# REPRESENTATIONS OF THE HECKE ALGEBRA ON A FAMILY OF RATIONAL FUNCTIONS

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Abstract. We give a representation of the Hecke algebra on the linear space spanned by a family of rational functions. This representation is isomorphic to the induced representation from a trivial representation. Our family of rational functions arose from the study of the quantum Knizhnik-Zamolodchikov equations.

The first author introduced a family of rational functions $\{\varphi_F\}$ in [8] for the study of the quantum Knizhnik-Zamolodchikov (QKZ) equations due to Cherednik [2]. These functions $\varphi_F$ are parametrized by the flags $F$ of indexing sets defined below, and the action of the symmetric groups on them was used to derive the QKZ equation. On the other hand, the QKZ equation is also formulated by the extended affine or double affine Hecke algebras besides the Weyl groups. See [2] and also [3],[4]. Hence it is important and indispensable to clarify the aspect from the Hecke algebras of these rational functions. Such problem was also proposed in [8]. This paper is devoted to this point: We show this family gives a representation of the Hecke algebra $H(\mathbb{S}_n)$ via Lusztig operator $T_i$. This representation is isomorphic to the induced representation from a one dimensional "trivial" representation $\mathfrak{t}$. We believe such aspects should be important for the study in these topics. Indeed, to prove the Macdonald conjecture about the integrality of the $(q,t)$-Kostka coefficients, Kirillov and the second author used our Corollary 10 effectively [5].

We treat in this paper the Hecke algebra associated with the symmetric group $\mathbb{S}_n$. The connection with the extended affine Hecke algebras will be discussed elsewhere in the context of the $q$-de Rham cohomology in the sense of Aomoto [1].

## 1. Preliminary and main theorem

Let $n$ be a positive integer and fix a sequence $\mathbf{m} = (m_0, m_1, \dots, m_d)$ of positive integers such that

$$n = m_0 \geq m_1 \geq \cdots \geq m_d > m_{d+1} = 0.$$

We say that a sequence $F = (F^{(0)}, F^{(1)}, \dots, F^{(d)})$ of subsets of the interval $[1,n] = \{1, \dots, n\}$ is a *flag of type* $\mathbf{m}$ if

$$[1,n] = F^{(0)} \supset F^{(1)} \supset \cdots \supset F^{(d)} \supset F^{(d+1)} = \emptyset \quad \text{and} \quad |F^{(i)}| = m_i.$$

We denote by $\mathcal{F}(\mathbf{m})$ the set of all flags of type $\mathbf{m}$ in $[1,n]$ and take the flag

$$B = (B^{(0)}, \dots, B^{(d)}), \quad B^{(i)} = [n - m_i + 1, n]$$

as the base point of $\mathcal{F}(\mathbf{m})$. The symmetric group $W = \mathbb{S}_n$ acts on this set $\mathcal{F}(\mathbf{m})$ transitively and its stablizer at $B$ is given by

$$W_0 = \mathbb{S}_{n_0} \times \mathbb{S}_{n_1} \times \cdots \times \mathbb{S}_{n_d}$$

where $n_i = m_i - m_{i+1}$ $(i = 0, 1, \dots, d)$, and so $n_0 + \cdots + n_d = n$. There is a bijective mapping

$$W/W_0 = \mathbb{S}_n/\mathbb{S}_{n_0} \times \mathbb{S}_{n_1} \times \cdots \mathbb{S}_{n_d} \xrightarrow{\sim} \mathcal{F}(\mathbf{m}).$$

Given a flag $F = (F^{(0)}, \dots, F^{(d)}) \in \mathcal{F}(\mathbf{m})$, for each $i = 1, \dots, d$, we define the strictly increasing mapping $a^{(i)} : [1, m_i] \to [1, n]$ by $\operatorname{Im} a^{(i)} = F^{(i)}$. Associated with the function $a^{(i)}$, we also define the strictly increasing mappings $p^{(i)} : [1, m_i] \to [1, m_{i-1}]$ so that the following diagram should be commutative.

$$\begin{array}{ccccccccccc}
 & & & \overset{p^{(1)}}{} & & & & \overset{p^{(i)}}{} & & \overset{p^{(d)}}{} & \\
[1,n] = [1,m_0] & \longleftarrow & [1,m_1] & \longleftarrow \cdots \longleftarrow & [1,m_{i-1}] & \longleftarrow & [1,m_i] & \longleftarrow \cdots \longleftarrow & [1,m_d] \\
a^{(0)} = id \downarrow & & a^{(1)} \downarrow & & a^{(i-1)} \downarrow & & a^{(i)} \downarrow & & a^{(d)} \downarrow & \\
[1,n] = F^{(0)} & \hookleftarrow & F^{(1)} & \hookleftarrow \cdots \hookleftarrow & F^{(i-1)} & \hookleftarrow & F^{(i)} & \hookleftarrow \cdots \hookleftarrow & F^{(d)} &
\end{array}$$

In other words, the sequence $\mathbf{p} = (p^{(1)}, \dots, p^{(d)})$ of strictly increasing mappings $p^{(i)} : [1, m_i] \to [1, m_{i-1}]$ is determined by the condition

$$F^{(i)} = \operatorname{Im}(p^{(1)} \circ \cdots \circ p^{(i)} : [1, m_i] \to [1, n]).$$

We introduce a family of rational functions $\varphi_F = \varphi_F(x^{(0)}; \dots; x^{(d)})$ indexed by $F \in \mathcal{F}(\mathbf{m})$ in the variables

$$x^{(0)} = (x_1^{(0)}, \dots, x_{m_0}^{(0)}),\ x^{(1)} = (x_1^{(1)}, \dots, x_{m_1}^{(1)}),\ \dots,\ x^{(d)} = (x_1^{(d)}, \dots, x_{m_d}^{(d)})$$

as

$$\varphi_F = \sum_{\sigma} \sigma \prod_{i=1}^{d} \left\{ \prod_{j=1}^{m_i} \frac{\displaystyle\prod_{k=p^{(i)}(j)+1}^{m_{i-1}} \left(1 - \frac{x_k^{(i-1)}}{x_j^{(i)}}\right)}{\displaystyle\prod_{k=p^{(i)}(j)}^{m_{i-1}} \left(1 - t\frac{x_k^{(i-1)}}{x_j^{(i)}}\right)} \prod_{1 \le j < k \le m_i} \frac{\left(1 - t\dfrac{x_k^{(i)}}{x_j^{(i)}}\right)}{\left(1 - \dfrac{x_k^{(i)}}{x_j^{(i)}}\right)} \right\}.$$

Here the summation is taken over all $\sigma = (\sigma^{(1)}, \dots, \sigma^{(d)}) \in \mathbb{S}_{m_1} \times \cdots \times \mathbb{S}_{m_d}$ and the action of $\sigma = (\sigma^{(1)}, \dots, \sigma^{(d)})$ is determined by $\sigma(x_j^{(i)}) = x_{\sigma^{(i)}(j)}^{(i)}$ for $i = 1, \dots, d$ and $j = 1, \dots, m_i$.

We denote by $V(\mathbf{m})$ the vector subspace of $\mathbb{C}(x^{(0)}, \dots, x^{(d)})$ spanned by all $\varphi_F = \varphi_F(x^{(0)}; \dots; x^{(d)})$ :

$$V(\mathbf{m}) = \sum_{F \in \mathcal{F}(\mathbf{m})} \mathbb{C}\,\varphi_F \subset \mathbb{C}(x^{(0)}, \dots, x^{(d)}).$$

Next we recall the definition of the Hecke algebra $H(\mathbb{S}_n)$ associated with the symmetric group $\mathbb{S}_n$. We denote by $H(\mathbb{S}_n)$ the $\mathbb{C}$-algebra generated by the elements

$T_1, \dots, T_{n-1}$ with fundamental relations

$$\begin{aligned} &(T_k - t)(T_k + 1) = 0 && (k = 1, \dots, n-1), \\ &T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1} && (k = 1, \dots, n-2), \\ &T_k T_l = T_l T_k && (|k - l| \geq 2), \end{aligned} \tag{1.1}$$

where $t$ is a fixed nonzero complex parameter.

We call the representation defined by $T_k \mapsto t$ the "trivial" representaion, and denote it by $\mathrm{t} = \mathrm{t}_{H(\mathbb{S}_n)}$. It is noted as a reference that the representation defined by $T_k \mapsto -1$ should be called the "parity" representation. These names are after the case of the symmetric groups.

It is known [6] that this algebra $H(\mathbb{S}_n)$ can be realized by the following *Lusztig operators* $T_k^y$ $(k = 1, \dots, n-1)$ acting on functions in the $n$ variables $y = (y_1, \dots, y_n)$ :

$$T_k^y = t + \frac{1 - t y_k / y_{k+1}}{1 - y_k / y_{k+1}} (s_k^y - 1) \quad (k = 1, \dots, n-1). \tag{1.2}$$

Here $s_k$ denotes the transposition $(k, k+1)$ in $\mathbb{S}_n$, so that $s_k^y$ interchanges the variables $y_k$ and $y_{k+1}$. The superscript $y$ here indicates the variables on which the symmetric group $\mathbb{S}_n$ acts. Similar notations will be frequently used below without further remarks.

In what follows we identify the variaibles $y = (y_1, \dots, y_n)$ with $x^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$.

**Theorem 1.** (1) *The vector space $V(\mathbf{m})$ spanned by the rational functions $\varphi_F$ $(F \in \mathcal{F}(\mathbf{m}))$ becomes a left $H(\mathbb{S}_n)$-module through the action of the Lusztig operators $T_k^y$ $(k = 1, \dots, n-1)$ in the variables $y = x^{(0)}$.*
(2) *The $H(\mathbb{S}_n)$-module $V(\mathbf{m})$ is isomorphic to the induced representation*

$$\mathrm{Ind}_{H(W_0)}^{H(W)} \mathrm{t}_{H(W_0)} = H(W) \otimes_{H(W_0)} \mathrm{t}_{H(W_0)}$$

*for the pair $(W, W_0) = (\mathbb{S}_n, \mathbb{S}_{n_0} \times \cdots \times \mathbb{S}_{n_d})$. In fact there is a $H(\mathbb{S}_n)$-isomorphism*

$$\mathrm{Ind}_{H(W_0)}^{H(W)} \mathrm{t}_{H(W_0)} = H(\mathbb{S}_n)/\mathcal{J} \xrightarrow{\sim} V(\mathbf{m})$$

*which sends the canonical generator to $\varphi_B$, where $\mathcal{J}$ is the left ideal of $H(\mathbb{S}_n)$ generated by $T_k - t$ $(k \neq n_0, n_0 + n_1, \dots, n_0 + \cdots + n_{d-1})$. The dimension of $V(\mathbf{m})$ is $n!/n_0!n_1! \cdots n_d!$.*

Moreover we will describe the action of the Hecke algebra on the rational functions $\{\varphi_F\}$, precicely. For that purpose, we prepare some notation.

With each sequence $\mathbf{p} = (p^{(1)}, \dots, p^{(d)})$ of strictly increasing mappings, we associate a mapping $\mu : [1, n] \to [0, d]$ defined by

$$\mu(k) = \max\{i \in [0, d]; k \in \mathrm{Im}(p_1 \circ \cdots \circ p_i)\} \quad \text{for} \quad k = 1, \dots, n,$$

and $\mu(k) = 0$ if $k \notin \mathrm{Im}\, p_1$. The set of all sequences $\mathbf{p}$ has a bijection to the set of all mappings $\mu$ such that $|\mu^{-1}(i)| = n_i = m_i - m_{i+1}$ for $i \in [0, d]$. As a result, we obtain a bijection between $\mathcal{F}(\mathbf{m})$ and the set of such $\mu$'s. Thus we can set

$$\varphi_\mu = \varphi_F.$$

Then we can state the following:

**Theorem 2.** *For each $l = 1, \dots, n-1$, we have*

$$\begin{aligned}
&(1) \quad T_l^y \varphi_\mu = \varphi_\mu && (\mu(l) = \mu(l+1)) \\
&(2) \quad T_l^y \varphi_\mu = t\,\varphi_{\mu s_l} && (\mu(l) < \mu(l+1)) \\
&(3) \quad T_l^y \varphi_\mu = \varphi_{\mu s_l} + (t-1)\varphi_\mu && (\mu(l) > \mu(l+1))
\end{aligned}$$

We note that the canonical generator of the induced representation coresponds to the $\varphi_{\mu_{\min}}(x^{(0)}, \dots, x^{(d)})$, where $\mu_{\min}$ is the unique non-decreasing mapping $\mu : [1,n] \to [0,d]$ such that $|\mu^{-1}(i)| = n_i$ for $i \in [0,d]$. In terms of the flag $F$, this corresponds to the case $F = B$, where $B$ is the base point of $\mathcal{F}(\mathbf{m})$ defined before.

**Corollary 3.** *The representation $V(n, n-1, \dots, 1)$ is isormorphic to the regular representation of $H(\mathbb{S}_n)$. In this case, we have*

$$T_w \varphi_{\mu_{\min}}(x^{(0)}, \dots, x^{(n-1)}) = t^{l(w)} \varphi_{\mu_{\min} w^{-1}}(x^{(0)}, \dots, x^{(n-1)}).$$

**Remark.** In [8], this rational function $\varphi_{\mu_{\min} w^{-1}}$ is denoted by $\varphi_w$, so that $T_w \varphi_e = t^{l(w)} \varphi_w$.

## 2. Proof of Theorems

First we consider the $d = 1$ case and set $m_1 = m$ and $x^{(1)} = x = (x_1, \dots, x_m)$ for brevity. For each injective mapping $p : [1,m] \to [1,n]$, we define a rational function $\psi_p(y;x)$ in $\mathbb{C}(y,x)$ to be

$$\psi_p(y;x) = \prod_{1 \le s \le m} \frac{1}{1 - t y_{p(s)}/x_s} \prod_{p(s) < r \le n} \frac{1 - y_r/x_s}{1 - t y_r/x_s} \prod_{1 \le p(r) < p(s) \le m} \frac{1 - t x_s/x_r}{1 - x_s/x_r}. \tag{2.1}$$

For each subset $K \subset [1,n]$ with $|K| = m$, we also use the symbol $\psi_K(y;x)$ to express $\psi_{p_K}(y;x)$, where $p_K : [1,m] \to [1,n]$ is a unique *strictly increasing* mapping such that $p_K([1,m]) = K$, namely $K = \{p_K(1) < \dots < p_K(m)\} \subset [1,n]$. Then we have

**Lemma 4.**

$$\begin{aligned}
&(1) \quad T_l^y(\psi_K(y;x)) = \left\{ t + (s_j^x - 1)\frac{1 - t x_j/x_{j+1}}{1 - x_j/x_{j+1}} \right\} (\psi_K(y;x)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (l, l+1 \in K; p_K(j) = l), \\
&(2) \quad T_l^y(\psi_K(y;x)) = t\,\psi_{s_l K}(y;x) \qquad\qquad (l \notin K, l+1 \in K), \\
&(3) \quad T_l^y(\psi_K(y;x)) = \psi_{s_l K}(y;x) + (t-1)\psi_K(y;x) \qquad (l \in K, l+1 \notin K), \\
&(4) \quad T_l^y(\psi_K(y;x)) = t\,\psi_K(y;x) \qquad\qquad (l, l+1 \notin K).
\end{aligned}$$

*Proof.* In the case (1), we have

$$\frac{1 - t y_l/y_{l+1}}{1 - y_l/y_{l+1}} (s_l^y - 1)(\psi_K) = \frac{(y_{l+1} - t y_l)(x_{j+1} - t x_j)}{(x_{j+1} - t y_l)(x_j - y_{l+1})} \psi_K$$

and

$$(s_j^x - 1) \left( \frac{1 - t x_j/x_{j+1}}{1 - x_j/x_{j+1}} \psi_K \right) = \frac{(y_{l+1} - t y_l)(x_{j+1} - t x_j)}{(x_{j+1} - t y_l)(x_j - y_{l+1})} \psi_K \,,$$

which leads to the formula.
The formulas for the other cases (2) – (4) are derived similarly. □

Let $\mathcal{K}$ be an arbitrary field containing $\mathbb{C}$ and regard $\mathcal{K}(x)$ as a left $H(\mathbb{S}_m)$-module through the Lusztig operators $T_j^x$ $(j=1,\dots,m-1)$, and $\mathcal{K}(y,x)$ as a left $H(\mathbb{S}_n)$-modules through $T_l^y$ $(l=1,\dots,n-1)$. Let $S^x=\sum_{\sigma\in\mathbb{S}_m}\sigma^x$ represent the *symmetrization* in the variables $x=(x_1,\dots,x_m)$. For each subset $K\subset[1,n]$ with $|K|=m$, we define a linear mapping $I_K:\mathcal{K}(x)\to\mathcal{K}(y,x)$ by

$$I_K(u)=S^x(\psi_K(y,x)u(x))=\sum_{\sigma\in\mathbb{S}_m}\psi_{p_K\sigma^{-1}}(y;x)\sigma^x(u(x))\quad(u\in\mathcal{K}(x)),$$

noting the relation $\sigma^x\psi_p=\psi_{p\sigma^{-1}}$ for $\sigma\in S_m$. Then Lemma 4 implies the following.

**Proposition 5.** *Let $K$ be a subset of $[1,n]$ with $|K|=m$, and $l=1,\dots,n-1$. For each $u\in\mathcal{K}(x)$, we have*

$$\begin{aligned}
&(1)\quad T_l^y I_K(u)=I_K(T_j^x u) && (l,l+1\in K;p_K(j)=l),\\
&(2)\quad T_l^y I_K(u)=t\,I_{s_lK}(u) && (l\notin K,l+1\in K),\\
&(3)\quad T_l^y I_K(u)=I_{s_lK}(u)+(t-1)\,I_K(u) && (l\in K,l+1\notin K),\\
&(4)\quad T_l^y I_K(u)=t\,I_K(u) && (l,l+1\notin K).
\end{aligned}$$

*Proof.* The equality

$$\begin{aligned}
&\sum_{\sigma^x}\sigma^x\left\{\left(s_j^x\cdot\frac{1-tx_j/x_{j+1}}{1-x_j/x_{j+1}}\cdot\psi_K(y;x)\right)u(x)\right\}\\
&=\sum_{\sigma^x}\sigma^x\left\{\psi_K(y;x)\cdot\left(\frac{1-tx_j/x_{j+1}}{1-x_j/x_{j+1}}\cdot(s_j^x u(x))\right)\right\}
\end{aligned}$$

shows the relation (1). The other relations directly follow from Lemma 4. □

**Remark.** The formula in case (3) is equivalent to $(T_l^y)^{-1}I_K(u)=t^{-1}I_{s_lK}(u)$.

The mapping $I_K$ can be shown to be injective. To see it we introduce an ordering on the set of all subsets $K\subset[1,n]$ with $|K|=m$.

Given two distinct subsets $K,L\subset[1,n]$ such that $|K|=|L|=m$, we take the minimal element $r$ of the symmetric difference $(K\backslash L)\cup(L\backslash K)$. We write $K\prec L$ or $L\prec K$ according as $r\in K$ or $r\in L$. In terms of the strictly increasing mappings $p_K,p_L:[1,m]\to[1,n]$ such that $K=p_K([1,m]),L=p_L([1,m])$, the condition "$K\preceq L$" is equivalent to saying that $(p_K(1),\dots,p_K(m))\preceq(p_L(1),\dots,p_L(m))$ under the lexicographic order of $[1,n]^m$.

Fix a subset $K$ of $[1,n]$ with $|K|=m$. When a rational function $f(y;x)\in\mathbb{C}(y,x)$ is regular along $x_i=y_{p_K(i)}$ for all $i=1,\dots,m$, we can consider the substitution $x_i\mapsto y_{p_K(i)}$. With the abbreviation

$$y_K=(y_{p_K(1)},\dots,y_{p_K(m)})\quad(K=\{p_K(1)<\dots<p_K(m)\}),$$

we simply denote by $f(y;y_K)$ the funtion obtained from $f(y;x)$ by this substitution.

**Lemma 6.** *Let $K$ be a subset of $[1,n]$ with $|K|=m$ and $q:[1,m]\to[1,n]$ any injective mapping with $q([1,m])=L$.*

(1) *If $L\prec K$, then $\psi_q(y;y_K)=0$.*

(2) *If $L=K$ and $q\neq p_K$, then $\psi_q(y;y_K)=0$.*

(3) *If* $L = K$ *and* $q = p_K$, *then*

$$\psi_K(y; y_K) = \frac{1}{(1-t)^m} \prod_{\substack{k<l \\ k\in K, l\notin K}} \frac{1 - y_l/y_k}{1 - t y_l/y_k}.$$

*Proof.* If $q(i) \geq p_K(i)$ for all $i = 1, \dots, m$, we have $p_K(i) \leq q(i)$ and $p_K < q(i) \quad (i < j \leq n)$ for each $i = 1, \dots, m$. Thus, under the condition $L \preceq K$, we have $p_K(i) \leq q(i) \quad (1 \leq i \leq m)$. This implies that if $L \prec K$ or $L = K$ with $p_K \neq q$ there exists a number $i$ such that $q(i) < p_K(i)$ and $1 \leq i \leq m$. It implies (1) and (2). The assertion (3) follows from a direct computation. □

Lemma 6 implies the injectivity as follows:

**Lemma 7.** $I_K : \mathcal{K}(x) \to \mathcal{K}(y, x)$ *is injective mapping for any* $K \in [1, n]$.

*Proof.* By Lemma 6, we have

$$I_K(u)(y; y_K) = \psi_K(y; y_K) u(y_K),$$

which implies the assertion. □

When $M$ is a left $H(\mathbb{S}_m)$-submodule of $\mathcal{K}(x)$, we have an isomorphism

$$M \simeq I_K(M)$$

as a left $H(\mathbb{S}_m)$-submodule by Lemma 7.

On the other hand, by Proposition 5 we have

$$T_l^y I_{[n-m+1,n]}(u) = t\, I_{[n-m+1,n]}(u) \qquad (l = 1, \dots, n-m-1),$$
$$T_{n-m+j}^y I_{[n-m+1,n]}(u) = I_{[n-m+1,n]}(T_j^x u) \qquad (j = 1, \dots, m-1),$$

which imply the vector subspace $I_{[n-m+1,n]}(\mathcal{K}(x)) \subset \mathcal{K}(y, x)$ is stable under the action of $T_l^y$ $(l \neq n-m)$. Combining these, we obtain the isomorphism

$$\mathrm{t}_{H(\mathbb{S}_{n-m})} \otimes_{\mathbb{C}} \mathcal{K}(x) \xrightarrow{\sim} I_{[n-m+1,n]}(\mathcal{K}(x))$$

as a $H(\mathbb{S}_{n-m} \times \mathbb{S}_m)$-module, via the embedding $H(\mathbb{S}_m) \hookrightarrow H(\mathbb{S}_{n-m} \times \mathbb{S}_m)$ which sends $T_j$ to $T_{n-m+j}$ for $j = 1, \dots, m-1$.

Hereafter, we regard

$$H(\mathbb{S}_{n-m} \times \mathbb{S}_m) = \mathbb{C}\langle T_1, \dots, T_{n-m-1}, T_{n-m+1}, \dots, T_{n-1} \rangle$$

as a subalgebra of $H(\mathbb{S}_n)$ and will investigate the left $H(\mathbb{S}_n)$-module structure of $I_K(\mathcal{K}(x)) \subset \mathcal{K}(y, x)$ via the Lusztig operator $T_j^y$.

The element $T_w \in H(\mathbb{S}_n)$ for a general $w \in \mathbb{S}_n$ is defined to be

$$T_w = T_{k_1} \cdots T_{k_l},$$

using a reduced expression $w = s_{k_1} \cdots s_{k_l}$. The length $l$ and the definition of $T_w$ do not depend on the choice of a reduced expression.

Let $K$ be a subset of $[1, n]$ with $|K| = m$, and $p_K : [1, m] \to [1, n]$ the strictly increasing mapping such that $p_K([1, m]) = K$. Then we denote by $w_K$ the following permutation:

$$w_K = \begin{pmatrix} 1 & \cdots & n-m & n-m+1 & \cdots & n \\ q_K(1) & \cdots & q_K(n-m) & p_K(1) & \cdots & p_K(m) \end{pmatrix},$$

where $q_K : [1, n-m] \to [1, n]$ is the strictly increasing mapping corresponding to the complement $[1,n]\backslash K$ of $K$: $q_K([1, n-m]) = [1,n]\backslash K$. Here a reduced expression of $w_K$ is given by

$$w_K = (s_{p_K(m)} s_{p_K(m)+1} \cdots s_{n-1})\,(s_{p_K(m-1)} \cdots s_{n-2}) \cdots \\ \cdots (s_{p_K(1)} s_{p_K(1)+1} \cdots s_{n-m})$$

and its length is

$$l(w_K) = \sum_{i=1}^{m} (n - m + i - p_K(i)) = mn - \binom{m}{2} - \sum_{k \in K} k.$$

For a fixed number $m$, each permutation $w \in S_n = \langle s_1, \ldots, s_{n-1} \rangle$ can be expressed uniquely in the form $w = w_K w_1 w_2$ with some $w_1 \in \langle s_1, \ldots, s_{n-m-1} \rangle$, $w_2 \in \langle s_{n-m+1}, \ldots, s_{n-1} \rangle$ and $K \subset [1,n]$ ($|K| = m$), so that $l(w) = l(w_K) + l(w_1) + l(w_2)$. It implies that $T_w = T_{w_K} T_{w_1} T_{w_2}$, and moreover that the Hecke algebra $H(\mathbb{S}_n)$ is a free right $H(\mathbb{S}_{n-m} \times \mathbb{S}_m)$-module of rank $\binom{n}{m}$ with basis $\{T_{w_K}\,;\, K \subset [1,n], |K| = m\}$.

The expression

$$T_{w_K} = \left(T_{p_K(m)} T_{p_K(m)+1} \cdots T_{n-1}\right)\,\left(T_{p_K(m-1)} \cdots T_{n-2}\right) \cdots \\ \cdots \left(T_{p_K(1)} T_{p_K(1)+1} \cdots T_{n-m}\right)$$

and the formula (2) in Proposition 5 imply the relation

$$T_{w_K} I_{[n-m+1,n]}(u) = t^{l(w_K)} I_K(u) \quad (u \in \mathcal{K}(x)).$$

If we define the vector subspace $I_m^n(M)$ of $\mathcal{K}(y,x)$ by

$$I_m^n(M) = \sum_{\substack{K \subset [1,n] \\ |K| = m}} I_K(M), \tag{2.2}$$

it is easily seen by Proposition 5 that $I_m^n(M)$ is a left $H(\mathbb{S}_n)$-submodule of $\mathcal{K}(y,x)$. Moreover we have the following:

**Proposition 8.** *There is a natural $H(\mathbb{S}_n)$-isomorphism*

$$H(\mathbb{S}_n) \otimes_{H(\mathbb{S}_{n-m} \times \mathbb{S}_m)} (\mathrm{t}_{H(\mathbb{S}_{n-m})} \otimes_{\mathbb{C}} M) \;\tilde{\to}\; I_m^n(M),$$

*which maps $T_{w_K} \otimes v \otimes u$ to $t^{l(w_K)} I_K(u) = T_{w_K} I_{[n-m+1,n]}(u)$, where $v$ is the canonical generator of the trivial module $\mathrm{t}_{H(\mathbb{S}_{n-m})}$ and $u$ is an element of $M$.*

*Proof.* It is enough to show that the sum in (2.2) is actually a direct sum.

Suppose that there exists a nontrivial relation

$$\sum_{\substack{L \subset [1,n] \\ |L| = m}} I_L(u_L) = 0 \quad (u_L \in M). \tag{2.3}$$

If we take the maximum $K$ under $\preceq$ among all $L$ such that $u_L \neq 0$, the substitution $x \mapsto y_K$ in (2.3) implies $I_K(u_K)(y; y_K) = \psi_K(y; y_K) u(y_K) = 0$, contrary to the choice of $K$. This completes the proof of Proposition 8. $\square$

In particular, our rational functions $\varphi_F(y;x)$ with $F=([1,n],K)$, which will be denoted by $\varphi_K(y;x)$ for simplicity, can be expressed as

$$\varphi_K(y;x)=I_K(1)=\sum_{p([1,m])=K}\psi_p(y;x).$$

Accordingly, the vector subspace $V(n,m)$ is expressed as

$$V(n,m)=\sum_{\substack{K\subset[1,n]\\|K|=m}}\mathbb{C}\varphi_K(y;x)\subset\mathbb{C}(y,x).$$

Then Proposition 8 implies the following, which is a special case ($d=1$) of Theorem 1.

**Corollary 9.** *There is an isomorphism*

$$V(n,m)\simeq H(\mathbb{S}_n)\otimes_{H(\mathbb{S}_{n-m}\times\mathbb{S}_m)}\mathrm{t}_{H(\mathbb{S}_{n-m}\times\mathbb{S}_m)}$$

*as a left $H(\mathbb{S}_n)$-module, whose dimension is $\binom{n}{m}$.*

The action of the Lusztig operator is given by Proposition 5 as follows, which is also a special case of Theorem 2.

**Corollary 10.**

(1) $T_l^y\varphi_K(y;x)=\varphi_{s_lK}(y;x)+(t-1)\varphi_K(y;x)\qquad(l\in K,l+1\notin K)$,

(2) $T_l^y\varphi_K(y;x)=t\,\varphi_{s_lK}(y;x)\qquad(l\notin K,l+1\in K)$,

(3) $T_l^y\varphi_K(y;x)=t\,\varphi_K(y;x)\qquad(l,l+1\notin K;l,l+1\in K)$.

As is noted in Introduction, these relations played a crucial role in the proof of the integrality of the $(q,t)$-Kostka coefficients (See Theorem 4.1 in [5]).

The remainder will be devoted to the general $d$ case.

Let $M$ be a $H(\mathbb{S}_{m_d})$-submodule of $\mathbb{C}(x^{(d)})$. Application of Proposition 8 to the variables $(x^{(d-1)},x^{(d)})$ gives a $H(\mathbb{S}_{m_{d-1}})$-submodule $I_{m_d}^{m_{d-1}}(M)$ of $\mathbb{C}(x^{(d-1)},x^{(d)})$. Similar application of Proposition 8 to the variables $(x^{(d-2)},x^{(d-1)})$ with $\mathcal{K}=\mathbb{C}(x^{(d)})$ gives a $H(\mathbb{S}_{m_{d-2}})$-submodule $I_{m_{d-1}}^{m_{d-2}}I_{m_d}^{m_{d-1}}(M)$ of $\mathbb{C}(x^{(d-2)},x^{(d-1)},x^{(d)})$. Repeating this procedure, we finally obtain a $H(\mathbb{S}_n)$-submodule ($n=m_0$)

$$I_F(M)=I_{m_1}^{m_0}\cdots I_{m_d}^{m_{d-1}}(M)\subset\mathbb{C}(x^{(0)},x^{(1)},\dots,x^{(d)}).$$

Proposition 8 guarantees there is a natural $H(\mathbb{S}_n)$-isomorphism

$$H(\mathbb{S}_n)\otimes_{H(\mathbb{S}_{n_0}\times\cdots\times\mathbb{S}_{n_d})}(\mathrm{t}_{H(\mathbb{S}_{n_0}\times\cdots\times\mathbb{S}_{n_{d-1}})}\otimes M)\xrightarrow{\sim}I_F(M),$$

where $n_i=m_i-m_{i+1}$ $(i=0,1,\dots,d)$ with $m_{d+1}=0$.

Proposition 5 implies the following:

**Proposition 11.** *For each $l=1,\dots,n-1$ and $u=u(x^{(d)})\in\mathbb{C}(x^{(d)})$, we have*

(1) $T_l^yI_\mu(u)=I_\mu(T_j^{x^{(d)}}u)$ *if* $\mu(l)=\mu(l+1)=d$, *and* $j\in[1,m-1]$ *such that* $s_lp_1\cdots p_d=p_1\cdots p_ds_j$ ,

(2) $T_l^yI_\mu(u)=t\,I_\mu(u)$ *if* $\mu(l)=\mu(l+1)<d$ ,

(3) $T_l^yI_\mu(u)=t\,I_{\mu s_l}(u)$ *if* $\mu(l)<\mu(l+1)$ ,

(4) $T_l^yI_\mu(u)=I_{\mu s_l}(u)+(t-1)I_\mu(u)$ *if* $\mu(l)>\mu(l+1)$ .

Noting $\varphi_F=I_F(1)$, the proof of our Theorems completes.

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