Soliton solutions of an isospectral flow on an energy dependent Schrödinger equation

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散乱問題

- Energy dependent Schrödinger equation

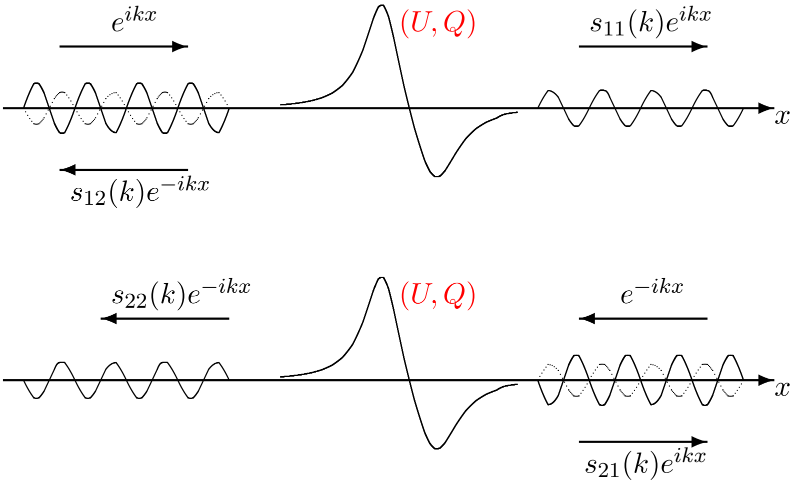
$$f'' + [k^2 - (U(x) + 2k\,Q(x))]f = 0, \quad -\infty < x < \infty,$$

$U(x)$, $Q(x)$: real-valued, decreasing at $x \to \pm\infty$.

- Scattering matrix（Heisenberg 1943〜1944)

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \qquad -\infty < k < \infty.$$

散乱行列



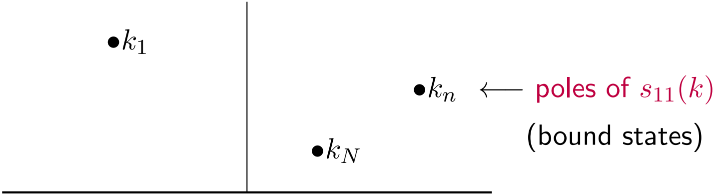
$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \qquad -\infty < k < \infty.$$

逆散乱問題 $\boxed{(U,Q)} \longleftarrow \boxed{S(k)}$

Scattering Matrix

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \qquad -\infty < k < \infty.$$

- Unitarity $\quad S(k)S(k)^* = I \implies |s_{11}(k)|^2 + |s_{21}(k)|^2 = 1$
- Symmetry $\quad s_{11}(k) = s_{22}(k)$
- Analiticity $\quad s_{11}(k)$ is analytically continued to $\boldsymbol{C}_+$



$$f_-(x,k_n) = d_n f_+(x,k_n), \qquad (f_\pm(x,k) \sim e^{\pm ikx} \text{ as } x \to \pm\infty)$$

Historical Note

Schrödinger equation $f'' + [k^2 - U(x)]f = 0$	Energy dependent equation $f'' + [k^2 - (U(x) + 2kQ(x))]f = 0$	
Marchenko, 1955	Jaulent-Jean, 1976 $(N = 0)$	
Faddeev, 1964	Sattinger-Szmigielski, 1995	$N > 0$
Deift-Trubowitz, 1979	Kamimura, 2008 $(N = 0)$	?

The inverse scattering for energy dependent equation in the presence of bound states $(N > 0)$ is still open.

無反射散乱と散乱データ

無反射散乱 $s_{12}(k) \equiv s_{21}(k) \equiv 0$ のときのポテンシャル (U, Q) を散乱行列から定めよう．

そのためのデータとして，次の定数を導入：

$$c_n := -i \operatorname{Res}_{k=k_n} s_{11}(k) \times d_n, \quad n = 1, \cdots, N.$$

定数 c_n は元祖 Schrödinger のときの規格化定数の拡張概念．

三つ組み

$$\{0, k_n, c_n\}$$

を散乱データとして用いる．．

鍵関数 $\Delta(x)$

散乱データ $\{0, k_n, c_n\}$ から，行列 B とベクトル $\boldsymbol{v}$ を

$$B := \begin{pmatrix} \overline{c_1}\,\frac{e^{(\overline{ik_1}+ik_1)x}}{\overline{ik_1}+ik_1} & \cdots & \overline{c_1}\,\frac{e^{(\overline{ik_1}+ik_N)x}}{\overline{ik_1}+ik_N} \\ \vdots & \ddots & \vdots \\ \overline{c_N}\,\frac{e^{(\overline{ik_N}+ik_1)x}}{\overline{ik_N}+ik_1} & \cdots & \overline{c_N}\,\frac{e^{(\overline{ik_N}+ik_N)x}}{\overline{ik_N}+ik_N} \end{pmatrix}, \quad \boldsymbol{v} := \begin{pmatrix} \frac{c_1}{ik_1}e^{ik_1x} \\ \vdots \\ \frac{c_N}{ik_N}e^{ik_Nx} \end{pmatrix}$$

と定め，次の関数 $\Delta(x)$ を導入する．

$$\Delta(x) := \det(I - B\,\overline{B}\,) + (e^{\overline{ik_1}x} \cdots e^{\overline{ik_N}x})(I - B\overline{B}\,)\tilde{}\,(B\boldsymbol{v} - \overline{\boldsymbol{v}}),$$

ただし，$(I - B\overline{B}\,)\tilde{}$ は $I - B\overline{B}$. の余因子行列.

主定理

Theorem 1 $\{0, k_n, c_n\}$: prescribed

- $\{0, k_n, c_n\}$ が $\exists(U, Q) \in \mathcal{S} \times \mathcal{S}$ の散乱データ

  $\Longleftrightarrow \Delta(x) \neq 0$ on $\boldsymbol{R}$.

- 上の条件の下で (U, Q) は次の逆公式で決定される

$$\begin{cases} Q(x) = -\frac{d}{dx} \arg \Delta(x), \\ U(x) + Q(x)^2 = -\frac{d^2}{dx^2} \log |\Delta(x)|. \end{cases}$$

Here $\mathcal{S}$ denotes the Schwartz class on $\boldsymbol{R}$.

The reflectionless scattering is completely controlled by $\Delta(x)$.

A data $\{0, k_n, c_n\}$ is said to be regular if $\Delta(x) \neq 0$ on $\boldsymbol{R}$.

Reduction

元祖 Schrödinger の場合は

$$c_n > 0, \ \ ik_n < 0.$$

である．このときは B は実行列で

$$\Delta(x) = (\det(I - B))^2 > 0.$$

となる．したがって， Theorem 1 より次を得る：:

Corollary $c_n > 0$, $ik_n < 0$ のとき，$Q(x) \equiv 0$,

$$U(x) = -2\frac{d^2}{dx^2}\log\det(I - B).$$

This is a well-known representation of reflectionless potentials in the Schrödinger equation via Hirota's transformation. In other words, Theorem 1 gives a (complex) generalization of reflectionless scattering theory for the Schrödinger equation.

まとめ１（無反射逆散乱理論）

	Schrödinger equation	Energy dependent equation
k_n	on imaginary axis	in $\boldsymbol{C}_+$
c_n	$c_n > 0$, any	$c_n \in \boldsymbol{C} \setminus \{0\}$, regular
(U, Q)	$(U, 0)$, single	(U, Q), system
Formula	Real	Inversion Formula

$$U(x) = -2\frac{d^2}{dx^2}\log\det(I - B), \quad \left\{ \begin{array}{l} Q(x) = \frac{d}{dx}\arg \varDelta(x) \\ U(x) + Q(x)^2 = -\frac{d^2}{dx^2}\log|\varDelta(x)| \end{array} \right.$$

非線形発展系

一般に，スペクトル問題で，束縛状態 k_n を不変に保つ非線形発展方程式をそのスペクトル問題の等スペクトル流という．エネルギー依存 Schrödinger 方程式の等スペクトル流は，1970 年代に，Jaulent and Miodek によって形式的に（Lax pair の形式的な運用で）見つけられており，変換

$$u = -4Q, \quad w = 4(U + Q^2),$$

のもとで，その最低次のものを書くと，次の形になる．

$$\begin{cases} u_t + w_x + uu_x = 0, \\ w_t - u_{xxx} + (uw)_x = 0. \end{cases}$$

これが，元祖 Schrödinger 方程式の等スペクトル流である KdV 方程式 $u_t - 6uu_x + u_{xxx} = 0$ の対応物となる．

逆散乱法 for $\begin{cases} u_t + w_x + uu_x = 0, \\ w_t - u_{xxx} + (uw)_x = 0. \end{cases}$

厳密な議論により，この非線形発展系は，確かにエネルギー依存 Schrödinger 方程式の等スペクトル流であることが証明できる．さらに，無反射を保存する．この考察と Theorem 1 を組み合わせることによって，この発展系の N-ソリトン解を求めるための次の逆散乱法が確立される．

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Cauchy data}\\ (u(x,0),w(x,0))\end{array}} \arrow[r, "\text{Thm 1}"] \arrow[d, "\text{Time Evolution}"] & \boxed{\begin{array}{c}\text{Scattering data at } t=0\\ \{0,k_n,c_n(0)\}\end{array}} \arrow[d, "\text{Time Evolution}"] \\
\boxed{\begin{array}{c}\text{Solution}\\ {\color{red}(u(x,t),w(x,t))}\end{array}} & \boxed{\begin{array}{c}\text{Scattering data at } t\\ \{0,k_n,c_n(0)e^{4ik_n^2t}\}\end{array}} \arrow[l, "\text{Thm 1}"']
\end{tikzcd}
$$

Theorem for the system $\begin{cases} u_t + w_x + uu_x = 0, \\ w_t - u_{xxx} + (uw)_x = 0. \end{cases}$

Theorem 2 :

$c_n(t) = c_n(0)e^{4ik_n^2 t}$ とするとき，関数

$$\begin{cases} u(x,t) = \frac{d}{dx} \arg \mathit{\Delta}(x,t), \\ w(x,t) = -\frac{d^2}{dx^2} \log |\mathit{\Delta}(x,t)|, \end{cases}$$

は発展系をみたす．

Here

$$B = \left(\overline{c_\ell(t)} \, \frac{e^{(\overline{ik_\ell} + ik_j)x}}{\overline{ik_\ell} + ik_j} \right), \quad \boldsymbol{v} := \left(\frac{c_\ell(t)}{ik_\ell} e^{ik_\ell x} \right)$$

$$\mathit{\Delta}(x,t) = \det(I - B\,\overline{B}\,) + (e^{\overline{ik_1}x} \cdots e^{\overline{ik_N}x})(I - B\overline{B}\,)^{\tilde{}}(B\boldsymbol{v} - \overline{\boldsymbol{v}}),$$

Example $(N=1)$ 1-soliton of $\begin{cases} u_t + w_x + uu_x = 0, \\ w_t - u_{xxx} + (uw)_x = 0. \end{cases}$

$$u(x,t) = 8b \times$$

$$\frac{(1-(\frac{a}{b})^2)\sinh\xi\sin\eta - 2\frac{a}{b}\,(\cosh\xi\cos\eta + 1)}{(1+(\frac{a}{b})^2)\cosh^2\xi + 2\left(\cosh\xi\cos\eta + \frac{a}{b}\sinh\xi\sin\eta\right) + 1 - (\frac{a}{b})^2},$$

$$w(x,t) = -16(a^2+b^2) \times$$

$$\Big\{(1-3(\tfrac{a}{b})^2)\cosh^3\xi\cos\eta + \tfrac{a}{b}(3-(\tfrac{a}{b})^2)\sinh^3\xi\sin\eta$$
$$+3(1-(\tfrac{a}{b})^2)\cosh^2\xi + 3(1+(\tfrac{a}{b})^2)\cosh\xi\cos\eta + \cos 2\eta + 3(\tfrac{a}{b})^2\Big\}\;/$$
$$\left((1+(\tfrac{a}{b})^2)\cosh^2\xi + 2\left(\cosh\xi\cos\eta + \tfrac{a}{b}\sinh\xi\sin\eta\right) + 1 - (\tfrac{a}{b})^2\right)^2,$$

$$\xi := 2b(x - x_0 + 4at),$$
$$\eta := 2a(x - x_0 + 4at) - (4(a^2+b^2)t - \Theta).$$
$$k_1 = a + bi, \quad x_0 = \frac{1}{2b}\log\frac{|c_1(0)|}{2b},$$
$$\Theta = \arg c_1(0) + 2\tan^{-1}\frac{a}{b} + \frac{a}{b}\log\frac{|c_1(0)|}{2b},\ \Theta \in (-\pi, \pi)$$

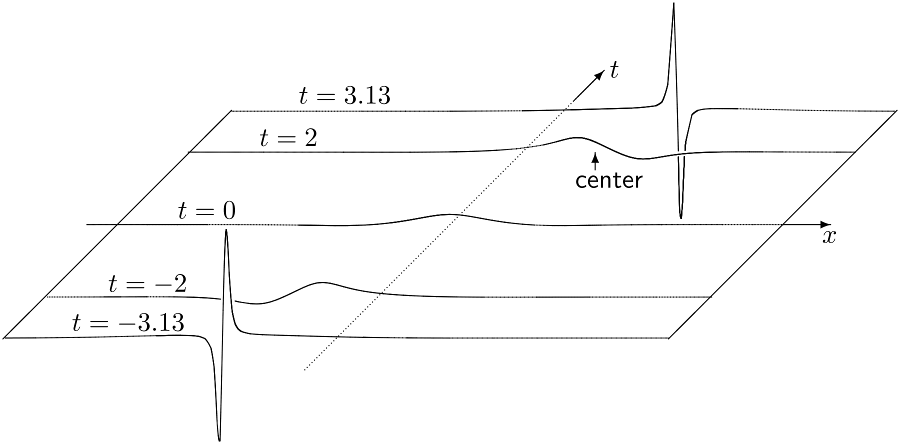


Figure: Profile of $u(x,t)$ for $-\pi < t < \pi$ in the case $a = -\frac{3}{10}$, $b = \frac{4}{10}$, $x_0 = 0$, $\Theta = 0$. The life span of the solution $(u(x,t), w(x,t))$ is finite; it exists only in the interval $t_{\min} := \frac{\Theta - \pi}{4(a^2+b^2)} < t < \frac{\Theta+\pi}{4(a^2+b^2)} =: t_{\max}$, since it has a singularity $x = x_0 - 4at$ at $t = t_{\max}, t_{\min}$ $(\Leftrightarrow \cos(\Theta - 4(a^2+b^2)) = -1)$.

まとめ　２　（非線形発展系）

$$\begin{cases} u_t + w_x + uu_x = 0, \\ w_t - u_{xxx} + (uw)_x = 0. \end{cases}$$

	Schrödinger equation	Energy dependent
Isospectral flow	single, the KdV eq $u_t - 6uu_x + u_{xxx} = 0$	system
$c_n(t)$	$c_n(0)e^{8ik_n{}^3t}$: positive	$c_n(0)e^{4ik_n{}^2t}$: complex
Life span	infinite	no longer infinite

Future Works

- To develop a complete inverse scattering theory for the energy dependent equation, involving the case with reflection.

- To develop a reflectionless inverse scattering theory for other energy dependent Schrödinger equations.

- To show the life span of N-soliton solutions of the system is necessarily finite.

- To obtain inverse scattering methods for higher order isospectral flow of the energy dependent equation.