)=\log\Gamma(x)+\log x$$

2003年06月23日　15時15分

$$\zeta(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\sum_{n=0}^{\infty} e^{-(n+x)t}\,dt$$

$$= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1} e^{-xt}\underbrace{\frac{1}{1-e^{-t}}}\,dt$$

$$\frac{1}{1-e^{-t}} = \frac{1}{t} + c_0 + c_1 t + \left\{\frac{1}{1-e^{-t}} - \left(\frac{1}{t}+c_0+c_1 t\right)\right\}$$

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\,dt \overset{t=ax}{=} a^s \int_0^{\infty} x^{s-1} e^{-ax}\,dx$$

$$\zeta_a(s) = \sum_{n=0}^{\infty} a^{-ns} = \frac{1}{1-a^{-s}}$$ ← $s=0$ で hol でない．

$\boxed{a>1}$ $\quad a = \{a^n\}_{n=0,1,\dots}$ $\quad \prod_{n=0}^{\infty} a^n = ?$

2003年06月23日　15時29分

<u>dotted product</u>

<u>Donburi-product</u>

$\zeta_a(s)$ が mero.
$\quad = \sum_{n\in I} a_n^{-s}$

$$\underset{n\in I}{\boxed{\bullet}}\, a_n = \exp\left(-\operatorname*{Res}_{s=0} \frac{\zeta_a(s)}{s^2}\right) \quad \left(\underset{\zeta_a(s)\text{ が } s=0\text{ で hol.}}{=\!=} \prod_{n\in I} a_n\right)$$

$$\left(\underset{n\in I}{\boxed{\bullet}}\, a_n\right)^c = \underset{n\in I}{\boxed{\bullet}}\, a_n^c$$

$c>0$

$\sum_{n\in I} a_n^{-cs}$

<u>Rep.</u> 示せ.

$$\frac{1}{1-z} = 1+z+z^2+\cdots$$

$$\frac{1}{1-a^{-s}} = \frac{1}{1-\left(1-s\log a+\frac{1}{2}(s\log a)^2 - \frac{1}{6}(s\log a)^3 - \cdots\right)}$$

s の1次の係数 $= -\frac{1}{6}(\log a)^2$

$$= \frac{1}{s\log a}\,\frac{1}{1-\frac{1}{2}(s\log a)+\frac{1}{6}(s\log a)^2 - \cdots} = \frac{1}{s\log a}\left[1+\left(\frac{s}{2}\log a - \frac{s^2}{6}(\log a)^2+\cdots\right) + \left(\frac{s}{2}\log a - \frac{s^2}{6}(\log a)^2+\cdots\right)^2 + \cdots\right]$$

$-\frac{1}{6}(\log a)^2 + \frac{1}{4}(\log a)^2 = \frac{1}{12}(\log a)^2$

$$\operatorname*{Res}_{s=0} \frac{\zeta_a(s)}{s^2} = \frac{1}{12}\log a = \log a^{\frac{1}{12}}$$

$$\therefore \prod_{n=0}^{\infty} a^{n} = a^{-\frac{1}{12}} = a^{\zeta(-1)}$$

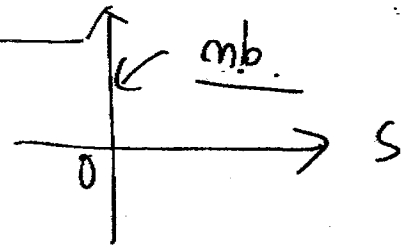
known

$$\zeta(-1) = 1+2+3+\cdots\cdots = -\frac{1}{12}$$

$$\prod_{p:\text{素数}} p \quad ? \quad \nexists$$

$$-\log(1-x) = \sum_{d=1}^{\infty} \frac{x^d}{d}$$

$$\varphi(s) = \sum_{p:\text{素数}} p^{-s}$$



$$\zeta(s) = \prod_p (1-p^{-s})^{-1}$$

$$\log\zeta(s) = -\sum_p \log(1-p^{-s}) = \sum_p \sum_d \frac{p^{-ds}}{d}$$

$$= \sum_d \varphi(ds)$$

2003年06月23日　15時40分

Möbius 反転公式

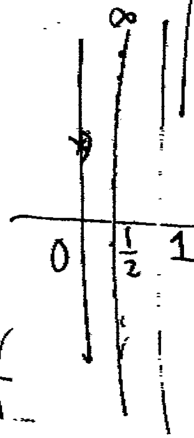
$$\varphi(s) = \sum_{m=1}^{\infty} \mu(m) \log \zeta(ms)$$

$$\mu(m) = \begin{cases} (-1)^d & : m = p_1 \cdots p_d \text{ 相異なる素数のべき} \\ 0 & : \text{otherwise} \end{cases}$$

$\mu(1) = 1$

$$\varphi'(s) = \sum_{m=1}^{\infty} m\mu(m) \frac{\zeta'(ms)}{\zeta(ms)}$$

$ms = \frac{1}{2} + ir$　$s = \frac{1}{2m} + i\frac{r}{m}$



$$\sum_d \varphi(ds)$$

$$d_m = n\frac{\sum a_n^{-s}}{1+\xi}$$

$$= \sum_d \sum_m \mu(m) \log \zeta(mds)$$

$$= \sum_n \underbrace{\sum_{d|n} \mu\left(\frac{n}{d}\right)}_{n=1\text{以外は}0} \cdot \log \zeta(ns)$$

$$= \log \zeta(s)$$

2003年06月23日　15時46分

$$\operatorname*{Res}_{s=0} \frac{\zeta_a(s)}{s^2} = \frac{1}{12}\log a = \log a^{\frac{1}{12}}$$

$$\therefore \prod_{n=0}^{\infty} a^{n} = a^{-\frac{1}{12}} = a^{\zeta(-1)}$$

known

$$\zeta(-1) = 1+2+3+\cdots\cdots = -\frac{1}{12}$$

$$\prod a^{n+x}$$

$$\prod_{n:\text{sq. free}} n$$

$$\rho(s) = \sum_{n:\text{sq.free}} n^{-s} = \prod_p (1+p^{-s})$$

$$= \prod_p \frac{1-p^{-2s}}{1-p^{-s}}$$

$$\rho'(s) = \frac{\zeta'(s)\zeta(2s) - 2\zeta(s)\zeta'(2s)}{\zeta(2s)^2}$$

$$\rho'(0) = \frac{-\zeta'(0)\zeta(0)}{\zeta(0)^2} = -\frac{\zeta'(0)}{\zeta(0)} = -\frac{-\log\sqrt{2\pi}}{-\frac{1}{2}} = -\log 2\pi$$

$$\left(\prod a_n\right)^{c} e^{-\rho'(0)} = 2\pi$$

$$\prod a_n^{c} = \left(\prod a_n\right)^{c}$$

$c>0$

$$\prod_{n:sq} n = \left(\prod_{n=1}^{\infty} n\right)^2 = 2\pi \qquad \frac{\zeta(s)}{\zeta(2s)}$$

$$\sum z^{n!} = \sum e^{-sn!}$$

$$z = e^{-s} \qquad \prod e^{n!}$$

2003年06月24日　13時47分

$$\prod_{m,n\geq 1}(mn-x)=\frac{1}{\sqrt{2\pi}}e^{-(\gamma^2+2\gamma_1)x}\prod_{m,n\geq 1}\left(1-\frac{x}{mn}\right)e^{\frac{x}{mn}}$$

$\exp\left(-\operatorname{Res}_{s=0}\frac{\varphi(s)}{s^2}\right)$

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)=0.57721\ldots$$

$$\zeta(s)=\frac{1}{s-1}+\gamma+\gamma_1(s-1)+\cdots\cdots$$

$0<x<1$

$$\binom{s+l-1}{l}=\frac{s(s+1)\cdots(s+l-1)}{l!}\quad l\geq 1$$

$\log$ をとる

$$-\log\prod_{m,n\geq 1}\left(1-\frac{x}{mn}\right)e^{\frac{x}{mn}}$$

$$=\sum_{m,n}\sum_{d\geq 2}\frac{1}{d}\left(\frac{x}{mn}\right)^d$$

$$=\sum_{d\geq 2}\frac{1}{d}\zeta(d)^2x^d$$

$$\varphi(s)=\sum_{m,n\geq 1}(mn-x)^{-s}=\sum(mn)^{-s}\left(1-\frac{x}{mn}\right)^{-s}=\sum_{m,n\geq 1}(mn)^{-s}\sum_{l=0}^{\infty}\binom{-s}{l}(-1)^l\left(\frac{x}{mn}\right)^l$$

$$=\sum_{m,n\geq 1}(mn)^{-s}+s\sum_{m,n\geq 1}(mn)^{-s}\left(\frac{x}{mn}\right)+\sum_{l\geq 2}\sum_{m,n\geq 1}\binom{s+l-1}{l}(mn)^{-(s+l)}x^l$$

$\sum_d\sigma(d)d^{-s}$

$$=\zeta(s)^2+s\,\zeta(s+1)^2x+s\sum_{l\geq 2}\frac{1}{l}\zeta(l)^2x^l+O(s^2)$$

$$=\Big(\underset{-\frac{1}{2}}{\zeta(0)}+\zeta'(0)s+\cdots\Big)^2+s\left(\frac{1}{s}+\gamma+\gamma_1s+\cdots\right)^2x+s\sum_{l\geq 2}\frac{1}{l}\zeta(l)^2x^l+O(s^2)$$

2003年06月24日　13時57分

<u>Remarks</u> ① $\dot{\prod} a_n b_n \neq \dot{\prod} a_n \dot{\prod} b_n$

② $\dot{\prod} |a_n| \neq \left|\dot{\prod} a_n\right|$

一致するくらい. $(m=1,2,\cdots$

$$\prod_{n=0}^{\infty}\left((n+x)^m - y^m\right) = \frac{(2\pi)^{\frac{m}{2}}}{\prod_{\xi^m=1}\Gamma(x-\xi y)}$$

がわかる

$\underset{\mathbb{a}}{\{a_n\}}, \underset{\mathbb{b}}{\{b_n\}} \rightsquigarrow \dot{\prod}_{n\in I}((a_n))^{b_n} = \exp\left(-\operatorname{Res}_{s=0}\frac{\zeta_{\mathbb{a},\mathbb{b}}(s)}{s^2}\right)$

$\zeta_{\mathbb{a},\mathbb{b}}(s) := \sum_{n\in I} b_n \cdot a_n^{-s}$ ($b_n \in \mathbb{Z}_{\geq 0}$ は a_n の multiplicity の役割)

A_N を $A_{b_1+\cdots+b_{n-1}+j} = a_n$ $(1\leq j\leq b_n)$ で定める

<u>Remark</u> $\dot{\prod}((a_n))^{b_n} \neq \dot{\prod} a_n^{b_n}$

$$\prod_{n=1}^{\infty}((n))^{n^x} = \exp(-\zeta'(-x))$$

"$1\times 2\times 2\times 3\times 3\times 3\times 4\times 4\times 4\times \cdots\cdots$"

$\dot{\prod}((n))^n \neq \dot{\prod} n^n$

$\dot{\prod}((a_n))^{b_n} = \dot{\prod} A_N$

2003年06月24日　14時04分

$$\sum_{n=1}^{\infty} |b_n| \cdot |a_n|^{-t}$$

μ：収束指数, i.e. $\forall \varepsilon > 0$ に対し

$\mathrm{Re}(t) = \mu + \varepsilon$　収束

$\mathrm{Re}(t) = \mu - \varepsilon$　発散

$$\zeta_{a,b}(s,x) = \sum_n b_n (a_n - x)^{-s} \qquad \mathrm{Re}(s) > \mu : \text{abs. conv.}$$

$\boxed{\text{仮定}}$ $\zeta_{a,b}(s,x)$: $s=0$ で meromorphic.

$$\prod_n ((a_n - x))^{b_n} = \exp\left(-\mathrm{Res}_{s=0} \frac{\zeta_{a,b}(s,x)}{s^2}\right)$$

Thm. $b_n \in \mathbb{Z}_{\geq 0}$

$\prod_n ((a_n - x))^{b_n}$ は $\mathbb{C}$ で entire

$x = a_n$ で order b_n の zero

$r := [\mu]$

$$\prod_{n=1}^{\infty} ((a_n - x))^{b_n} = e^{\exists F(x)} \prod_{n=1}^{\infty} P_r\left(\frac{x}{a_n}\right)^{b_n}$$

$F(x)$ は高々 r 次の polyn.

但し. $P_r(u) = (1-u) \exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right)$

2003年06月24日　14時08分

$$\sum_{n=1}^{\infty} |b_n|\cdot|a_n|^{-t}$$

μ：収束指数, i.e. $\forall\varepsilon>0$に対し

$\mathrm{Re}(t)=\mu+\varepsilon$　収束

$\mathrm{Re}(t)=\mu-\varepsilon$　発散

$$\zeta_{a,b}(s,x)=\sum_n b_n(a_n-x)^{-s} \qquad \mathrm{Re}(s)>\mu : \text{abs. conv.}$$

$\boxed{\text{仮定}}$ $\zeta_{a,b}(s,x)$: $s=0$ で meromorphic.

$$\prod_n ((a_n-x))^{b_n} = \exp\left(-\mathrm{Res}_{s=0}\frac{\zeta_{a,b}(s,x)}{s^2}\right)$$

<u>Thm.</u> $b_n\in\mathbb{Z}_{\geq 0}$

$\prod_n ((a_n-x))^{b_n}$ は $\mathbb{C}$ で <u>entire</u>

$x=a_n$ で order b_n の <u>zero</u>

$$r := [\mu]$$

$$\prod_{n=1}^{\infty}((a_n-x))^{b_n} = e^{\exists F(x)}\prod_{n=1}^{\infty} P_r\left(\frac{x}{a_n}\right)^{b_n} = \Delta(x)$$

但し、$P_r(u)=(1-u)\exp\left(u+\frac{u^2}{2}+\cdots+\frac{u^r}{r}\right)$

$$\frac{a_n}{1-\frac{x}{a_n}}$$

$F(x)$は次数が高々rのpolyn.

2003年06月24日　14時19分

$$\eta_{a,\mathbb{b}}(s,x) := \Gamma(s)\zeta_{a,\mathbb{b}}(s,x)$$

• $\frac{\partial}{\partial x}\eta_{a,\mathbb{b}}(s,x) = \eta_{a,\mathbb{b}}(s+1,x)$

◎ $\frac{d^{r+1}}{d^{r+1}x}\log\Delta(x) = -\eta_{a,\mathbb{b}}(r+1,x)$

$$F_{a,\mathbb{b}}(x) := \exp\Big(-\underset{\substack{\uparrow \\ s=0}}{\underline{\mathrm{FP}}}\,\eta_{a,\mathbb{b}}(0,x)\Big)$$

FP $\leftrightarrow \frac{\partial}{\partial x}$ 可換

$$\frac{d^{r+1}}{dx^{r+1}}\log F_{a,\mathbb{b}}(x) = -\eta_{a,\mathbb{b}}(r+1,x)$$

$f(z)$: $z=a$ で mero.

$$\mathrm{FP}f(a) = \begin{cases} f(a) & \text{hol.} \\ \lim_{z\to a}\big(f(z) - \text{principal term of } f(z)\big) \end{cases}$$

$$F_{a,\mathbb{b}}(x) = e^{\exists P(x)}\Delta(x)$$

$\uparrow$ poly. degree $\leq r$

$$\Gamma(s) = \frac{1}{s} + \gamma_0 + \gamma_1 s + \gamma_2 s^2 + \cdots\cdots$$

$$\zeta_{a,\mathbb{b}}(s,x) = \frac{b_{-N}(x)}{s^N} + \cdots + \frac{b_{-1}(x)}{s} + b_0(x) + b_1(x)s + O(s^2)$$

2003年06月24日　14時23分

$$\mathrm{FP}\ \eta_a(0,x)$$

$$= \underline{b_1(x)} + \sum_{j=0}^{N} \gamma_j b_{-j}(x)$$

$$\| $$

$$\operatorname*{Res}_{s=0} \frac{\zeta_{a,b}(s,x)}{s^2}$$

$$\prod_n ((G(a_n-x)))^{b_n}$$

Ⅱ/res.

$G(x)=x$,

$G(x)=$ 三角 : q-analog の x

$s=0$ で hol.

$\downarrow$

$$\frac{\partial^{r+1}}{\partial x^{r+1}} \zeta_{a,b}(s,x) = s(s+1)\cdots(s+r)\zeta_{a,b}(s+r+1,x)$$

abs. conv.

$s=0$ で 消えている。

$\Downarrow$

$$\frac{d^{r+1}}{dx^{r+1}} b_{-j}(x) \equiv 0$$

Thm. $b_n \in \mathbb{Z}_{\geq 0}$

$$\frac{1}{a_n} \Big/ \left(1-\frac{x}{a_n}\right)$$

$\prod_n ((a_n-x))^{b_n}$ は $\mathbb{C}$ で <u>entire</u>

$x=a_n$ で order b_n の <u>zero</u>

$r := [\mu]$

$$\prod_{n=1}^{\infty} ((a_n-x))^{b_n}$$

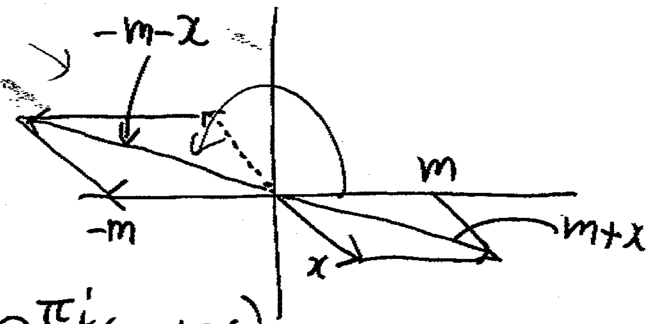
$$= e^{\exists F(x)} \prod_{n=1}^{\infty} P_r\left(\frac{x}{a_n}\right)^{b_n} \fallingdotseq \Delta(x)$$

$= \exp(-b_1(x))$ $F(x)$ は高々 r 次 の polyn.

但し. $P_r(u) = (1-u)\exp\left(u+\frac{u^2}{2}+\cdots+\frac{u^r}{r}\right)$

∵) $\mathrm{Im}(x)<0$

$$\varphi(s,x)=\sum_{m\in\mathbb{Z}}(m-x)^{-s}=\sum_{m=0}^{\infty}(m-x)^{-s}+\sum_{m=1}^{\infty}(-m-x)^{-s}$$



$$=\zeta(s,-x)+e^{-s\pi i}\zeta(s,x+1)$$

$$-m-x=e^{\pi i}(m+x)$$

$-1+1$

$$\varphi'(0,x)=\zeta'(0,-x)-\pi i\,\zeta(0,x+1)+\zeta'(0,x+1)$$

$$\zeta'(0,-x)=\log\frac{\Gamma(-x)}{\sqrt{2\pi}}$$

$$\zeta(0,x+1)=\frac{1}{2}-(x+1)=-\left(\frac{1}{2}+x\right)$$

$$\zeta'(0,x+1)=\log\frac{\Gamma(x+1)}{\sqrt{2\pi}}$$

$$\boxed{\frac{2\pi}{\Gamma(x)\Gamma(1-x)}=2\sin\pi x}$$

2003年06月24日　14時42分

$$S_{\mathbb{Z}}(x) := \prod_{m\in\mathbb{Z}} (m-x)$$

$$S_{\mathbb{Z}[\tau]}(x) = \prod_{m,n\in\mathbb{Z}} (m+n\tau-x)$$
$(\operatorname{Im}\tau>0)$

$$S_{\mathbb{Z}}(x) = \begin{cases} 1-e^{2\pi i x} & \operatorname{Im}(x)>0 \\ 1-e^{2\pi i x} & (0<x<1) \\ 1-e^{-2\pi i x} & \operatorname{Im}(x)<0. \end{cases}$$

$$\boxed{-\pi \leq \arg(m-x) < \pi}$$

K : nf　$^{\exists} f_K(x)$

$$K^{ab} = K(f_K(K))$$

Kroneckerの青春の夢

$$\mathbb{Q}^{ab} = \mathbb{Q}(\underbrace{e^{2\pi i x}}_{f_{\mathbb{Q}}(x)})$$

虚2次　Takagi

Krokawaの中年の夢　$sn(x,\tau)$

$$\text{"}\prod_{a\in\mathcal{O}} (a-x)\text{"}$$

2003年06月24日　14時46分

$$S_{\mathbb{Z}}(x) := \prod_{m\in\mathbb{Z}} (m-x)$$

$$S_{\mathbb{Z}[\tau]}(x) = \prod_{m,n\in\mathbb{Z}} (m+n\tau-x)$$
$(\mathrm{Im}\,\tau > 0)$

答 $\left(1-q^{-\frac{x}{\tau}}\right)\prod_{n=1}^{\infty}\left(1-q^{-(n+\frac{x}{\tau})}\right)\left(1-q^{-(n-\frac{x}{\tau})}\right)$

$(0 < \mathrm{Im}\,x < \mathrm{Im}\,\tau)$

$q = e^{-2\pi i\tau}$

2重Hurwitz を使って計算.

$$S_{\mathbb{Z}}(x) = \begin{cases} 1-e^{2\pi i x} & \mathrm{Im}(x) > 0 \\ 1-e^{2\pi i x} & (0 < x < 1) \\ 1-e^{-2\pi i x} & \mathrm{Im}(x) < 0. \end{cases}$$

レポート

$-\pi \leq \arg(m-x) < \pi$

2003年06月24日　14時56分

$$\Gamma_q(x) = \frac{\prod_{n=1}^{\infty}(1-q^{-n})}{\prod_{n=0}^{\infty}(1-q^{-(n+x)})}\left(q^{\frac12}-q^{-\frac12}\right)^{1-x} q^{x(x-1)/4}$$: Jackson q-gamma. $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$

$[x]_q = \dfrac{q^{\frac{x}{2}}-q^{-\frac{x}{2}}}{q^{\frac12}-q^{-\frac12}}$

$\dfrac{1-q^{x+n}}{1-q}$

$$[\infty]_q! := \prod_{n=1}^{\infty}[n]_q = q^{-\frac{1}{24}}\left(q^{\frac12}-q^{-\frac12}\right)^{-\log(1-q^{-1})/\log q}\prod_{n=1}^{\infty}(1-q^{-n})$$

$(q>1)$

$\prod_{m=0}^{\infty}[m+x]_q \qquad \prod_{m\in\mathbb{Z}}[m-x]_q$ を計算したい。

$$\zeta_q(s,x) := \sum_{n=0}^{\infty}[n+x]_q^{-s} \quad (q\text{-Hurwitz})$$

<u>Lemma</u> $0<\mathrm{Re}\,x<1$, $-\dfrac{2\pi}{\log q}\le \mathrm{Im}\,x<\dfrac{2\pi}{\log q}$ $\Rightarrow$ $s=0$ での Laurent expansion は次で与えられる

$$\zeta_q(s,x) = \frac{2}{\log q}\frac{1}{s} + \frac{2\log\left(q^{\frac12}-q^{-\frac12}\right)}{\log q} - \frac12(2x-1) + s\log\frac{\Gamma_q(x)}{[\infty]_q!} + O(s^2)$$

2003年06月24日　15時07分

$$\Gamma_q(x) = \frac{\prod_{n=1}^{\infty}(1-q^{-n})}{\prod_{n=0}^{\infty}(1-q^{-(n+x)})}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{1-x} q^{x(x-1)/4}$$

: Jackson q-gamma. $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$

$$[x]_q = \frac{q^{\frac{x}{2}}-q^{-\frac{x}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$$

$$\frac{1-q^{x+n}}{1-q}$$

$$[\infty]_q! := \prod_{n=1}^{\infty}[n]_q = q^{-\frac{1}{24}}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{-\log(1-q^{-1})/\log q} \prod_{n=1}^{\infty}(1-q^{-n})$$

q-Lerch の公式

$$\prod_{m=0}^{\infty}[m+x]_q \overset{?}{=} \frac{[\infty]_q!}{\Gamma_q(x)}$$

$$\prod_{m\in\mathbb{Z}}[m-x]_q$$ を計算したい。

$(q>1)$

Lemma $0<\mathrm{Re}\,x<1$

$$-\frac{2\pi}{\log q} \leq \mathrm{Im}\,x < \frac{2\pi}{\log q} \Rightarrow$$

$$\zeta_q(s,x) = \sum_{n=0}^{\infty}[n+x]_q^{-s} \quad (q\text{-Hurwitz})$$

$s=0$ での Laurent expansion は次で与えられる

$$\zeta_q(s,x) = \frac{2}{\log q}\frac{1}{s} + \frac{2\log(q^{\frac{1}{2}}-q^{-\frac{1}{2}})}{\log q} - \frac{1}{2}(2x-1) + s\log\frac{\Gamma_q(x)}{[\infty]_q!} + O(s^2)$$

2003年06月24日　15時09分

$n \geq 0$: $\left(q^{\frac{n+x}{2}} - q^{-\frac{n+x}{2}}\right)^{-s} = \left(q^{\frac{n+x}{2}}(1-q^{-n-x})\right)^{-s}$

$$= q^{-\frac{s(n+x)}{2}}\left(1-q^{-n-x}\right)^{-s}$$

$\binom{s+l-1}{l} \underset{l \geq 1}{=} \frac{s(s+1)\cdots(s+l-1)}{l!} = \frac{s}{l} + O(s^2)$

$$\zeta_q(s,x) = \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^s \sum_{n=0}^{\infty} q^{-\frac{s(n+x)}{2}} \sum_{l=0}^{\infty} \binom{s+l-1}{l} q^{-(n+x)l}$$

$$= \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)^s \sum_{l=0}^{\infty} \binom{s+l-1}{l} \frac{q^{-(\frac{s}{2}+l)x}}{1-q^{-(\frac{s}{2}+l)}}$$

$$= \left\{1 + s\log\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) + \frac{s^2}{2}\left\{\log\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)\right\}^2 + \cdots\right\} \times \left\{\frac{q^{\frac{s}{2}(1-x)}}{q^{\frac{s}{2}}-1} + s\sum_{l=1}^{\infty} \frac{1}{l} \frac{q^{l(1-x)}}{q^l - 1} + O(s^2)\right\}$$

sの1次の係数　$\left(\frac{x^2}{4} - \frac{x}{4} + \frac{1}{24}\right)\log q + \sum_{l=1}^{\infty} \frac{1}{l} \frac{q^{l(1-x)}}{q^l - 1} - \frac{1}{2}(2x-1)\log\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right) + \frac{\left\{\log\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)\right\}^2}{\log q}$

2003年06月24日　15時21分

$$S_{\mathbb{Z}}^{q}(x)=\prod_{n\in\mathbb{Z}}[n+x]_q=\begin{cases} i\,e^{-\pi i x+\pi^2/\log q}\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)^{\frac{2\pi i}{\log q}}\dfrac{([\infty]!)^2}{\Gamma_q(x)\Gamma_q(1-x)} & \left(-\dfrac{2\pi}{\log q}\leq \operatorname{Im}x<0\right)\\ -i\,e^{\pi i x+\pi^2/\log q}\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)^{-\frac{2\pi i}{\log q}}\;\;〃 & \left(0\leq \operatorname{Im}x<\dfrac{2\pi}{\log q}\right)\end{cases}$$

$$S_{\mathbb{Z}}(x)\overset{q}{\longmapsto}S_{\mathbb{Z}}^{q}(x)$$

sin

$$S_{\mathbb{Z}[\tau]}(x)\;/\!/\!\wr$$

$$\because\;\zeta_q(s,x)=\sum_{n\in\mathbb{Z}}[n-x]_q^{-s}=\sum_{n=0}^{\infty}[n+(1-x)]_q^{-s}+\sum_{n=0}^{\infty}[-n-x]_q^{-s}$$

$$@\;\; q^{\pm\frac{n+x}{2}}=e^{\pm\frac{n+x}{2}\log q}=e^{\pm\frac{n+\operatorname{Re}(x)}{2}\log q}\underbrace{e^{\pm i\frac{\operatorname{Im}(x)}{2}\log q}}_{\cos(\;)\pm i\sin\underbrace{\frac{\operatorname{Im}(x)}{2}\log q}}$$

$$\operatorname{Im}[-n-x]=\operatorname{Im}\frac{q^{-\frac{n+x}{2}}-q^{\frac{n+x}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$$

$$=\frac{-2}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\cosh\left(\frac{n+\operatorname{Re}(x)}{2}\log q\right)\sin\left(\frac{\operatorname{Im}(x)}{2}\log q\right)\begin{cases}>0 & \left(-\dfrac{2\pi}{\log q}\leq\operatorname{Im}x<0\right)\\ \leq 0 & (\text{[illegible]})\end{cases}$$

$$\Rightarrow [-n-x]_q=e^{\pi i}[n+x]_q$$

$$[n+x]_q$$

2003年06月24日　15時32分

―― の場合

$$\Xi_q(s,x) = \xi_q(s,1-x) + \underbrace{e^{-s\pi i}}_{=\,1 - s\pi i + \frac{s^2}{2}(\pi i)^2 + O(s^3)} \xi_q(s,x)$$

$$\xi_q(s,x) = \frac{1}{s}\frac{2}{\log q} + \cdots$$

$$\frac{\cos \sin^{-s-1}}{\cos^2 \sin^{-s-2}} \quad \frac{\sin^{-s}}{} \quad \cos^2 = 1 - \sin^2$$

$$\Xi_q(s,x) = \sum_{n\in\mathbb{Z}} [n-x]_q^{-s} \quad \Rightarrow \quad \Xi_q(2,x) : \wp\text{-ft.}$$

Weierstrass の

$s=0$ で hol

$$\Downarrow$$

$$\frac{\partial^2}{\partial x^2}\Xi_q(s,x) = \left(\frac{s\log q}{2}\right)^2 \Xi_q(s,x) + s(s+1)\left(\frac{\log q}{q^{\frac12}-q^{-\frac12}}\right)^2 \Xi_q(s+2,x)$$

$$\frac{\partial^2}{\partial x^2} \operatorname*{Res}_{s=0} \frac{\Xi_q(s,x)}{s^2} = \left(\frac{\log q}{2}\right)^2 \operatorname*{Res}_{s=0}\Xi_q(s,x) + \left(\frac{\log q}{q^{\frac12}-q^{-\frac12}}\right)^2 \Xi_q(2,x)$$

shita

$$\operatorname*{Res}_{s=0}\frac{\Xi_q(s,x)}{s^2} = -\log\prod_{n\in\mathbb{Z}}[n-x]_q, \qquad \operatorname*{Res}_{s=0}\Xi_q(s,x) = \frac{4}{\log q}$$

2003年06月24日　15時44分

$$S_{\mathbb{Z}}^{q}(x)=\prod_{n\in\mathbb{Z}}[n+x]_q=\begin{cases} i\,e^{-\pi i x+\pi^2/\log q}\left(q^{\frac12}-q^{-\frac12}\right)^{\frac{2\pi i}{\log q}}\dfrac{([\infty]_q!)^2}{\Gamma_q(x)\Gamma_q(1-x)} & \left(-\frac{2\pi}{\log q}\le \mathrm{Im}\,x<0\right)\\ -i\,e^{\pi i x+\pi^2/\log q}\left(q^{\frac12}-q^{-\frac12}\right)^{-\frac{2\pi i}{\log q}}\;\text{〃} & \left(0\le \mathrm{Im}\,x<\frac{2\pi}{\log q}\right)\end{cases}$$

$$S_{\mathbb{Z}}(x) \xmapsto{q} S_{\mathbb{Z}}^{q}(x)$$

sin

$$S_{\mathbb{Z}[\tau]}(x) \quad /\!/\!\wr$$

$$\tau=-\frac{\log q}{2\pi i}$$

$$S_{\mathbb{Z}}^{q}\left(\frac{x}{\tau}\right)=\left(\frac{q^{-\frac{1}{2\tau^2}-\frac{1}{4\tau}}\left(q^{\frac12}-q^{-\frac12}\right)^{1+\frac1\tau}([\infty]_q!)^2}{\prod_{n=1}^{\infty}(1-q^{-n})^2}\right)q^{x(x-\tau-1)/2\tau}\times S_{\mathbb{Z}[\tau]}(x)$$

$$S_{\mathbb{Z}[\tau]}(x)=\prod_{m,n\in\mathbb{Z}}(m+n\tau-x)$$

$$\prod_{(m,n)\neq(0,0)}\frac{|mz+n|}{\sqrt{y}}=2\pi\left(y^6|\Delta(z)|\right)^{\frac{1}{12}}$$

$$z=x+iy \quad \underline{y>0} \qquad \Delta(z)=q^{-1}\prod_{n=1}^{\infty}(1-q^{-n})^{24} \qquad q=e^{-2\pi i z}$$

2003年06月24日　15時44分

— の場合

$$\xi_q(s,x) = \xi_q(s,1-x) + \underbrace{e^{-s\pi i}}_{\| \atop 1 - s\pi i + \frac{s^2}{2}(\pi i)^2 + O(s^3)} \xi_q(s,x)$$

$$\xi_q(s,x) = \frac{1}{s}\frac{2}{\log q} + \cdots$$

$$\frac{\cos \sin^{-s-1}}{\cos^2 \sin^{-s-2}} \qquad \frac{\sin^{-s}}{} \qquad \cos^2 = 1 - \sin^2$$

$$\xi_q(s,x) = \sum_{n\in\mathbb{Z}} [n-x]_q^{-s} \quad \Rightarrow \quad \xi_q(2,x) : \wp\text{-ft.}$$

Weierstrass の

$$\Downarrow$$

$$\frac{\partial^2}{\partial x^2}\xi_q(s,x) = \left(\frac{s\log q}{2}\right)^2 \xi_q(s,x) + s(s+1)\left(\frac{\log q}{q^{\frac12}-q^{-\frac12}}\right)^2 \xi_q(s+2,x)$$

$s=0$ で hol

$$\prod [n-x]_q$$

‖ ずれる

$$\prod [n-x]_q$$

$$\frac{\partial^2}{\partial x^2} \underbrace{\operatorname*{Res}_{s=0} \frac{\xi_q(s,x)}{s^2}}_{-\log \prod_{n\in\mathbb{Z}} [n-x]_q} = \left(\frac{\log q}{2}\right)^2 \underbrace{\operatorname*{Res}_{s=0} \xi_q(s,x)}_{= \frac{4}{\log q}} + \left(\frac{\log q}{q^{\frac12}-q^{-\frac12}}\right)^2 \xi_q(2,x)$$

ϑ-theta

RH　q のべき Σ

2003年06月25日　13時37分

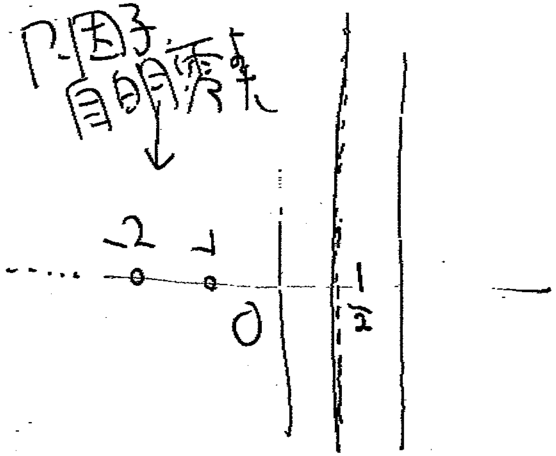
$$\left(\begin{array}{c} \text{Riemann面} \quad g \geq 2 \\ \shortparallel \\ \Gamma \backslash H \end{array} \right) \qquad \begin{array}{c} SL_2(\mathbb{Z}) \backslash H \\ \shortparallel \\ \Gamma \end{array}$$

Selberg zeta ft.

$$Z_\Gamma(s) = \prod_{n=0}^{\infty} \prod_{\substack{P:\text{素元} \\ (\text{双曲的})}} \left(1 - N(P)^{-s-n}\right) \qquad (\mathrm{Re}\, s > 1)$$

$$P \sim \begin{pmatrix} \alpha(P) & 0 \\ 0 & \beta(P) \end{pmatrix}$$

$$N(P) = \max(|\alpha(P)|, |\beta(P)|)$$



Γ因子 $\det(\Delta_\Gamma + s(1-s))$

$\Gamma_1 \Gamma_2$

RHのanalogueが成立

∵ 位数が $\underline{\underline{2}}$

2003年06月25日　13時45分

(Riemann面 $g \geq 2$ $= \Gamma\backslash H$) $SL_2(\mathbb{Z})\backslash H = \Gamma$

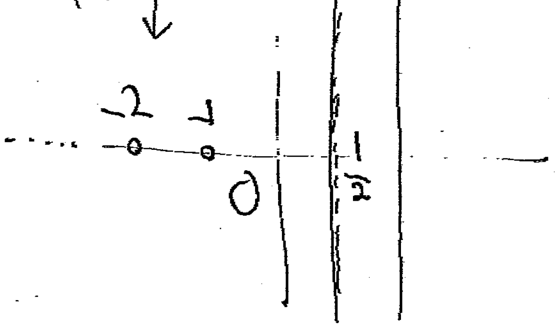
Selberg zeta ft.

$$Z_\Gamma(s) = \prod_{n=0}^{\infty} \underbrace{\prod_{P\ \text{素元}\ (\text{双曲的})} \left(1 - N(P)^{-s-n}\right)}_{=\ \zeta_\Gamma(s+n)} \qquad (\mathrm{Re}\, s > 1)$$

$$P \sim \begin{pmatrix} \alpha(P) & 0 \\ 0 & \beta(P) \end{pmatrix}$$

$$N(P) = \max\left(|\alpha(P)|^2, |\beta(P)|^2\right)$$

Γ因子 自明零点



Γ因子 $\det(\Delta_\Gamma + s(1-s))$

$\Gamma_1 \Gamma_2$

RH の analogue が成立

∵ 位数が 2

2003年06月25日　13時57分

$\zeta(s) = \prod (1-p^{-s})^{-1}$ (Re $s>1$)

p: 素数

$\hat{\zeta}_l(s) \cdot \hat{\zeta}_l(l+1-s) = 1$

位数 1

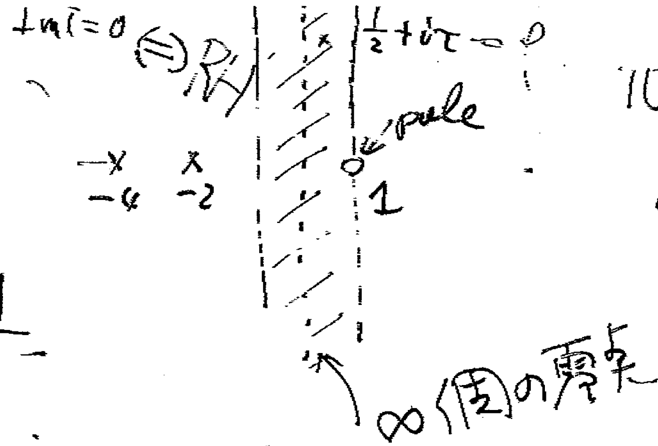
$l = 1, 2, 3 \ldots$

$$\zeta_l(s) := \prod_{n=0}^{\infty} \zeta(s+nl) \quad (\mathrm{Re}\, s > 1)$$

$$\zeta_l(s+l) = \zeta(s)^{-1} \zeta_l(s) \cdots$$ 解析接続できる。

位数 2

Γ-factor ($\Gamma_1\ \Gamma_2$ を用いてかける) + "$\prod_{\mathrm{Im}\,\rho > 0} \sin \frac{2\pi}{l}(\rho - x)$"



$\pi(x) = \#\{p : \text{prime},\ p \le x\}$

$\sim \frac{x}{\log x}$

$\hat{\zeta}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$

$\hat{\zeta}(1-s) = \hat{\zeta}(s)$

$\zeta(\rho) = 0$

non-trivial

2003年06月25日　14時12分

"$\prod \sin\alpha(\rho - x)$" $(\alpha > 0)$ を考えたい。

もし、$L_\alpha(s,x)$ が $s=0$ で mero

$$\prod_{\mathrm{Im}\rho>0} \sin\alpha(\rho - x) = \exp\left(-\operatorname*{Res}_{s=0} \frac{L_\alpha(s,x)}{s^2}\right)$$

$$L_\alpha(s,x) := \sum_{\mathrm{Im}\rho>0} \{\sin\alpha(\rho - x)\}^{-s} \qquad 0 \leq \mathrm{Re}(x) < \frac{2\pi}{\alpha}$$

$$= \sum_{\mathrm{Im}\rho>0} \left\{\frac{e^{i\alpha(\rho-x)} - e^{-i\alpha(\rho-x)}}{2i}\right\}^{-s} = (-2i)^s \sum_{\mathrm{Im}\rho>0} e^{i\alpha(\rho-x)s}\left(1 - e^{2i\alpha(\rho-x)}\right)^{-s}$$

$$= e^{s(-i\alpha x + \log(-2i))} \sum_{\mathrm{Im}\rho>0} e^{i\alpha\rho s} \sum_{n=0}^{\infty} \binom{-s}{n} (-1)^n e^{2ni\alpha(\rho - x)}$$

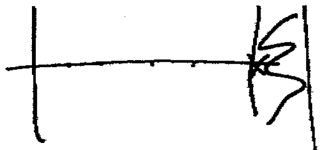
∘ Cramér (1919)

$$V(w) = \sum_{\mathrm{Im}\rho>0} e^{\rho w} \quad (\mathrm{Im}(w) > 0) \Rightarrow V(w) - \frac{1}{2\pi i}\left(\frac{\log w}{1 - e^{-w}} + \frac{\gamma + \log 2\pi - \frac{\pi i}{2}}{w}\right)$$

single valued, $w=0$ で hol

2003年06月25日　14時16分

$\mathfrak{a} = \{a_n\}_{n\in I}$



$\prod\limits_{n\in I} a_n := \exp\left(-\mathop{\mathrm{LT}}\limits_{s=0} \zeta_{\mathfrak{a}}(s)\right)$

$\zeta_{\mathfrak{a}}(s) = \sum\limits_{n\in I} a_n^{-s}$　　$\mathrm{Re}(s) \gg 1$

$\mathrm{Re}(s)>0$ で hol.

$\exists$ 有限個 mero. ft. $\{Q_m(s;\mathfrak{a})\}_{m=1}^{M}$

s.t. $P(s;\mathfrak{a}) := \zeta_{\mathfrak{a}}(s) - \sum\limits_{m=1}^{M} Q_m(s;\mathfrak{a})(\log s)^m$

が single valued ft として $\mathbb{C}$ に解析接続できる
(meroに)

$\prod\limits \sinh(a_n - x)$

を考えたい。

$\mathop{\mathrm{LT}}\limits_{s=0} \zeta_{\mathfrak{a}}(s) := \mathop{\mathrm{Res}}\limits_{s=0} \frac{P(s;\mathfrak{a})}{s^2}$

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Thm $\sum_{n\in I} a_n^{-p-1}$ が abs. conv. ような最小の整数を p とする。

$\zeta_a^{\mathrm{trig}}(s,x) := \sum_{n\in I} \sinh(a_n - x)^{-s}$ が左の仮定をみたしている。

$$\exists \prod_{n\in I} \sinh(a_n - x) = e^{\underline{f_a(x)}} \prod_{\substack{n\in I \\ k\in\mathbb{Z}}} \left(1 - \frac{x}{a_n + k\pi i}\right) \exp\left(\sum_{m=1}^{p+1} \frac{1}{m}\left(\frac{x}{a_n+k\pi i}\right)^m\right)$$

← poly. degree $\leq p$

$$L_\alpha(s,x) = e^{s(-i\alpha x + \log(-2i))} \sum_{\underline{\operatorname{Im}\rho>0}} e^{i\alpha\rho s} \sum_{n=0}^{\infty} \binom{-s}{n} (-1)^n e^{2ni\alpha(\rho - x)}$$

∘ Cramér (1919)

$$V(w) := \sum_{\operatorname{Im}\rho>0} e^{\rho w} \quad (\operatorname{Im}(w) > 0) \Rightarrow V(w) - \frac{1}{2\pi i}\left(\frac{\log w}{1-e^{-w}} + \frac{\gamma + \log 2\pi - \frac{\pi i}{2}}{w}\right)$$

single valued, $w=0$ で hol

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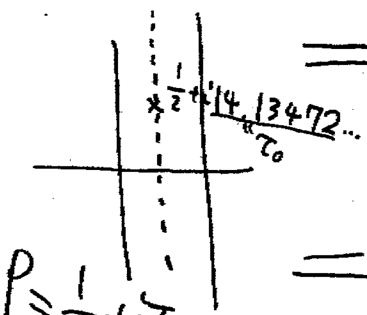
$$L_\alpha(s,x) = e^{f_\alpha(x)s} \sum_{\mathrm{Im}\rho>0} e^{i\alpha s\rho} \sum_{n=0}^{\infty} \binom{s+n-1}{n} e^{2ni\alpha(\rho-x)}$$

$$\binom{s+n-1}{n} \overset{n\geq 1}{=} \frac{s}{n} + O(s^2)$$

$$\boxed{f_\alpha(x) := -i\alpha x + \log(-2i)}$$



$$= e^{f_\alpha(x)s}\left\{V(i\alpha s) + \sum_{n=1}^{\infty} \binom{s+n-1}{n} V(i\alpha(s+2n))\, e^{-2ni\alpha x}\right\}$$

$\rho = \frac{1}{2} + i\tau$

$$= e^{f_\alpha(x)}\left\{V(i\alpha s) + s\sum_{n=1}^{\infty} \frac{V(i\alpha n)}{n} e^{-2ni\alpha x} + O(s^2)\right\}$$

$$P_\alpha(x) := \sum_{n=1}^{\infty} \frac{\Phi(2\alpha n)}{n} e^{ni\alpha(1-2x)}$$

但し、$\Phi(t) := \sum_{\mathrm{Re}\,\tau>0} e^{-\tau t}$

$$\boxed{L_\alpha(s,x) = e^{f_\alpha(x)s}\left\{e^{\frac{i\alpha s}{2}}\Phi(\alpha s) + sP_\alpha(x) + O(s^2)\right\}}$$

$V(it) = e^{\frac{it}{2}}\Phi(t) \quad (t>0)$

$$\mathcal{L}\mathrm{T}_{s=0}\, L_\alpha(s,x) = \mathcal{L}\mathrm{T}_{s=0}\, e^{s\left(f_\alpha(x)+\frac{i\alpha}{2}\right)} \cdot \Phi(\alpha s) + P_\alpha(x)$$

$$= \underbrace{F_\alpha(x)}_{} + P_\alpha(x)$$

$$\llcorner\ A_\alpha x^2 + B_\alpha x + C_\alpha ,$$

$$A_\alpha = \frac{\alpha(\gamma+\log 2\pi\alpha)}{4\pi}$$

$$B_\alpha = -i\alpha\left\{\frac{\gamma+\log 2\pi\alpha}{2\pi} - \left(\frac{\log(-2i)}{\alpha}+\frac{i}{2}\right) - \frac{7}{8}\right.$$

Thm　$S_\alpha(x) = \prod_{\mathrm{Im}\,\rho>0} \sin\alpha(\rho - x)$:　$\left(0 \le \mathrm{Re}\,x < 2\pi/\alpha\right)$

但し．$(x;q)_\zeta := \prod_{\mathrm{Im}\,\rho>0}\left(1 - x q^{-\rho}\right)$

$$S_\alpha(x) = e^{-F_\alpha(x)}\left(e^{2\alpha i x}; e^{-2\alpha i}\right)_\zeta = e^{-F_\alpha(x)} \prod_{\mathrm{Im}\,\rho>0}\left(\sin\alpha(\rho - x)\right) e^{i\alpha(\rho-x)+\log(-2i)}$$

$x \to i\infty$

Thm. $R_\alpha(u) = -\frac{\gamma + \log 2\pi\alpha}{4\pi\alpha}\left(\left(\alpha u + \frac{\pi}{2}\right)^2 - \left(\alpha u + \frac{\pi}{2}\right)\alpha - \frac{\alpha^2}{4}\right)$ とおく.

$$\prod_{\mathrm{Im}\rho>0}\left|\sin\alpha(\rho - x)\right| = e^{-R_\alpha(\mathrm{Re}(x))}\left|\prod_{\mathrm{Im}\rho>0}\sin\alpha(\rho - x)\right|$$

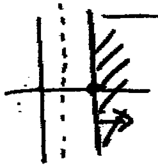
holds for $\forall\alpha > 0$

$\left(0 \geqq \mathrm{Im}x > -\frac{2\pi}{\alpha}\right)$

$\left(0 \leqq \mathrm{Re}\alpha < \frac{2\pi}{\alpha}\right)$

$\Updownarrow$

RH



$\pi(x) \sim \frac{x}{\log x}$

$\Big\downarrow$

$Li(x) + O(x^{\frac{1}{2}}\log x)$

$\overset{\shortparallel}{\int_0^x \frac{dt}{\log t}} = \lim_{\varepsilon \downarrow 0}\left(\int^{1-\varepsilon} + \int_{1+\varepsilon}^{x}\frac{dt}{\log t}\right)$

$$\tilde{L}_\alpha(s, x) = \sum_{\mathrm{Im}\rho>0}\left|\sin\alpha(\rho - x)\right|^{-s}$$

$$= e^{s\tilde{f}_\alpha(x)}\sum_{\mathrm{Im}\rho>0} e^{-s\alpha\mathrm{Im}(\rho)}\left|1 - e^{2i\alpha(\rho - x)}\right|^{-s}$$

ここで $\tilde{f}_\alpha(x) = \alpha\,\mathrm{Im}(x) + \log 2$
$= \mathrm{Re} f_\alpha(x)$

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$$\left|1-e^{2i\alpha(\rho-x)}\right|^{-s}=\left(1-e^{2i\alpha(\rho-x)}\right)^{-\frac{s}{2}}\left(1-e^{-2i\alpha(\bar\rho-\bar x)}\right)^{-s/2}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\underbrace{\binom{-\frac{s}{2}}{m}\binom{-\frac{s}{2}}{n}(-1)^{m+n}}_{\binom{\frac{s}{2}+m-1}{m}\binom{\frac{s}{2}+n-1}{n}}e^{2mi\alpha(\rho-x)}\cdot e^{-2ni\alpha(\bar\rho-\bar x)}$$

$$=1+\frac{s}{2}\left\{\sum_{m=1}^{\infty}\frac{1}{m}e^{2mi\alpha(\rho-x)}+\sum_{m=1}^{\infty}\frac{1}{m}e^{-2mi\alpha(\bar\rho-\bar x)}\right\}+O(s^2)$$

$\rho=\frac{1}{2}+i\tau$

$\boxed{RH \Leftrightarrow \mathrm{Im}\,\rho=\mathrm{Re}\,\tau=\tau}$

$$RH \Rightarrow \sum_{\mathrm{Im}\rho>0}e^{-s\alpha\,\mathrm{Im}(\rho)}=\Phi(s\alpha)$$

$$\tilde{L}_\alpha(s,x)=e^{s\tilde{f}_\alpha(x)}\left\{\Phi(s\alpha)+s\,\mathrm{Re}\,P_\alpha(x)+O(s^2)\right\}$$

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$$\underset{s=0}{\mathcal{L}T}\ \tilde{L}_\alpha(s,x) \underset{\substack{\uparrow\\ \text{計算すると}}}{=} R_e\Big(\underset{s=0}{\mathcal{L}T}\ L_\alpha(s,x)\Big) + R_\alpha\big(R_e(x)\big)$$ がわかる。

<u>Remark.</u>

$$\prod_{n\geq 0}[n+x]_q = \frac{[\infty]_q!}{\Gamma_q(x)}$$

$$\prod_{n\geq 0}\big|[n+x]_q\big| \underset{?}{\gtrless} q^{\frac{(x-\bar{x})^2}{16}}\left|\frac{[\infty]_q!}{\Gamma_q(x)}\right| = q^{\frac{(x-\bar{x})^2}{16}}\left|\prod_{n\geq 0}[n+x]_q\right|$$

$$\prod_{n\in\mathbb{Z}}\big|[n-x]_q\big| \underset{?}{\gtrless} e^{-\frac{\pi^2}{\log q}}\, q^{\frac{(x-\bar{x})^2}{8}}\left|\prod_{n\in\mathbb{Z}}[n-x]_q\right|$$

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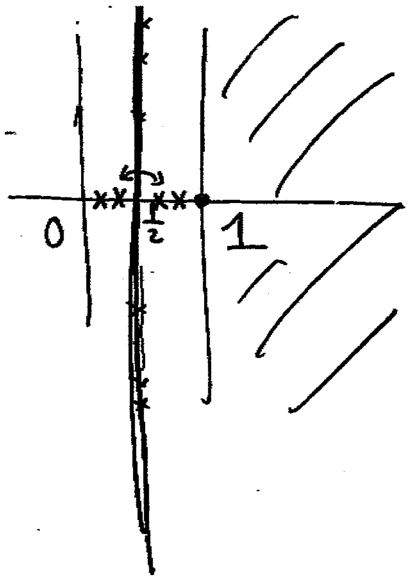

△

$\rho_n = \frac{1}{2} + i\lambda_n$

Cartier-Voros.

Remark $\Gamma_2, \Gamma_3, \dots$

$$\Gamma_2(z+1)^{-1} = (2\pi)^{z/2} e^{-\frac{(1+\gamma)z^2+z}{2}} \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^{k} e^{-z+\frac{z^2}{2k}} \right.$$

= 正規化積表示

$$\Gamma_2(x) \;?\; \prod_n^{☆} \Gamma(x+n) \quad ?$$

$$\prod_n ((a_n - x))^{b_n}$$

"$\prod^{?} \sin(a_n - x)$"

$$\boxed{\prod^{☆} \varphi(a_n - x)}$$

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$\boxed{\text{試験的}}$ $\psi(s)$: hol. $\psi(0)=0$

$\zeta_a(s)$ が $\underline{\text{asymptotically}}$ $\underline{\psi\text{-regularizable}}$

$= \sum_n a_n^{-s}$

$\Updownarrow$ def.

$\exists Z_a(s)$ $\cancel{(\#)}$ $Z_a(s) - \zeta_a(\psi(s)) = \varepsilon(s) \xrightarrow{\mathrm{Re}(s)>0}$ $s=0$ で $\underline{\text{mero}}$.

$\psi \prod_{n\in I} a_n = \exp\left(-\operatorname{Res}_{s=0} \frac{Z_a(s)}{s^2}\right)$

$\Rightarrow$ $\displaystyle \prod_{k\in\mathbb{Z}}^{s^2} \vartheta(x+kt, t) = \sqrt{\frac{\pi}{t}} \exp(-x^2/t)\, \vartheta(x,t)$

$\vartheta(x,t) = \sum_{n\in\mathbb{Z}} \exp(-n^2 t - 2nx)$

$= \vartheta_3(ix/\pi, it/\pi)$

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@ $f(x)$: given　$F(x+1) = f(x) F(x)$
$\exists F$?
Beta-extension

$\Gamma_{m+1}(x) = \prod_{k_i \geq 0} (k_1 + \cdots + k_{m+1} + x)$
$= \Gamma_m(x) \Gamma_{m+1}(x+1)$

@ $\sin_\zeta^\alpha(x) := \prod_{\substack{\mathrm{Im}\,\rho > 0 \\ m \geq 0}} \sin\alpha(\rho + im - x)$　$\prod_{IU}$

$\left|\prod \Gamma(n+x)_\zeta\right|$
$= e^{-\frac{(x-\bar{x})^2}{16}} \prod \left|\Gamma(n+x)_\zeta\right|$

$\Rightarrow \sin_\zeta^\alpha(x) = S_\alpha(x) \sin_\zeta^\alpha($
$S_\alpha(x) = \prod \sin\alpha(\rho - x)$

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$$Z_\Gamma(s) = \prod_{n=0} \prod_{P:\text{prim}} \left(1-N(P)^{-s-n}\right)$$

$\uparrow$ zeta ext.

$$\zeta_\Gamma(s) = \prod_{P} \left(1-N(P)^{-s}\right)^{-1}$$

$$N(P) = e^{\ell(P)}$$

$$\zeta_\Gamma(s)\,\zeta_\Gamma(-s) = (2\sin\pi s)^{2}$$

Selberg [illegible] ζ_2

ℓ ℓ ℓ

$$\prod_{(m,n)=1} \left(1-N(P)^{-s}\right)^{-1}$$

$P(m,n)$

$$\ell(P) = \sqrt{m^2+n^2}$$

$\sqrt{mn}$

$\boxed{m+n}$

m,n

$m = dm'$
$n = dn'$

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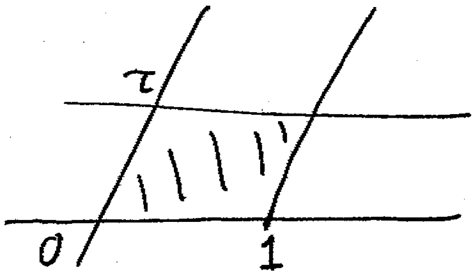
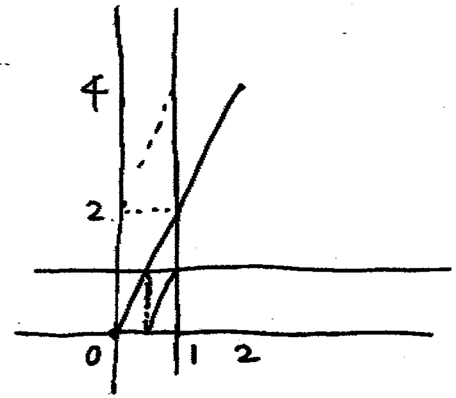
$$\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z} = \mathbb{C}/L$$



L の素元   互いに素

$$L \ni m+\tau n = (m,n)$$

$\gcd(m,n)=1$



$$\zeta_L(s) = \prod_{P\in \mathrm{Prim}(L)} (1-N(P)^{-s})^{-1}$$

norm

$N: L \to \mathbb{R}_{\geq 1}$

$(\mathrm{Prim}(L) \to \mathbb{R}_{>1})$

- $N(P^{l}) = N(P)^{l}$

$\forall \gamma \in L \quad \gamma = d(m_0+\tau n_0)$

$"(m,n)$

$d=\gcd(m,n)$, $m_0+\tau n_0 \in \mathrm{Prim}(L)$ 素元全体

$N(P) = e^{l(P)}$

l: length

l: 1次

$$l(P)=l(m+n\tau)=\begin{cases}\sqrt{mn} & (\sqrt{|mn|}) \\ m+n & (|m|+|n|) \\ \sqrt{m^2+n^2} \end{cases} \dashrightarrow \begin{array}{l} \zeta_{\sqrt{x}}(s)=\prod_{(m,n)=1}(1-e^{-\sqrt{mn}s})^{-1} \\ \zeta_{+}(s)=\prod(1-e^{-(m+n)s})^{-1} \\ \zeta_{\sqrt{+^2}}(s)=\prod(1-e^{-\sqrt{m^2+n^2}s})^{-1} \end{array}$$

① $\zeta_{\sqrt{x}}(s)$

Lemma. Euler積は $\mathrm{Re}(s)>0$ で絶対収束

$$\text{☆}=\sum_{(m,n)=1}e^{-\sqrt{mn}\,s}$$ $(s>0)$ のとき絶対収束する.

Lemma $\{a_n\}$: non-increasing seq. of non-neg. real numbers

$\sum_{n=1}^{\infty}a_n$:収束 $\Leftrightarrow$ $\sum_{n=0}^{\infty}2^n a_{2^n}$:収束

Cauchy criterion

$n>m$ $\quad n=dm+r\ (1<r<m)$ $\quad -\sqrt{mn}<-m\sqrt{d}$

$$\text{☆}<2\sum_{m=1}^{\infty}\sum_{d=1}^{\infty}me^{-m\sqrt{d}s}<{}^{\exists}C\sum_{m=1}^{\infty}\sum_{d=1}^{\infty}e^{-m\sqrt{d}s/2}=C\sum_{d=1}^{\infty}\frac{1}{e^{\sqrt{d}s/2}-1}\to\text{収}$$

$$<{}^{\exists}C'\sum_{d=1}^{\infty}e^{-\sqrt{d}s/2}$$ HW

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Re(s)>0 $$\log \xi_{\sqrt{\cdot}}(s) = -\sum_{(m,n)=1} \log\left(1-e^{-\sqrt{mn}\,s}\right)$$

$$= \sum_{(m,n)=1} \sum_{d=1}^{\infty} \frac{1}{d} e^{-\sqrt{mn}\,ds}$$

$(m,n)=l(m',n')$

$$= \sum_{m,n=1}^{\infty} (m,n)^{-1} e^{-\sqrt{mn}\,s} = \sum_{n=1}^{\infty} H(n)\, e^{-\sqrt{n}\,s}$$

いま $(n \in \mathbb{N})$ $$H(n) := \sum_{lm=n} (l,m)^{-1}$$

square-free

Lemma : $H(n)$: multiplicative i.e. $(m,n)=1 \Rightarrow H(mn)=H(m)H(n)$

$(H(1)=1)$

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∵) $(p, N) = 1$ のとき $H(p^l N) = H(p^l) H(N)$ を示せばよい。
p prime.

$n_0 = 1 < n_1 < n_2 < \cdots < n_d = N$: distinct な約数.

$$H(N) = \sum_{j=0}^{d} (n_j, N/n_j)^{-1}$$

$$H(p^l N) = \sum_{k=0}^{l} \sum_{j=0}^{d} (p^k n_j, p^{l-k} N/n_j)^{-1}$$

$(p^k, n_j) = 1$
$(p^k, N/n_j) = 1$

$$= \sum_{k=0}^{l} (p^k, p^{l-k})^{-1} \sum_{j=0}^{d} (n_j, N/n_j)^{-1} = H(p^l) H(N)$$

$$\sum_{k=1}^{\infty} H(p^{2k-1})p^{-(2k-1)s} = \frac{2}{1-p^{-1}}\left(\sum_{k=1}^{\infty}(1-p^{-k})p^{-(2k-1)s}\right) = \frac{2}{1-p^{-1}}\left\{\frac{p^{-s}}{1-p^{-2s}} + \frac{p^{-1-s}}{1-p^{-2s-1}}\right\}$$

同様に計算する.

$$\sum_{n=0}^{\infty} H(p^n)p^{-ns} = \frac{\overset{\zeta(2s)^{-1}}{1-p^{-2s}}}{(1-p^{-s})^2(1-p^{-2s-1})} \quad //$$

Rem. $H(n,x) \overset{\text{def}}{=} \sum_{lm=n} (l,m)^{-x}$ $(x \in \mathbb{R})$ も multiplicative

$$\sum_{n=1}^{\infty} H(n,x)n^{-s} = \zeta(2s)^{-1}\zeta(2s+x)\zeta(s)^2$$

・$I(n,x) = \sum_{lm=n} \mathrm{lcm}(l,m)^{-x}$

$\parallel$

$n^{-x}H(n,-x)$

$$\sum_{n=1}^{\infty} I(n,x)n^{-s} = \zeta(2s+2x)^{-1}\zeta(2s+3x)\zeta(s+x)^2$$

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Rem.

$$\zeta_{\sqrt{X}}(s) = \prod_{l=1}^{\infty} \left(1 - e^{-\sqrt{l}s}\right)^{-h(l)}$$

$$h(l) = \#\{d>0 \mid d|l \ (d, l/d)=1\}$$

multiplicative

$$= 2^{\omega(l)}$$

$\omega(l) = \#$ l の distinct な prime 因子の個数

$$\log \zeta_{\sqrt{X}}(s) = \sum_{l=1}^{\infty} h(l) \sum \frac{e^{-d\sqrt{l}s}}{d}$$

先の表示とくらべると

$$\rho(k) = \begin{cases} \frac{1}{\sqrt{k}} & : k = \text{squ} \\ 0 & : \text{otherwise} \end{cases}$$

$$H(n) = \sum_{ld^2=n} \frac{1}{d} h(l) = (\rho * h)(n)$$

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◦ f, g : multiplicative

$$(f*g)(n) = \sum_{l|n} f(l)\, g\left(\frac{n}{l}\right) \implies f*g : \text{multiplicative}.$$

$\because)$ $(m,n)=1$.

$$(f*g)(mn) = \sum_{l|mn} f(l)\, g\left(\frac{mn}{l}\right) = \sum_{\substack{l_1|m \\ l_2|n}} \underbrace{f(l_1 l_2)}_{= f(l_1)f(l_2)}\, \underbrace{g\left(\frac{m}{l_1}\frac{n}{l_2}\right)}_{= g\left(\frac{m}{l_1}\right)g\left(\frac{n}{l_2}\right)}$$

$$= (f*g)(m)(f*g)(n) \quad //$$

$$D_f(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \qquad D_{f*g}(s) = D_f(s) D_g(s)$$

(たしかめよ)

$$D_H(s) = \zeta(2s+1)\, D_h(s)$$

$$D_H(s) = \zeta(2s)^{-1}\zeta(s)^2\zeta(2s+1)$$

$$D_\rho(s) = \sum_l \frac{1}{\sqrt{l^2}} (l^2)^{-s} = \sum l^{-2s-1} = \zeta(2s+1)$$

$$D_h(s) = \zeta(2s)^{-1}\zeta(s)^2 \quad //$$

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$$\sum_{n\le x} H(n) \longrightarrow \text{どのくらい} \quad (x\to\infty) \qquad \pi(x)=\sum_{p\le x} 1 \sim$$

Tauberian theorem

Ram Murty - Kumar Murty

Non-vanishing of L-functions and Applications. Prog. in Math 157 ('97)

Prop. $F(s)=\sum_{n=1}^{\infty} a_n n^{-s}$ $\quad a_n \geqq 0$

$\mathrm{Re}(s)>\beta \Rightarrow F(s)$ は絶対収束

$\mathrm{Re}(s)\geqq\beta$ で $F(s)$: mero. $s=\beta$ で 位数 $d+1$ の pole 持つ

$$c := \frac{1}{d!}\lim_{s\to\beta}(s-\beta)^{d+1}F(s) \Rightarrow \sum_{n\le X} a_n = (c+o(1))X^{\beta}(\log X)^{d} \quad (X\to\infty)$$

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Thm $x \in \mathbb{R}$.

$$\sum_{n \le X} H(n,x) = \begin{cases} \left\{ \dfrac{\zeta\left(\frac{1-x}{2}\right)^2}{2\zeta(1-x)} + o(1) \right\} X^{\frac{1-x}{2}} & (x < -1) \\ \left\{ \dfrac{6\zeta(2+x)}{\pi^2} + o(1) \right\} X \log X & (x > -1) \\ \left\{ \dfrac{3}{2\pi^2} + o(1) \right\} X (\log X)^2 & (x = -1) \end{cases}$$

$(X \to \infty)$

$\sum_{lm=n} (l,m)^{-1}$

$l+m=n$ ……

$l^2+m^2=n$ ……

$\boxed{x = 1}$

$$\sum_{n \le X} H(n) = \left\{ \frac{6\zeta(3)}{\pi^2} + o(1) \right\} \underline{\underline{X \log X}}$$

$$\frac{\pi^2}{6} = \frac{\zeta(3)}{\zeta(2)}$$

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$$D_H(s,x) = \sum_{n=1}^{\infty} H(n,x) n^{-s} = \zeta(2s)^{-1} \zeta(2s+x) \zeta(s)^2$$

$x=-1$, $\alpha=2$, $\beta=1$

$$\mathrm{Re}(s) > \max\left(\frac{1-x}{2}, 1\right)$$

$$\zeta(s) = \frac{1}{s-1} + \gamma + \cdots$$

$\boxed{x<-1}$ $\mathrm{Re}(s) > \frac{1-x}{2}$ (収束) (>1)

$\alpha=0$, $\beta=\frac{1-x}{2}$

$$C = \lim_{s\to\frac{1-x}{2}} D_H(s,x)\left(s-\frac{1-x}{2}\right) = \left(\lim_{s\to\frac{1-x}{2}} \zeta(2s+x)\left(s-\frac{1-x}{2}\right)\right) \zeta(1-x)^{-1} \zeta\left(\frac{1-x}{2}\right)^2$$

$$= \frac{1}{2}\zeta(1-x)^{-1}\zeta\left(\frac{1-x}{2}\right)^2$$

$\boxed{x>-1}$ $\underline{\mathrm{Re}(s)>1}$

$\alpha=1$, $\beta=1$

$C = \lim_{s\to 1} D_H(s,x)(s-1)^2$

$\zeta(2+x)/\zeta($

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$$\zeta_+(s) = \prod_{(m,n)=1} (1-e^{-(m+n)s})^{-1} \quad (\mathrm{Re}(s)>0)$$

$$F(n,x) = \begin{cases} \sum_{l+m=n} (l,m)^{-x} & (n \geq 2) \\ 0 & (n=1) \end{cases}$$

$$\log \zeta_+(s) = \sum_{n=1}^{\infty} F(n,1) e^{-ns} \quad (\mathrm{Re}(s)>0)$$

$$(\mathbb{Z}/n\mathbb{Z})^\times \overset{1:1}{\longleftrightarrow} \left\{ (l,m) \in \mathbb{Z}^2 \;\middle|\; \begin{matrix} (l,m)=1 \\ l+m=n \\ l,m \geq 1 \end{matrix} \right\}$$

$$d \longmapsto (d, n-d)$$

$$\because)\ (n,d)=1$$

<u>Lemma</u> $$F(n,x) = \sum_{d|n,\, d<n} d^{-x} \varphi\left(\frac{n}{d}\right) = \sum_{d|n} d^{-x} \varphi\left(\frac{n}{d}\right) - n^{-x}$$

$$F(n,x) = \sum_{1 \leq d < n} \sum_{\substack{(l,m)=d \\ l+m=n}} (l,m)^{-x} = \sum_{\substack{d|n \\ d<n}} \sum_{\substack{l'+m'=\frac{n}{d} \\ (l',m')=1}} d^{-x}$$

$$\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^\times = n \prod_{p|n} \left(1-\frac{1}{p}\right)$$

<u>Rem.</u> $\hat{\xi}_+(s) := (1-e^{-s})\,\zeta_+(s)$

$$\log \hat{\xi}_+(s) = \sum_{n=1}^{\infty} \left\{ \sum_{d\mid n} d^{-x} \varphi\left(\frac{n}{d}\right) \right\} e^{-ns}$$

$$D_F(s,x) = \sum_{n=1}^{\infty} F(n,x)\, n^{-s} = \underline{\zeta(s+x)}\ \underbrace{\frac{\zeta(s-1)}{\zeta(s)}}_{\uparrow\ \varphi} - \zeta(s+x) \quad /\!/$$

$$\Downarrow$$

$x=1$

$$\sum_{n\leq X} F(n) = \left\{ \frac{6\zeta(3)}{\pi^2} + o(1) \right\} X^2 \quad (X\to\infty)$$

2003年06月26日　15時05分

$$\zeta_+(s) = \prod_{(m,n)=1} (1-e^{-(m+n)s})^{-1} \quad (\mathrm{Re}(s)>0)$$

$\mathrm{Re}(s)=0$??

$$\log \zeta_+(s) = \sum_{n=1}^{\infty} F(n) e^{-ns}$$

$$D_F(s,1) = \zeta(s+1)\left(\frac{\zeta(s-1)}{\zeta(s)} - 1\right) = \sum_{n=1}^{\infty} F(n) n^{-s} = \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} \log \zeta_+(t)\, dt$$

Mellin 逆変換

$$\log \zeta_+(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-t} \zeta(s+1)\left(\frac{\zeta(s-1)}{\zeta(s)} - 1\right) dt$$

$(c>2)$.

2003年06月26日　15時09分

$$\zeta_+(s) = \prod_{(m,n)=1} \left(1 - e^{-(m+n)s}\right)^{-1} \quad (\mathrm{Re}(s) > 0)$$

$\mathrm{Re}(s)=0$??

$$\log \zeta_+(s) = \sum_{n=1}^{\infty} F(n) e^{-ns}$$

$$D_F(s,1) = \zeta(s+1)\left(\frac{\zeta(s-1)}{\zeta(s)} - 1\right) = \sum_{n=1}^{\infty} F(n) n^{-s} = \Gamma(s)^{-1} \int_0^{\infty} t^{s-1} \log \zeta_+(t)\, dt$$

Mellin 逆変換

$$Z_\Gamma(s) = \prod_n \prod_p \left(1 - N(p)^{-s-n}\right)$$

$$\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} + \frac{Z_\Gamma'(1-s)}{Z_\Gamma(1-s)} = (s^2 \tanh \pi s)\, \mathrm{vol}(R)$$

R: cpt. Riem 面.

$$\log \zeta_+(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-t} \zeta(s+1)\left(\frac{\zeta(s-1)}{\zeta(s)} - 1\right) dt$$

$(c > 2)$.

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Thm. $s \downarrow 0$

$$\log \zeta_+(s) \sim \frac{6\zeta(3)}{\pi^2 s^2} - \frac{5}{6} + \sum_{m=1}^{\infty} \frac{2^{2m+1}\pi^{2m}(-1)^m}{(2m)!} \frac{\zeta(1-2m)\zeta(-1-2m)}{\zeta(2m+1)} s^{2m}.$$

$\boxed{\text{Rem}}$ $\log \zeta_{\sqrt{x}}(s)$ 　　$\log s$ が現れる

$$G_+(s) := \prod_{l,m=1}^{\infty} \left(1-e^{-(l+m)s}\right)^{-1} = \prod_{n=1}^{\infty} \left(1-e^{-ns}\right)^{-(n-1)}$$

$(\mathrm{Re}(s)>0)$

2003年06月26日　15時16分

$$\log G_+(s) = -\sum_{l,m=1} \log\left(1-e^{-(l+m)s}\right) = \sum_{l,m=1} \sum_{d=1} \frac{1}{d} e^{-(l+m)ds} \leftarrow (*)$$

$$= \sum_{j=1}^{\infty} \sum_{(l,m)=j} \sum_{d=1}^{\infty} \text{---}\,//\,\text{---} = \sum_{j=1}^{\infty} \sum_{(l,m)=1} \sum_{d=1}^{\infty} \frac{1}{d} e^{-dj(l+m)s}$$

$$= \sum_{j=1}^{\infty} \log \zeta_+(jS) \implies$$

Möbius inversion formula

$$\log \zeta_+(S) = \sum_{m=1}^{\infty} \mu(m) \log G_+(mS)$$

ちがう

$$Z_\Gamma(s) = \prod_n \prod_p \left(1-N(p)^{-s-n}\right)$$

$$\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} + \frac{Z_\Gamma'(1-s)}{Z_\Gamma(1-s)} = (s^2 \tanh \pi s)\,\mathrm{vol}(R)$$

R: cpt. Riem. 面

Prop.

$$\zeta_+(S) = \prod_{m=1}^{\infty} G_+(mS)^{\mu(m)} \quad (\mathrm{Re}(S)>0)$$

$$\mu(m) = \begin{cases} (-1)^k & m = p_1 \cdots p_k \\ 0 & \text{otherwise} \end{cases}$$

$$\mu(1) = 1$$

2003年06月26日　15時23分

(*)より $\log G_+(s) = \sum_{d=1} \frac{1}{d}\left(\frac{1}{e^{ds}-1}\right)^2$ | $\xi_+(s) = \prod_{m=1} \exp\left(\mu(m)\sum_{d=1}\frac{1}{d}\left(\frac{1}{e^{dms}-1}\right)^2\right)$

Prop. $\frac{d}{ds}\log G_+(s) = \frac{G_+'(s)}{G_+(s)}$ "Eisenstein series of wt $\underline{\underline{2}}$ & $\underline{\underline{3}}$??" で表される.

$$\boxed{q = e^{-s}} \qquad \frac{G_+'}{G_+}(s) = -\sum_{j=1}^{\infty} \frac{j^2 q^j}{1-q^j} + \underline{\sum_{j=1}^{\infty} \frac{j q^j}{1-q^j}}$$

∵ $$\frac{G_+'}{G_+}(s) = -2\sum_{d=1}^{\infty} \frac{e^{ds}}{(e^{ds}-1)^3} = -2\sum \frac{e^{-2ds}}{(1-e^{-ds})^3} = -2\sum_{d=1}^{\infty} e^{-2ds}\sum_{l=0}^{\infty}\binom{-3}{l}(-1)^l e^{-lds}$$

$$= -2\sum_{l=0}^{\infty}\binom{l+2}{l}\sum_{d=1}^{\infty} e^{-(l+2)ds}$$

2003年06月26日　15時29分

$$= -2\sum_{l=0}^{\infty}\binom{l+2}{l}\frac{e^{-(l+2)s}}{1-e^{-(l+2)s}} \qquad 2\binom{l+2}{l} = (l+2)^2-(l+2)$$

$$= -\sum_{l=0}^{\infty}\frac{(l+2)^2}{e^{(l+2)s}-1} + \sum_{l=0}^{\infty}\frac{(l+2)}{e^{(l+2)s}-1}$$

$$= -\sum_{j=2}^{\infty}\frac{j^2}{e^{js}-1} + \sum_{j=2}^{\infty}\frac{j}{e^{js}-1}$$

$$j=1 \qquad\qquad j=1$$

Rem. $E_2(z) \underset{(q=e^{2\pi iz})}{=\!=\!=} 1-24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}$ $\qquad E_2\left(-\frac{1}{z}\right) = z^2E_2(z)+\frac{6z}{\pi i}$

2003年06月26日　15時36分

Thm. $\mathrm{Re}(s)>0$

$$\frac{G_+'}{G_+}(s)+\frac{G_+'}{G_+}(-s)=-\frac{1}{2}\sum_{d=1}^{\infty}\frac{1}{\sinh^2\left(\frac{ds}{2}\right)}=e^{-s/2}\frac{Q_0'}{Q_0}\left(e^{-s/2}\right)$$

$$Q_0=Q_0(q)=\prod_{n=1}^{\infty}(1-q^{2n})=\prod_{n=1}^{\infty}(1-e^{-ns})$$

$$\left(q=e^{-s/2}\right)$$

$$\downarrow$$

$$\frac{\xi_+'}{\xi_+}(s)+\frac{\xi_+'}{\xi_+}(-s)=\sum_{m=1}^{\infty}m\mu(m)e^{-sm/2}\frac{Q_0'}{Q_0}\left(e^{-sm/2}\right)$$

2003年06月26日　15時44分

$\therefore$ $$\frac{G_+'}{G_+}(s) + \frac{G_+'}{G_+}(-s) = -2\sum_{d=1} \frac{e^{2ds} - e^{ds}}{(e^{ds}-1)^3} = -2\sum_{d=1}^{\infty} \frac{e^{ds}}{(e^{ds}-1)^2}$$

$\log \zeta_{\sqrt{2}}(s)$

$$Q_0(e^{-s/2}) = \prod_{n=1}^{\infty}(1-e^{-ns})$$

$$= -2\sum_{d=1}^{\infty} \frac{1}{(e^{\frac{ds}{2}} - e^{-\frac{ds}{2}})^2} = -\frac{1}{2}\sum_{d=1}^{\infty}\frac{1}{\sinh^2\left(\frac{ds}{2}\right)}$$

$$-\log Q_0(e^{-s/2}) = \sum_{d=1}^{\infty} \frac{1}{d(e^{ds}-1)} = \frac{1}{2}\sum_{d=1}^{\infty} \frac{\coth\left(\frac{ds}{2}\right)-1}{d}$$

$$2q^{\frac{1}{4}} Q_0^3(q) = \theta_2 \cdot \theta_3 \cdot \theta_4(q)$$

$$e^{-s/2}\frac{d}{ds}\log Q_0(e^{-s/2}) = -\frac{1}{2}\sum \frac{1}{\sin^2\left(\frac{ds}{2}\right)}$$

$L = \{m + n\tau\}$

$\log \zeta_{\sqrt{x}}(s)$

$\log \zeta_+(s)$

$$G_+^{\tau}(s) = \prod_{m,n=1}^{\infty}(1-e^{-s(m\tau+n)})^{-1}$$

$\ni i$

2003年06月27日　13時38分

$\ell(m+n\tau) = \sqrt{mn}$ $\quad$ $\max(m,n)$ …… $\zeta_{max}(s) = \prod_{(m,n)=1} (1-e^{-s\max(m,n)})$

$m+n$

$\sqrt{m^2+n^2}$

$= \prod (1-e^{-s\ell})^{2\varphi(\ell)}$

$\{\zeta_+(s)(1-e^{-s})\}^2$

$$\zeta_{\sqrt{t^2}}(s) = \prod_{(m,n)=1} (1-e^{-\sqrt{m^2+n^2}s})^{-1}$$

$\underline{\mathrm{Re}\, s>0}$

$$\tilde{\zeta}_{\sqrt{t^2}}(s) = \prod_{\substack{(m,n)=1 \\ m,n\in\mathbb{Z}}} (1-e^{-\sqrt{m^2+n^2}s})^{-1} = \{(1-e^{-s})\zeta_{\sqrt{t^2}}(s)\}^4$$

$m,n\in\mathbb{Z}_{\neq 0}$

$(m,n)=(|m|,|n|)$,

$(\pm 1,0),(0,\pm 1)=1$

$$\Downarrow \quad \log \tilde{\zeta}_{\sqrt{t^2}}(s) = \sum_{n=1}^{\infty} \tilde{G}(n) e^{-\sqrt{n}s}$$

$\tilde{G}(n) = \tilde{G}(n,1)$

$\underline{\tilde{G}(n,x)} = \sum_{\substack{\ell^2+m^2=n \\ \ell,m\in\mathbb{Z}}} (\ell,m)^{-x}$

2003年06月27日　13時39分

$\ell(m+n\tau)=\sqrt{mn}$ 　 $\max(m,n)$ …… $\xi_{\max}(s)=\prod_{(m,n)=1}(1-e^{-s\max(m,n)})$

$m+n$ 　 $=\prod(1-e^{-s\ell})^{2\varphi(\ell)}$

$\sqrt{m^2+n^2}$ 　 $\{\xi_+(s)(1-e^{-s})\}^2$

$$\xi_{\sqrt{t^2}}(s)=\prod_{(m,n)=1}(1-e^{-\sqrt{m^2+n^2}s})^{-1}$$

$\mathrm{Re}\,s>0$

$m,n\in\mathbb{Z}_{\geq 0}$

$(m,n)=(|m|,|n|)$,

$(\pm 1,0),(0,\pm 1)=1$

$(0,m)=|m|$

$(m,0)$

$$\tilde{\xi}_{\sqrt{t^2}}(s)=\prod_{\substack{(m,n)=1\\ m,n\in\mathbb{Z}}}(1-e^{-\sqrt{m^2+n^2}s})^{-1}=\{(1-e^{-s})\xi_{\sqrt{t^2}}(s)\}^4$$

$$\Downarrow$$

$$\log\tilde{\xi}_{\sqrt{t^2}}(s)=\sum_{n=1}^{\infty}\tilde{G}(n)e^{-\sqrt{n}s}$$

$\tilde{G}(n)=\tilde{G}(n,1)$

$\tilde{G}(n,x)=\sum_{\ell^2+m^2=n}(\ell,m)^{-x}$

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$$L(s) = L(s, \chi_4) := \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \cdots$$

$$= \prod_{p:\text{prime}} (1 - \chi_4(p) p^{-s})^{-1}$$

$$\chi_4(n) = \begin{cases} +1 & n \equiv 1 \mod 4 \\ -1 & n \equiv -1 \mod 4 \\ 0 & n \equiv 0 \mod 2 \end{cases}$$

$$r(n) := \#\{(l, m) \in \mathbb{Z}^2 ;\ n = l^2 + m^2\}$$

$\boxed{\text{fact}}$ $\frac{r(n)}{4}$: multiplicative.

$$\frac{1}{4} \sum_{n=1}^{\infty} \frac{r(n)}{n^s} = \zeta(s) L(s)$$

$$\tilde{G}(n, x) = \sum_{d=1}^{\infty} d^{-x} \sum_{\substack{(l,m)=d \\ l^2+m^2=n,\ l,m\in\mathbb{Z}}} 1$$

$$= \sum_{d^2 | n} d^{-x} R\left(\frac{n}{d^2}\right)$$

但し,

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$$R(n) = \#\{(l,m)\in\mathbb{Z}^2 \mid (l,m)=1,\ l^2+m^2=n\}.$$

<u>Lemma</u> $\frac{1}{4}R(n)$: multiplicative

$$R(p^m) = r(p^m) - r(p^{m-2}) \qquad m \geq 0$$

(但し、$m<2 \Rightarrow r(p^{m-2})=0$)

∵)

$$r(n) = \sum_{d=1}^{\infty} \#\{(l,m)\in\mathbb{Z}^2 \mid \underbrace{(l,m)=d,\ l^2+m^2=n}_{(l,m)=1\ \ l^2+m^2=\frac{n}{d^2}}\}$$

$$= \sum_{d=1}^{\infty} R\left(\frac{n}{d^2}\right) = \sum_{d^2\mid n} R\left(\frac{n}{d^2}\right)$$

$$r(p^m) = \sum_{j=0}^{[\frac{m}{2}]} R\left(\frac{p^m}{p^{2j}}\right) = \sum_{j=0}^{[\frac{m}{2}]} R(p^{m-2j})$$

$$r(p^{m-2}) = \sum_{j=0}^{[\frac{m-2}{2}]} R(p^{m-2-2j})$$

①をひくと

$$r(p^m) - r(p^{m-2}) = R(p^m)$$

$\tilde{r}(n) = \frac{1}{4} r(n)$

$\tilde{R}(n) = \frac{1}{4} R(n)$ とおく.

$n = \underline{p_1^{e_1}} \cdots p_l^{e_l} q_1 \cdots q_k$ $\left(p_i, q_j : \text{primes}, \ e_i \geq 2 \right.$

$$\tilde{r}(n) = \sum_{j=0}^{[\frac{e_1}{2}]} \sum_{d^2|K} \tilde{R}\left(\frac{p_1^{e_1} K}{p_1^{2j} d^2} \right)$$

但. $K = p_2^{e_2} \cdots p_l^{e_l} q_1 \cdots q_l = p_1^{-e_1} n$

$\qquad = \underbrace{p_2^{e_2}}_{} \cdot J$

$$-) \quad \tilde{r}(n p_1^{-2}) = \sum_{j=0}^{[\frac{e_1-2}{2}]} \sum_{d^2|K} \tilde{R}\left(\frac{p_1^{e_1-2} K}{p_1^{2j} d^2} \right)$$

$$\tilde{r}(n) - \tilde{r}(n p_1^{-2}) = \sum_{d^2|K} \tilde{R}\left(\frac{K}{d^2} p_1^{e_1} \right)$$

$\qquad \| $

$\tilde{r}(p_1^{e_1} K) - \tilde{r}(p_1^{e_1-2} K)$

$\tilde{r}$: multi $\longrightarrow \|$

$(\tilde{r}(p_1^{e_1}) - \tilde{r}(p_1^{e_1-2})) \tilde{r}(K)$

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$$\frac{(\tilde r(p_1^{e_1})-\tilde r(p_1^{e_1-2}))(\tilde r(K)-\tilde r(p_2^{-2}K))}{(r(p_2^{e_2})-r(p_2^{e_2-2}))\tilde r(J)}=\sum_{d^2|J}\tilde R\left(\frac{J}{d^2}p_1^{e_1}p_2^{e_2}\right)$$

Thm. $x\in\mathbb{R}$. $D_{\tilde G}(s,x)=\sum_{n=1}^{\infty}\tilde G(n,x)n^{-s}$ $\quad Re(s)>\max\left(\frac{1-x}{2},1\right)$

$$D_{\tilde G}(s,x)=4\zeta(2s)^{-1}\zeta(2s+x)\zeta(s)L(s)$$

$\because$) $g_x(m)=\begin{cases}m^{-\frac{x}{2}} & m\text{ squre}\\ 0 & \text{otherwise}\end{cases}$ $\quad \tilde G(n,x)=\sum_{d|n}g_x(d)R\left(\frac{n}{d}\right)=(g_x*R)(n)$

$$\therefore D_{\tilde G}(s,x)=4D_{g_x}(s)D_{\tilde R}(s)$$

- $D_{g_x}(s) = \sum g_x(n) n^{-s} = \sum (m^2)^{-\frac{x}{2}} (m^2)^{-s} = \zeta(2s+x)$

- $\sum_{m=0}^{\infty} \tilde{R}(p^m) p^{-sm} = \sum_{m=0}^{\infty} \{\tilde{r}(p^m) - \tilde{r}(p^{m-2})\} p^{-ms}$

$$= \sum_{m=0}^{\infty} \tilde{r}(p^m) p^{-ms} - \sum_{m=0}^{\infty} \tilde{r}(p^m) p^{-(m+2)s}$$

$$= (1-p^{-2s}) \sum \tilde{r}(p^n) p^{-ms}$$

$$D_{\tilde{R}}(s) = \prod_{p:\mathrm{prime}} \left\{ \sum_{m=0}^{\infty} \tilde{R}(p^n) p^{-sm} \right\} = \prod_p (1-p^{-2s}) \times \prod_p \left\{ \sum \tilde{r}(p^m) p^{-ms} \right\}$$

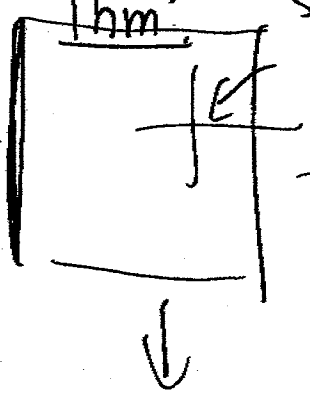
$$= \zeta(2s)^{-1} \zeta(s) L(s) \,/\!/$$

2003年06月27日　14時16分

Cor. $\underbrace{D_{\tilde{G}}(s,x)\zeta(s) = 4D_H(s,x)L(s)}_{x=1}$ $\left((\tilde{G} * 1)(n) = 4(H * \chi_4)(n) \right)$

Cor. $$\sum_{n \le X} \tilde{G}(n) = \left\{ \frac{6\zeta(3)}{\pi^2} + o(1) \right\} X \quad (X \to \infty)$$

Thm.



$s \to 0$ $\log \tilde{\zeta}_{\mathbb{H}^2}(s) \sim -\frac{\pi}{4} \underline{\underline{\log s}} + T(s)$

$$T(s) = \frac{3\zeta(3)}{\pi^2 s^2} + \frac{\pi}{4} + \log\left(\Gamma\left(\frac{1}{4}\right)^2 2^{-\frac{25}{16}} \pi^{-\frac{17}{16}}\right) + \sum_{m=1}^{\infty} \frac{\zeta(3-4m)\zeta(1-2m)\zeta'(-4m+2)^{-1}L'(1-2m)}{2(4m-2)!} s^{4m-2}$$

$$\log \zeta_{\sqrt{X}}(s), \ \log \zeta_+(s) \qquad + \sum_{m=1}^{\infty} \frac{\zeta(1-4m)L(-2m)\zeta'(-4m)^{-1}\zeta'(-2m)}{2(4m)!} s^{4m}$$

2003年06月27日　14時24分

$$\tilde{\zeta}_{\sqrt{t^2}}(s) = \prod_{\mathbb{P}:\text{prime}} \left(1 - e^{-|\mathbb{P}|s}\right)^{-1}$$

$\mathbb{P} = (p_0, p_1 \cdots p_r)$

$\gcd \mathfrak{m} = \gcd(|m_0|, |m_1| \cdots |m_r|)$

$\gcd \mathbb{P} = 1$　　$(r=1$, 生のは$)$

$$\log \tilde{\zeta}_{\sqrt{t^2}}(s) = \sum_{\mathbb{P}} \sum_{d=1}^{\infty} \frac{1}{d} e^{-|\mathbb{P}|sd}$$

$$\varphi(s) := \sum_{\mathfrak{m}} e^{-|\mathfrak{m}|s}$$

$$\log G(s) = \sum_{d=1}^{\infty} \frac{1}{d} \varphi(ds),$$

$$G(s) = \prod_{\mathfrak{m} \in \mathbb{Z}^{r+1}} \left(1 - e^{-|\mathfrak{m}|s}\right)^{-1}$$

- $\log G(s) = \sum_{\mathfrak{m}} \sum_{d=1}^{\infty} \frac{1}{d} e^{-|\mathfrak{m}|sd} = \sum_{l \geq 1} \log \tilde{\zeta}_{\sqrt{t^2}}(sl)$

- $\log \tilde{\zeta}_{\sqrt{t^2}}(s) = \sum_{m \geq 1} \mu(m) \log G(ms)$

2003年06月27日　14時34分

Poisson's summation formula.

$$\sum_m f(m) = \sum_m \hat{f}(m)$$

$$\hat{f}(y) = \int_{\mathbb{R}^{r+1}} f(x) e^{-ix\cdot y}\, dx$$

$$\left(e^{-|x|s}\right)^{\wedge}(y) = \frac{s}{\left(s^2+|y|^2\right)^{\frac{r+2}{2}}} \times 2^{r+1}\pi^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)$$

Res>0

$$\int_{-\infty}^{\infty} e^{-s|x|} e^{-ixy}\, dx$$

$$= \int_0^{\infty} e^{-sx-ixy}\, dx + \int_0^{\infty} e^{-sx+ixy}\, dx$$

$$= \frac{-1}{-s-iy} + \frac{-1}{-s+iy}$$

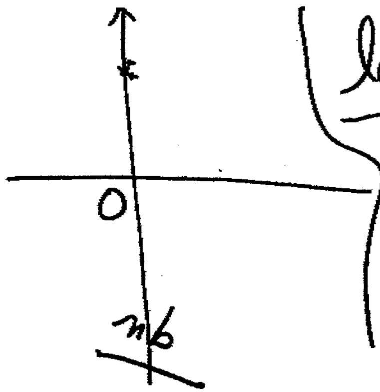
$$= \frac{2s}{s^2+y^2}$$

$r=0$

$$\varphi(s) = \sum_m e^{-|m|s} - sC\sum_m \frac{1}{\left(s^2+|m|^2\right)^{\frac{r+2}{2}}}$$

$$\log\zeta(s) = C\sum_{d=1}^{\infty} \frac{1}{d}(ds)\sum_m \frac{1}{\left((ds)^2+|m|^2\right)^{\frac{r+2}{2}}}$$

$$s = \frac{\pm i\,|m|}{d}$$



$\log \tilde{\zeta}_{\sqrt{-2}}(s)$

$$= \sum_{l \geq 1} \mu(l) \sum \frac{1}{d} \varphi(dls)$$

$$= \sum_{n=1}^{\infty} \underbrace{\left\{ \sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{1}{d} \right\}}_{= \frac{(1-p_1)\cdots(1-p_r)}{n}} \varphi(ns)$$

$n = p_1^{e_1} \cdots p_r^{e_r}$

$\mathbb{P} = (p_0, p_1 \cdots p_r)$

$\gcd m = \gcd(|m_0|, |m_1| \cdots |m_r|)$

$\gcd \mathbb{P} = 1$ 　　($r=1$, 空のは)

$$\varphi(s) := \sum_{m} e^{-|m|s}$$

$$\log G(s) = \sum_{d=1}^{\infty} \frac{1}{d} \varphi(ds)$$

$$G(s) = \prod_{m \in \mathbb{Z}^{r+1}} \left(1 - e^{-|m|s}\right)^{-1}$$

- $\log G(s) = \sum_{m} \sum_{d=1}^{\infty} \frac{1}{d} e^{-|m|sd} = \sum_{l \geq 1} \log \tilde{\zeta}_{\sqrt{-2}}(sl)$
- $\log \tilde{\zeta}_{\sqrt{-2}}(s) = \sum_{l \geq 1} \mu(l) \log G(ls) \neq \sum$

2003年06月27日　14時48分

$$\rho : \mathbb{Z}\oplus\tau\mathbb{Z} \xrightarrow[\text{rep.}]{} U(1)$$

$$\rho(1)=\alpha,\ \rho(\tau)=\beta$$

$$\rho(l+\tau m)=\alpha^l\beta^m$$

$$\zeta_L(s,\rho)=\prod_{(l,m)=1}\Big(1-\alpha^l\beta^m\,\overbrace{e^{-l(l+\tau m)s}}^{N(l+\tau m)^{-s}}\Big)^{-1}$$

H.J.S Smith. 1875/76

$$\det\big(\Delta(-1)_{1\le k,l\le N}\big)=\varphi(1)\varphi(2)\cdots\varphi(N)$$

$$\Delta_{kl}(x)=(k,l)^{-x}$$

$$\Delta(x)=\big(\Delta_{kl}(x)\big)_{k,l=1,2}$$

$$\operatorname{tr}\Delta(x)=\zeta(x)$$

$$\begin{pmatrix} & & \\ & & \end{pmatrix}\quad l+m=n$$

$$\begin{pmatrix} & & \\ & & \end{pmatrix}\quad lm=n$$

$$\begin{pmatrix} & & \\ & & \end{pmatrix}\quad l^2+m^2=n$$

2003年06月27日　14時57分

$$\mathcal{S}_r(x) = e^{\frac{x^{r-1}}{r-1}} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} P_r\left(\frac{x}{n}\right)^{n^{r-1}} \qquad P_r(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots \frac{u^r}{r}\right)$$

← (basic)

多重三角、黒川

$$\left(\mathcal{C}_r(x) = \cdots \right)$$

normalized 多重三角

$$\Gamma_m(x) = \prod_{k_1, \dots, k_m \geq 0} (k_1 + \cdots + k_m + x)$$

$$\vdots$$

$$\Gamma_{m+1}(x+1) = \Gamma_m(x)\Gamma_{m+1}(x)$$

$$\mathcal{S}_m(x) = \Gamma_m(x)\Gamma_m(m-x)^{(-1)^{m-1}}$$

$$\zeta(3) = \frac{8\pi^2}{7}\log\left(\mathcal{S}_3\left(\frac{1}{2}\right)^{-1} 2^{\frac{1}{4}}\right)$$

極値

2003年06月27日　15時04分

$$G_m(x)^{-1} := \prod_{n=1}^{\infty} ((n+x))^{n^m} \Rightarrow \mathcal{S}_{2m}(x) = G_{2m-1}(x)\, G_{2m-1}(-x)^{-1}$$

$$\mathcal{S}_{2m+1}(x) = \exp\left((-1)^m \frac{\zeta(2m+1)\,(2m)!}{2^{2m}\pi^{2m}}\right) G_{2m}(x)^{-1} G_{2m}(-x)$$

$\exists$

$$\circledcirc \prod_{n=1}^{\infty} [[n+x]]_q^{n^m} \left(= \prod_{n=1}^{\infty} (([n+x]_q))^{n^m} \right) \qquad [x] = \frac{q^{\frac{x}{2}} - q^{-\frac{x}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

$$\Downarrow$$

$$\mathbb{G}^q_m(x)^{-1}$$

$$\wr\wr$$

$\frac{\log q}{B_k}$

$$\text{|||} \prod_{n=1}^{\infty} (1 - q^{n+x})^{n^m}$$

$$\mathbb{G}^q_m(x+1) = \left\{ \prod_{k=0}^{m-1} \mathbb{G}^q_k(x)^{\binom{m}{k}(-1)^{m-k}} \right\} \mathbb{G}^q_m(x)$$

$\Gamma_q(x)$

$m=1$

$$\mathbb{G}^q_1(x+1) = \mathbb{G}^q_0(x)\, \mathbb{G}^q_1(x)$$

2003年06月27日　15時09分

$$\frac{G_m^{q\,\prime}}{G_m^{q}}(x) = (\log q)\left\{\frac{(-1)^{m-1}B_{m+1}}{2(m+1)} + \frac{(-1)^{m}2^{m}}{(\log q)^{m+1}(m+1)}\left\{\log\left(q^{-\frac{1}{2}}-q^{\frac{1}{2}}\right)+\frac{1}{2}(\log q)x\right\}^{m+1} - \sum_{j=1}^{\infty}\frac{j^{m}q^{x+j}}{1-q^{x+j}}\right\}$$

$$\sigma_k(j)=\sum_{d|j}d^{k}$$

$$q=e^{2\pi i z}$$

$$\frac{G_m^{q\,\prime}}{G_m^{q}}(0) = \frac{\log q}{4m}\left\{B_{2m}\,E_{2m}(z) - \left(\frac{\log\left(q^{-\frac{1}{2}}-q^{\frac{1}{2}}\right)^{2}}{\log q}\right)^{2m}\right\}$$

$$E_{2m}(z) = \frac{1}{2}\sum_{(c,d)=1}\frac{1}{(cz+d)^{2m}} = 1-\frac{4m}{B_{2m}}\sum_{j=1}^{\infty}\sigma_{2m-1}(j)q^{j}$$

$$\Theta_m(x;q) = \gamma_m^q(x)\,\gamma_m^q(-x)^{(-1)^m} \qquad \prod(1-q^{n+x})^{n^m} = \gamma_m^q(x)$$

$$\left(\ \delta_m^q(x) = \boxed{}\ \Theta_{m-1}(x,q)^{-1}\right.$$

$$\Theta_{2m}(x,q) = \prod_{j=1}^{\infty}\left\{1-(q^x+q^{-x})q^j+q^{2j}\right\}^{j^{2m}}$$

$$\Theta_{2m-1}(x,q) = \prod_{j=1}^{\infty}\left\{\frac{1-q^{x+j}}{1-q^{-x+j}}\right\}^{j^{2m-1}}$$

とくに

$$\Theta_0(x,q) = i\,q^{-\frac{1}{8}}\prod_{j=1}^{\infty}(1-q^j)^{-1}\cdot q^{-\frac{x}{2}} \cdot (1-q^{-x})^{-1}\cdot Ш\left(\frac{\log q}{2\pi i}x,\ \right)$$

2003年06月27日　15時21分

$$\textcircled{H}_m(x;q) = \gamma_m^q(x)\,\gamma_m^q(-x)^{(-1)^m}$$

$$\prod(1-q^{n+x})^{n^m} = \gamma_m^q(x)$$

$$\left(\underline{\delta_m^q(x)} = \boxed{}\ \textcircled{H}_{m-1}(x,q)^{-1} \right.$$

$$\textcircled{H}_{2m}(x,q) = \prod_{j=1}^{\infty}\{1-(q^x+q^{-x})q^j+q^{2j}\}^{j^{2m}}$$

$$\left(e^{-t\sqrt{x_0^2+\cdots+x_r^2}}\right)^{\wedge}(y_0,\cdots,y_r)$$

$$\textcircled{H}_{2m-1}(x,q) = \prod_{j=1}^{\infty}\left\{\frac{1-q^{x+j}}{1-q^{-x+j}}\right\}^{j^{2m-1}}$$

とくに

$$\textcircled{H}_0(x,q) = i q^{-\frac{1}{8}}\prod_{j=1}^{\infty}(1-q^j)^{-1} q^{-\frac{x}{2}} \cdot (1-q^{-x})^{-1} \cdot \mathscr{Q}^{\frac{\log q}{2\pi i}x}$$