E, E & E

(Elliptic Quantum Groups, Elliptic Weight Functions and Elliptic Stable Envelopes)

or

(Elliptic Quantum Groups, Elliptic Hypergeometric Integrals and Elliptic Cohomologies)

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- H.K, "Elliptic Weight Functions and Elliptic q-KZ Equation", J.Int.Systems 2 (2017)
- H.K, "Elliptic Stable Envelopes and Finite-dim. Reps of Elliptic Quantum Group", J.Int.Systems 3 (2018)
- H.K and K.Oshima, in preparation

Weight Functions & Elliptic q-KZ Equation

Face Type Elliptic q-KZ equation (Foda-Jimbo-Miki-Miwa-Nakayashiki '94 Felder '94)

$F(z_1,\cdots,z_n;\Pi)$: meromorphic func. valued in $V_1\otimes\cdots\otimes V_n$

$$
\begin{aligned}
&F(z_1,\cdots,q^\kappa z_i,\cdots,z_n;\Pi)\\
&=R^{(ii+1)}\left(\frac{q^{-\kappa}z_{i+1}}{z_i},\Pi q^{-2\sum_{k=1}^{i-1}h^{(k)}}\right)\cdots R^{(in)}\left(\frac{q^{-\kappa}z_n}{z_i},\Pi q^{-2\sum_{\substack{k=1\\ \neq i}}^{n-1}h^{(k)}}\right)\\
&\qquad\times\Gamma_i R^{(i1)}\left(\frac{z_1}{z_i},\Pi\right)\cdots R^{(ii-1)}\left(\frac{z_{i-1}}{z_i},\Pi q^{-2\sum_{k=1}^{i-2}h^{(k)}}\right)F(z_1,\cdots,z_i,\cdots,z_n;\Pi).
\end{aligned}
$$

where

- $R^{(ij)}(z,\Pi)\in\mathrm{End}(V_i\otimes V_j)$: elliptic dynamical R-matrix
  (⇔ face type Boltzmann weight of the SOS model)
- Π : dynamical parameter
- $\Gamma_i f(\Pi)=f(\Pi q^{2h^{(i)}})$, $\quad h^{(i)}v_\mu^{(i)}=\mathrm{wt}(v_\mu)v_\mu^{(i)}$

Cf. (Trigonometric) q-KZ equation $\cdots$ Vertex type

Smirnov '92, Frenkel-Reshetikhin '92, Jimbo-Miwa '95

Weight Functions & Elliptic q-KZ Equation

Formal Integral solutions: (Felder-Tarasov-Varchenko '97 : $\widehat{\mathfrak{sl}}_2$ case)

$$F_{\mu_1,\cdots,\mu_n}(z_1,\cdots,z_n;\Pi)=\oint_{\mathcal{C}} dt_1\cdots dt_\lambda\ \Phi(t,z)\ \omega_{\mu_1,\cdots,\mu_n}(t,z;\Pi)$$

where $z=(z_1,\cdots,z_n),\ t=(t_1,\cdots,t_\lambda)$

- $\Phi(t,z)$: phase function, symmetric in t and z, respectively
  - $b(t,z)=\Phi(q^\kappa t,z)/\Phi(t,z)$ defines
    the q-difference twisted de Rham cohomology
    $$f(t,z;\Pi)-f(q^\kappa t,z;\Pi)b(t,z)\ \sim\ 0$$ (Cf. Aomoto '90 : trig.case)
- $\omega_{\mu_1,\cdots,\mu_n}(t,z:\Pi)$: the weight function
  - $\omega(\cdots,z_{i+1},z_i,\cdots)=R\,(z_i/z_{i+1},\Pi)\,\omega(\cdots,z_i,z_{i+1},\cdots)$
  - Triangularity w.r.t. a certain specialization in t
  - The shuffle algebra structure, etc.

Cf. Trigonometric cases:

Tarasov-Varchenko '94,'97, Matsuo '93 : $\widehat{\mathfrak{sl}}_2$ case, Mimachi '96: $\widehat{\mathfrak{sl}}_N$ case

New Interests in the Weight Functions

Gorbounov-Rimanyi-Tarasov-Varchenko '13 showed in the rational case:

- the weight functions = Maulik-Okounkov's stable envelops

$$\mathrm{Stab}_{\mathfrak{C}} : H_T^*((T^*\mathcal{F}_\lambda)^A) \to H_T^*(T^*\mathcal{F}_\lambda), \qquad \mathfrak{C} : \text{chamber of } \mathrm{Lie}A, \quad T = A \times \mathbb{C}^\times$$

  s.t.

  - Triangularity w.r.t. the restriction to the fixed point classes $\{[I]\}_{I \in \mathcal{I}_\lambda}$
  - $\ldots$

- Finite dim. tensor product rep. of $Y(\mathfrak{gl}_N)$ on the Gelfand-Tsetlin basis
  $\rightsquigarrow$ geometric rep. on $H_T^*(T^*\mathcal{F}_\lambda)$

Rimanyi-Tarasov-Varchenko '13 extended to the trigonometric case

Finite dim. tensor product rep. of $U_q(\widehat{\mathfrak{gl}}_N)$ on the Gelfand-Tsetlin basis
$\rightsquigarrow$ geometric rep. on $\mathrm{K}_T(T^*\mathcal{F}_\lambda)$

Why not the elliptic case ?

Aganagic-Okounkov '16

- X: Nakajima quiver variety
- $\mathrm{Ell}_T(X)$: T-equivariant elliptic cohomology, $T = A \times \mathbb{C}^\times_{q^2}$
- $\mathcal{E}_{\mathrm{Pic}_T(X)} := \mathrm{Pic}_T(X) \otimes_{\mathbb{Z}} \mathrm{E}$
- Elliptic stable envelopes $\mathrm{Stab}_{\mathfrak{C}}$ ($\mathfrak{C}$: chamber of $\mathrm{Lie}A$)

$$\mathrm{Stab}_{\mathfrak{C}} : \begin{matrix}\text{sheaves on}\\ \mathrm{Ell}_T(X^A) \times \mathcal{E}_{\mathrm{Pic}_T(X)}\end{matrix} \to \begin{matrix}\text{sheaves on}\\ \mathrm{Ell}_T(X) \times \mathcal{E}_{\mathrm{Pic}_T(X)}\end{matrix}$$

  – Triangularity w.r.t. the restriction to the fixed point classes $\{[I]\}_{I \in \mathcal{I}_\lambda}$
  – $\cdots$

- $\mathrm{Stab}_{\mathfrak{C}}$ are the face type (i.e. dynamical) ones !

  Kähler parameter of $\mathcal{E}_{\mathrm{Pic}_T(X)} \Leftrightarrow$ dynamical parameter
  $\mathrm{Stab}^{-1}_{\mathfrak{C}'} \circ \mathrm{Stab}_{\mathfrak{C}} = R_{\mathfrak{C}'\mathfrak{C}}$: elliptic dynamical R-matrix !

We show : elliptic dynamical weight functions $\Leftrightarrow$ elliptic stable envelopes, geometric action of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ on $\mathrm{E}_T(X) = \mathrm{Ell}_T(X) \times \mathcal{E}_{\mathrm{Pic}_T(X)}$

SUSY Gauge Theories

4d, 5d, 6d (Moduli sp. of Instantons or VEV's (Higgs, Coulomb))

Nekrasov-Shatashvili Corresp. ⟺

Quantum Int. Systems

XXX,	rRS model,	Toda
XXZ,	RS model,	q-Toda
XYZ	Ruijsenaars	??
∼8VSOS,	model,	

AGT Corresp. ⇘

Jimbo-Miwa R-matrix, q-KZ eq. ⇗

Modules of Quantum Groups

$\mathcal{D}Y(\mathfrak{g})$

$U_q(\mathfrak{g})$

$U_{q,p}(\mathfrak{g})$

Nakajima ⇕

Givental et.al. ⇕

Geom.Rep. Theory ⇗

?? ⇖⇘ Quantum Diff. eq. (?)

Equiv. Cohom.

$\mathrm{H}^*_T(X)$

$\mathrm{K}_T(X)$

$\mathrm{E}_T(X)$

X: Quiver Var.

Maulik-Okounkov ⇄

Kähler parameters $\to 0$

Quantum Equiv. Cohom.

$Q\mathrm{H}^*_T(X)$

$Q\mathrm{K}_T(X)$

$Q\mathrm{E}_T(X)$??

SUSY Gauge Theories

4d, 5d, 6d: (Moduli sp. of Instantons or VEV's (Higgs, Coulomb))

Nekrasov-Shatashvili Corresp. $\Longleftrightarrow$

Quantum Int. Systems

XXX, rRS model, Toda

XXZ, RS model, q-Toda

XYZ Ruijsenaars ??

$\sim$ 8VSOS, model,

AGT Corresp. $\Leftrightarrow$

Jimbo-Miwa $\Leftrightarrow$

R-matrix, q-KZ eq.

Nakajima $\Updownarrow$

Modules of Quantum Groups

$\mathcal{D}Y(\mathfrak{g})$

$U_q(\mathfrak{g})$

$U_{q,p}(\widehat{\mathfrak{sl}}_N), U_{q,p}(\mathfrak{gl}_{N,tor})$

Givental et.al. $\Updownarrow$

Geom.Rep. Theory $\Leftrightarrow$

?? $\Leftrightarrow$ Quantum Diff. eq. (?)

Equiv. Cohom.

$\mathrm{H}^*_T(X)$

$\mathrm{K}_T(X)$

$\mathrm{E}_T(X)$

$X = T^*\mathcal{F}_\lambda, \mathrm{Hilb}$

Maulik-Okounkov $\rightleftarrows$

Kähler parameters $\to 0$

Quantum Equiv. Cohom.

$Q\mathrm{H}^*_T(X)$

$Q\mathrm{K}_T(X)$

$Q\mathrm{E}_T(X)$??

This talk : part 1

- Derivation of the elliptic weight functions of type $\widehat{\mathfrak{sl}}_N$ by using representation theory of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_N)$
- Properties of the elliptic weight functions such as triangular property, orthogonality, transition property, shuffle alg. str.
- Elliptic hypergeometric integral solution to the elliptic q-KZ eq.
- Connection to the nested Bethe ansatz for the $A^{(1)}_{N-1}$ type face model
- Tensor product rep. of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ on the Gelfand-Tsetlin basis.

This talk : part 2

- Identification of the elliptic weight functions with the elliptic stable envelopes

- Correspondence between the Gelfand-Tsetlin bases and the fixed point classes, and finite dimensional reps. of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ on $\mathrm{E}_T(X) = \mathrm{Ell}_T(X) \times \mathcal{E}_{\mathrm{Pic}_T(X)}$, $X = T^*\mathcal{F}_\lambda$

- Drinfeld coproduct Δ^D

- Triality among Hopf algebroid twistors, weight functions and stable envelopes

This talk : part 3

- Elliptic quantum toroidal algebras $U_{q,p}(\mathfrak{g}_{tor})$, $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{gl}_1$
- q-Fock reps. (semi-infinite tensor product reps.) of $U_{q,p}(\mathfrak{g}_{tor})$ and their geometric interpretation
- VO's of $U_{q,p}(\mathfrak{g})$ and deformed W-algebras $W_{p,p^*}(\bar{\mathfrak{g}})$
- VO's of $U_{q,p}(\mathfrak{gl}_{1,tor})$ and deformed affine quiver W-algebra $W_{p,p^*}(\Gamma(\widehat{A}_0))$

Elliptic Dynamical R-matrix : the $\widehat{\mathfrak{sl}}_N$ example

(Jimbo-Miwa-Okado'87)

$$R^+(z,s) = \rho^+(z)\overline{R}(z,s), \quad s = P \text{ or } P+h$$

$$\overline{R}(z,s) = \sum_{j=1}^{N} E_{jj}\otimes E_{jj} + \sum_{1\leq j<l\leq N}\Big(b(z,s_{j,l})E_{jj}\otimes E_{ll} + \bar{b}(z)E_{ll}\otimes E_{jj}$$
$$+c(z,s_{j,l})E_{jl}\otimes E_{lj} + c(z,-s_{j,l})E_{lj}\otimes E_{jl}\Big),$$

$$b(z,s) = q\frac{\theta_p(q^2q^{2s})\theta_p(q^{-2}q^{2s})\theta_p(z)}{\theta_p(q^{2s})^2\theta_p(q^2z)}, \qquad \in \mathcal{M}_{H^*}[[p]][[z,z^{-1}]]$$

$$\bar{b}(z) = q\frac{\theta_p(z)}{\theta_p(q^2z)}, \quad c(z,s) = \frac{\theta_p(q^2)\theta_p(q^{2s}z)}{\theta_p(q^{2s})\theta_p(q^2z)},$$

$\mathcal{M}_{H^*}$: field of merom. fnc. of $P, P+h$

$$\theta_p(z) = (z;p)_\infty(p/z;p)_\infty, \qquad (z;p)_\infty = \prod_{n=0}^{\infty}(1-zp^n)$$

Dynamical Yang-Baxter eq. $\quad (\Pi = q^{2P})$

$$R^{+(12)}(z_1/z_2, P+h^{(3)})R^{+(13)}(z_1,P)R^{+(23)}(z_2,P+h^{(1)})$$
$$= R^{+(23)}(z_2,P)R^{+(13)}(z_1,P+h^{(2)})R^{+(12)}(z_1/z_2,P)$$

Definition of $U_{q,p}(\widehat{\mathfrak{g}})$, $\widehat{\mathfrak{g}}$: untwisted affine Lie alg.

- $H = \bar{\mathfrak{h}} \oplus P_{\bar{\mathfrak{h}}} \oplus \mathbb{C}c, \quad H^* = \bar{\mathfrak{h}}^* \oplus Q_{\bar{\mathfrak{h}}} \oplus \mathbb{C}\Lambda_0$
- $\mathbb{F} = \mathcal{M}_{H^*}$: the field of meromorphic functions on H^*
- $U_{q,p}(\widehat{\mathfrak{g}})$ is a topological algebra over $\mathbb{F}[[p]]$ generated by $e_{j,m}, f_{j,m}$, $\alpha_{j,n}$, $K^{\pm}_{\varepsilon_j}$ $(j \in \{1, 2, \cdots, l = \mathrm{rank}\mathfrak{g}\}, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0})$, d and the central element c.

Set the elliptic currents :

$$e_j(z) = \sum_{m\in\mathbb{Z}} e_{j,m} z^{-m}, \quad f_j(z) = \sum_{m\in\mathbb{Z}} f_{j,m} z^{-m}, \quad K^{\pm}_j = K^{\pm}_{\varepsilon_j}(K^{\pm}_{\varepsilon_{j+1}})^{-1}\text{etc.},$$

$$\psi^{\pm}_j(q^{\mp\frac{c}{2}} z) = K^{\pm}_j : \exp\left\{ \pm(q - q^{-1}) \sum_{n\neq 0} \frac{\alpha_{j,n} p^{\pm n}}{1 - p^{\pm n}} z^{-n} \right\} : .$$

Set also $p^* = pq^{-2c}$, $q_j = q^{d_j}$, $D = \mathrm{diag}.(d_1, \cdots, d_l)$, $B = (b_{ij}) = DA$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

Defining Relations:

$$g(P,P+h) \in \mathcal{M}_{H^*}$$

$$g(P+h)e_j(z) = e_j(z)g(P+h), \quad g(P)e_j(z) = e_j(z)g(P - < Q_{\alpha_j}, P >),$$

$$g(P+h)f_j(z) = f_j(z)g(P+h - < Q_{\alpha_j}, P+h >), \quad g(P)f_j(z) = f_j(z)g(P),$$

$$g(P+h)K_j^{\pm} = K_j^{\pm}g(P+h - < Q_{\alpha_j}, P+h >), \quad g(P)K_j^{\pm} = K_j^{\pm}g(P - < Q_{\alpha_j}, P >),$$

$$[\alpha_{i,m}, \alpha_{j,n}] = \delta_{m+n,0}\frac{[b_{ij}m]_q[cm]_q}{m}\frac{1-p^m}{1-p^{*m}}q^{-cm} \qquad : \text{the elliptic bosons},$$

$$[\alpha_{i,m}, e_j(z)] = \frac{[b_{ij}m]_q}{m}\frac{1-p^m}{1-p^{*m}}q^{-cm}z^m e_j(z), \qquad [\alpha_{i,m}, f_j(z)] = -\frac{[b_{ij}m]_q}{m}z^m f_j(z),$$

$$z_1\frac{(q^{b_{ij}}z_2/z_1;p^*)_\infty}{(p^*q^{-b_{ij}}z_2/z_1;p^*)_\infty}e_i(z_1)e_j(z_2) = -z_2\frac{(q^{b_{ij}}z_1/z_2;p^*)_\infty}{(p^*q^{-b_{ij}}z_1/z_2;p^*)_\infty}e_j(z_2)e_i(z_1),$$

$$z_1\frac{(q^{-b_{ij}}z_2/z_1;p)_\infty}{(pq^{b_{ij}}z_2/z_1;p)_\infty}f_i(z_1)f_j(z_2) = -z_2\frac{(q^{-b_{ij}}z_1/z_2;p)_\infty}{(pq^{b_{ij}}z_1/z_2;p)_\infty}f_j(z_2)f_i(z_1),$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}}\left(\delta(q^{-c}z_1/z_2)\psi_j^-(q^{\frac{c}{2}}z_2) - \delta(q^c z_1/z_2)\psi_j^+(q^{-\frac{c}{2}}z_2)\right),$$

$+$ Serre relations

The coefficients in z_1, z_2 are well defined in the p-adic topology.

Remark 2.1

$U_{q,p}(\widehat{\mathfrak{g}})$ is a face type elliptic analogue of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld's new realization.

Remark 2.2

One can formulate the central extension of Felder's elliptic quantum group $E_{\tau,\eta}(\mathfrak{gl}_N)$ as a topological algebra $E_{q,p}(\widehat{\mathfrak{gl}}_N)$ over $\mathbb{F}[[p]]$.

Then

Theorem 2.3 (H.K '16)

$U_{q,p}(\widehat{\mathfrak{gl}}_N) \cong E_{q,p}(\widehat{\mathfrak{gl}}_N)$

Remark 2.4

In practical calculations, we take c as a real number, say $k \in \mathbb{R}$ (the level-k rep.), and also treat q, p as $|q| < 1, |p| < 1$. Then in the analytic continuation

$$f_i(z_1)f_j(z_2) = -\frac{z_2}{z_1}\frac{\theta_p(q^{-b_{ij}} z_1/z_2)}{\theta_p(q^{-b_{ij}} z_2/z_1)} f_j(z_2)f_i(z_1),$$

$$e_i(z_1)e_j(z_2) = -\frac{z_2}{z_1}\frac{\theta_{p^*}(q^{b_{ij}} z_1/z_2)}{\theta_{p^*}(q^{b_{ij}} z_2/z_1)} e_j(z_2)e_i(z_1), \quad p^* = pq^{-2k}$$

Moreover introducing r, r^* by $p = q^{2r}, p^* = q^{2r^*}$ $(r^* = r - k \in \mathbb{R}_{>0})$, we set

$$E_i(z) = e_i(z)z^{-(P_{\alpha_i}-1)/r^*}, \quad F_i(z) = f_i(z)z^{((P+h)_{\alpha_i}-1)/r}$$

Then we have

$$F_i(z_1)F_j(z_2) = \frac{[u_1 - u_2 - b_{ij}/2]}{[u_1 - u_2 + b_{i,j}/2]} F_j(z_2)F_i(z_1),$$

$$E_i(z_1)E_j(z_2) = \frac{[u_1 - u_2 + b_{ij}/2]^*}{[u_1 - u_2 - b_{i,j}/2]^*} E_j(z_2)E_i(z_1)$$

where $z_i = q^{2u_i}$ $(i = 1, 2)$,

$$[u] = \vartheta_1\left(\frac{u}{r}\middle|\tau\right), \qquad [u]^* = \vartheta_1\left(\frac{u}{r^*}\middle|\tau^*\right), \qquad p = e^{-2\pi i/\tau}, p^* = e^{-2\pi i/\tau^*}.$$

The L-operator & the Half-Currents : $\widehat{\mathfrak{sl}}_N$ case

Dynamical RLL-relation $\quad L^+(z) \in \mathrm{End}(\mathbb{C}^N) \otimes U_{q,p}$

$$R^{+(12)}(z_1/z_2, P+h)L^{+(1)}(z_1)L^{+(2)}(z_2) = L^{+(2)}(z_2)L^{+(1)}(z_1)R^{+*(12)}(z_1/z_2, P)$$

Define the half currents $E^+_{l,j}(z),\ F^+_{j,l}(z),\ K^+_j(z)$ by $\quad (R^{+*} = R^+|_{p\to p^*})$

$$L^+(z) = \begin{pmatrix} 1 & F^+_{1,2}(z) & F^+_{1,3}(z) & \cdots & F^+_{1,N}(z) \\ 0 & 1 & F^+_{2,3}(z) & \cdots & F^+_{2,N}(z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F^+_{N-1,N}(z) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K^+_1(z) & 0 & \cdots & 0 \\ 0 & K^+_2(z) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^+_N(z) \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E^+_{2,1}(z) & 1 & \ddots & & \vdots \\ E^+_{3,1}(z) & E^+_{3,2}(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E^+_{N,1}(z) & E^+_{N,2}(z) & \cdots & E^+_{N,N-1}(z) & 1 \end{pmatrix}.$$

Realization of the Half-Currents (Kojima-H.K '03)

$$K_j^+(z) =: \exp\left(\sum_{m\neq 0}\frac{(q^m-q^{-m})^2}{1-p^m}p^m\mathcal{E}_m^{-j}(q^{-j}z)^{-m}\right): e^{-Q_{\bar{\epsilon}_j}}(q^{-j}z)^{\frac{P_{\bar{\epsilon}_j}}{r^*}-\frac{(P+h)_{\bar{\epsilon}_j}}{r}} \quad (1\le j\le N)$$

where $\mathcal{E}_m^{-j}$ denote the orthonormal basis type elliptic bosons defined by

$$\alpha_{j,m} = -[m]^2(q-q^{-1})\left(\mathcal{E}_m^{-j}-q^m\mathcal{E}_m^{-(j+1)}\right) \quad (1\le j\le N-1), \quad \sum_{j=1}^{N} q^{-(j-1)m}\mathcal{E}_m^{-j}=0.$$

$$\begin{aligned}
F_{j,l}^+(z) &= a_{j,l}\oint_{C(j,l)}\prod_{m=j}^{l-1}\frac{dt_m}{2\pi i t_m}F_{l-1}(t_{l-1})F_{l-2}(t_{l-2})\cdots F_j(t_j)\\
&\times\frac{[u-v_{l-1}+(P+h)_{j,l}+\frac{l-1}{2}-1][1]}{[u-v_{l-1}+\frac{l-1}{2}][(P+h)_{j,l}-1]}\prod_{m=j}^{l-2}\frac{[v_{m+1}-v_m+(P+h)_{j,m+1}-\frac{1}{2}][1]}{[v_{m+1}-v_m+\frac{1}{2}][(P+h)_{j,m+1}]},\\
E_{l,j}^+(z) &= a_{j,l}^*\oint_{C^*(j,l)}\prod_{m=j}^{l-1}\frac{dt_m}{2\pi i t_m}E_j(t_j)E_{j+1}(t_{j+1})\cdots E_{l-1}(t_{l-1})\\
&\times\frac{[u-v_{l-1}-P_{j,l}+\frac{l-1}{2}+\frac{c}{2}+1]^*[1]^*}{[u-v_{l-1}+\frac{l-1}{2}+\frac{c}{2}]^*[P_{j,l}-1]^*}\prod_{m=j}^{l-2}\frac{[v_{m+1}-v_m-P_{j,m+1}+\frac{1}{2}]^*[1]^*}{[v_{m+1}-v_m+\frac{1}{2}]^*[P_{j,m+1}-1]^*}
\end{aligned}$$

Hopf Algebroid Structure

(Etingof-Varchenko'98, Koelink-Rosengren'01, H.K'08)

- Modified tensor product $\widetilde{\otimes}$ defined by adding the extra condition:

$$f(P,p^*)a \,\widetilde{\otimes}\, b = a \,\widetilde{\otimes}\, f(P+h,p)b \qquad (p = p^* q^{2c})$$

- Two moment maps $\mu_l, \mu_r : \mathcal{M}_{H^*} \hookrightarrow (U_{q,p})_{0,0}$

$$\mu_l(f) = f(P+h,p), \quad \mu_r(f) = f(P,p^*)$$

Theorem 2.5 (H.K '08, '16)

The following (Δ, ε, S) gives an H-Hopf algebroid str. of $U_{q,p}(\widehat{\mathfrak{g}})$.

- $\Delta(L^+_{ij}(z)) = \sum_k L^+_{ik}(z)\widetilde{\otimes}L^+_{kj}(z)$,

  $\Delta(\mu_l(f)) = \mu_l(f)\widetilde{\otimes}1, \quad \Delta(\mu_r(f)) = 1\widetilde{\otimes}\mu_r(f)$,
- $\varepsilon(L^+_{ij}(z)) = \delta_{ij} e^{-Q_{\varepsilon_j}}, \quad \varepsilon(\mu_l(f)) = f(P+h,p), \quad \varepsilon(\mu_r(f)) = f(P,p^*)$
- $S(L^+(z)) = L^+(z)^{-1}, \quad S(\mu_l(f)) = \mu_r(f), \quad S(\mu_r(f)) = \mu_l(f)$

The Vertex Operators (Type I)

Let λ, λ' be level-k weights of $\widehat{\mathfrak{g}}$

- $\mathcal{V}(\lambda,\nu), \mathcal{V}(\lambda',\nu)$: level-k irr. h.w. $U_{q,p}(\widehat{\mathfrak{g}})$-modules
- W: fin. dim. rep. of $U_{q,p}$, $W_z = W[[z, z^{-1}]]$

$H = \bar{\mathfrak{h}} \oplus P_{\bar{\mathfrak{h}}} \oplus \mathbb{C}c$

$\cdot\ f(P)v = f(< \nu, P >)v$

$\cdot\ f(P+h)v$

$= f(< \lambda, P+h >)v$

$$\Phi_W(z)\ :\ \mathcal{V}(\lambda,\nu) \to\ W_z \widetilde{\otimes}\, \mathcal{V}(\lambda',\nu)$$

$$\text{s.t.} \quad \Phi(z)x = \Delta(x)\Phi(z) \qquad \forall x \in U_{q,p}(\widehat{\mathfrak{g}}) \quad \cdots\ (\star)$$

Note that

$$(\star) \quad \Leftrightarrow \quad \Phi_W(z)L^+(w) = R^+_{VW}(w/z, P+h)L^+(w)\Phi_W(z), \quad \cdots$$

Examples :

1. $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, $\mathcal{V}$: level-1 irr. h.w. rep. & $W = V$: N-dim.rep.
2. $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, $\mathcal{V}$: level-k irr. h.w. rep. & $W = V^{(l)}$: $l+1$-dim.rep.

Example 1. The level-1 irr. h.w. $U_{q,p}(\widehat{\mathfrak{sl}}_N)$-modules

Let Λ_a $(a = 0, \cdots, N-1)$: the fundamental weights of $\widehat{\mathfrak{sl}}_N$

Theorem 2.6 (Kojima-K'03, Farghly-K-Oshima '13)

The following realizes the level-1 irr. h.w. rep. of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$.

$$E_j(z) = \exp\left\{\sum_{n>0} \frac{\alpha_{j,-n}}{[n]_q} z^n\right\} \exp\left\{-\sum_{n>0} \frac{\alpha_{j,n}}{[n]_q} z^{-n}\right\} \otimes \mathcal{Z}_j^+(z),$$

$$F_j(z) = \exp\left\{-\sum_{n>0} \frac{\alpha'_{j,-n}}{[n]_q} z^n\right\} \exp\left\{\sum_{n>0} \frac{\alpha'_{j,n}}{[n]_q} z^{-n}\right\} \otimes \mathcal{Z}_j^-(z),$$

$$\mathcal{Z}_j^+(z) = \mathrm{id} \otimes e^{\alpha_j} z^{h_j+1-\frac{P_j}{r^*}} \otimes e^{-Q_{\alpha_j}}, \quad \mathcal{Z}_j^-(z) = \mathrm{id} \otimes e^{-\alpha_j} z^{-h_j+1+\frac{(P+h)_j}{r}} \otimes 1.$$

The level-1 irr. h.w. $U_{q,p}(\widehat{\mathfrak{sl}}_N)$*-module with the h.w.* $(\Lambda_a + \nu, \nu)$ $(\nu \in \mathfrak{h}^*)$:

$$\mathcal{V}(\Lambda_a + \nu, \nu) = \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes e^{\Lambda_a}\mathbb{C}[\mathcal{Q}]) \otimes e^{Q_\nu}\mathbb{C}[\mathcal{R}_Q] = \bigoplus_{\xi,\eta \in \mathcal{Q}} \mathcal{F}_{a,\nu}(\xi, \eta)$$

where $\mathcal{Q} = \oplus_j \mathbb{Z}\alpha_j$, $\mathcal{R}_Q = \oplus_j \mathbb{Z}Q_{\alpha_j}$. *The h.w. vec. is given by* $1 \otimes e^{\Lambda_a} \otimes e^{Q_\nu}$.

The Vertex Operators of the Level-1 $U_{q,p}(\widehat{\mathfrak{sl}}_N)$-modules

Theorem 2.7 (Kojima-H.K '03 Cf. Asai-Jimbo-Miwa-Pugai '96)

The intertwiner $\Phi_V(z) : \mathcal{F}_{a,\nu}(\xi,\eta) \to V_z \widetilde{\otimes}\, \mathcal{F}_{a',\nu}(\xi,\eta)$, *where* a' =*cyclic permutation of* $a \in \{0,1,\cdots,N-1\}$, *is realized by*

$$\Phi_V(z) = \sum_{\mu=1}^{N} v_\mu \widetilde{\otimes}\, \Phi_\mu(z), \qquad V = \oplus_{\mu=1}^{N} \mathbb{C} v_\mu\ , \quad V_z = V \otimes \mathbb{C}[[z,z^{-1}]]$$

$$\Phi_N(z) =: \exp\left(\sum_{m\neq 0}(q^m - q^{-m})\mathcal{E}'^{-N}_m (q^{N-1}z)^{-m}\right) : e^{-\bar{\varepsilon}_N} z^{h_{\bar{\varepsilon}_N}} z^{-\frac{1}{r}(P+h)_{\bar{\varepsilon}_N}},$$

$$\Phi_\mu(z) = F^{+}_{\mu,N}(q^{-1}z)\Phi_N(z) \qquad (\mu = 1,\cdots,N-1)$$

$$\alpha_{j,m} = -[m]_q^2 (q-q^{-1})\left(\mathcal{E}^{-j}_m - q^m \mathcal{E}^{-(j+1)}_m\right), \quad \sum_{j=1}^{N} q^{-(j-1)m}\mathcal{E}^{-j}_m = 0$$

Explicitly,

$$\Phi_\mu(z) = \oint_{\mathbb{T}^{N-\mu}} \prod_{m=\mu}^{N-1} \frac{dt_m}{2\pi i t_m} \Phi_N(z) F_{N-1}(t_{N-1}) F_{N-2}(t_{N-2}) \cdots F_\mu(t_\mu)$$
$$\times \varphi_\mu(z, t_\mu, \cdots, t_{N-1}; \{\Pi_{\mu,m}\}),$$

where $\Pi_{\mu,m} = q^{2(P+h)_{\mu,m}}$, $t_m = q^{2v_m}, z = q^{2u}$,

$$\varphi_\mu(z, t_\mu, \cdots, t_{N-1}; \{\Pi_{\mu,m}\}) = \prod_{m=\mu}^{N-1} \frac{[v_{m+1} - v_m + (P+h)_{\mu,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_m + \frac{1}{2}][(P+h)_{\mu,m+1}]},$$
$$v_N = u$$

Proposition 2.8 (Kojima-H.K '03)

$$\Phi_{\mu_2}(z_2)\Phi_{\mu_1}(z_1) = \sum_{\mu_1'\mu_2'} R(z_1/z_2, P+h)_{\mu_1\mu_2}^{\mu_1'\mu_2'} \Phi_{\mu_1'}(z_1)\Phi_{\mu_2'}(z_2)$$

Combinatorial Notations

For $\quad \Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n) \qquad (\mu_j \in \{1,\cdots,N\})$

- For $l \in \{1,\cdots,N\}$, $I_l := \{\ i \in [1,n] \mid \mu_i = l\}$, $\quad \lambda_l := |I_l| \in \mathbb{Z}_{\geq 0}$, $\lambda := (\lambda_1,\cdots,\lambda_N)$. Then $I = (I_1,\cdots,I_N)$ is a partition of $[1,n]$

  i.e. $I_1 \cup \cdots \cup I_N = [1,n], \quad I_k \cap I_l = \emptyset\ (k \neq l)$.
- We often denote resulting I as $I_{\mu_1\cdots\mu_n}$
- For $\lambda = (\lambda_1,\cdots,\lambda_N)$, $|\lambda| = \lambda_1 + \cdots + \lambda_N = n$,
  $\mathcal{I}_\lambda$: the set of all partitions $I = (I_1,\cdots,I_N)$ of $[1,n]$ with $|I_l| = \lambda_l$.
- Set also $\lambda^{(l)} := \lambda_1 + \cdots + \lambda_l$, $I^{(l)} := I_1 \cup \cdots \cup I_l =: \{i_1^{(l)} < \cdots < i_{\lambda^{(l)}}^{(l)}\}$.

Remark 3.1

Each partition $I \in \mathcal{I}_\lambda$ specifies the coordinate flag for the partial flag variety $\mathcal{F}_\lambda$ consisting of $0 = V_0 \subset V_1 \subset \cdots \subset V_N = \mathbb{C}^n \quad$ with $\dim V_l/V_{l-1} = \lambda_l$.

Example: $\widehat{\mathfrak{sl}}_3$, $n = 5$ case

$$\Phi_2(z_1)\Phi_1(z_2)\Phi_3(z_3)\Phi_1(z_4)\Phi_2(z_5)$$

$$\rightsquigarrow I_1 = \{2, 4\},\ I_2 = \{1, 5\},\ I_3 = \{3\}, \quad \lambda = (2, 2, 1)$$

$$I^{(1)} = I_1 = \{2, 4\}, \qquad \begin{array}{cc} 2 & 4 \\ \| & \| \\ i^{(1)}_1 & i^{(1)}_2 \end{array}$$

$$I^{(2)} = I_1 \cup I_2 = \{1, 2, 4, 5\}, \qquad \begin{array}{cccc} 1 & 2 & 4 & 5 \\ \| & \| & \| & \| \\ i^{(2)}_1 & i^{(2)}_2 & i^{(2)}_3 & i^{(2)}_4 \\ & \| & \| & \\ & i^{(1)}_1 & i^{(1)}_2 & \end{array}$$

$$I^{(3)} = I_1 \cup I_2 \cup I_3 = \{1,\ 2,\ 3,\ 4,\ 5\} \qquad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \| & \| & \| & \| & \| \\ i^{(3)}_1 & i^{(3)}_2 & i^{(3)}_3 & i^{(3)}_4 & i^{(3)}_5 \\ \| & \| & & \| & \| \\ i^{(2)}_1 & i^{(2)}_2 & & i^{(2)}_3 & i^{(2)}_4 \\ & \| & & \| & \\ & i^{(1)}_1 & & i^{(1)}_2 & \end{array}$$

$$\rightsquigarrow \quad 0 \subset V_1 =< v_2, v_4 > \subset V_2 =< v_1, v_2, v_4, v_5 > \subset \mathbb{C}^5 \ \in \mathcal{F}_{(2,2,1)}$$

Label of the Integration Variables

$$\Phi_2(z_1)\Phi_1(z_2)\Phi_3(z_3)\Phi_1(z_4)\Phi_2(z_5)$$

$$I^{(1)} = I_1 = \{\underset{\substack{\| \\ i_1^{(1)}}}{2}, \underset{\substack{\| \\ i_2^{(1)}}}{4}\}, \qquad I^{(2)} = I_1 \cup I_2 = \{\underset{\substack{\| \\ i_1^{(2)}}}{1}, \underset{\substack{\| \\ i_2^{(2)} \\ \| \\ i_1^{(1)}}}{2}, \underset{\substack{\| \\ i_3^{(2)} \\ \| \\ i_2^{(1)}}}{4}, \underset{\substack{\| \\ i_4^{(2)}}}{5}\},$$

$$I^{(3)} = I_1 \cup I_2 \cup I_3 = \{\underset{\substack{\| \\ i_1^{(3)} \\ \| \\ i_1^{(2)}}}{1},\ \underset{\substack{\| \\ i_2^{(3)} \\ \| \\ i_2^{(2)} \\ \| \\ i_1^{(1)}}}{2},\ \underset{\substack{\| \\ i_3^{(3)}}}{3},\ \underset{\substack{\| \\ i_4^{(3)} \\ \| \\ i_3^{(2)} \\ \| \\ i_2^{(1)}}}{4},\ \underset{\substack{\| \\ i_5^{(3)} \\ \| \\ i_4^{(2)}}}{5}\}$$

Assign $t_k^{(l)}$ to the argument of F_l appearing in the $i_k^{(l)}$-th vertex op.

$$\Phi_2(z_1) = \oint \frac{dt_1^{(2)}}{2\pi i t_1^{(2)}} \Phi_3(z_1) F_2(t_1^{(2)}) \varphi(z_1, t_1^{(2)}; \Pi)$$

$$\Phi_1(z_2) = \oint \oint \frac{dt_2^{(2)}}{2\pi i t_2^{(2)}} \frac{dt_1^{(1)}}{2\pi i t_1^{(1)}} \Phi_3(z_2) F_2(t_2^{(2)}) F_1(t_1^{(1)}) \varphi(z_2, t_1^{(1)}, t_2^{(2)}; \Pi),$$

etc.

Theorem 3.2

$$\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)=\oint_{\mathbb{T}^M}\underline{d\mathbb{t}}\ \widetilde{\Phi}(\mathbb{t},\mathbb{z})\omega_{\mu_1,\cdots,\mu_n}(\mathbb{t},\mathbb{z},\Pi),$$

- $$\widetilde{\Phi}(\mathbb{t},\mathbb{z})=:\Phi_N(z_1)\cdots\Phi_N(z_n):$$
$$\times :F_{N-1}(t_1^{(N-1)})\cdots F_{N-1}(t^{(N-1)}_{\lambda^{(N-1)}}):\cdots:F_1(t_1^{(1)})\cdots F_1(t^{(1)}_{\lambda^{(1)}}):$$
$$\times\prod_{1\le l<m\le n}<\Phi_N(z_l)\Phi_N(z_m)>^{Sym}\prod_{l=1}^{N-1}\prod_{1\le a<b\le\lambda^{(l)}}<F_l(t_a^{(l)})F_l(t_b^{(l)})>^{Sym}$$

  : symmetric in $\mathbb{z}$ and $\{\mathbb{t}^{(l)}\}$, for each l, respectively

- $$\omega_{\mu_1,\cdots,\mu_n}(\mathbb{t},\mathbb{z},\Pi)=\prod_{1\le l<m\le n}z_l^{r^*(N-1)/rN}\frac{\Gamma(q^2z_l/z_k;p,q^{2N})}{\Gamma(q^{2N}z_l/z_k;p,q^{2N})}\times\widetilde{W}_I(\mathbb{t},\mathbb{z},\Pi)$$

$$\widetilde{W}_I(\mathbb{t},\mathbb{z},\Pi)$$
$$=\mathrm{Sym}_{t^{(1)}}\cdots\mathrm{Sym}_{t^{(N-1)}}\left[\prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}}\left(\left.\frac{[v_b^{(l+1)}-v_a^{(l)}+(P+h)_{\mu_s,l+1}-C_{\mu_s,l+1}][1]}{[v_b^{(l+1)}-v_a^{(l)}+1][(P+h)_{\mu_s,l+1}-C_{\mu_s,l+1}]}\right|_{i_s^{(N)}=i_b^{(l+1)}=i_a^{(l)}}\right.\right.$$
$$\left.\left.\times\prod_{\substack{b=1\\ i_b^{(l+1)}>i_a^{(l)}}}^{\lambda^{(l+1)}}\frac{[v_b^{(l+1)}-v_a^{(l)}]}{[v_b^{(l+1)}-v_a^{(l)}+1]}\prod_{b=a+1}^{\lambda^{(l)}}\frac{[v_b^{(l)}-v_a^{(l)}+1]}{[v_b^{(l)}-v_a^{(l)}]}\right)\right],$$

where $v_s^{(N)}=u_s,\quad C_{\mu_s,l+1}=\sum_{j=s+1}^{n}<\bar{\epsilon}_{\mu_j},h_{\mu_s,l+1}>.$

Combinatorial Formula for $C_{\mu_s,l+1}=\sum_{j=s+1}^{n} <\bar{\epsilon}_{\mu_j}, h_{\mu_s,l+1}>$

For a partition $I=(I_1,\cdots,I_N)\in\mathcal{I}_\lambda$, let $I_k=\{i_{k,1}<\cdots<i_{k,\lambda_k}\}$ $(k=1,\cdots,N)$.

Proposition 3.3

$$C_{\mu_s,l+1}=\begin{cases}\lambda_{\mu_s}-\lambda_{l+1}-\tilde{s}+m_{l+1}(s) & \textit{if}\ \ s<i_{l+1,\lambda_{l+1}}\\ \lambda_{\mu_s}-\tilde{s} & \textit{if}\ \ s>i_{l+1,\lambda_{l+1}}\end{cases}$$

where $\tilde{s}$ and $m_{l+1}(s)$ are defined by $i_{\mu_s,\tilde{s}}=s$ and

$$m_{l+1}(s)=\mathit{min}\{1\le j\le\lambda_{l+1}\ |\ s<i_{l+1,j}\ \}.$$

Remark 3.4

In the trig. and non-dynamical limit $\widetilde{W}_I(t\!\!t, z\!\!z;\Pi)$ coincides with the one obtained by Mimachi '96 and used in Rimanyi-Tarasov-Varchenko '14.

Example 2. The level-$k(\in \mathbb{Z}_{>0})$ irr. h.w. $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-modules

Let
$\lambda_a = (k-a)\Lambda_0 + a\Lambda_1 \ (a = 0, 1, \cdots, k)$: the level-k dominant int. weights of $\widehat{\mathfrak{sl}}_2$.

Theorem 3.5 (K'98, Kojima-K-Weston '05)

The following realizes the level-k irr. h. w. rep. of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

$$E(z) = \exp\left\{\sum_{n>0} \frac{\alpha_{-n}}{[kn]_q} z^n\right\} \exp\left\{-\sum_{n>0} \frac{\alpha_n}{[kn]_q} z^{-n}\right\} \otimes \mathcal{Z}^+(z),$$

$$F(z) = \exp\left\{-\sum_{n>0} \frac{\alpha'_{-n}}{[kn]_q} z^n\right\} \exp\left\{\sum_{n>0} \frac{\alpha'_n}{[kn]_q} z^{-n}\right\} \otimes \mathcal{Z}^-(z), \quad \alpha'_n = \frac{1-p^{*n}}{1-p^n}\alpha_n$$

$$\mathcal{Z}^+(z) = \Psi(z) \otimes e^{\alpha} z^{\frac{h}{k} - \frac{P}{r^*}} \otimes e^{-Q},$$

$$\mathcal{Z}^-(z) = \Psi^\dagger(z) \otimes e^{-\alpha} z^{-\frac{h}{k} + \frac{P+h}{r}} \otimes 1,$$

$\Psi(z), \Psi^\dagger(z)$: q-$\mathbb{Z}_k$ *parafermions* , $\quad e^{\pm\alpha} \in \mathbb{C}[\mathcal{Q}],\ e^{\pm Q} \in \mathbb{C}[\mathcal{R}_Q],$

$\mathcal{Q} = \mathbb{Z}\alpha,\ \mathcal{R}_Q = \mathbb{Z}Q$

The level-k irr. h.w. $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-module with the h.w. $(\lambda_a+\nu,\nu)$:

$$\mathcal{V}(\lambda_a+\nu,\nu)=\mathbb{F}\otimes_{\mathbb{C}}\left(\mathcal{F}_{\alpha,k}\otimes\bigoplus_{n\in\mathbb{Z}}\bigoplus_{\substack{M=0 \bmod 2k\\ M\equiv a \bmod 2}}^{2k-1}\mathcal{H}^{PF}_{a,M}\otimes e^{(M+2kn)\alpha/2}\mathbb{C}[\mathcal{Q}]\right)\otimes e^{Q_\nu}\mathbb{C}[\mathcal{R}_Q],$$

$\mathcal{F}_{\alpha,k}$: the Fock space of $\{\alpha_n\}$

$\mathcal{H}^{PF}_{a,M}$: the irr. q-Parafermion modules

The h.w. vector is given by $1\otimes e^{\lambda_a}\otimes e^{Q_\nu}$.

We write

$$\begin{aligned}\mathcal{V}(\lambda_a+\nu,\nu) &= \bigoplus_{m\in\mathbb{Z}}\bigoplus_{n\in\mathbb{Z}}\bigoplus_{\substack{M=0 \bmod 2k\\ M\equiv a \bmod 2}}^{2k-1}\mathcal{F}^{(M)}_{a,\nu}(m,n),\\ \mathcal{F}^{(M)}_{a,\nu}(m,n) &= \mathcal{F}_{\alpha,k}\otimes\mathcal{H}^{PF}_{a,M}\otimes e^{(M+2kn)\alpha/2}\otimes e^{Q_\nu-mQ}.\end{aligned}$$

Remark 3.6 For generic r

$$\bigoplus_{\substack{M=0 \bmod 2k\\ M\equiv a \bmod 2}}^{2k-1}\mathcal{F}^{(M)}_{a,\nu}(m,n) \cong$$ Verma module of the coset Virasoro alg. ass. with $(\widehat{\mathfrak{sl}}_2)_k\oplus(\widehat{\mathfrak{sl}}_2)_{r-k-2}/(\widehat{\mathfrak{sl}}_2)_{r-2}$

Let $V^{(l)} = \bigoplus_{\mu\in\{-l,-l+2,\cdots,l\}} \mathbb{C}v_\mu^{(l)}$: $l+1$-dim. rep. of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ $(l = 1,\cdots,k)$

Theorem 3.7 (K '98, Jimbo-K-Odake-Shiraishi '99, Kojima-K-Weston '05)

The vertex op. $\Phi_{V^{(l)}}(z) : \mathcal{F}^{(M)}_{a,\nu}(m,n) \to V_z^{(l)} \widetilde{\otimes}\, \mathcal{F}^{(M)}_{k-a,\nu}(m,n)$ *is realized by*

$$\Phi_{V^{(l)}}(z) = \sum_{\mu\in\{-l,-l+2,\cdots,l\}} v_\mu^{(l)} \widetilde{\otimes}\, \Phi_\mu(z),$$

$$\Phi_l(z) = \phi_{l.l}(z) : \exp\left(\sum_{n\neq 0} \frac{[lm]_q \alpha'_m}{[2m]_q[km]_q} z^{-m}\right) : e^{\frac{l}{2}\alpha}(-z)^{\frac{lh}{2k}} z^{-\frac{l}{2r}(P+h)},$$

$$\Phi_\mu(z) = F^+_{1,2}(q^l z)^m \Phi_k(z)$$

$$= \oint_{\mathcal{C}_\mu} \prod_{j=1}^m \frac{dt_j}{2\pi i t_j} \Phi_l(z) F(t_1)\cdots F(t_m) \prod_{j=1}^m \frac{[u-v_j+P+h-\frac{l}{2}-1+\mu+2j]}{[u-v_j-\frac{l}{2}]}$$

where $m = (l-\mu)/2$, $z = q^{2u},\ t_j = q^{2v_j}$.

The Vertex Operators of the Level-k $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-modules

$$\Phi_{\mu_n}(z_n)\cdots\Phi_{\mu_1}(z_1):\mathcal{V}(\lambda_a+\nu,\nu)\to V_{z_n}^{(l_n)}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_1}^{(l_1)}\widetilde{\otimes}\mathcal{V}(\lambda_{a'}+\nu,\nu)$$

$$(\mu_j\in\{-l_j,-l_j+2,\cdots,l_j\}),$$

Let

- $m_j=(l_j-\mu_j)/2\in\{0,1,\cdots,l_j\}$: # of $F(t)$'s attached to $\Phi_{\mu_j}(z_j)$
- $m:=m_1+\cdots+m_n$: the total # of $F(t)$'s in $\Phi_{\mu_n}(z_n)\cdots\Phi_{\mu_1}(z_1)$
- $I=(I_1,\cdots,I_n)$ is a partition of $[1,m]=\{1,\cdots,m\}$ such that $I_1\cup\cdots\cup I_n=[1,m]$, $|I_j|=m_j$.

Theorem 3.8

$$\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)=\oint \underline{d\mathbb{t}}\ \widetilde{\Phi}(\mathbb{z},\mathbb{t})\ \omega_{\mu_n,\cdots,\mu_1}(\mathbb{z},\mathbb{t};\Pi)$$

Here

$$\widetilde{\Phi}(\mathbb{z},\mathbb{t})=:\Phi_{l_n}(z_n)\cdots\Phi_{l_1}(z_1)::F(t_1)\cdots F(t_m): \times\prod_{1\le s<t\le n}<\Phi_{l_s}(z_s)\Phi_{l_t}(z_t)>^{Sym}\prod_{1\le a<b\le m}<F(t_a)F(t_b)>^{Sym}$$

$$\omega_{\mu_n,\cdots,\mu_1}(\mathbb{z},\mathbb{t};\Pi)=\prod_{i<j}G_{l_il_j}(z_i/z_j)\prod_{1\le a<b\le m}\frac{[v_a-v_b]}{[v_a-v_b-1]}\times\frac{1}{m!}\sum_{\substack{I_1\sqcup\cdots\sqcup I_n=[1,m]\\ |I_s|=m_s\ (s=1,\cdots,n)}}\prod_{s=1}^{n}\prod_{a\in I_s}\left(\frac{[u_s-v_a+K_s+m_s]}{[u_s-v_a+l_s/2]}\prod_{j=1}^{s-1}\frac{[u_j-v_a-l_j/2]}{[u_j-v_a+l_j/2]}\right)\times\prod_{\substack{1\le s<t\le n\\ a\in I_s,b\in I_t}}\frac{[v_a-v_b-1]}{[v_a-v_b]}$$

with $K_s=P+h-l_s/2+\sum_{j=1}^{s}\mu_j$.

The weight function $\omega_{\mu_n,\cdots,\mu_1}(\mathbb{z},\mathbb{t};\Pi)$ agrees with Felder-Tarasov-Varchenko '97.

- $$< F(t_i)F(t_j) >= t_i^{-2/r}\frac{(q^2t_j/t_i;p)_\infty(t_j/t_i;p)_\infty}{(pt_j/t_i;p)_\infty(pq^{-2}t_j/t_i;p)_\infty}$$
$$= t_i^{-2/r}\frac{(q^2t_j/t_i;p)_\infty(pq^{-2}t_i/t_j;p)_\infty}{(pt_j/t_i;p)_\infty(t_i/t_j;p)_\infty}\frac{(t_i/t_j;p)_\infty(t_j/t_i;p)_\infty}{(pq^{-2}t_i/t_j;p)_\infty(pq^{-2}t_j/t_i;p)_\infty}$$
$$= t_i^{-2/r}\frac{\theta_p(pq^{-2}t_i/t_j)}{\theta_p(t_i/t_j)}\frac{(t_i/t_j;p)_\infty(t_j/t_i;p)_\infty}{(pq^{-2}t_i/t_j;p)_\infty(pq^{-2}t_j/t_i;p)_\infty}$$
symm. by $t_i \leftrightarrow t_j$

- $$< \Phi_{l_i}(z_i)\Phi_{l_j}(z_j) >= G_{l_il_j}(z_i/z_j)< \Phi_{l_i}(z_i)\Phi_{l_j}(z_j) >^{Sym},$$

where

$$G_{l_il_j}(z_i/z_j) = z_i^{-l_il_j/2r}\frac{1}{\Gamma(q^{l_i+l_j+2}z_i/z_j;p,q^4)}\frac{(pq^{-l_i+l_j+2}z_j/z_i;p,q^4)_\infty}{(q^{-l_i+l_j+2}z_i/z_j;p,q^4)_\infty}\Big/(p\to q^{2(k+2)})$$

$$< \Phi_{l_i}(z_i)\Phi_{l_j}(z_j) >^{Sym}= \frac{(q^{l_i-l_j+2}z_j/z_i, q^{-l_i+l_j+2}z_i/z_j;p,q^4)_\infty}{(q^{l_i+l_j+2}z_j/z_i, q^{l_i+l_j+2}z_i/z_j;p,q^4)_\infty}\Big/(p\to q^{2(k+2)})$$

symm. by $z_i \leftrightarrow z_j$ with $l_i \leftrightarrow l_j$

Properties of the Elliptic Weight Functions

It is convenient to consider

$$\mathcal{W}_I(\mathbb{t},\mathbb{z},\Pi) := \frac{\widetilde{W}_I(\mathbb{t},\mathbb{z},\Pi)H_\lambda(\mathbb{t},\mathbb{z})}{E_\lambda(\mathbb{t})}$$

$$= \mathrm{Sym}_{t^{(1)}}\cdots\mathrm{Sym}_{t^{(N-1)}}\left[\frac{\prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}} u_I^{(l)}(t_a^{(l)},\mathbb{t}^{(l+1)},\Pi_{\mu_{i_a^{(l)}},l+1}q^{-2C_{\mu_{i_a^{(l)}},l+1}})}{\prod_{l=1}^{N-1}\prod_{1\le a<b\le\lambda^{(l)}}[v_a^{(l)}-v_b^{(l)}][v_b^{(l)}-v_a^{(l)}-1]}\right],$$

where

$$H_\lambda(\mathbb{t},\mathbb{z}) = \prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}}\prod_{b=1}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}+1], \qquad E_\lambda(\mathbb{t}) = \prod_{l=1}^{N-1}\prod_{a,b=1}^{\lambda^{(l)}}[v_b^{(l)}-v_a^{(l)}+1],$$

and

$$u_I^{(l)}(t_a^{(l)},\mathbb{t}^{(l+1)},\Pi_{j,k})$$

$$= \prod_{\substack{b=1\\ i_b^{(l+1)}>i_a^{(l)}}}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}]\times\left.\frac{[v_b^{(l+1)}-v_a^{(l)}+(P+h)_{j,k}]}{[(P+h)_{j,k}]}\right|_{i_b^{(l+1)}=i_a^{(l)}}\times\prod_{\substack{b=1\\ i_b^{(l+1)}<i_a^{(l)}}}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}+1]$$

- For each l $(1\le l\le N-1)$, $\mathcal{W}_I$ is symmetric in $t_a^{(l)}=q^{2v_a^{(l)}}$ $(1\le a\le\lambda^{(l)})$.

Quasi-periodicity

Proposition 4.1

For $I \in \mathcal{I}_\lambda$, the weight functions $\mathcal{W}_I(t, z, \Pi)$ have the following quasi-periodicity.

$$\mathcal{W}_I(\cdots, pt_a^{(l)}, \cdots, z, \Pi) = (-1)^{\lambda^{(l+1)}+\lambda^{(l-1)}} \mathcal{W}_I(\cdots, t_a^{(l)}, \cdots, z, \Pi),$$

$$\begin{aligned}
&\mathcal{W}_I(\cdots, e^{-2\pi i} t_a^{(l)}, \cdots, z, \Pi) \\
&= (-e^{-\pi i \tau})^{\lambda^{(l+1)}-2\lambda^{(l)}+\lambda^{(l-1)}+2} \\
&\times \exp\Bigg\{ -\frac{2\pi i}{r} \Bigg((\lambda^{(l+1)} - 2\lambda^{(l)} + \lambda^{(l-1)}) v_a^{(l)} - \sum_{b=1}^{\lambda^{(l+1)}} v_b^{(l+1)} + 2\sum_{b=1}^{\lambda^{(l)}} v_b^{(l)} - \sum_{b=1}^{\lambda^{(l-1)}} v_b^{(l-1)} \\
&\qquad\qquad - (P+h)_{l,l+1} - \lambda_{l+1} \Bigg) \Bigg\} \\
&\quad \times \mathcal{W}_I(\cdots, t_a^{(l)}, \cdots, z, \Pi) \qquad\qquad (1 \le l \le N-1, 1 \le a \le \lambda^{(l)})
\end{aligned}$$

- For each l, $\mathcal{W}_I$ has the same quasi-periodicity for all $t_a^{(l)}$.
- This and symm. property indicate that $\mathcal{W}_I$'s are merom. sections of certain line bundle over $\mathrm{E}^{(\lambda^{(1)})} \times \cdots \times \mathrm{E}^{(\lambda^{(N-1)})}$, where $\mathrm{E}^{(k)} = \mathrm{E}^k / \mathfrak{S}_k$.

- This is consistent to the structure

$$\mathrm{Ell}_T(X) \to \mathrm{Ell}_T(\mathrm{pt}) \times \mathrm{E}^{(\lambda^{(1)})} \times \cdots \times \mathrm{E}^{(\lambda^{(N-1)})},$$

arising from the construction of $X = T^*\mathcal{F}_\lambda$ as a hyper-Kähler quotient by $\prod_{l=1}^{N-1} GL(\lambda^{(l)})$. (Aganagic-Okounkov '16)

This suggests :

(1) $\{t_a^{(l)}\} \Leftrightarrow$ the Chern roots of the tautological vec. bundles $\{V_l\}$ over X
(2) $z_1, \cdots, z_n, q^2 \Leftrightarrow$ the equivariant parameters in $\mathrm{Ell}_T(\mathrm{pt}) = \mathrm{E}^n \times \mathrm{E}$
(3) $\mathcal{W}_I(\mathbb{t}, \mathbb{z}, \Pi)$'s are meromorphic sections of certain line bundle over $\mathrm{Ell}_T(X)(\times \mathcal{E}_{\mathrm{Pic}_T(X)})$

Remark 4.2

Felder, Rimanyi, Varchenko '17 and Rimanyi,Tarasov, Varchenko '17 based themselves on a different tautological bundles $\{V_l/V_{l-1}\}$ and structure

$$\mathrm{Ell}_T(\mathcal{F}_\lambda) \to \mathrm{Ell}_T(\mathrm{pt}) \times \mathrm{E}^{(\lambda_1)} \times \cdots \times \mathrm{E}^{(\lambda_N)},$$

where $\lambda_l = \lambda^{(l)} - \lambda^{(l-1)}$, given in Ginzburg, Kapranov, Vasserot '95.

Triangular Property :

Define the partial ordering for $I, J \in \mathcal{I}_\lambda$ by

$$I \le J \ \Leftrightarrow\ i_a^{(l)} \le j_a^{(l)} \quad \text{for } l = 1, \cdots, N,\ a = 1, \cdots, \lambda^{(l)}$$

Then the specialization $\mathbb{t} = \mathbb{z}_I$ i.e. $t_a^{(l)} = z_{i_a^{(l)}}$ yields

$$(\star) \qquad \mathcal{W}_J(\mathbb{z}_I, \mathbb{z}, \Pi) = 0 \text{ unless } I \le J.$$

$$\mathcal{W}_I(\mathbb{z}_I, \mathbb{z}, \Pi) = \prod_{1 \le k < l \le N} \prod_{a \in I_k} \left(\prod_{\substack{b \in I_l \\ a < b}} [u_b - u_a] \prod_{\substack{b \in I_l \\ a > b}} [u_b - u_a + 1] \right)$$

Remark 4.3

Noting $\{I\}_{I \in \mathcal{I}_\lambda}$ labels the $(\mathbb{C}^\times)^n$-fixed points of $T^*\mathcal{F}_\lambda$, $(\star)$ suggests

$$\mathbb{t} = \mathbb{z}_I \Leftrightarrow \text{the restriction to the fixed point } I$$

and $(\star)$ should correspond to the triangular property of $\mathrm{Stab}_{\mathfrak{C}}(F_J)$ w.r.t the restriction to the fixed points.

Transition Property :

For $I = I_{\cdots\ \mu_i\mu_{i+1}\cdots}$

$$\widetilde{W}_{s_i(I)}(\mathbb{t},\cdots,z_{i+1},z_i,\cdots,\Pi) = \sum_{\mu'_i,\mu'_{i+1}} R(z_i/z_{i+1},\Pi q^{2\sum_{j=1}^{i-1}h^{(j)}})_{\mu_i\mu_{i+1}}^{\mu'_i\mu'_{i+1}}\ \widetilde{W}_I(\mathbb{t},\cdots,z_i,z_{i+1},\cdots,\Pi)$$

$$\because\quad \Phi_{\mu_n}(z_n)\cdots\Phi_{\mu_1}(z_1) = \oint \underline{d\mathbb{t}}\ \widetilde{\Phi}(\mathbb{t},\mathbb{z})\ \omega_{\mu_n,\cdots,\mu_1}(\mathbb{t},\mathbb{z};\Pi)$$

- LHS: $\Phi_\mu(z_{i+1})\Phi_\nu(z_i) = \sum_{\mu',\nu'} R(z_i/z_{i+1},\Pi)_{\nu\mu}^{\nu'\mu'}\ \Phi_{\nu'}(z_i)\Phi_{\mu'}(z_{i+1})$
- RHS: $\widetilde{\Phi}(\mathbb{t},\mathbb{z})$ is symmetric in $z_i \leftrightarrow z_{i+1}$

Matrix Form

For $\sigma \in \mathfrak{S}_n$, define

$$\mathcal{W}_{\sigma,I}(\mathbb{t},\mathbb{z},\Pi) := \mathcal{W}_{\sigma^{-1}(I)}(\mathbb{t},\sigma(\mathbb{z}),\Pi)$$

and the matrix

$$\widehat{W}_\sigma(\mathbb{z},\Pi) := \left(\mathcal{W}_{\sigma,J}(\mathbb{z}_I,\mathbb{z},\Pi)\right)_{I,J\in\mathcal{I}_\lambda}.$$

We put the matrix elements in the order

$$I^{max} \equiv I_{\underbrace{N\ldots N}_{\lambda_N}\cdots\underbrace{1\ldots 1}_{\lambda_1}} > \cdots > I^{min} \equiv I_{\underbrace{1\ldots 1}_{\lambda_1}\cdots\underbrace{N\ldots N}_{\lambda_N}}.$$

Then

- the triangular property $\Leftrightarrow \widehat{W}_{\mathrm{id}}(\mathbb{z},\Pi)$ is lower triangular
- the transition prop. $\Leftrightarrow {}^t\widehat{W}_{s_i}(\mathbb{z},\Pi) = \mathcal{R}(z,\Pi q^{-2\sum_j h^{(j)}})\,{}^t\widehat{W}_{\mathrm{id}}(\mathbb{z},\Pi)$

Hence ${}^t\widehat{W}_{s_i}(\mathbb{z},\Pi)\left({}^t\widehat{W}_{\mathrm{id}}(\mathbb{z},\Pi)\right)^{-1} = \mathcal{R}(z,\Pi q^{-2\sum_j h^{(j)}})$

This should correspond to $\mathrm{Stab}_{\mathfrak{C}'}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}} = \mathcal{R}_{\mathfrak{C}',\mathfrak{C}}$

Orthogonality

From the transition property

$$ {}^t\widehat{W}_{\sigma_0}(\mathbb{z},\Pi) = \mathcal{R}(\mathbb{z},\Pi q^{-2\sum_j h^{(j)}})\ {}^t\widehat{W}_{\mathrm{id}}(\mathbb{z},\Pi) $$

Due to the property of the elliptic dynmical R matrix, $b(z,\Pi^{-1}) = b(z,\Pi)$ and $c(z,\Pi^{-1}) = \bar{c}(z,\Pi)$, the matrix $\mathcal{R}(\mathbb{z},\Pi)$ satisfies

$$ {}^t\mathcal{R}(\mathbb{z},\Pi q^{-2\sum_{j=1}^n h^{(j)}}) = \mathcal{R}(\mathbb{z},\Pi^{-1}). $$

Therefore

$$ \underbrace{\widehat{W}_{\sigma_0}(\mathbb{z},\Pi)\ {}^t\widehat{W}_{\mathrm{id}}(\mathbb{z},\Pi^{-1}q^{2\sum_j h^{(j)}})}_{\text{upper triangular}} = \underbrace{\widehat{W}_{\mathrm{id}}(\mathbb{z},\Pi)\ {}^t\widehat{W}_{\sigma_0}(\mathbb{z},\Pi^{-1}q^{2\sum_j h^{(j)}})}_{\text{lower triangular}} $$

$=$ diagonal matrix

Orthogonality :

For $J, K \in \mathcal{I}_\lambda$, $\sigma_0 \in \mathfrak{S}_n$: the longest element

$$\sum_{I \in \mathcal{I}_\lambda} \frac{\mathcal{W}_J(\mathbb{z}_I, \mathbb{z}, \Pi^{-1} q^{2\sum_{j=1}^{n} <\bar{\epsilon}_{\mu_j}, h>}) \mathcal{W}_{\sigma_0(K)}(\mathbb{z}_I, \sigma_0(\mathbb{z}), \Pi)}{Q(\mathbb{z}_I) R(\mathbb{z}_I)} = \delta_{J,K}$$

where

$$Q(\mathbb{z}_I) = \prod_{1 \le k < l \le N} \prod_{a \in I_k} \prod_{b \in I_l} [u_b - u_a + 1],$$

$$R(\mathbb{z}_I) = \prod_{1 \le k < l \le N} \prod_{a \in I_k} \prod_{b \in I_l} [u_b - u_a].$$

Remark 4.4

Felder, Rimanyi, Varchenko '17 and Rimanyi,Tarasov, Varchenko '17 missed the dynamical shift. This shift will turn out to be important to get a consistent geometric picture.

Shuffle (Feigin-Odesskii) Algebra Structure :

Let $\lambda, \lambda' \in \mathbb{N}^N$, $|\lambda| = m, |\lambda'| = n$, $I \in \mathcal{I}_\lambda, I' \in \mathcal{I}_{\lambda'}$.

The following $\star$-product of $\widetilde{W}_I(t, z, \Pi_I)$ and $\widetilde{W}_{I'}(t', z', \Pi'_{I'})$ gives again an elliptic weight function $\widetilde{W}_{I+I'}(t \cup t', z \cup z', \Pi_{I+I'})$.

$$(\widetilde{W}_I \star \widetilde{W}_{I'})(t \cup t', z \cup z', \Pi_I \cup \Pi'_{I'})$$
$$= \frac{1}{\prod_{l=1}^{N-1} \lambda^{(l)}! \lambda'^{(l)}!}$$
$$\times \mathrm{Sym}_{t^{(1)}} \cdots \mathrm{Sym}_{t^{(N-1)}} \left[\widetilde{W}_I(t, z, \Pi_I q^{-2\sum_{j=1}^{n} <\bar{\epsilon}_{\mu'_j}, h>})\ \widetilde{W}_{I'}(t', z', \Pi'_{I'})\ \Xi(t, t', z, z') \right],$$

where $I' = I'_{\mu'_1, \cdots, \mu'_n}$ and

$$\Xi(t, t', z, z') = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \left(\prod_{b=1}^{\lambda'^{(l+1)}} \frac{[v'^{(l+1)}_b - v^{(l)}_a]}{[v'^{(l+1)}_b - v^{(l)}_a + 1]} \prod_{c=1}^{\lambda'^{(l)}} \frac{[v'^{(l)}_c - v^{(l)}_a + 1]}{[v'^{(l)}_c - v^{(l)}_a]} \right).$$

This probably correspond to

$$\mathrm{Stab}_{\mathfrak{C}} = \mathrm{Stab}_{\mathfrak{C}'} \circ \tau(q^{-2} \det \mathrm{ind}_{X^{A'}})^* \mathrm{Stab}_{\mathfrak{C}/\mathfrak{C}'},$$

$\mathfrak{C}' \subset \mathfrak{C}$ as $A' = (\mathbb{C}^\times)^{n'} \subset A = (\mathbb{C}^\times)^n$ (Aganagic-Okounkov '16)

Derivation

$$
\begin{aligned}
&\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_m}(z_m)\Phi_{\mu'_1}(z'_1)\cdots\Phi_{\mu'_n}(z'_n)\\
&=\oint \underline{dt}\oint \underline{dt'}\Phi(t,z)\omega_I(t,z,\Pi_I)\Phi(t',z')\omega_{I'}(t',z',\Pi'_{I'})\\
&=\oint \underline{dt}\oint \underline{dt'}\ \Phi(t,z)\Phi(t',z')\omega_I(t,z,\Pi_I q^{-2\sum_j <\bar{\epsilon}_{\mu'_j} j,h>})\omega_{I'}(t',z',\Pi'_{I'})\\
&=\oint \underline{dt}\oint \underline{dt'}\ \Phi(t\cup t',z\cup z')\Xi(t,t',z,z')\\
&\qquad\qquad\times\omega_I(t,z,\Pi_I q^{-2\sum_j <\bar{\epsilon}_{\mu'_j},h>})\omega_{I'}(t',z',\Pi'_{I'})
\end{aligned}
$$

Elliptic q-KZ Equation (Foda-Jimbo-Miki-Miwa-Nakayashiki '94, Felder '94)

$F(z_1,\cdots,z_n;\Pi)$: meromorphic func. valued in $V_1\otimes\cdots\otimes V_n$

$$
\begin{aligned}
&F(z_1,\cdots,q^{\kappa}z_i,\cdots,z_n;\Pi)\\
&=R^{(ii+1)}\left(\frac{q^{-\kappa}z_{i+1}}{z_i},\Pi q^{2\sum_{k=1}^{i-1}h^{(k)}}\right)\cdots R^{(in)}\left(\frac{q^{-\kappa}z_n}{z_i},\Pi q^{2\sum_{\substack{k=1\\ \neq i}}^{n-1}h^{(k)}}\right)\\
&\quad\times\Gamma_i R^{(i1)}\left(\frac{z_1}{z_i},\Pi\right)\cdots R^{(ii-1)}\left(\frac{z_{i-1}}{z_i},\Pi q^{2\sum_{k=1}^{i-2}h^{(k)}}\right)F(z_1,\cdots,z_i,\cdots,z_n;\Pi).
\end{aligned}
$$

where

- $R^{(ij)}(z,\Pi)$: elliptic dynamical R-matrix
- Π : dynamical parameter
- $\Gamma_i f(\Pi)=f(\Pi q^{2h^{(i)}})$, $\quad h^{(i)}v_\mu^{(i)}=\mathrm{wt}(v_\mu)v_\mu^{(i)}$

Solutions

(Foda-Jimbo-Miwa-Miki-Nakayashiki '94)

Let $$F^a_{\mu_1,\cdots,\mu_n}(z_1,\cdots,z_n;\Pi) := \mathrm{tr}_{\mathcal{F}_{a,\nu}}(q^{-\kappa d}\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)),$$

where $\mathcal{F}_{a,\nu} = \mathcal{F}_{a,\nu}(\xi,\eta)$ (level-1 $U_{q,p}(\widehat{\mathfrak{sl}}_N)$-module)

Then $F^a_{\mu_1,\cdots,\mu_n}(z;\Pi)$ satisfies the elliptic q-KZ eq.

Lemma 5.1

$$(1)\quad F^a_{\mu_1,\mu_2,\cdots,\mu_n}(z_1,z_2,\cdots,q^\kappa z_n;\Pi) = F^{a'}_{\mu_n,\mu_1,\cdots,\mu_{n-1}}(z_n,z_1,\cdots,z_{n-1};\Pi q^{2h^{(n)}}),$$

$$(2)\quad F^a_{\cdots,\mu_{i+1},\mu_i,\cdots}(\cdots,z_{i+1},z_i,\cdots;\Pi)$$
$$= \sum_{\mu'_i\mu'_{i+1}} R\,(z_i/z_{i+1},\Pi)^{\mu'_i\mu'_{i+1}}_{\mu_i\mu_{i+1}}\, F^a_{\cdots,\mu'_i,\mu'_{i+1},\cdots}(\cdots,z_i,z_{i+1},\cdots;\Pi).$$

(1): cyclic property of trace, $\Pi\Phi_{\mu_j}(z) = \Phi_{\mu_j}(z)\Pi q^{2h^{(j)}}$,
$\Phi_\mu(q^\kappa z) = q^{-\kappa d}\Phi_\mu(z)q^{\kappa d}$

(2): $\Phi_V(z_2)\Phi_V(z_1) = R(z_1/z_2,\Pi)\Phi_V(z_1)\Phi_V(z_2)$

Elliptic Hypergeometric Integral Solution

Theorem 5.1 (H.K '17)

$$\mathrm{tr}_{\mathcal{F}_{a,\nu}}\left(q^{-\kappa d}\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)\right)=\oint_{\mathbb{T}^M}\underline{d\mathbb{t}}\;\Phi(\mathbb{t},\mathbb{z})\omega_{\mu_1,\cdots,\mu_n}(\mathbb{t},\mathbb{z},\Pi),$$

where *with* $\sum_{j=1}^{n}\bar{\epsilon}_{\mu_j}=0$

$$\Phi(\mathbb{t},\mathbb{z})=\mathrm{tr}_{\mathcal{F}_{a,\nu}}\left(q^{-\kappa d}\widetilde{\Phi}(\mathbb{t},\mathbb{z})\right)$$

$$\sim\exp\left\{\frac{1}{\log p}\sum_{l=1}^{N-1}\log\Pi_{\alpha_l}\log\left(\prod_{a=1}^{\lambda^{(l)}}t_a^{(l)}\right)\right\}\times\prod_{l=1}^{N-1}\left(\prod_{a=1}^{\lambda^{(l)}}t_a^{(l)}\right)^{\lambda_l-h_{\alpha_l}}$$

$$\times\prod_{l=1}^{N-1}\left[\prod_{a=1}^{\lambda^{(l)}}\prod_{b=1}^{\lambda^{(l+1)}}\frac{\Gamma(t_a^{(l)}/t_b^{(l+1)};p,q^\kappa)}{\Gamma(p^*t_a^{(l)}/t_b^{(l+1)};p,q^\kappa)}\prod_{1\le a<b\le\lambda^{(l)}}\frac{\Gamma(p^*t_a^{(l)}/t_b^{(l)},p^*t_b^{(l)}/t_a^{(l)};p,q^\kappa)}{\Gamma(t_a^{(l)}/t_b^{(l)},t_b^{(l)}/t_a^{(l)};p,q^\kappa)}\right],$$

$$\Gamma(z;p,q)=\frac{(pq/z;p,q)_\infty}{(z;p,q)_\infty}$$

This is an elliptic and dynamical analogue of Mimachi '96. Geometrically, this can be identified with Okounkov's vertex function with descendent .

Cf. Quantum q-Langlands Correspondence
(Aganagic-Frankel-Okounkov '17)

Correspondence between

Solutions to the q-KZ eq. ass.w. $U_{\tilde{q}}(\widehat{\mathfrak{g}})$	and	Conformal block of $W_{q,t}({}^L\mathfrak{g})$

Geometrically a vertex function with descendent (trig.case) yields

$$\mathcal{F}_i(z) = \int_\gamma dt\ t^{\eta-1}\Phi(t,z)\mathrm{Stab}_i(t,z)$$

This corresponds to our formula

$$\mathrm{tr}_{\mathcal{F}_{a,\nu}}\left(q^{-\kappa d}\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)\right) \sim \oint \underline{d\mathbb{t}}\ \Phi(\mathbb{t},\mathbb{z})\widetilde{W}_I(\mathbb{t},\mathbb{z},\Pi)$$

In fact

- $\Phi(\mathbb{t},\mathbb{z}) \Leftrightarrow$"elliptic conf. block" (Iqbal et.al.'15, Nieli '15, Kimura-Pestun '16)
- $\widetilde{W}_I(\mathbb{t},\mathbb{z},\Pi) \ \Leftrightarrow\ \mathrm{Stab}(\mathbb{t},\mathbb{z})$

Dictionary of $U_{q,p}$ - $W_{q,t}$

level-1 $U_{q,p}(\widehat{\mathfrak{sl}}_N)$		$W_{q,t}(\mathfrak{sl}_N)$ $\quad \beta = \frac{r-1}{r}$	
$p = q^{2r}$	$\leftrightarrow$	q	
$p^* = q^{2(r-1)}$	$\leftrightarrow$	t	
$q^2 (= p/p^*)$	$\leftrightarrow$	q/t	
$F_j(z)$	$\leftrightarrow$	$S_j^+(z)$	
$E_j(z)$	$\leftrightarrow$	$S_j^-(z)$	
$\prod_{m=1}^{a} \Phi_m^D(zq^{A_m})$	$\leftrightarrow$	$V_+^a(z)$	(Awata-Yamada '10)
$\prod_{m=1}^{a} \Psi_m^{*D}(zq^{A'_m})$	$\leftrightarrow$	$V_-^a(z)$	$a = 1, \cdots, N-1$
$\prod_{m=1}^{a} \prod_{k=1}^{l-1} \Phi_m^D(zq^{A_m} p^{*B_k})$	$\leftrightarrow$	$V_u^a(z)$	$u = p^l$

Relation to the Nested Bethe Ansatz

Cf. Tarasov-Varchenko '94 trig.case

The elliptic q-KZ eq. :

$$\mathbb{K}_i(q^\kappa)F(z_1,\cdots,z_i,\cdots,z_n;\Pi)=F(z_1,\cdots,q^\kappa z_i,\cdots,z_n;\Pi),$$

where

$$\begin{aligned}\mathbb{K}_i(q^\kappa) &= R^{(ii+1)}\left(\frac{q^{-\kappa}z_{i+1}}{z_i},\Pi q^{2\sum_{k=1}^{i-1}h^{(k)}}\right)\cdots R^{(in)}\left(\frac{q^{-\kappa}z_n}{z_i},\Pi q^{2\sum_{\substack{k=1\\ \neq i}}^{n-1}h^{(k)}}\right)\\ &\quad\times\Gamma_i R^{(i1)}\left(\frac{z_1}{z_i},\Pi\right)\cdots R^{(ii-1)}\left(\frac{z_{i-1}}{z_i},\Pi q^{2\sum_{k=1}^{i-2}h^{(k)}}\right)\end{aligned}$$

On the other hand, the transfer matrix of the $A^{(1)}_{N-1}$ type face model:

$$T(z)=\mathrm{tr}_{(\mathbb{C}^N)^{(0)}}\left(\Gamma_0 R^{(01)}\left(\frac{z_1}{z},\Pi\right)R^{(02)}\left(\frac{z_2}{z},\Pi q^{2h^{(1)}}\right)\cdots R^{(0n)}\left(\frac{z_n}{z},\Pi q^{2\sum_{k=1}^{n-1}h^{(k)}}\right)\right)$$

We have

$$\operatorname*{Res}_{z=z_i}T(z)\frac{dz}{z}=\mathrm{const.}\prod_{\substack{k=1\\ \neq i}}^{n}\frac{[u_k-u_i]}{[u_k-u_i+1]}\times\mathbb{K}_i(1)$$

$q^\kappa \to 1$ Limit:

$$F(z_1,\cdots,z_n;\Pi) = \oint_{\mathbb{T}^M} \underline{d\mathbb{t}}\ \Phi(\mathbb{t},\mathbb{z}) \sum_I \omega_I(\mathbb{t},\mathbb{z},\Pi) v_I,$$

$$\Phi(\mathbb{t},\mathbb{z}) \sim e^{-\frac{1}{1-q^\kappa}\mathscr{W}(\mathbb{t},\mathbb{z})},$$

where $v_I = v_{\mu_1}\otimes\cdots\otimes v_{\mu_n}$ for $I = I_{\mu_1,\cdots,\mu_n}$.

Saddle points:

$$\frac{\partial}{\partial t_a^{(l)}}\mathscr{W}(\mathbb{t},\mathbb{z}) = 0 \quad (1\le l\le N, 1\le a\le \lambda^{(l)}) \Leftrightarrow$$ nested Bethe eqs.

Then

$$F(z_1,\cdots,z_n;\Pi) \sim \sum_I \omega_I(\mathbb{t},\mathbb{z},\Pi)\Big|_{\mathbb{t}=\mathbb{t}_0} v_I$$

where $\mathbb{t}_0$ denotes a Bethe root.

$q^\kappa \to 1$ Limit:

Similarly,

$$F(z_1,\cdots,q^\kappa z_i,\cdots,z_n;\Pi)$$
$$=\oint_{\mathbb{T}^M} \underline{d\mathbb{t}}\ \frac{\Phi(\mathbb{t},\cdots,q^\kappa z_i,\cdots)}{\Phi(\mathbb{t},\mathbb{z})}\Phi(\mathbb{t},\mathbb{z})\sum_I \omega_I(\mathbb{t},\cdots,q^\kappa z_i,\cdots,\Pi)v_I,$$

- $\displaystyle\lim_{q^\kappa\to 1}\frac{\Phi(\mathbb{t},\cdots,q^\kappa z_i,\cdots)}{\Phi(\mathbb{t},\mathbb{z})}=\prod_{a=1}^{\lambda^{(N-1)}}\frac{[v_a^{(N-1)}-u_i-1]}{[v_a^{(N-1)}-u_i]}=: E(\mathbb{t}^{(N-1)},z_i),$
- $\displaystyle\lim_{q^\kappa\to 1}\omega_I(\mathbb{t},\cdots,q^\kappa z_i,\cdots,\Pi)=\omega_I(\mathbb{t},\cdots,z_i,\cdots,\Pi)$

Hence

$$F(z_1,\cdots,q^\kappa z_i,\cdots,z_n;\Pi)\sim E(\mathbb{t}_0^{(N-1)},z_i)\sum_I \omega_I(\mathbb{t},\mathbb{z},\Pi)\Big|_{\mathbb{t}=\mathbb{t}_0} v_I.$$

Therefore

$$\mathbb{K}_i(q^\kappa)F(z_1,\cdots,z_i,\cdots,z_n;\Pi)=F(z_1,\cdots,q^\kappa z_i,\cdots,z_n;\Pi)$$

$$\Downarrow \quad q^\kappa\to 1$$

$$\left(\prod_{\substack{k=1\\ \neq i}}^{n}\frac{[u_k-u_i+1]}{[u_k-u_i]}\ \mathop{\mathrm{Res}}_{z=z_i}T(z)\frac{dz}{z}\right)\sum_I \omega_I(\mathbb{t},\mathbb{z},\Pi)\Big|_{\mathbb{t}=\mathbb{t}_0} v_I$$

$$=E(\mathbb{t}_0^{(N-1)},z_i)\sum_I \omega_I(\mathbb{t},\mathbb{z},\Pi)\Big|_{\mathbb{t}=\mathbb{t}_0} v_I$$

$$(i=1,\cdots,n)$$

Representations of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ on $V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_n}$

The vector representation : $V_z = V\otimes\mathbb{C}[[z,z^{-1}]], \quad V=\oplus_{\mu=1}^N\mathbb{C}v_\mu$

Proposition 7.1 (Kojima-K '03)

The following gives a level-0 (i.e. $c=0$) $U_{q,p}(\widehat{\mathfrak{sl}}_N)$-module str. on V_z.

$$\psi_j^{\pm}(q^jw)v_\mu = \begin{cases} \left.\frac{[v-u+1]}{[v-u]}\right|_{\pm} e^{-Q_{\alpha_j}}v_\mu & (\mu=j)\\ \left.\frac{[v-u-1]}{[v-u]}\right|_{\pm} e^{-Q_{\alpha_j}}v_\mu & (\mu=j+1)\\ v_\mu & (\mu\neq j,j+1)\end{cases}$$

$$e_j(q^jw)v_\mu = \begin{cases} C_+\delta(z/w)e^{-Q_{\alpha_j}}v_{\mu-1} & (\mu=j+1)\\ 0 & (\mu\neq j+1)\end{cases}$$

$$f_j(q^jw)v_\mu = \begin{cases} C_-\delta(z/w)v_{\mu+1} & (\mu=j)\\ 0 & (\mu\neq j+1)\end{cases}$$

where

$$C_\pm = \frac{(pq^{\pm 2};p)_\infty}{(p;p)_\infty}$$

Representation on $V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_n}$

Problem : Find an level 0 (i.e. $c=0$) action of the elliptic currents

$$E_j(z), F_j(z), \psi_j^{\pm}(z) \text{ of } U_{q,p}(\widehat{\mathfrak{sl}}_N) \text{ on } V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_n}$$

- Dynamical L-operator : $L^+(z) = L^+(z,P)e^{-\sum_j h_{\epsilon_j}Q_{\bar{\epsilon}_j}}$

$$[P_{\bar{\epsilon}_j}, Q_{\bar{\epsilon}_k}] = \delta_{j,k} - 1/N$$

- Vector representation: $V = \oplus_{\mu=1}^N \mathbb{C}v_\mu$

$$\pi_z(L_{ij}^+(w))v_\nu = \pi_z(L_{ij}^+(w,P)e^{-Q_{\bar{\epsilon}_j}})v_\nu = \sum_\mu \bar{R}(z/w,P)_{i\mu}^{j\nu}v_\mu$$

- Then $L^+(w)$ acts on $V_w\widetilde{\otimes}V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_n}$ by

$$\pi_{z_1}\otimes\cdots\otimes\pi_{z_n}\Delta'^{(n-1)}(L^+(w))$$
$$= \bar{R}^{(0n)}(z_n/w, P+\sum_{j=1}^{n-1}h^{(j)})\bar{R}^{(0n-1)}(z_{n-1}/w, P+\sum_{j=1}^{n-2}h^{(j)})\cdots\bar{R}^{(01)}(z_1/w,P)$$

Δ' : the opposite coproduct: $\Delta'(L_{ij}^+(z)) = \sum_k L_{kj}^+(z)\widetilde{\otimes}L_{ik}^+(z)$

The Half Currents & the Quantum Minor Determinants

Define also $L^-(z) := L^+(zp) \quad \rightsquigarrow \quad E^-_{l,j}(z),\ F^-_{j,l}(z),\ K^-_j(z)$

Theorem 7.2 (H.K'16)

$$K^{\pm}_j(z) = const.\frac{q\text{-}\det L^{\pm}(z)_{j,j}}{q\text{-}\det L^{\pm}(zq^{-2})_{j+1,j+1}}$$

$$E^{\pm}_{k,j}(z) = \frac{q\text{-}\det L^{\pm}(z)_{k,j}}{q\text{-}\det L^{\pm}(z)_{k,k}}$$

$$F^{\pm}_{j,k}(z) = \frac{q\text{-}\det L^{\pm}(z)_{j,k}}{q\text{-}\det L^{\pm}(z)_{k,k}} \qquad (j<k)$$

Theorem 7.3 (H.K'16)

Set

$$\psi^{\pm}_j(z) := const.K^{\pm}_j(z)K^{\pm}_{j+1}(z)^{-1}e^{-Q_{\alpha_j}},$$

$$E_j(zq^{j-c/2}) := const.e^{Q_{\bar{\epsilon}_{j+1}}}(E^{+}_{j+1,j}(zq^{c/2},P) - E^{-}_{j+1,j}(zq^{-c/2},P))e^{-Q_{\bar{\epsilon}_j}},$$

$$F_j(zq^{j-c/2}) := const.(F^{+}_{j,j+1}(zq^{-c/2},P) - F^{-}_{j,j+1}(zq^{c/2},P)).$$

Then $\psi^{\pm}_j(z), E_j(z), F_j(z)$ satisfy the defining relations of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$.

The Gelfand-Tsetlin Basis

- The Gelfand-Tsetlin basis
  $\overset{\text{def}}{\Leftrightarrow}$ the eigenbasis of the Gelfand-Tsetlin subalgebra

- The Gelfand-Tsetlin subalgebra $\mathfrak{G}$ of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ at level 0:
  a unital commutative subalgebra generated by $K_j^+(z)\ \ (j=1,\cdots,N)$

Remark 7.4 (H.K '16)

Even $c \neq 0$,

$$K_1^+(z)\cdots K_l^+(zq^{-2(l-1)}) \in \mathcal{Z}(U_{q,p}(\widehat{\mathfrak{gl}}_l))\ (l=1,\cdots,N)$$

Construction of the GT Basis in $V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_n}$,

(Gorbounov-Rimanyi-Tarasov-Varchenko'13,'14 : rational & trig. cases)

- Realization of $\mathfrak{S}_n$ in terms of the elliptic dynamical R:

  Define $\widetilde{S}_i(P)$ by $\widetilde{S}_i(P) := \mathcal{P}^{(ii+1)}R^{(ii+1)}(z_i/z_{i+1}, P + \sum_{j=1}^{i-1} h^{(j)})s_i^z$,

  $$\mathcal{P} : v\widetilde{\otimes}w \mapsto w\widetilde{\otimes}v, \qquad s_i^z : z_i \leftrightarrow z_{i+1}$$

  Then DYBE and the unitarity of R yields

  $$\widetilde{S}_i(P)\widetilde{S}_{i+1}(P)\widetilde{S}_i(P) = \widetilde{S}_{i+1}(P)\widetilde{S}_i(P)\widetilde{S}_{i+1}(P),$$
  $$\widetilde{S}_i(P)\widetilde{S}_j(P) = \widetilde{S}_j(P)\widetilde{S}_i(P) \qquad (|i-j|>1)$$
  $$\widetilde{S}_i(P)^2 = 1$$

- For $\lambda = (\lambda_1, \cdots, \lambda_N)$, $I = I_{\mu_1\cdots\mu_n} \in \mathcal{I}_\lambda$, set $v_I := v_{\mu_1}\widetilde{\otimes}\cdots\widetilde{\otimes}v_{\mu_n}$.

  Define GT bases $\{\xi_I\}_{I\in\mathcal{I}_\lambda}$ by

  $$\xi_{I^{max}} := v_{I^{max}}, \qquad \text{where} \quad I^{max} = I_{\underbrace{N\cdots N}_{\lambda_N}\cdots\underbrace{1\cdots 1}_{\lambda_1}}$$

  $$\xi_{s_i(I)} := \widetilde{S}_i(P)\xi_I$$

Explicit Realization :

Theorem 7.5 (H.K '18)

$$\xi_I = \sum_{J\in\mathcal{I}_\lambda} \Xi_{JI}(z,P)v_J,$$
$$\Xi_{JI}(z,P) = \widetilde{W}_J(z_I^{-1}, z^{-1}, \Pi q^{2\sum_{j=1}^{n} <\bar{\epsilon}_{\mu_j},h>}).$$

$\because$ the transition property of $\widetilde{W}_J(z_I, z, \Pi)$

$\Leftrightarrow$ recursion formula for Ξ obtained from $\xi_{s_i(I)} = \widetilde{S}_i(P)\xi_I$

Remark 7.6

Su '17 obtained the same recursion formula for $Stab_{\mathfrak{C}}(F_J)|_{F_I}$ for $H_T^*(T^*(G/P))$.

Key Property:

Proposition 7.7

$$\widetilde{S}_i(P){\Delta'}^{(n-1)}(L^{\pm}(w,P)) = {\Delta'}^{(n-1)}(L^{\pm}(w,P))\widetilde{S}_i(P+h^{(0)})$$

$\because$ DYBE.

Hence it suffices to construct an action of $\Delta^{(n-1)}(L^{\pm}(w,P))$ on $\xi_{I^{max}}$.

Action of the Half Currents on the GT Basis

Theorem 7.8 (H.K '18)

$$K_j^{\pm}(w)\xi_I = \prod_{k=1}^{j-1}\prod_{a\in I_k}\left.\frac{[u_a-v]}{[u_a-v+1]}\right|_{\pm}\prod_{l=j+1}^{N}\prod_{b\in I_l}\left.\frac{[u_b-v-1]}{[u_b-v]}\right|_{\pm}\xi_I$$

$$E_{j+1,j}^{\pm}(w,P)\xi_I = \sum_{i\in I_{j+1}}\left.\frac{[P_{j,j+1}-u_i+v][1]}{[P_{j,j+1}][u_i-v]}\right|_{\pm}\prod_{\substack{k\in I_{j+1}\\ \neq i}}\frac{[u_i-u_k+1]}{[u_i-u_k]}\xi_{I^{i'}}$$

$$F_{j,j+1}^{\pm}(w,P)\xi_I = \sum_{i\in I_j}\left.\frac{[P_{j,j+1}+\lambda_j-\lambda_{j+1}+u_i-v-1][1]}{[P_{j,j+1}+\lambda_j-\lambda_{j+1}-1][u_i-v]}\right|_{\pm}\prod_{\substack{k\in I_j\\ \neq i}}\frac{[u_k-u_i+1]}{[u_k-u_i]}\xi_{I'^{i}}$$

where $w=q^{2v}, z_i=q^{2u_i}\ (i=1,\cdots,n)$, *and* $I=(I_1,\cdots,I_N)$

$$(I^{i'})_j = I_j\cup\{i\},\quad (I^{i'})_{j+1}=I_{j+1}-\{i\},\quad (I^{i'})_k=I_k\ \ (k\neq j,j+1),$$
$$(I'^{i})_j = I_j-\{i\},\quad (I'^{i})_{j+1}=I_{j+1}\cup\{i\},\quad (I'^{i})_k=I_k\ \ (k\neq j,j+1)$$

Since $L^-(w,P)=L^+(pw,P)$ (at $c=0$),

$K_j^-(w)=K_j^+(pw), E^-_{j+1,j}(w,P)=E^+_{j+1,j}(pw,P),\ F^-_{j,j+1}(w,P)=F^+_{j,j+1}(pw,P)$.

We define $|_\pm$ by

$$\left.\frac{[s+u]}{[s][u]}\right|_+ = w^{\frac{s}{r}}\frac{\theta_p(q^{2s}w)}{\theta_p(q^{2s})\theta_p(w)} \qquad \text{expand in } w$$

$$\left.\frac{[s+u]}{[s][u]}\right|_- = (pw)^{\frac{s}{r}}\frac{\theta_p(pq^{2s}w)}{\theta_p(q^{2s})\theta_p(pw)}$$

$$= -w^{\frac{s}{r}}\frac{\theta_p(q^{-2s}/w)}{\theta_p(q^{-2s})\theta_p(1/w)} \qquad \text{expand in } 1/w \quad \text{etc.}$$

where $w=q^{2u}$, $p=q^{2r}$.

Furthermore note the formula

$$\left.\frac{[s+u]}{[s][u]}\right|_+ - \left.\frac{[s+u]}{[s][u]}\right|_- = \frac{1}{[0]'}\delta(w)$$

Action of the Elliptic Currents on the GT Basis

Corollary 7.1 (H.K '18)

$$\psi_j^{\pm}(w)\xi_I=\prod_{a\in I_j}\frac{[u_a-v+1]}{[u_a-v]}\Bigg|_{\pm}\prod_{b\in I_{j+1}}\frac{[u_b-v-1]}{[u_b-v]}\Bigg|_{\pm}e^{-Q_{\alpha_j}}\xi_I$$

$$E_j(w)\xi_I=a^*\sum_{i\in I_{j+1}}\delta(z_i/w)\prod_{\substack{k\in I_{j+1}\\ \neq i}}\frac{[u_i-u_k+1]}{[u_i-u_k]}\,e^{-Q_{\alpha_j}}\xi_{I^{i'}}$$

$$F_j(w)\xi_I=a\sum_{i\in I_j}\delta(z_i/w)\prod_{\substack{k\in I_j\\ \neq i}}\frac{[u_k-u_i+1]}{[u_k-u_i]}\,\xi_{I'^{i}}$$

where $aa^*=-\dfrac{1}{q-q^{-1}}\dfrac{[0]'}{[1]}$.

Remark 7.9

In the trig. and non-dynamical limit, the combinatorial str. coincides with the geom. rep. of $U_q(\widehat{\mathfrak{sl}}_N)$ on the equiv. K-theory of the quiver variety of type A_{N-1} obtained by Ginzburg, Vasserot '98 and Nakajima '00.

Equivariant Elliptic Cohomology

- $X=T^*\mathcal{F}_\lambda, \quad \lambda=(\lambda_1,\cdots,\lambda_N),\ |\lambda|=n$ (Aganagic, Okounkov '16)
- $T=A\times\mathbb{C}^\times$, $A=(\mathbb{C}^\times)^n$
- $X^A=\sqcup_{I\in\mathcal{I}_\lambda}F_I$: A-fixed points locus with connected comp. F_I
- $\mathrm{E}=\mathbb{C}^\times/p^{\mathbb{Z}}$: elliptic curve, a group scheme over $\mathbb{C}$.
- $\mathrm{Ell}_T(X)$: T-equiv. elliptic cohomology, a scheme over

$$\mathrm{Ell}_T(\mathrm{pt})\cong T/p^{\mathrm{cochar(T)}}=\mathrm{E}^n\times\mathrm{E}.$$

Due to a construction of X as a hyper-Kähler quotient

$$T^*\left(\bigoplus_{l=1}^{N-1}\mathrm{Hom}(\mathbb{C}^{\lambda^{(l)}},\mathbb{C}^{\lambda^{(l+1)}})\right)//\prod_{l=1}^{N-1}GL(\lambda^{(l)},\mathbb{C}),$$

$\exists$ tautological vector bundles $\{V_l\}$ of rank $\lambda^{(l)}$ $(l=1,\cdots,N-1)$ over X and a map

$$\mathrm{Ell}_T(X)\to\mathrm{Ell}_T(\mathrm{pt})\times\mathrm{E}^{(\lambda^{(1)})}\times\cdots\times\mathrm{E}^{(\lambda^{(N-1)})}$$

embedding near the origin of $\mathrm{Ell}_T(\mathrm{pt})$ (McGerty-Nevins'16)

- $t_a^{(l)}$ $(a=1,\cdots,\lambda^{(l)})$: the Chern roots of V_l

Kähler Parameters (= Dynamical Parameters)

Let

$$\mathcal{E}_{\mathrm{Pic}_T(X)} := \mathrm{Pic}_T(X) \otimes_{\mathbb{Z}} \mathrm{E}.$$

Define

$$\mathrm{E}_T(X) := \mathrm{Ell}_T(X) \times \mathcal{E}_{\mathrm{Pic}_T(X)}$$

as a scheme over

$$\mathcal{B}_{T,X} := \mathrm{Ell}_T(\mathrm{pt}) \times \mathcal{E}_{\mathrm{Pic}_T(X)}.$$

$$\begin{array}{cc} \nearrow & \uparrow \\ z_1, \cdots, z_n, q^2 & \Pi_{j,j+1}\ (1 \le j \le N-1) \end{array}$$

(the equiv. parameters) (the Kähler parameters)

Universal Line Bundle

In general, an equiv. rank r complex vector bundle V over X defines Chern class

$$c : \mathrm{Ell}_T(X) \to \mathrm{Ell}_{GL(r)}(\mathrm{pt}) \cong \mathrm{E}^{(r)}$$

coordinates on the target are symm. funcs on E^r

For line bundles ($r = 1$ case), this Chern class gives a group hom.

$$\mathrm{Pic}_T(X) \to \mathrm{Maps}(\mathrm{Ell}_T(X) \to \mathrm{E}),$$

which can be viewed as a map

$$\tilde{c} : \mathrm{Ell}_T(X) \to \mathcal{E}^{\vee}_{\mathrm{Pic}_T(X)},$$

where

$$\mathcal{E}^{\vee}_{\mathrm{Pic}_T(X)} = \mathrm{Hom}(\mathrm{Pic}_T(X), \mathrm{E})$$

is the dual abelian variety of $\mathcal{E}_{\mathrm{Pic}_T(X)} = \mathrm{Pic}_T(X) \otimes_{\mathbb{Z}} \mathrm{E}$.

Note $\exists$ a universal line bundle $\mathcal{U}_{\text{Poincaré}}$ over $\mathcal{E}^{\vee}_{\mathrm{Pic}_T(X)} \times \mathcal{E}_{\mathrm{Pic}_T(X)}$.
e.g. Sections of $\mathcal{U}_{\text{Poincaré}}$ on $\mathrm{E}^{\vee} \times \mathrm{E}$ are analytic functions of the form

$$\frac{\theta_p(zw)}{\theta_p(z)\theta_p(w)}.$$

Therefore

$$\mathrm{E}_T(X) = \mathrm{Ell}_T(X) \times \mathcal{E}_{\mathrm{Pic}_T(X)},$$

$$\tilde{c} : \mathrm{Ell}_T(X) \to \mathcal{E}^{\vee}_{\mathrm{Pic}_T(X)}$$

$\rightsquigarrow$ $\mathcal{U} = (\tilde{c} \times 1)^* \, \mathcal{U}_{\text{Poincaré}}$ is a line bundle on $\mathrm{E}_T(X)$.
Hence

$$\frac{\theta_p(z\Pi)}{\theta_p(z)\theta_p(\Pi)} \sim \frac{[u+P+h]}{[u][P+h]}$$

can be regarded as a part of a section of $\mathcal{U}$

Elliptic Stable Envelopes

- $\mathfrak{C} \in \mathrm{Lie}A \otimes \mathbb{R}$ be a chamber of $\mathrm{Lie}A$ s.t. $z_j/z_i > 0 \ (i < j)$.
- For F_K in $X^A = \sqcup_{I \in \mathcal{I}_\lambda} F_I$, define an attracting manifold

$$\mathrm{Attr}_{\mathfrak{C}}(F_K) = \{x \in X \mid \lim_{t \to 0} \rho(t)x \in F_K, \ \rho(t) \in \mathfrak{C}\ \}$$

and a partial ordering for $F_I, F_J \in X^A$ by

$$F_J \leq F_I \quad \Leftrightarrow \quad \mathrm{Attr}_{\mathfrak{C}}(F_I) \cap F_J \neq \emptyset.$$

Definition 8.1

The elliptic stable envelope is a map of $\mathcal{O}_{\mathcal{B}_{T,X}}$-modules

$$\mathrm{Stab}_{\mathfrak{C}} : \Theta(T^{1/2}X^A) \otimes \mathcal{U}' \to \Theta(T^{1/2}X) \otimes \mathcal{U} \otimes \cdots$$

s.t.

(i) (triangularity) If $F_K < F_I$, $\mathrm{Stab}_{\mathfrak{C}}(F_K)|_{F_I} = 0$

(ii) (normalization) Near the diagonal in $X \times F_K$, we have

$\mathrm{Stab}_{\mathfrak{C}} = (-1)^{\mathrm{rk\ ind}} j_* \pi^*$, where $F_K \xleftarrow{\pi} \mathrm{Attr}_{\mathfrak{C}}(F_K) \xrightarrow{j} X$

are the natural projection and inclusion maps.

Direct Comparison Between $\mathcal{W}_I$ and $\mathrm{Stab}_{\mathfrak{C}}(F_I)$

Abelianization of X:

- M : a vector space $\curvearrowleft G$: a reductive group
- $S \subset G$: the maximal torus
- μ_G : the moment map
- $\mu_S := \pi_S \circ \mu_G$ by a projection $\pi_S : (\mathrm{Lie}\, G)^* \to (\mathrm{Lie}\, S)^*$

For the hyper-Kähler quotient

$$X = \mu_G^{-1}//G,$$

the abelian quotient

$$X_S = \mu_S^{-1}//S$$

is called the abelianization of X.

Successive Construction of $\mathrm{Stab}_{\mathfrak{C}}$ (Shenfeld'13, Aganagic-Okounkov'16)

- $T^*\mathbb{P}(\mathbb{C}^n)$ case: $F_a = \mathbb{C}v_a$ $(a = 1, \cdots, n)$: $A = \mathrm{diag}(z_1, \cdots, z_n)$-fixed pts.

$$\mathrm{Stab}_{\mathfrak{C}^{(n)}}^{T^*\mathbb{P}(\mathbb{C}^n)}(F_a) = \prod_{b<a}[u_b - v] \times \frac{[-u_a + v + (P+h)_{1,2} + n - a]}{[(P+h)_{1,2} + n - a]} \times \prod_{b>a}[u_b - v + 1]$$

- $X = T^*\mathrm{Gr}(k,n)$ case : $S^{(k)} \subset GL(k)$, $\mu : [1,k] \to [1,n]$,

$I^{(k)} = \{\mu(1) < \cdots < \mu(k)\}$

$\rightsquigarrow X_{S^{(k)}} = (T^*\mathbb{P}(\mathbb{C}^n))^k$

$$\mathrm{Stab}_{\mathfrak{C}^n}^{(T^*\mathrm{Gr})_S}(F_{I^{(k)}}) = \prod_{a=1}^{k} \text{“} \left.\mathrm{Stab}_{\mathfrak{C}^{(n)}}^{T^*\mathbb{P}(\mathbb{C}^n)}(F_{\mu(a)})\right|_{v \mapsto v_a} \text{ with dynamical shift ”}$$

- $X = T^*\mathcal{F}_\lambda$ case: $S = \prod_{l=1}^{N-1} S^{(\lambda^{(l)})} \subset \prod_{l=1}^{N-1} GL(\lambda^{(l)})$, $I = (I_1, \cdots, I_N)$

$\rightsquigarrow X_S =$ “ $\prod_{l=1}^{N-1}(T^*\mathbb{P}(\mathbb{C}^{\lambda^{(l+1)}}))^{\lambda^{(l)}}$ ”

$$\mathrm{Stab}_{\mathfrak{C}}^{X_S}(F_I) = \prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}} \text{“} \left.\mathrm{Stab}_{\mathfrak{C}^{\lambda^{(l+1)}}}^{T^*\mathbb{P}(\mathbb{C}^{(\lambda^{(l+1)})})}(F_{i^{(l+1)}_{\mu(a)}})\right|_{v \mapsto v_a^{(l)}} \text{ with dynamical shift ”}$$

Abelianization formula yields $\mathrm{Stab}_{\mathfrak{C}}$ for $X = T^*\mathcal{F}_\lambda$ as

$$\mathrm{Stab}_{\mathfrak{C}}(F_I)$$
$$= \mathrm{Sym}_{t^{(1)}} \cdots \mathrm{Sym}_{t^{(N-1)}} \left[\frac{\mathrm{Stab}_{\mathfrak{C}}^{X_S}(F_I)}{\prod_{l=1}^{N-1} \prod_{1\le a<b\le \lambda^{(l)}} [v_a^{(l)} - v_b^{(l)}][v_b^{(l)} - v_a^{(l)} - 1]} \right]$$

Then we find

(1) $\mathrm{Stab}_{\mathfrak{C}}(F_I) = \mathcal{W}_{\sigma_0(I)}(\tilde{\mathbb{t}}, \sigma_0(\mathbb{z}^{-1}), \Pi^{-1})$, $\quad \tilde{t}^{(l)}_{\sigma_0^{(l)}(a)} = t_a^{(l)}$,

(2) $\mathrm{Stab}_{\mathfrak{C}}(F_I)|_{F_J} = \mathcal{W}_{\sigma_0(I)}(\mathbb{z}_J^{-1}, \sigma_0(\mathbb{z}^{-1}), \Pi^{-1})$

- $\mathbb{t} = (t_a^{(l)})$ $(l = 1, \cdots, N-1, a = 1, \cdots, \lambda^{(l)})$: the Chern roots of the tautological vec. b'dles on X
- $\mathbb{z} = (z_1, \cdots, z_n), q^2$: the equiv. parameters
- $\Pi = (\Pi_{l,l+1})$ $(l = 1, \cdots, N-1)$: the Kähler parameters

$$\mathcal{W}_I(\mathbb{t},\mathbb{z},\Pi)$$

$$= \mathrm{Sym}_{t^{(1)}}\cdots\mathrm{Sym}_{t^{(N-1)}}\left[\frac{\prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}} u_I^{(l)}(t_a^{(l)},\mathbb{t}^{(l+1)},\Pi_{\mu_{i_a^{(l)}},l+1}q^{-2C_{\mu_{i_a^{(l)}},l+1}})}{\prod_{l=1}^{N-1}\prod_{1\le a<b\le\lambda^{(l)}}[v_a^{(l)}-v_b^{(l)}][v_b^{(l)}-v_a^{(l)}-1]}\right],$$

$$u_I^{(l)}(t_a^{(l)},\mathbb{t}^{(l+1)},\Pi_{j,k})$$

$$= \prod_{\substack{b=1\\ i_b^{(l+1)}>i_a^{(l)}}}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}]\times\left.\frac{[v_b^{(l+1)}-v_a^{(l)}+(P+h)_{j,k}]}{[(P+h)_{j,k}]}\right|_{i_b^{(l+1)}=i_a^{(l)}}\times\prod_{\substack{b=1\\ i_b^{(l+1)}<i_a^{(l)}}}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}+1]$$

The Stable Classes and the Fixed Point Classes in $\mathrm{E}_T(X)$

By definition, "the stable classes" $\mathrm{Stab}_{\mathfrak{C}}(F_J)$ are triangular w.r.t the fixed point classes $\{[I]\}_{I\in\mathcal{I}_\lambda}$

$$\mathrm{Stab}_{\mathfrak{C}}(F_J) = \sum_{I\in\mathcal{I}_\lambda} \frac{\mathrm{Stab}_{\mathfrak{C}}(F_J)|_{F_I}}{R(z_I^{-1})}[I]$$ Cf. A.Smirnov '14

Here we take $R(z_I^{-1}) = \prod_{1\le k<l\le N}\prod_{a\in I_k}\prod_{b\in I_l}[u_a-u_b]$.

Under the identification

$$\mathrm{Stab}_{\mathfrak{C}}(F_J)|_{F_I} = \mathcal{W}_{\sigma_0(J)}(\mathbb{z}_I^{-1}, \sigma_0(\mathbb{z}^{-1}), \Pi^{-1}),$$

the orthogonality of $\mathcal{W}_J$ yields

$$[I] = \sum_{J\in\mathcal{I}_\lambda} \widetilde{W}_J(\mathbb{z}_I^{-1}, \mathbb{z}^{-1}, \Pi q^{2\sum_j<\bar{\epsilon}_{\mu_j},h>})\ \mathrm{Stab}_{\mathfrak{C}}(F_J).$$

This is identical to $\xi_I = \sum_{J\in\mathcal{I}_\lambda} \widetilde{W}_J(\mathbb{z}_I^{-1}, \mathbb{z}^{-1}, \Pi q^{2\sum_j<\bar{\epsilon}_{\mu_j},h>})v_J$ (Thm 7.5) !

Hence

$$\text{the Gelfand-Tsetlin base } \xi_I \quad \Leftrightarrow \quad \text{the fixed point class } [I],$$
$$\text{the standard base } v_J \quad \Leftrightarrow \quad \text{the stable class } \mathrm{Stab}_{\mathfrak{C}}(F_J)$$

Remark 8.2

For different algebras and geometries, the same correspondence has been studied by Nagao'09, Feigin, Finkelberg, Frenkel, Negut, Rybnikov'11, Tsymbaliuk'10.

Theorem 8.3 (H.K '18)

The following gives an action of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ *on* $\mathrm{E}_T(X)$.

$$\psi_j^{\pm}(w)[I] = \prod_{a\in I_j} \frac{[u_a - v + 1]}{[u_a - v]}\Bigg|_{\pm} \prod_{b\in I_{j+1}} \frac{[u_b - v - 1]}{[u_b - v]}\Bigg|_{\pm} \; e^{-Q_{\alpha_j}}[I]$$

$$E_j(w)[I] = a^* \sum_{i\in I_{j+1}} \delta(z_i/w) \prod_{\substack{k\in I_{j+1}\\ \neq i}} \frac{[u_i - u_k + 1]}{[u_i - u_k]} \; e^{-Q_{\alpha_j}}[I^{i'}]$$

$$F_j(w)[I] = a \sum_{i\in I_j} \delta(z_i/w) \prod_{\substack{k\in I_j\\ \neq i}} \frac{[u_k - u_i + 1]}{[u_k - u_i]} \; [I'^{i}]$$

2nd Hopf Algebroid Structure : the Drinfeld Copro. Δ^D

Theorem 9.1 (H.K '14)

The following $(\Delta^D, S^D, \varepsilon^D)$ *gives an another* H*-Hopf algebroid str. of* $U_{q,p}(\widehat{\mathfrak{g}})$.

$$\Delta^D(\mu_l(f)) = \mu_l(f)\widetilde{\otimes}1, \quad \Delta^D(\mu_r(f)) = 1\widetilde{\otimes}\mu_r(f),$$

$$\Delta^D(\psi_j^{\pm}(z)) = \psi_j^{\pm}(q^{\pm c^{(2)}/2}z)\widetilde{\otimes}\psi_j^{\pm}(q^{\mp c^{(1)}/2}z) \qquad \cdots \quad \textit{group like !}$$

$$\Delta^D(e_j(z)) = 1\widetilde{\otimes}e_j(z) + e_j(zq^{-c^{(2)}})\widetilde{\otimes}\psi_j^-(q^{-c^{(2)}/2}z),$$

$$\Delta^D(f_j(z)) = f_j(z)\widetilde{\otimes}1 + \psi_j^+(q^{-c^{(1)}/2}z)\widetilde{\otimes}f_j(zq^{-c^{(1)}})$$

$$\varepsilon^D(q^c) = 1,\ \varepsilon^D(g(\Pi^*, p^*)) = g(\Pi^*, p^*),\ \varepsilon^D(g(\Pi, p)) = g(\Pi, p),$$
$$\varepsilon^D(\psi_j^{\pm}(z)) = e^{-Q_{\alpha_j}},\ \varepsilon^D(e_j(z)) = \varepsilon^D(f_j(z)) = 0$$

$$S^D(q^c) = q^{-c}, \quad S^D(g(\Pi^*, p^*)) = g(\Pi, p), \quad S^D(g(\Pi, p)) = g(\Pi^*, p^*),$$
$$S^D(\psi_j^{\pm}(z)) = \psi_j^{\pm}(q^{-c}z)^{-1},$$
$$S^D(e_j(z)) = -\psi_j^-(q^{c/2}z)e_j(q^c z),$$
$$S^D(e_j(z)) = -f_j(q^c z)\psi_j^+(q^{c/2}z)^{-1}$$

Hopf Algebroid Twistor

(Drinfeld '89, Khoroshkin-Tolstoy '93, Jimbo-K-Odake-Shiraishi '99)

$(U_{q,p}, \Delta, \varepsilon, S, R(\Pi))$ and $(U_{q,p}, \Delta^D, \varepsilon^D, S^D, R^D(\Pi))$ give two different quasi-triangular co-associative H-Hopf algebroids.

$\exists$ a twistor $\mathcal{F}(\Pi) \in U_{q,p}\widetilde{\otimes}U_{q,p}$, invertible, s.t.

$$\mathcal{F}^{(12)}(\Pi)(\Delta^D\widetilde{\otimes}\mathrm{id})\mathcal{F}(\Pi) = \mathcal{F}^{(23)}(\Pi)(\mathrm{id}\widetilde{\otimes}\Delta^D)\mathcal{F}(\Pi)$$

and

$$\Delta(x) = \mathcal{F}(\Pi)\Delta^D(x)\mathcal{F}^{-1}(\Pi),$$
$$R(\Pi) = \mathcal{F}^{(21)}(\Pi)R^D(\Pi)\mathcal{F}(\Pi)^{-1}.$$

Two Actions of $U_{q,p}$ on $V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_n}$

It turns out that the action

$(\Delta^D)^{(n-1)}(x(w))$ on $\widetilde{v}_I$ coincides with $\Delta^{(n-1)}(x(w))$ on ξ_I,

for $x=\psi_j^{\pm}, e_j, f_j$. Here

$$\widetilde{v}_I = N_I(\mathbb{z})v_I,$$

where $N_I(\mathbb{z})$ satisfies, for $i\in I_{j+1}$

$$\frac{N_I(\mathbb{z})}{N_{I^{i'}}(\mathbb{z})} = \prod_{\substack{k\in I_j\\ i+1\le k}} \frac{\theta(q^2 z_i/z_k)}{\theta(z_i/z_k)} \prod_{\substack{k\in I_{j+1}\\ k\le i-1}} \frac{\theta(z_i/z_k)}{\theta(q^{-2}z_i/z_k)}.$$

$\because$

$\psi_j^{\pm}(w)$ are diagonal on v_μ, $\Delta^D(\psi_j^{\pm}(w)) = \psi_j^{\pm}(w)\widetilde{\otimes}\psi_j^{\pm}(w)$.
Hence $(\Delta^D)^{(n-1)}(\psi_j^{\pm}(w)$ are diagonal on $v_I = v_{\mu_1}\widetilde{\otimes}\cdots\widetilde{\otimes}v_{\mu_n}$.

Weight Function as the Hopf Algebroid Twistor

One finds

$$\xi_I \quad = \quad (\pi_{V_{z_1}} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{V_{z_n}}) \mathcal{F}^{(n-1)}(\Pi) N_I(\mathbb{z}) v_I,$$

where $\mathcal{F}^{(n-1)}(\Pi) \in (U_{q,p})^{\widetilde{\otimes} n}$ is the Hopf algebroid twistor s.t.

$$\Delta^{(n-1)}(x) = \mathcal{F}^{(n-1)}(\Pi) (\Delta^D)^{(n-1)}(x) \mathcal{F}^{(n-1)}(\Pi)^{-1}.$$

Comparing this with $\xi_I = \sum_{J \in \mathcal{I}_\lambda} \widetilde{W}_J(\mathbb{z}_I^{-1}, \quad \cdots \quad) \, v_J$ (Theorem 7.5),

one finds

$$\widetilde{W}_J(\mathbb{z}_I^{-1}, \mathbb{z}^{-1}, \Pi q^{2 \sum_{j=1}^{n} <\bar{\epsilon}_{\mu_j}, h>}) = \Big((\pi_{V_{z_1}} \widetilde{\otimes} \cdots \widetilde{\otimes} \pi_{V_{z_n}}) \mathcal{F}^{(n-1)}(\Pi) \Big)_{JI} N_I(\mathbb{z})$$

Triality in Hypergeometric Integrals, Hopf Algebras (-roids) and Equiv. Cohomologies

Weight Function

- a basis of Aomoto twisted de Rham cohomology
- $\xi_I = \sum_J \widetilde{W}_J(z_I^{-1}, \cdots)v_J$

⬃ ⬂

Hopf Algebra (-roid) Twistor

- $\Delta(x) = \mathcal{F}\Delta^D(x)\mathcal{F}^{-1}$
- $\xi_I = \sum_J (\mathcal{F})_{JI} N_I \; v_J$

⇔

Stable Envelope

- a “basis” of equiv. cohomology
- $[I] = \sum_J \mathrm{Stab}_{\mathfrak{C}}^{-1}(F_J)\Big|_{F_I} \mathrm{Stab}_{\mathfrak{C}}(F_J)$