研究集会「q,q qnd q」
2020年2年19日～21日，於神戸大理学部

微分方程式に付随する接続問題への交叉理論の応用

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1 Rigid systems of Fuchsian differential equations

Rigid local system

\# of irreducible rigid Fuchsian differential systems with 3 singularities on $\mathbb{P}^1$

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	1	3	5	13	20	45	74	142	212	421	588	1004

\# of irreducible rigid Fuchsian differential systems on $\mathbb{P}^1$

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	2	6	11	28	44	96	157	306	441	857	1117	2032

Simpson's list

	rank	spectral type
${}_nF_{n-1}$	n	$1^n\ ;\quad 1^n\ ;\quad n-1,1$
Even family	$2m$	$1^{2m}\ ;\quad m,m-1,1\ ;\quad m,m$
Odd family	$2m+1$	$1^{2m+1}\ ;\quad m,m,1\ ;\quad m+1,m$
Extra case	6	$1^6\ ;\quad 2^3\ ;\quad 4,2$

1.1 The differential equation ${}_{n+1}E_n$

The generalized hypergeometric function ${}_{n+1}F_n$ is tha analytic continuation of the generalized hypergeometric series

$$ {}_{n+1}F_n\left(\begin{matrix}\alpha_1,\alpha_2,\ \ldots,\ \alpha_{n+1}\\ \beta_1,\ \ldots\ldots,\ \beta_n\end{matrix};x\right)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_{n+1})_k}{(\beta_1)_k\ \cdots\cdots\ (\beta_n)_k\,k!}\,x^k. $$

This satisfies

$$ ({}_{n+1}E_n)\quad \left\{\prod_{1\le i\le n+1}(\theta_x+\beta_i-1)-x\prod_{1\le i\le n+1}(\theta_x+\alpha_i)\right\}F=0, $$

where $\beta_{n+1}=1$ and $\theta_x=xd/dx$.

- The rank of ${}_{n+1}E_n$ is $n+1$.

- The characteristic exponents are

$$1-\beta_1,\ 1-\beta_2,\ \ldots,\ 1-\beta_n,\ 0, \quad \text{at} \quad x=0,$$

$$0,\ 1,\ \ldots,\ n-1,\ \sum_{i=1}^{n}\beta_i-\sum_{i=1}^{n+1}\alpha_i \quad \text{at} \quad x=1,$$

$$\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{n+1} \quad \text{at} \quad x=\infty.$$

- Connection formulas are

$$f_i^{(\infty)}(z)=\sum_{j=1}^{n+1}\prod_{s\neq i}\frac{\Gamma(\alpha_i-\alpha_s+1)}{\Gamma(\beta_j-\alpha_s)}\prod_{s\neq j}\frac{\Gamma(\beta_j-\beta_s)}{\Gamma(\alpha_i-\beta_s+1)}\times f_j^{(0)}(z),$$

$$f_1^{(1)}(z)=\sum_{j=1}^{n+1}\prod_{s\neq i}\frac{\Gamma(1+\sum_{s=1}^{n}\beta_s-\sum_{s=1}^{n+1}\alpha_s)\prod_{\substack{1\le s\le n+1\\ s\neq i}}\Gamma(\beta_j-\beta_s)}{\prod_{1\le s\le n+1}\Gamma(\beta_j-\alpha_s)}\times f_j^{(0)}(z),$$

where $f_i^{(\infty)}(z)=(-z)^{-\alpha_i}(1+O(z^{-1}))$, $f_j^{(0)}(z)=(-z)^{1-\beta_j}(1+O(z))$ and $f_1^{(1)}(z)=(1-z)^{\sum_{i=1}^{n}\beta_i-\sum_{i=1}^{n+1}\alpha_i}(1+O(1-z))$.

- **Integral representations of $f_i^{(0)}(z)$ and $f_i^{(\infty)}(z)$:**

$$\int_{D_i^{(0)}} u_{D_i^{(0)}}(t)\,dt_1\cdots dt_n = \prod_{\substack{1\le s\le n+1\\ s\neq i}} B(\alpha_s-\beta_i+1,\beta_s-\alpha_s)\times f_i^{(0)}(z),$$

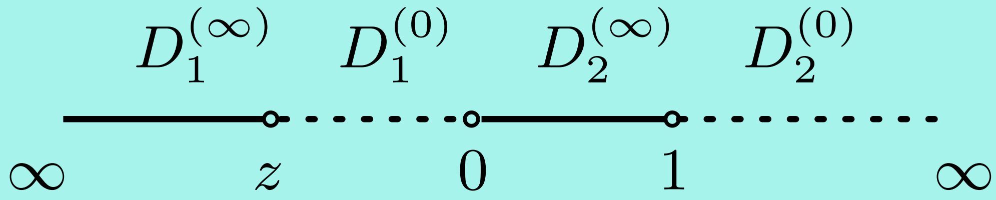
$$\int_{D_i^{(\infty)}} u_{D_i^{(\infty)}}(t)\,dt_1\cdots dt_n = \prod_{\substack{1\le s\le n+1\\ s\neq i}} B(\alpha_i-\beta_s+1,\beta_s-\alpha_s)\times f_i^{(\infty)}(z),$$

where $u_D(t)=\prod_i(\epsilon_i\, f_i(t))^{\lambda_i}$ with $\epsilon_i=\pm$ being determined so that $\epsilon_i\, f_i(t)>0$ on D for

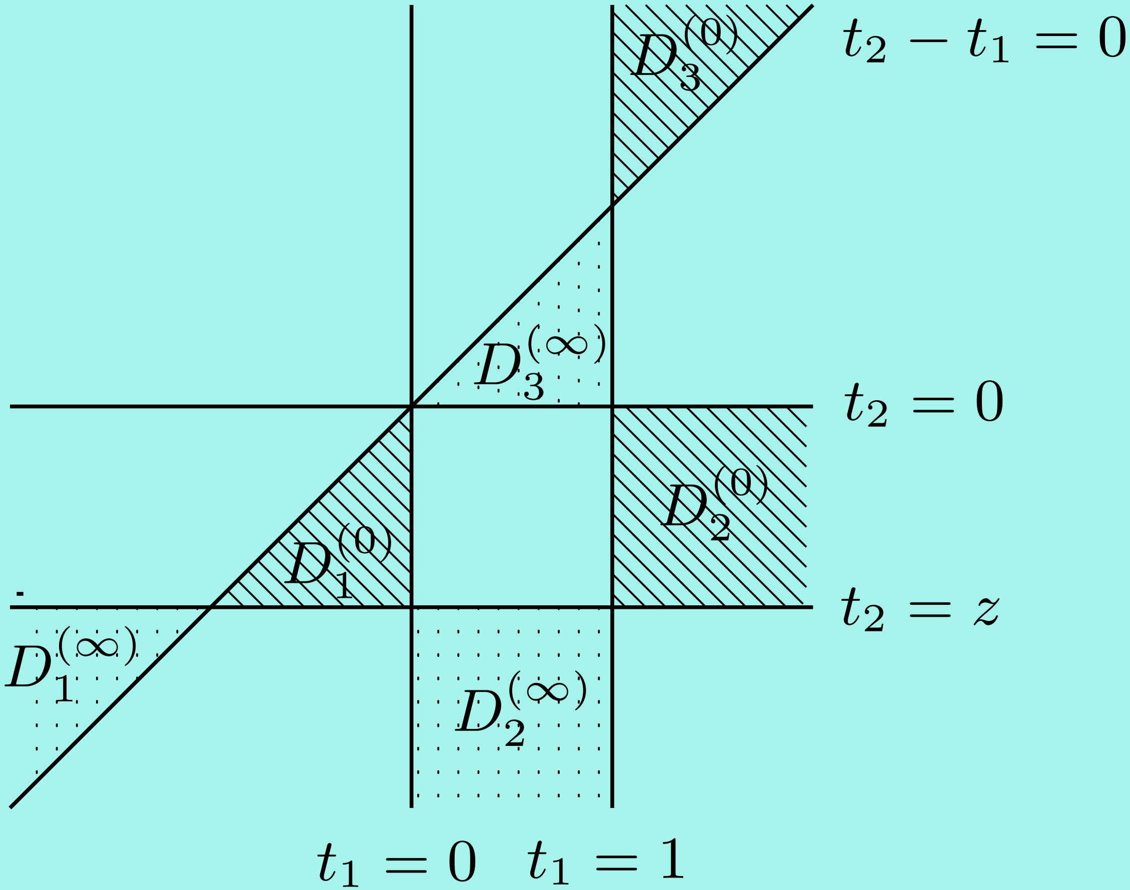
$$u(t)=\prod_i f_i(t)^{\lambda_i}$$

$$=\prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i}\prod_{i=1}^{n+1}(t_i-t_{i-1})^{\beta_i-\alpha_i-1},\quad (\beta_{n+1}=1,t_0=1,t_{n+1}=z).$$

$n = 1$


$D_1^{(\infty)}$
$D_1^{(0)}$
$D_2^{(\infty)}$
$D_2^{(0)}$
∞
z
0
1
∞

$n = 2$


$D_3^{(0)}$
$t_2 - t_1 = 0$
$D_3^{(\infty)}$
$t_2 = 0$
$D_2^{(0)}$
$D_1^{(0)}$
$t_2 = z$
$D_1^{(\infty)}$
$D_2^{(\infty)}$
$t_1 = 0$
$t_1 = 1$

- For any cycle C,

$$\int_C \prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\beta_i-\alpha_i-1}\, dt \quad (\beta_{n+1} = 1, t_0 = 1, t_{n+1} = z)$$

gives a solution of ${}_{n+1}E_n$ (Mimachi-Noumi (2016)).

- $H_n^{\mathrm{lf}}(T,\mathcal{L})$ or $H_n(T,\mathcal{L})$, where $\mathcal{L}$ is determined by

$$u(t)=\prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i}\prod_{i=1}^{n+1}(t_i - t_{i-1})^{\beta_i-\alpha_i-1}, \quad (\beta_{n+1}=1, t_0=1, t_{n+1}=z)$$

on

$$T=\mathbb{C}^n \setminus \cup_{i=1}^{n}\{\, t_i=0 \,\} \cup \cup_{i=1}^{n+1}\{\, t_i - t_{i-1}=0 \,\}.$$

- $\mathrm{rank} H_n^{\mathrm{lf}}(T,\mathcal{L}) = \mathrm{rank} H_n(T,\mathcal{L}) = n+1.$

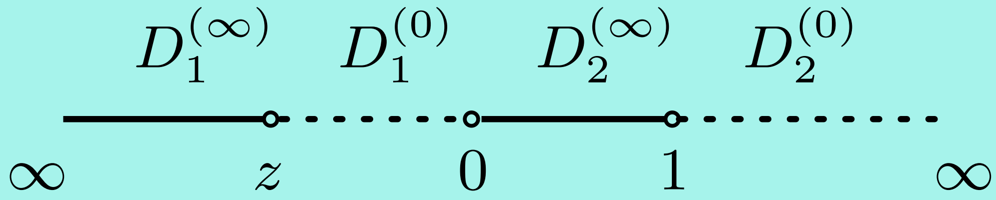
In what follows, z is fixed to be $\infty < z < 0$.

- **Bases of $H_n^{\mathrm{lf}}(T,\mathcal{L})$:**

$$\left\{ D_1^{(0)}, D_2^{(0)}, \ldots, D_{n+1}^{(0)} \;\middle|\; D_i^{(0)} = \begin{pmatrix} \infty < z < t_n < \cdots < t_i < 0 \\ 1 < t_1 < \cdots < t_{i-1} < \infty \end{pmatrix} \right\},$$

$$\left\{ D_1^{(\infty)}, D_2^{(\infty)}, \ldots, D_{n+1}^{(\infty)} \;\middle|\; D_i^{(\infty)} = \begin{pmatrix} \infty < t_i < \cdots < t_n < z \\ 0 < t_{i-1} < \cdots < t_1 < 1 \end{pmatrix} \right\}.$$

$n = 1$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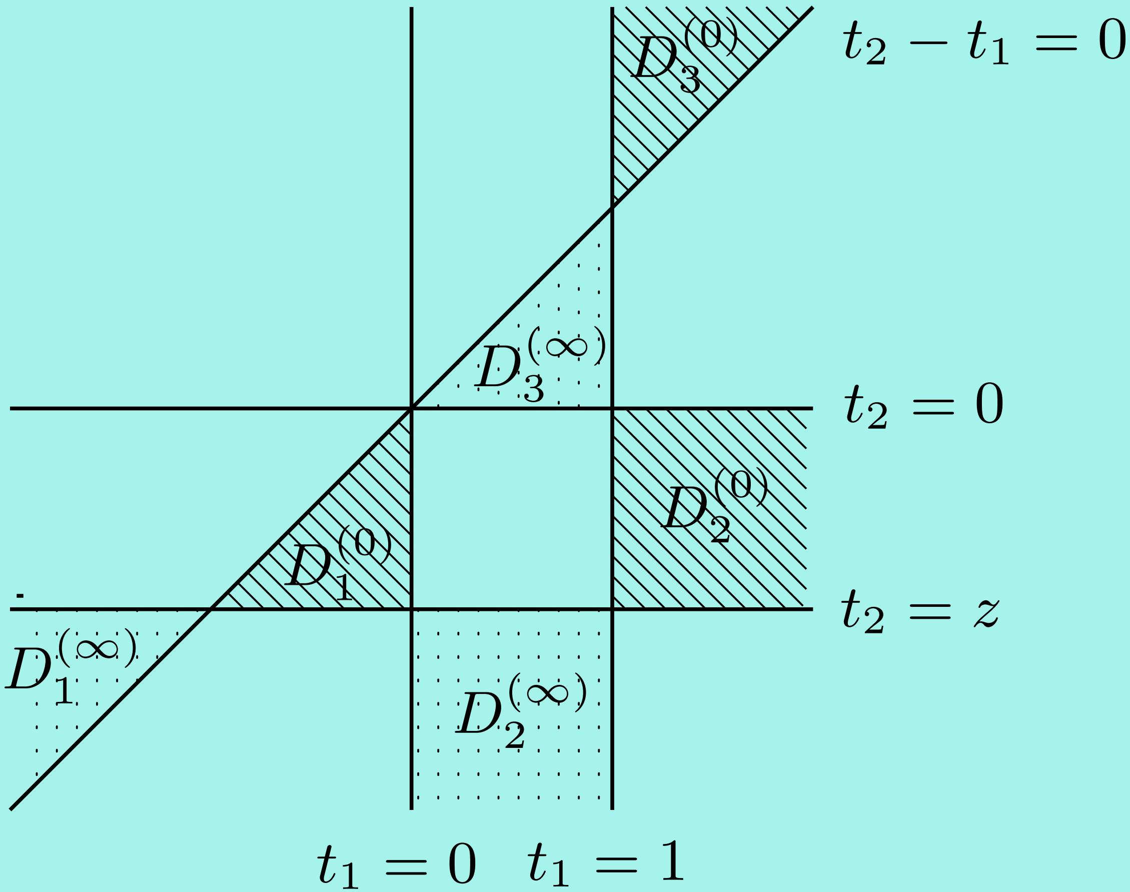
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• **Bases of $H_n^{\mathrm{lf}}(T,\mathcal{L})$:**

$$\left\{ D_1^{(0)}, D_2^{(0)}, \ldots, D_{n+1}^{(0)} \;\middle|\; D_i^{(0)} = \begin{pmatrix} \infty < z < t_n < \cdots < t_i < 0 \\ 1 < t_1 < \cdots < t_{i-1} < \infty \end{pmatrix} \right\},$$

$$\left\{ D_1^{(\infty)}, D_2^{(\infty)}, \ldots, D_{n+1}^{(\infty)} \;\middle|\; D_i^{(\infty)} = \begin{pmatrix} \infty < t_i < \cdots < t_n < z \\ 0 < t_{i-1} < \cdots < t_1 < 1 \end{pmatrix} \right\}.$$

$\Longrightarrow \exists\, c_{ij}$ such that

$$D_i^{(\infty)} = \sum_{1 \le j \le n+1} c_{ij}\, D_j^{(0)}.$$

$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ij} \, D_j^{(0)} .$$

Intersection numbers

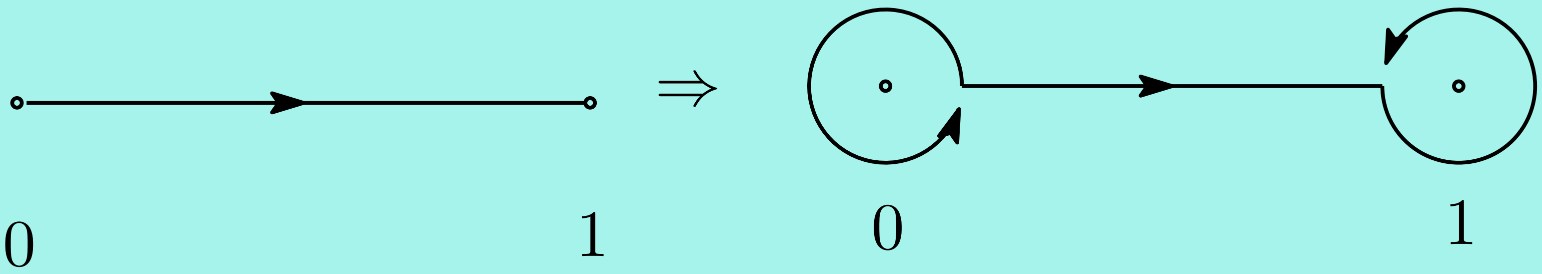
The *intersection form* $\bullet : H^{\mathrm{lf}}_m(T,\mathcal{L}) \times H^{\mathrm{lf}}_m(T,\mathcal{L}) \longrightarrow \mathbb{C}$ is the Hermitian form defined by

$$(C, C') \longmapsto C \bullet C' = \sum_{\rho,\,\sigma} a_\rho\, \overline{a'_\sigma} \sum_{t\in\rho\cap\sigma} I_t(\rho,\sigma) v_\rho(t) \overline{v'_\sigma(t)}/|u|^2,$$

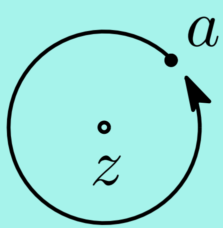
for $C,\ C' \in H^{\mathrm{lf}}_m(T,\mathcal{L})$, if reg $C = \sum_\rho a_\rho\, \rho \otimes v_\rho,\ C' = \sum_\sigma a'_\sigma\, \sigma \otimes v'_\sigma$, where $a_\rho, a'_\sigma \in \mathbb{C}$, each ρ or σ is an m-simplex, v_ρ or v'_σ a section of $\mathcal{L}$ on ρ or σ, $^-$ the complex conjugation, $I_t(\rho,\sigma)$ the topological intersection number of ρ and σ at t and the map $\mathrm{reg} : H^{\mathrm{lf}}_m(T,\mathcal{L}) \longrightarrow H_m(T,\mathcal{L})$ is defined as an inverse of the natural map $\iota : H_m(T,\mathcal{L}) \longrightarrow H^{\mathrm{lf}}_m(T,\mathcal{L})$.
The value $C \bullet C'$ of the intersection form for $C, C' \in H^{\mathrm{lf}}_m(T,\mathcal{L})$ is called the *intersection number* of C and C'.

An example of regularization. $T = \mathbb{C}\backslash\{0,1\}$, $u(t) = t^{\alpha}(1-t)^{\beta}$.

$$\overrightarrow{(0,1)} \quad \Rightarrow \quad \mathrm{reg}\,\overrightarrow{(0,1)} = \left\{ \frac{1}{d_\alpha} S(0\,;\epsilon) + \overrightarrow{[\epsilon, 1-\epsilon]} - \frac{1}{d_\beta} S(1\,;1-\epsilon) \right\}$$



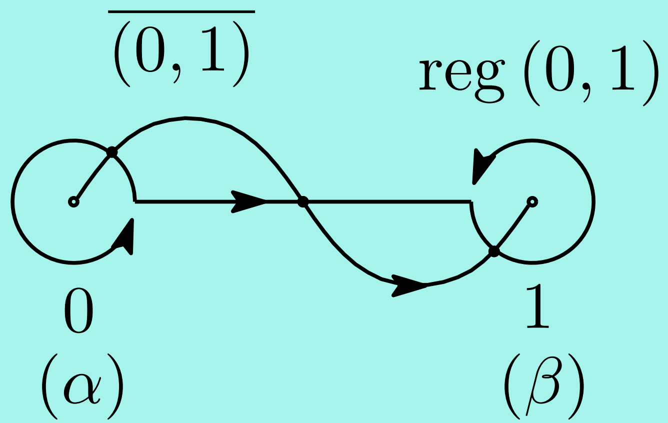
Here $d_a = e(a) - 1,\ e(a) = \exp(2\pi\sqrt{-1}a)$. The symbol $S(z\,;a)$ stands for the positively oriented circle centered at the point z with starting and ending point a, ϵ is a small positive number and the argument of each factor of $u(t)$ on the oriented circle $S(0\,;\epsilon)$ or $S(1\,;1-\epsilon)$ is defined so that $\arg t$ takes value from 0 to 2π on $S(0\,;\epsilon)$, and $\arg(1-t)$ from 0 to 2π on $S(1-\epsilon\,;1)$.



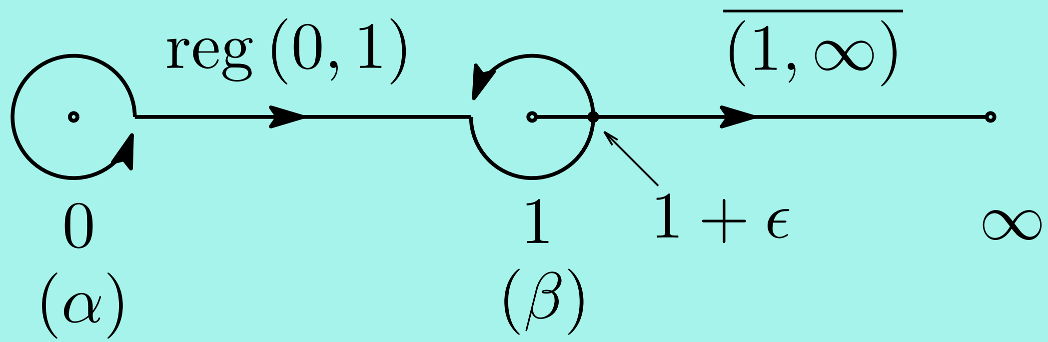
Examples of intersection numbers. $T = \mathbb{C}\backslash\{0,1\}, u(t) = t^\alpha(1-t)^\beta.$

$$\overrightarrow{(0,1)} \bullet \overrightarrow{(0,1)} = -\frac{1}{d_\alpha} - 1 + \frac{-1}{d_\beta}$$
$$= -\frac{d_{\alpha+\beta}}{d_\alpha d_\beta} = -\frac{s(\alpha+\beta)}{s(\alpha)s(\beta)},$$

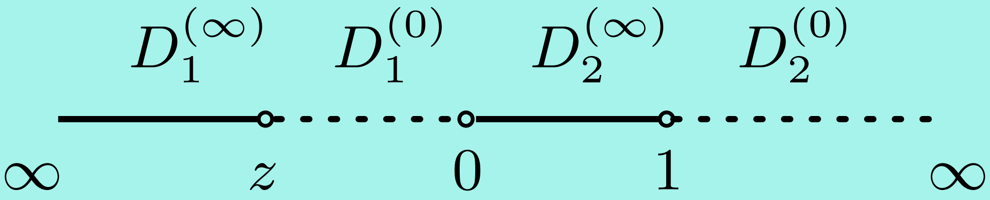
where $s(a) = \sin(\pi a)$.



$$\overrightarrow{(0,1)} \bullet \overrightarrow{(1,\infty)} = \frac{e(\beta/2)}{e(\beta)-1}.$$



$n = 1$



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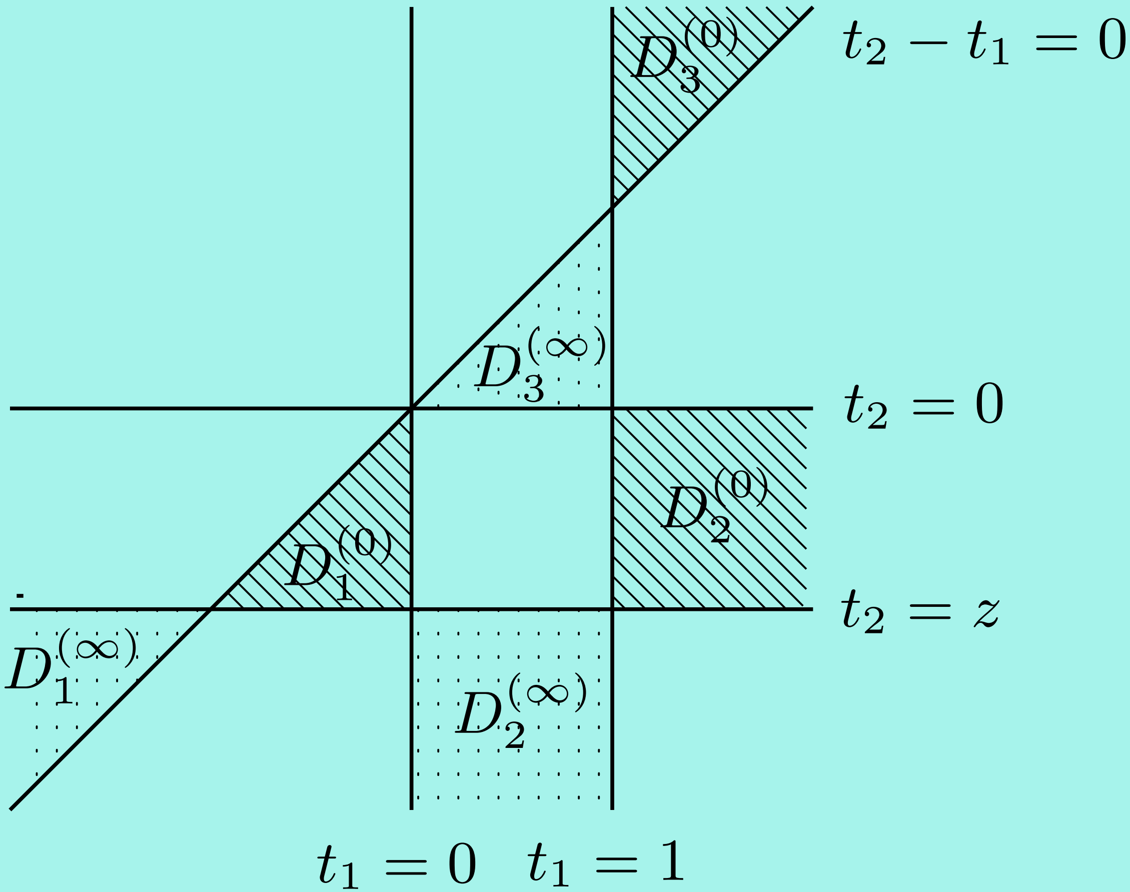

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$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\frac{\sqrt{-1}}{2} \right)^n \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \frac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s) \sin(\alpha_s - \beta_j)},$$

$$D_i^{(\infty)} \bullet D_j^{(0)} = \left(\frac{\sqrt{-1}}{2} \right)^n \frac{1}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \le s \le n+1 \\ s \ne i,j}} \frac{1}{\sin(\beta_s - \alpha_s)}.$$

$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ik} \, D_k^{(0)} .$$

$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ik}\, D_k^{(0)}.$$

$$D_j^{(0)} \bullet D_i^{(\infty)} = D_j^{(0)} \bullet \Big(\sum_{1 \leq k \leq n+1} c_{ik}\, D_k^{(0)} \Big).$$

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$$D_j^{(0)} \bullet D_i^{(\infty)} = \sum_{1\leq k\leq n+1} c_{ik}\, D_j^{(0)} \bullet \; D_k^{(0)}.$$

$$\Longrightarrow \quad c_{ij} = \frac{D_i^{(\infty)} \bullet D_j^{(0)}}{D_j^{(0)} \bullet D_j^{(0)}} = \frac{\sin(\beta_i - \alpha_i)}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{\sin(\alpha_s - \beta_j)}{\sin(\beta_s - \beta_j)}$$

$$\Longrightarrow \quad c_{ij} = \frac{D_i^{(\infty)} \bullet D_j^{(0)}}{D_j^{(0)} \bullet D_j^{(0)}} = \frac{\sin(\beta_i - \alpha_i)}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{\sin(\alpha_s - \beta_j)}{\sin(\beta_s - \beta_j)}$$

$$\Longrightarrow \quad f_i^{(\infty)}(z) = \sum_{j=1}^{n+1} \prod_{s \neq i} \frac{\Gamma(\alpha_i - \alpha_s + 1)}{\Gamma(\beta_j - \alpha_s)} \prod_{s \neq j} \frac{\Gamma(\beta_j - \beta_s)}{\Gamma(\alpha_i - \beta_s + 1)} \times f_j^{(0)}(z)$$

Master decomposition formula for cycles:

If $\{C_1, \ldots, C_\nu\}$ is a basis of $H_n^{\mathrm{lf}}(T, \mathcal{L})$,

$$C = \sum_{1 \leq j,k \leq \nu} C \bullet C_k ((C_p \bullet C_q)^{-1}_{1 \leq p,q \leq \nu})_{kj} C_j$$

for $\forall C \in H_n^{\mathrm{lf}}(T, \mathcal{L})$.

$\Longrightarrow$

$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ij}\, D_j^{(0)},$$

where

$$c_{ij} = \sum_{k=1}^{n+1} D_i^{(\infty)} \bullet D_k^{(0)} \left((D_p^{(0)} \bullet D_q^{(0)})^{-1}_{1 \leq p,q \leq n+1} \right)_{kj} .$$

1.2 The Even family and the Odd family

- The characteristic exponents of the Even-family:

$$0, 1, \ldots, m-1,\ a_1, a_1+1, \ldots, a_1+m-2,\ a_2 \quad \text{at} \quad z=0,$$

$$0, 1, \ldots, m-1,\ b, b+1, \ldots, b+m-1, \quad \text{at} \quad z=1,$$

$$c_1, c_2, \ldots, c_{2m}, \quad \text{at} \quad z=\infty.$$

- A connection formula of the Even-Odd family:

$$f_1^{(0)}(z) = \sum_{j=1}^{n} \frac{\Gamma(1-a_1+a_2, 1+a_2)}{\Gamma(1-b-c_j, 1-c_j)} \prod_{\substack{1 \le s \le n \\ s \ne j}} \frac{\Gamma(c_s - c_j)}{\Gamma(2-a_1-b-c_j-c_s)} \times f_j^{(\infty)}(z),$$

where $f_1^{(0)}(z) = (-z)^{a_2}(1+O(z)),\ f_j^{(\infty)}(z) = (-z)^{-c_j}(1+O(z^{-1}))$ and $n=2m$ in the case of the Even family and $n=2m+1$ in the case of the Odd family. (Even four case は Haraoka-Mimachi (2011))

2 Lauricella's differential equations

Lauricella's hypergeometric functions F_D, F_A, F_B are the analytic continuation of Lauricella's hypergeometric series

$$F_D(\alpha, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n)$$
$$= \sum_{m_1,\dots,m_n \geq 0} \frac{(\alpha)_{m_1+\dots+m_n}(\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n},$$
$$F_A(\alpha, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n; x_1, \dots, x_n)$$
$$= \sum_{m_1,\dots,m_n \geq 0} \frac{(\alpha)_{m_1+\dots+m_n}(\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n},$$
$$F_B(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n)$$
$$= \sum_{m_1,\dots,m_n \geq 0} \frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n}(\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n},$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ and $(\alpha)_0 = 1$.

- When $n = 2$, F_D, F_A, F_B are Appell's F_1, F_2, F_3.
- When $n = 1$, all F_D, F_A, F_B are Gauss HF ${}_2F_1$.

Lauricella's F_D, F_A, F_B satisfy

$$(E_D)\ \begin{cases} \theta_{x_i}(\theta_{x_1}+\cdots+\theta_{x_n}+\gamma-1)F = x_i(\theta_{x_i}+\beta_i)(\theta_{x_1}+\cdots+\theta_{x_n}+\alpha)F, \\ x_i(\theta_{x_i}+\beta_i)\theta_{x_j}F = x_j(\theta_{x_j}+\beta_j)\theta_{x_i}F, \end{cases}$$

$$(E_A)\quad \theta_{x_i}(\theta_{x_i}+\gamma_i-1)F = x_i(\theta_{x_i}+\beta_i)(\theta_{x_1}+\cdots+\theta_{x_n}+\alpha)F,$$

$$(E_B)\quad x_i(\theta_{x_i}+\alpha_i)(\theta_{x_i}+\beta_i)F = \theta_{x_i}(\theta_{x_1}+\cdots+\theta_{x_n}+\gamma-1)F,$$

where $1 \leq i, j \leq n$ and $\theta_{x_j} = x_j\partial/\partial x_j$.

- Each rank of E_D, E_A, E_B is $n+1, 2^n, 2^n$.
- When $n = 2$, E_D, E_A, E_B are E_1, E_2, E_3.
- When $n = 1$, all E_D, E_A, E_B are GHE ${}_2E_1$.

2.1 Lauricella's E_D

•A fundamental set of solutions around $(0,\dots,0)$: For $j=1,\dots,n+1$,

$$F_{D,j}(\underline{\alpha},\beta_1,\dots,\beta_n,\underline{\gamma};x_1,\dots,x_n)$$
$$=\sum_{m_1,\dots,m_n\geq 0}\frac{(\alpha)_{-m_1-\cdots-m_{j-1}+m_j+\cdots+m_n}(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{-m_1-\cdots-m_{j-1}+m_j+\cdots+m_n}m_1!\cdots m_n!}x_1^{m_1}\cdots x_n^{m_n}.$$

Gelfand-Zelevinsky-Kapranov(1989), by applying the theory of A-hypergeometric functions.

- In the case $j=1$, $F_{D,1}=F_D$.
- In the case $n=2$, $F_{D,1}=F_1(\alpha,\beta_1,\beta_2,\gamma;x,y)$,
  $F_{D,2}=G_2(\beta_1,\beta_2,\alpha,1-\gamma;-x,-y)$,
  $F_{D,3}=F_1(1-\gamma,\beta_1,\beta_2,1-\alpha;x,y)$.

$$\left((A)_m=\frac{(-1)^m}{(1-A)_{-m}}\right)$$

• $f_{n+1}^{(1)}(x) = F_D(\alpha, \beta_1, \dots, \beta_n, 1 + \alpha - \gamma + \sum_{i=1}^{n} \beta_i; 1 - x_1, \dots, 1 - x_n)$ is the holomorphic solution aroud $(1, \dots, 1)$.

• $\{f_j^{(0)}(x) \mid j = 1, \dots, n + 1\}$ is a fundamental set of solutions around $(0, \dots, 0)$ for $0 < x_1 < \dots < x_n < 1$. Here $f_{n+1}^{(0)}(x) = F_D(\alpha, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n)$ and $f_j^{(0)}(x) = x_j^{1-\gamma+\sum_{i=j+1}^{n} \beta_i} \prod_{i=j+1}^{n} x_i^{-\beta_i} \times F_{D,j}\Big(\underline{1 - \gamma + \sum_{i=j}^{n} \beta_i}, \beta_1, \cdots, \beta_{j-1}, 1 + \alpha - \gamma, \beta_{j+1}, \dots, \beta_n, \underline{2 - \gamma + \sum_{i=j+1}^{n} \beta_i}; \frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_j}{x_{j+1}}, \dots, \frac{x_j}{x_n}\Big)$, and $F_{D,j}(\underline{\alpha}, \beta_1, \dots, \beta_n, \underline{\gamma}; x_1, \dots, x_n) = \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha)_{-m_1 - \dots - m_{j-1} + m_j + \dots + m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{-m_1 - \dots - m_{j-1} + m_j + \dots + m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}$, where $(A)_m = \Gamma(A + m)/\Gamma(a)$ for any integer $m \in \mathbb{Z}$.

- A connection formula is

$$
\begin{aligned}
& f_{n+1}^{(1)}(x) \\
& = \sum_{j=1}^{n} \frac{\Gamma(-1+\gamma-\sum_{i=j+1}^{n}\beta_i, 1-\gamma+\sum_{i=j}^{n}\beta_i, 1+\alpha-\gamma+\sum_{i=1}^{n}\beta_i)}{\Gamma(\alpha, \beta_j, 1-\gamma+\sum_{i=1}^{n}\beta_i)} \\
& \times f_j^{(0)}(x) \\
& + \frac{\Gamma(1-\gamma, 1+\alpha-\gamma+\sum_{i=1}^{n}\beta_i)}{\Gamma(1+\alpha-\gamma, 1-\gamma+\sum_{i=1}^{n}\beta_i)} \times f_{n+1}^{(0)}(x),
\end{aligned}
$$

where $\Gamma(A_1, \ldots, A_l) = \prod_{j=1}^{l} \Gamma(A_j)$.

- $$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_D(\alpha, \beta_1, \ldots, \beta_n, \gamma; x_1, \ldots, x_n)$$
$$= \int_1^\infty t^{\beta_1+\cdots+\beta_n-\gamma}(t-1)^{\gamma-\alpha-1} \prod_{j=1}^n (t-x_j)^{-\beta_j}\, dt.$$

- $$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_D(\alpha, \beta_1, \ldots, \beta_n, \gamma; x_1, \ldots, x_n)$$
$$= \int_1^\infty t^{\beta_1+\cdots+\beta_n-\gamma}(t-1)^{\gamma-\alpha-1} \prod_{j=1}^{n} (t-x_j)^{-\beta_j}\, dt.$$

- For any cycle C,

$$\int_C t^{\beta_1+\cdots+\beta_n-\gamma}(t-1)^{\gamma-\alpha-1} \prod_{j=1}^{n} (t-x_j)^{-\beta_j}\, dt$$

gives a solution of E_D (Mimachi-Sasaki (2012), Mimachi-Noumi (2016)).

- For any cycle C,

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gives a solution of E_D (Mimachi-Sasaki (2012), Mimachi-Noumi (2016)).

- $H_1^{\mathrm{lf}}(T, \mathcal{L})$ or $H_1(T, \mathcal{L})$, where $\mathcal{L}$ is determined by

$$u(t) = t^{\beta_1+\cdots+\beta_n-\gamma}(t-1)^{\gamma-\alpha-1}\prod_{j=1}^{n}(t-x_j)^{-\beta_j}$$

$$= t^{\mu_0}(t-1)^{\mu_{n+1}}\prod_{j=1}^{n}(t-x_j)^{\mu_j}$$

on $T = \mathbb{C}\backslash\{x_0, x_1, \cdots, x_{n+1}\}$.

- $\mathrm{rank} H_1^{\mathrm{lf}}(T, \mathcal{L}) = \mathrm{rank} H_1(T, \mathcal{L}) = n + 1.$

In what follows, $x_0 < x_1 < \cdots < x_{n+1} = 1$ are fixed.

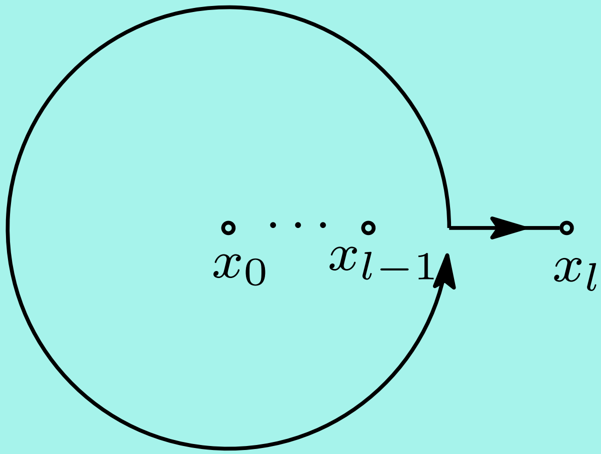
•Integral representations:

$$f_{n+1}^{(1)}(x) = (\text{const.}) \int_{-\infty}^{0} u_{(-\infty,0)}(t)\, dt$$

$$f_{l}^{(0)}(x) = (\text{const.}) \int_{(\{x_0,\ldots,x_{l-1}\},x_l)} u_{(x_{l-1},x_l)}(t)\, dt, \quad 1 \le l \le n+1,$$

where

$(\{x_0, \ldots, x_{l-1}\}, x_l)=$



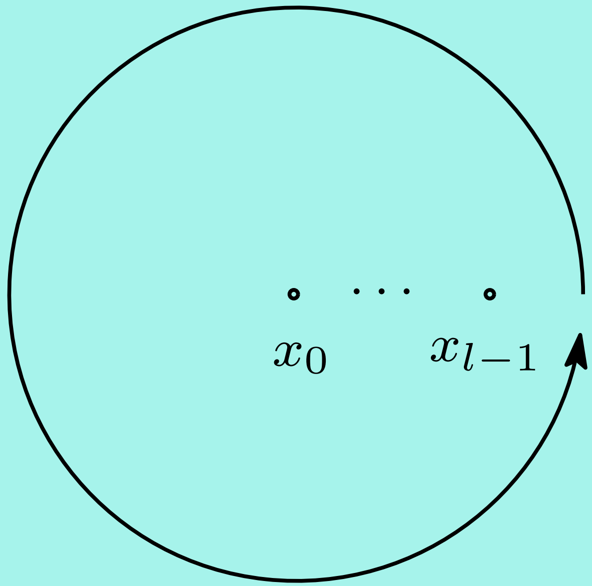
,

which is callled a *Erdelyi cycle*.

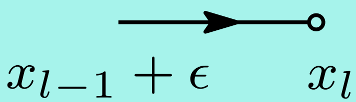
$\{ (\{x_0, \ldots, x_{l-1}\}, x_l) \mid l = 1, \ldots, n+1 \}$: a basis of $H_1^{\mathrm{lf}}(T_x, \mathcal{L})$,

where

$$(\{x_0, \ldots, x_{l-1}\}, x_l)$$
$$= \frac{1}{d_{01\cdots l-1}} S(x_0; x_{l-1} + \epsilon) \otimes u_{(x_{l-1}, x_l)}(t) + \overrightarrow{[x_{l-1} + \epsilon, x_l)} \otimes u_{(x_{l-1}, x_l)}(t)$$

$= \dfrac{1}{d_{01\cdots l-1}} \Bigg\{$ 

$\Bigg\} \otimes u_{(x_{l-1}, x_l)}(t)$

$+ \{$ 

$\} \otimes u_{(x_{l-1}, x_l)}(t),$

where $\mu_{ij\ldots k} = \mu_i + \mu_j + \cdots + \mu_k$ and $d_{ij\ldots k} = e(\mu_{ij\ldots k}) - 1$.

$\{\,(\{x_0, \ldots, x_{l-1}\}, x_l) \mid l = 1, \ldots, n+1\}$: a basis of $H_1^{\mathrm{lf}}(T_x, \mathcal{L})$,

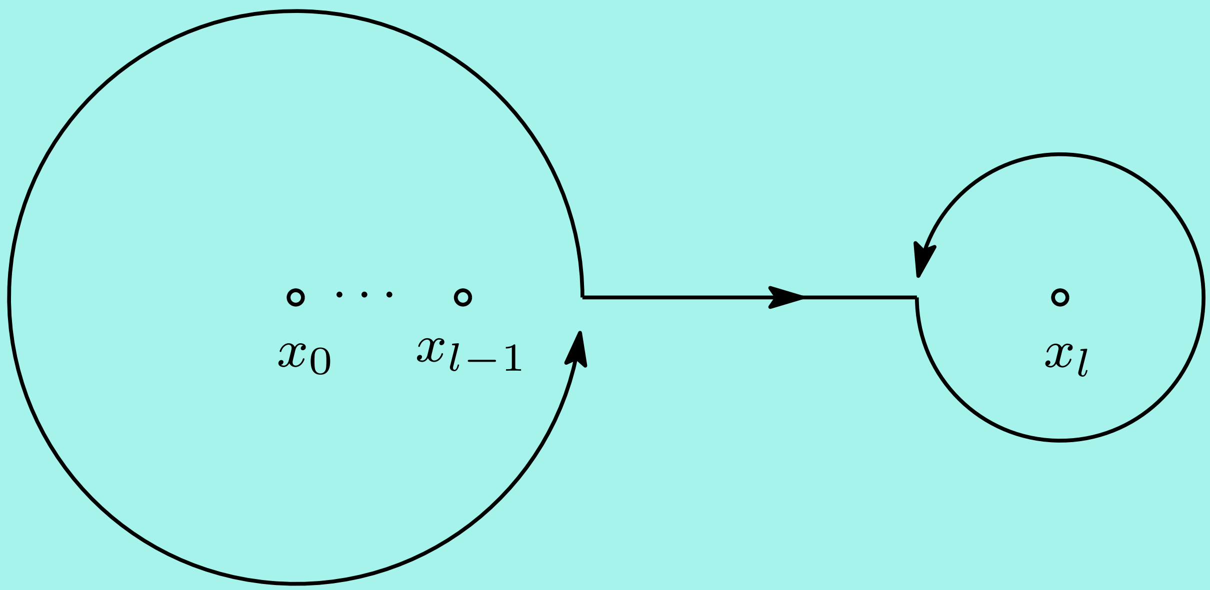
$(-\infty, 0) \in H_1^{\mathrm{lf}}(T_x, \mathcal{L})$.

$$\Downarrow$$

$$\exists q_l \in \mathbb{C} \quad \text{s.t.} \quad (-\infty, 0) = \sum_{l=1}^{n+1} q_l\,(\{x_0, \ldots, x_{l-1}\}, x_l).$$

The regularization or the compactification of $(\{x_0, \dots, x_{l-1}\}, x_l)$:

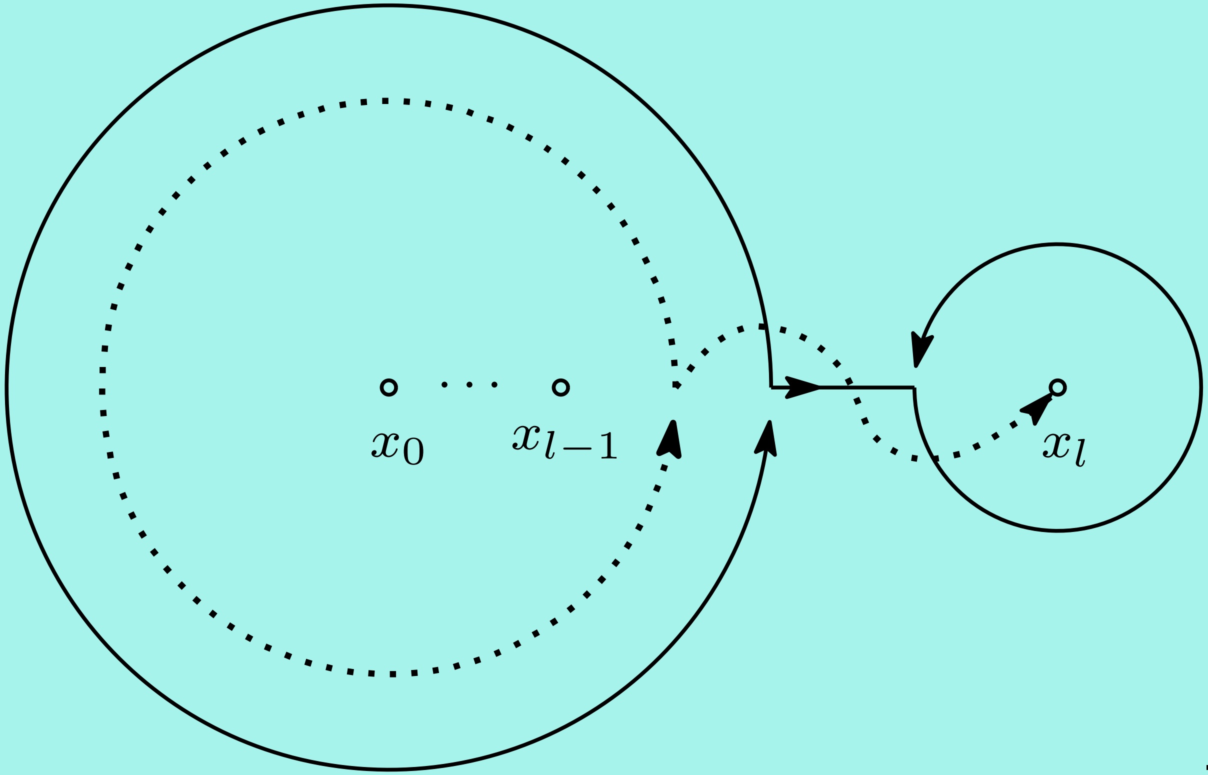
$$\operatorname{reg}(\{x_0, \dots, x_{l-1}\}, x_l) = \frac{1}{d_{01\cdots l-1}} S(x_0; x_{l-1} + \epsilon) \otimes u_{x_{l-1}+\epsilon}(t)$$
$$+ \overrightarrow{[x_{l-1} + \epsilon, x_l - \epsilon]} \otimes u_{x_{l-1}+\epsilon}(t) - \frac{1}{d_l} S(x_l; x_l - \epsilon) \otimes u_{x_l - \epsilon}(t).$$



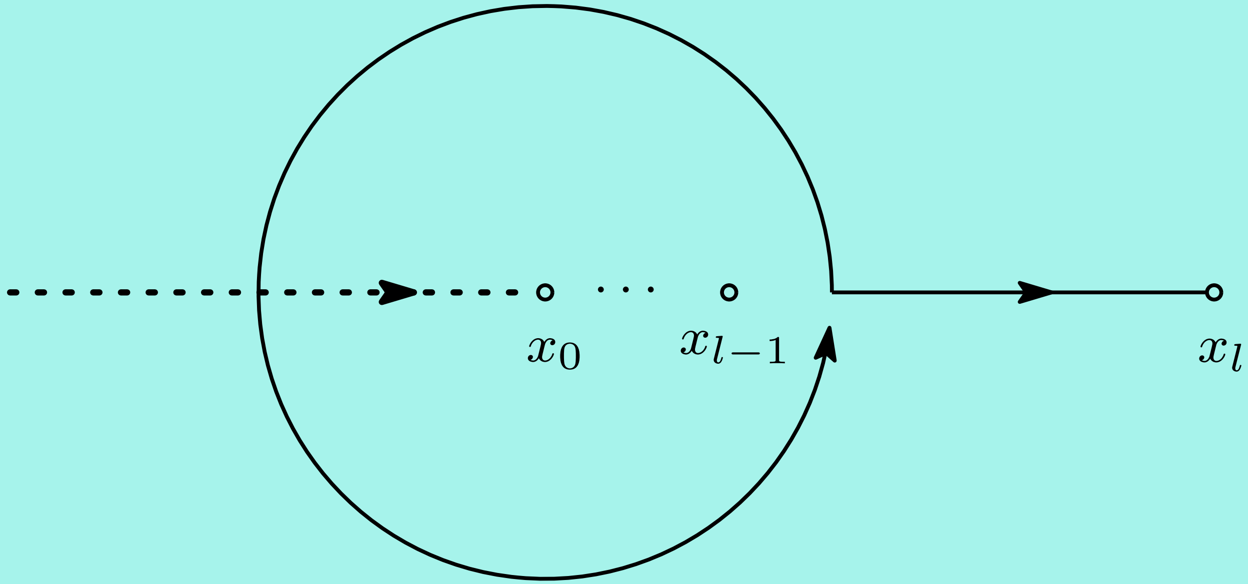
The cycle $\operatorname{reg}(\{x_0, \dots, x_{l-1}\}, x_l)$

The self-intersection number of $(\{x_0, \ldots, x_{l-1}\}, x_l)$:

$$(\{x_0, \ldots, x_{l-1}\}, x_l) \bullet (\{x_0, \ldots, x_{l-1}\}, x_l)$$
$$= -\frac{1}{d_{01\cdots l-1}} - 1 + \frac{-1}{d_l} = -\frac{d_{01\cdots l}}{d_{01\cdots l-1} d_l}.$$



$$\{(-\infty, 0) \otimes u_{0-\epsilon}\} \bullet (\{x_0, \ldots, x_{l-1}\}, x_l) = \frac{1}{\langle e_{01\cdots l-1} \rangle},$$



and moreover

$$(\{x_0, \ldots, x_{l-1}\}, x_l) \bullet (\{x_0, \ldots, x_{m-1}\}, x_m) = -\delta_{l,m} \frac{d_{01\cdots l}}{d_{01\cdots l-1} d_l}.$$

- Intersection numbers:

$$(-\infty, 0) \bullet (\{x_0, \ldots, x_{m-1}\}, x_m) = \frac{1}{\langle e_{01\cdots m-1} \rangle},$$

$$(\{x_0, \ldots, x_{l-1}\}, x_l) \bullet (\{z_0, \ldots, z_{m-1}\}, z_m) = -\delta_{l,m} \frac{d_{01\cdots m}}{d_{01\cdots m-1} d_m}.$$

$$\exists q_l \in \mathbb{C} \quad \text{s.t.} \quad (-\infty, 0) = \sum_{l=1}^{n+1} q_l \left(\{x_0, \dots, x_{l-1}\}, x_l\right).$$

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$$\exists q_l \in \mathbb{C} \quad \text{s.t.} \quad (-\infty, 0) \otimes u_{0-\epsilon}(t) = \sum_{l=1}^{n+1} q_l \left(\{x_0, \ldots, x_{l-1}\}, x_l\right).$$

$$\Downarrow$$

$$q_l = -\frac{\langle e_l \rangle}{\langle e_{01\cdots l} \rangle}$$

2.2 Lauricella's E_A

$$\prod_{i=1}^{n}(-x_i)^{-\beta_i} F_B\left(\begin{matrix} 1+\beta_1-\gamma_1,\dots,1+\beta_n-\gamma_n\,;\,\beta_1,\dots,\beta_n \\ 1-\alpha+\sum_{i=1}^{n}\beta_i \end{matrix}\,;\,\frac{1}{x_1},\dots,\frac{1}{x_n}\right)$$

$$=\sum_{r=0}^{n}\sum_{\substack{i_1<\cdots<i_r \\ j_1<\cdots<j_{n-r}}}\frac{\Gamma(1-\alpha+\sum_{i=1}^{n}\beta_i)}{\Gamma(1-\alpha+\sum_{p=1}^{r}(\gamma_{i_p}-1))}\prod_{p=1}^{r}\frac{\Gamma(\gamma_{i_p}-1)}{\Gamma(\beta_{i_p})}\prod_{q=1}^{n-r}\frac{\Gamma(\gamma_{j_q}-1)}{\Gamma(1+\beta_{j_q}-\gamma_{j_q})}$$

$$\times\prod_{p=1}^{r}(-x_{i_p})^{1-c_{i_p}}$$

$$\times F_A\left(\begin{matrix}\alpha-\sum_{p=1}^{r}(\gamma_{i_p}-1);1+\beta_{i_1}-\gamma_{i_1},\dots,1+\beta_{i_r}-\gamma_{i_r},\beta_{j_1},\dots,\beta_{j_{n-r}} \\ 2-\gamma_{i_1},\dots,2-\gamma_{i_p},\gamma_{j_1},\dots,\gamma_{j_{n-r}}\end{matrix}\right.$$

$$\left. x_{i_1},\dots,x_{i_p},x_{j_1},\dots,x_{j_q}\right),$$

where $\{i_1<\cdots<i_r\}\cup\{j_1<\cdots<j_{n-r}\}=\{1,\dots,n\}$.

which is a generalization of

$$(-x)^{-b_1}(-y)^{-b_2}F_3(b_1,b_2,1+b_1-c_1,1+b_2-c_2,1+b_1+b_2-a;x^{-1},y^{-1}) \leftarrow {\color{red}(\infty,\infty)}$$

$$= \frac{\Gamma(1-c_1,1-a+b_1+b_2,1-c_2)}{\Gamma(1+b_1-c_1,1-a,1+b_2-c_2)}(-x)^{b_1}(-y)^{b_2}F_2(a,b_1,b_2,c_1,c_2;x,y) \quad \leftarrow \quad {\color{red}(0,0)}$$

$$+ \frac{\Gamma(c_1-1,1-a+b_1+b_2,1-c_2)}{\Gamma(b_1,c_1-a,1+b_2-c_2)}$$

$$\times (-x)^{1+b_1-c_1}(-y)^{b_2}F_2(1+a-c_1,1+b_1-c_1,b_2,2-c_1,c_2;x,y) \quad \leftarrow \quad {\color{red}(0,0)}$$

$$+ \frac{\Gamma(1-c_1,1-a+b_1+b_2,c_2-1)}{\Gamma(b_2,c_2-a,1+b_1-c_1)}$$

$$\times (-x)^{b_1}(-y)^{1+b_2-c_2}F_2(1+a-c_2,b_1,1+b_2-c_2,c_1,2-c_2;x,y) \quad \leftarrow \quad {\color{red}(0,0)}$$

$$+ \frac{\Gamma(c_1-1,1-a+b_1+b_2,c_2-1)}{\Gamma(b_1,c_1+c_2-a-1,b_2)}$$

$$\times (-x)^{1+b_1-c_1}(-y)^{1+b_2-c_2}$$

$$\times F_2(2+a-c_1-c_2,1+b_1-c_1,1+b_2-c_2,2-c_1,2-c_2;x,y). \quad \leftarrow \quad {\color{red}(0,0)}$$

- $$\prod_{j=1}^{n} \frac{\Gamma(\beta_j)\Gamma(\gamma_j - \beta_j)}{\Gamma(\gamma_j)} F_A \left(\begin{array}{c} \alpha, \beta_1, \ldots, \beta_n \\ \gamma_1, \ldots, \gamma_n \end{array} ; x_1, \ldots, x_n \right)$$
$$= \int_{(0,1)^n} (1 - \sum_{j=1}^{n} x_j t_j)^{-\alpha} \prod_{1 \le i \le n} t_i^{\beta_i - 1} (1 - t_i)^{\gamma_i - \beta_i - 1} \, dt_1 \cdots dt_n .$$

- $$\prod_{j=1}^{n} \frac{\Gamma(\beta_j)\Gamma(\gamma_j - \beta_j)}{\Gamma(\gamma_j)} F_A \left(\begin{matrix} \alpha, \beta_1, \ldots, \beta_n \\ \gamma_1, \ldots, \gamma_n \end{matrix} ; x_1, \ldots, x_n \right)$$
$$= \int_{(0,1)^n} (1 - \sum_{j=1}^{n} x_j t_j)^{-\alpha} \prod_{1 \le i \le n} t_i^{\beta_i - 1} (1 - t_i)^{\gamma_i - \beta_i - 1} \, dt_1 \cdots dt_n.$$

- For any cycle C,

$$\int_C (1 - \sum_{j=1}^{n} x_j t_j)^{-\alpha} \prod_{1 \le i \le n} t_i^{\beta_i - 1} (1 - t_i)^{\gamma_i - \beta_i - 1} \, dt_1 \cdots dt_n$$

gives a solution of E_A (Mimachi-Noumi (2016)).

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ご清聴ありがとうございました

野海さん,
ひとまず，お疲れ様でした！

野海さん,
退職してから，ますます，ご活躍を！！