Reduction to discrete Painlevé equations from CACO lattice equations: δ-$E_6^{(1)}$ type

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Supported by

JSPS KAKENHI: JP19K14559 and JP17J00092

Australian Research Council: FL120100094 and DP160101728

研究集会「q, q and q」

神戸大学,　2020 年 2 月 20 日

Aim

Adelr-Bobenko-Suris **達による consistency around a cube (CAC) property を用いた** quad-equations **の分類が知られている** [ABS2003&2009,Boll2011]**.**

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Adelr-Bobenko-Suris 達による **consistency around a cube (CAC) property** を用いた quad-equations の分類が知られている [ABS2003&2009,Boll2011]．分類により得られた quad-equations から，可積分な 2 次元偏差分方程式 (ABS 方程式) とそのベックルンド変換が導出できる．また，ABS 方程式から離散パンルヴェ方程式，パンルヴェ方程式，および，その高階化が周期簡約によって得られることが知られている．

本講演では， CAC property **ではなく，consistency around a cuboctahedron (CACO) property と呼ばれるコンシステンシーを持つ格子方程式から，周期簡約によって** $E_6^{(1)}$ **型 (初期値空間は** $A_2^{(1)}$ **型)** **の加法型離散パンルヴェ方程式** $\delta\text{-}P(E_6^{(1)})$ **が得られることを示す．**

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  - Consistency around a cube (CAC) property
  - ABS equations

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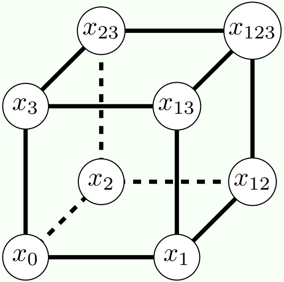
Review of previous works about CAC property

Consistency around a cube (CAC) property

Let us assign the eight variables: x_0, ..., x_{123} on the vertices and the quad-equations on the faces. Here, equation $Q(x,y,z,w)=0$ is said a quad-equation, if $Q(x,y,z,w)$ is an irreducible multi-affine polynomial.



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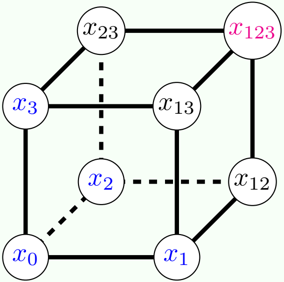

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Definition (Nijhoff-Walker 1999)

There are three ways to determine the value of x_{123} by the initial values $\{x_0,x_1,x_2,x_3\}$. If x_{123} can be uniquely expressed in terms of the initial values, then the cube is said to have the CAC property.

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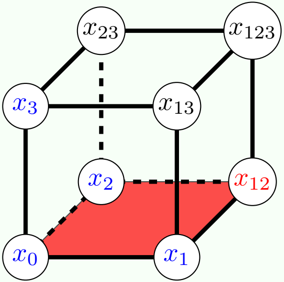

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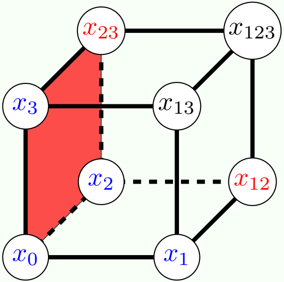

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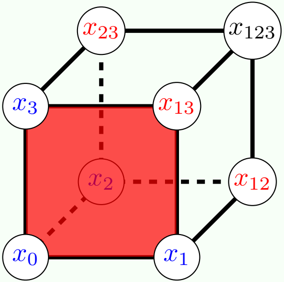

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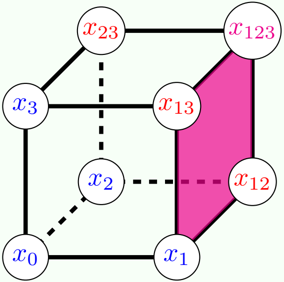

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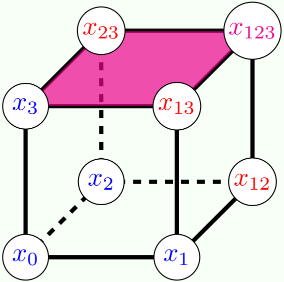

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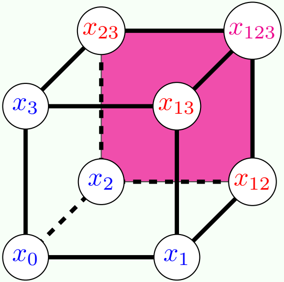

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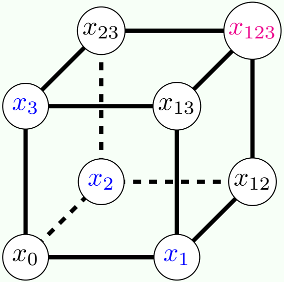
Tetrahedron property

Let us assign the eight variables: $x_0, \ldots, x_{123}$ on the vertices and the quad-equations on the faces. Here, equation $Q(x, y, z, w) = 0$ is said a quad-equation, if $Q(x, y, z, w)$ is an irreducible multi-affine polynomial.



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Definition

When the result for x_{123} turns out to depend only on $\{x_1, x_2, x_3\}$, and x_0 depends only on $\{x_{12}, x_{23}, x_{31}\}$, the CAC cube is said to have the tetrahedron property.

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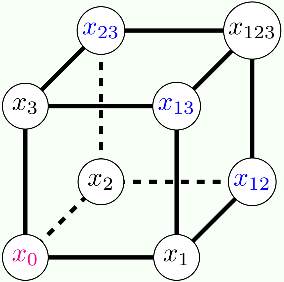

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ABS equations

By using the CAC and tetrahedron properties, Adler-Bobenko-Suris *et al.* classified quad-equations on a cube [ABS2003&2009,Boll2011]. Tessellating such CAC cubes to $\mathbb{Z}^3$, we can obtain various integrable PΔEs (ABS equations), e.g.

Discrete Schwarzian KdV equation [Nijhoff-Capel-Wiersma-Quispel 1984]

$$\frac{(u_{l,m}-u_{l+1,m})(u_{l,m+1}-u_{l+1,m+1})}{(u_{l,m}-u_{l,m+1})(u_{l+1,m}-u_{l+1,m+1})}=\frac{\alpha_l}{\beta_m}$$

Lattice modified KdV equation [Nijhoff-Quispel-Capel 1983]

$$\frac{u_{l+1,m+1}}{u_{l,m}}=\frac{\alpha_l u_{l+1,m}-\beta_m u_{l,m+1}}{\alpha_l u_{l,m+1}-\beta_m u_{l+1,m}}$$

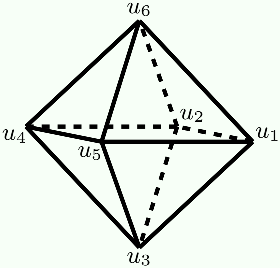
Lattice potential KdV equation [Hirota 1977]

$$(u_{l,m}-u_{l+1,m+1})(u_{l+1,m}-u_{l,m+1})=\alpha_l-\beta_m$$

Main result

Consistency around an octahedron (CAO) property

Let us consider an octahedron on whose vertices the six variables: u_1, $\ldots$, u_6 are assigned.



We impose the relations to the variables by the following quad-equations:

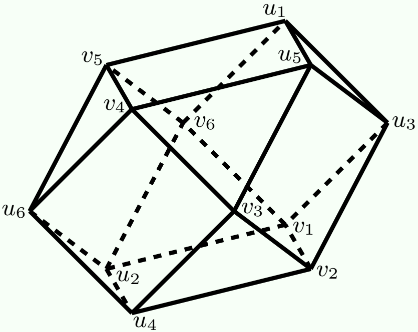
$$Q_1\left(u_4,u_2,u_1,u_5\right)=0,\quad Q_2\left(u_2,u_6,u_5,u_3\right)=0,\quad Q_3\left(u_6,u_4,u_3,u_1\right)=0.$$

Definition (Joshi-Nakazono)

The octahedron with quad-equations $\{Q_1,Q_2,Q_3\}$ is said to have a consistency around an octahedron (CAO) property, if each quad-equation can be obtained from the other two equations.

Consistency around a cuboctahedron property

Let us consider a cuboctahedron on whose vertices the twelve variables: $u_1, \ldots, u_6, v_1, \ldots, v_6$ are assigned.



We impose the relations to the variables by the following quad-equations:

$$Q_1(u_5, u_1, v_5, v_4) = 0, \quad Q_2(v_2, v_1, u_2, u_4) = 0, \quad Q_3(u_3, u_5, v_3, v_2) = 0,$$
$$Q_4(v_6, v_5, u_6, u_2) = 0, \quad Q_5(u_1, u_3, v_1, v_6) = 0, \quad Q_6(v_4, v_3, u_4, u_6) = 0,$$

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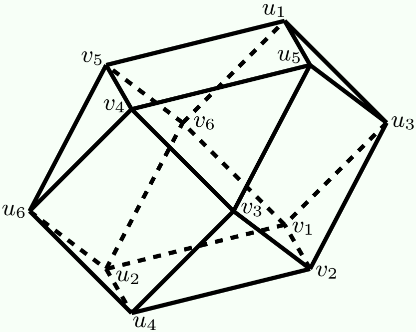

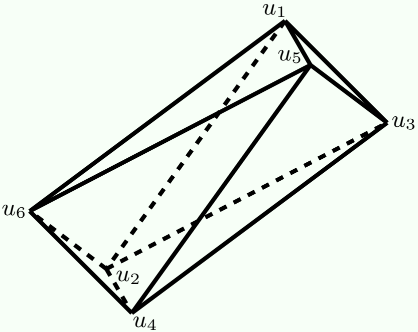

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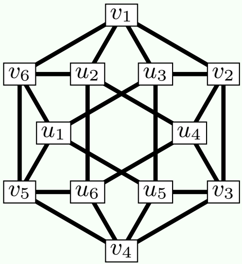
$$Q_4(v_6, v_5, u_6, u_2) = 0, \quad Q_5(u_1, u_3, v_1, v_6) = 0, \quad Q_6(v_4, v_3, u_4, u_6) = 0,$$
$$Q_7(u_4, u_2, u_1, u_5) = 0, \quad Q_8(u_2, u_6, u_5, u_3) = 0, \quad Q_9(u_6, u_4, u_3, u_1) = 0.$$

Definition (Joshi-Nakazono)

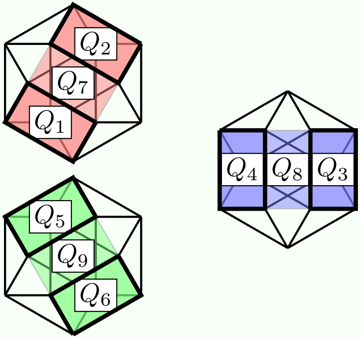
The cuboctahedron with quad-equations $\{Q_1, \dots, Q_9\}$ is said to have a consistency around a cuboctahedron (CACO) property, if the following properties hold.

(i) The octahedron with quad-equations $\{Q_7, Q_8, Q_9\}$ has the CAO property.

(ii) Assume that the variables $u_1, \dots, u_6$, are given so as to satisfy $Q_i = 0$, $i = 7, 8, 9$, and the variable v_1 is given. Then, by using quad-equations Q_i, $i = 1 \dots, 6$, the variable v_4 is uniquely determined.



Orthogonal projection of the cuboctahedron centered on the triangular face.



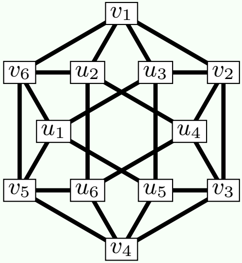
Definition (Joshi-Nakazono)

The CACO cuboctahedron with $\{Q_1, \ldots, Q_9\}$ is said to have a square property, if there exist polynomials $K_i = K_i(x, y, z, w)$, $i = 1, 2, 3$, where

$$\deg_x K_i = \deg_w K_i = 1, \quad 1 \leq \deg_y K_i,\ \deg_z K_i,$$

satisfying

$$K_1(v_1, u_1, u_4, v_4) = 0, \quad K_2(v_2, u_2, u_5, v_5) = 0, \quad K_3(v_3, u_3, u_6, v_6) = 0.$$



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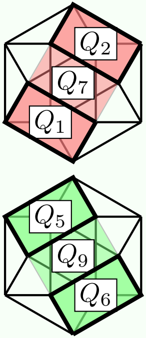
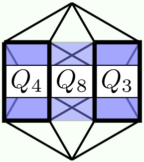




CACO and square properties for lattice equations

Consider the following system of PΔEs:

$$
\begin{aligned}
&P_1\left(u_{\overline{1}\overline{3}}, u_{\overline{2}\overline{3}}, u_{\underline{1}\overline{3}}, u_{\underline{2}\overline{3}}\right)=0, \qquad && P_2\left(u_{\overline{1}\overline{2}}, u_{\overline{1}\overline{3}}, u_{\overline{1}\underline{2}}, u_{\overline{1}\underline{3}}\right)=0,\\
&P_3\left(u_{\overline{2}\overline{3}}, u_{\overline{1}\overline{2}}, u_{\overline{2}\underline{3}}, u_{\underline{1}\overline{2}}\right)=0, \qquad && P_4\left(u_{\underline{2}\underline{3}}, u_{\underline{1}\underline{3}}, u_{\overline{2}\overline{3}}, u_{\overline{1}\overline{3}}\right)=0,\\
&P_5\left(u_{\underline{1}\underline{3}}, u_{\underline{1}\underline{2}}, u_{\overline{1}\overline{3}}, u_{\overline{1}\overline{2}}\right)=0, \qquad && P_6\left(u_{\underline{1}\underline{2}}, u_{\underline{2}\underline{3}}, u_{\overline{1}\overline{2}}, u_{\overline{2}\overline{3}}\right)=0,
\end{aligned}
$$

where $u=u(\boldsymbol{l})$ and $\boldsymbol{l}\in\Omega$, where

$$
\Omega=\left\{\sum_{i=1}^{3} l_i\epsilon_i \;\middle|\; l_i\in\mathbb{Z},\ l_1+l_2+l_3\in 2\mathbb{Z}\right\}.
$$

Here, P_i, $i=1,\dots,6$, are quad-equations, and subscripts $\overline{i}$ and $\underline{j}$ mean $\boldsymbol{l}\to\boldsymbol{l}+\epsilon_i$ and $\boldsymbol{l}\to\boldsymbol{l}-\epsilon_j$, respectively.

Then, given $\boldsymbol{l} \in \Omega$, we obtain the cuboctahedron centered around $\boldsymbol{l}$. We refer to its quad-equations as before by

$$Q_1(\boldsymbol{l}) = P_1\left(u_{\overline{13}}, u_{\overline{23}}, u_{\underline{1}\overline{3}}, u_{\underline{2}\overline{3}}\right) = 0, \quad Q_2(\boldsymbol{l}) = P_1\left(u_{\overline{1}\underline{3}}, u_{\overline{2}\underline{3}}, u_{\underline{13}}, u_{\underline{23}}\right) = 0,$$

$$Q_3(\boldsymbol{l}) = P_2\left(u_{\overline{12}}, u_{\overline{13}}, u_{\overline{1}\underline{2}}, u_{\overline{1}\underline{3}}\right) = 0, \quad Q_4(\boldsymbol{l}) = P_2\left(u_{\underline{1}\overline{2}}, u_{\underline{1}\overline{3}}, u_{\underline{12}}, u_{\underline{13}}\right) = 0,$$

$$Q_5(\boldsymbol{l}) = P_3\left(u_{\overline{23}}, u_{\overline{12}}, u_{\overline{2}\underline{3}}, u_{\underline{1}\overline{2}}\right) = 0, \quad Q_6(\boldsymbol{l}) = P_3\left(u_{\underline{2}\overline{3}}, u_{\overline{1}\underline{2}}, u_{\underline{23}}, u_{\underline{12}}\right) = 0,$$

$$Q_7(\boldsymbol{l}) = P_4\left(u_{\underline{23}}, u_{\underline{13}}, u_{\overline{23}}, u_{\overline{13}}\right) = 0, \quad Q_8(\boldsymbol{l}) = P_5\left(u_{\underline{13}}, u_{\underline{12}}, u_{\overline{13}}, u_{\overline{12}}\right) = 0,$$

$$Q_9(\boldsymbol{l}) = P_6\left(u_{\underline{12}}, u_{\underline{23}}, u_{\overline{12}}, u_{\overline{23}}\right) = 0.$$

Then, given $\boldsymbol{l} \in \Omega$, we obtain the cuboctahedron centered around $\boldsymbol{l}$. We refer to its quad-equations as before by

$$Q_1(\boldsymbol{l}) = P_1\left(u_{\overline{1}\overline{3}}, u_{\overline{2}\overline{3}}, u_{\underline{1}\overline{3}}, u_{\underline{2}\overline{3}}\right) = 0, \quad Q_2(\boldsymbol{l}) = P_1\left(u_{\overline{1}\underline{3}}, u_{\overline{2}\underline{3}}, u_{\underline{1}\underline{3}}, u_{\underline{2}\underline{3}}\right) = 0,$$

$$Q_3(\boldsymbol{l}) = P_2\left(u_{\overline{1}\overline{2}}, u_{\overline{1}\overline{3}}, u_{\overline{1}\underline{2}}, u_{\overline{1}\underline{3}}\right) = 0, \quad Q_4(\boldsymbol{l}) = P_2\left(u_{\underline{1}\overline{2}}, u_{\underline{1}\overline{3}}, u_{\underline{1}\underline{2}}, u_{\underline{1}\underline{3}}\right) = 0,$$

$$Q_5(\boldsymbol{l}) = P_3\left(u_{\overline{2}\overline{3}}, u_{\overline{1}\overline{2}}, u_{\overline{2}\underline{3}}, u_{\underline{1}\overline{2}}\right) = 0, \quad Q_6(\boldsymbol{l}) = P_3\left(u_{\underline{2}\overline{3}}, u_{\overline{1}\underline{2}}, u_{\underline{2}\underline{3}}, u_{\underline{1}\underline{2}}\right) = 0,$$

$$Q_7(\boldsymbol{l}) = P_4\left(u_{\underline{2}\underline{3}}, u_{\underline{1}\underline{3}}, u_{\overline{2}\overline{3}}, u_{\overline{1}\overline{3}}\right) = 0, \quad Q_8(\boldsymbol{l}) = P_5\left(u_{\underline{1}\underline{3}}, u_{\underline{1}\underline{2}}, u_{\overline{1}\overline{3}}, u_{\overline{1}\overline{2}}\right) = 0,$$

$$Q_9(\boldsymbol{l}) = P_6\left(u_{\underline{1}\underline{2}}, u_{\underline{2}\underline{3}}, u_{\overline{1}\overline{2}}, u_{\overline{2}\overline{3}}\right) = 0.$$

Moreover, the overlapped region gives an octahedron centred around $\boldsymbol{l} + \epsilon_3$, and we label its quad-equations by

$$\hat{Q}_1(\boldsymbol{l}) = P_1\left(u_{\overline{1}\overline{3}}, u_{\overline{2}\overline{3}}, u_{\underline{1}\overline{3}}, u_{\underline{2}\overline{3}}\right) = 0, \quad \hat{Q}_2(\boldsymbol{l}) = P_2\left(u_{\overline{2}\overline{3}}, u_{\overline{3}\overline{3}}, u_{\underline{2}\overline{3}}, u\right) = 0,$$

$$\hat{Q}_3(\boldsymbol{l}) = P_3\left(u_{\overline{3}\overline{3}}, u_{\overline{1}\overline{3}}, u, u_{\underline{1}\overline{3}}\right) = 0.$$

We are now in a position to define the CACO property for PΔEs.

Definition (Joshi-Nakazono)

We transfer the definitions of CACO and square properties to the system of PΔEs as follows.

(i) The cuboctahedra with quad-equations $\{Q_1(\boldsymbol{l}),\dots,Q_9(\boldsymbol{l})\}$ have the CACO and square properties, and the square equations $K_i=0$, $i=1,2,3$, are consistent with the PΔEs $P_i=0$, $i=1,2,3$.

(ii) The octahedra with quad-equations $\{\hat{Q}_1(\boldsymbol{l}),\hat{Q}_2(\boldsymbol{l}),\hat{Q}_3(\boldsymbol{l})\}$ have the CAO property.

The following system of PΔEs has the CACO and square properties:

$$
\begin{aligned}
P_1 &= \mathrm{N3}\left(u_{\overline{13}}, u_{\overline{23}}, u_{\underline{1}\overline{3}}, u_{\underline{2}\overline{3}}; a_2, a_1, a_3, a_4\right) = 0,\\
P_2 &= \mathrm{N3}\left(u_{\overline{12}}, u_{\overline{13}}, u_{\overline{1}\underline{2}}, u_{\overline{1}\underline{3}}; a_6, a_5, a_7, a_8\right) = 0,\\
P_3 &= \mathrm{N3}\left(u_{\overline{23}}, u_{\overline{12}}, u_{\overline{2}\underline{3}}, u_{\underline{1}\overline{2}}; a_{10}, a_9, a_{11}, a_{12}\right) = 0,\\
P_4 &= \mathrm{N3}\left(u_{\underline{23}}, u_{\underline{13}}, u_{\overline{23}}, u_{\overline{13}}; a_1, a_2, a_3, a_4\right) = 0,\\
P_5 &= \mathrm{N3}\left(u_{\underline{13}}, u_{\underline{12}}, u_{\overline{13}}, u_{\overline{12}}; a_5, a_6, a_7, a_8\right) = 0,\\
P_6 &= \mathrm{N3}\left(u_{\underline{12}}, u_{\underline{23}}, u_{\overline{12}}, u_{\overline{23}}; a_9, a_{10}, a_{11}, a_{12}\right) = 0,
\end{aligned}
$$

where $u = u(\boldsymbol{l})$, $\boldsymbol{l} = \displaystyle\sum_{i=1}^{3} l_i \epsilon_i \in \Omega$ and

$\mathrm{N3}(X, Y, Z, W; A_1, A_2, A_3, A_4) = A_1 XY + A_2 ZW + A_3 XW + A_4 YZ,$

$$
\begin{aligned}
a_1 &= \alpha_{12} + (-1)^{l_2+l_3}\delta_2 - (-1)^{l_1+l_3}\delta_3, & a_2 &= \alpha_{12} - (-1)^{l_2+l_3}\delta_2 + (-1)^{l_1+l_3}\delta_3,\\
a_3 &= \alpha_{21} - c + (-1)^{l_1+l_2}\delta_1, & a_4 &= \alpha_{21} + c - (-1)^{l_1+l_2}\delta_1,\\
a_5 &= \alpha_{23} + (-1)^{l_1+l_3}\delta_3 - (-1)^{l_1+l_2}\delta_1, & a_6 &= \alpha_{23} - (-1)^{l_1+l_3}\delta_3 + (-1)^{l_1+l_2}\delta_1,\\
a_7 &= \alpha_{32} - c + (-1)^{l_2+l_3}\delta_2, & a_8 &= \alpha_{32} + c - (-1)^{l_2+l_3}\delta_2,\\
a_9 &= \alpha_{31} + (-1)^{l_1+l_2}\delta_1 - (-1)^{l_2+l_3}\delta_2, & a_{10} &= \alpha_{31} - (-1)^{l_1+l_2}\delta_1 + (-1)^{l_2+l_3}\delta_2,\\
a_{11} &= \alpha_{13} - c + (-1)^{l_1+l_3}\delta_3, & a_{12} &= \alpha_{13} + c - (-1)^{l_1+l_3}\delta_3,\\
\alpha_{ij} &= \alpha_i(l_i) - \alpha_j(l_j), \quad i, j \in \{1, 2, 3\}, & \alpha_i(k) &= \alpha_i(0) + k, \quad i \in \{1, 2, 3\},\ k \in \mathbb{Z}.
\end{aligned}
$$

Reduction to δ-$P(E_6^{(1)})$

Lemma (Joshi-Nakazono)

By imposing the $(1,1,1)$*-periodic condition:*

$$u(\boldsymbol{l}+\epsilon_1+\epsilon_2+\epsilon_3)=u(\boldsymbol{l}),$$

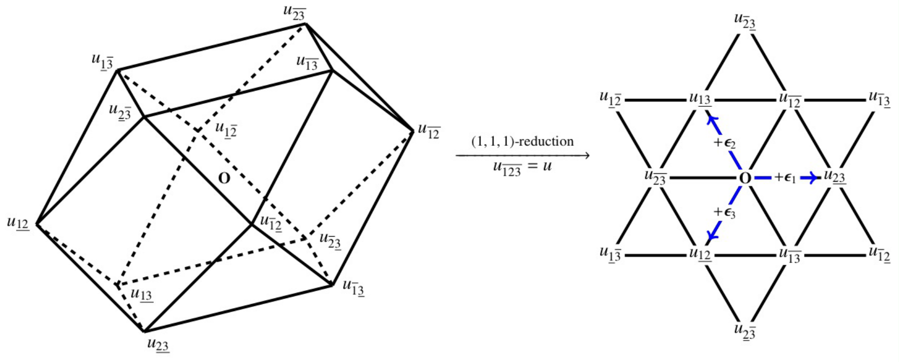
for $\boldsymbol{l}\in\Omega$*, the system of P∆Es can be reduced to*

$$\frac{u_{\overline{1}}}{u_{1}}=\frac{\left(\alpha_{12}-c+(-1)^{l_1+l_2}\delta_1\right)u_{\overline{2}}-\left(\alpha_{12}-(-1)^{l_2+l_3}\delta_2+(-1)^{l_1+l_3}\delta_3\right)u_{\underline{2}}}{\left(\alpha_{12}+(-1)^{l_2+l_3}\delta_2-(-1)^{l_1+l_3}\delta_3\right)u_{\overline{2}}-\left(\alpha_{12}+c-(-1)^{l_1+l_2}\delta_1\right)u_{\underline{2}}},$$

$$\frac{u_{\overline{2}}}{u_{\underline{2}}}=\frac{\left(\alpha_{23}-c+(-1)^{l_2+l_3}\delta_2\right)u_{\overline{3}}-\left(\alpha_{23}-(-1)^{l_1+l_3}\delta_3+(-1)^{l_1+l_2}\delta_1\right)u_{\underline{3}}}{\left(\alpha_{23}+(-1)^{l_1+l_3}\delta_3-(-1)^{l_1+l_2}\delta_1\right)u_{\overline{3}}-\left(\alpha_{23}+c-(-1)^{l_2+l_3}\delta_2\right)u_{\underline{3}}},$$

$$\frac{u_{\overline{3}}}{u_{\underline{3}}}=\frac{\left(\alpha_{31}-c+(-1)^{l_1+l_3}\delta_3\right)u_{\overline{1}}-\left(\alpha_{31}-(-1)^{l_1+l_2}\delta_1+(-1)^{l_2+l_3}\delta_2\right)u_{\underline{1}}}{\left(\alpha_{31}+(-1)^{l_1+l_2}\delta_1-(-1)^{l_2+l_3}\delta_2\right)u_{\overline{1}}-\left(\alpha_{31}+c-(-1)^{l_1+l_3}\delta_3\right)u_{\underline{1}}},$$

where $u=u(\boldsymbol{l})$ *and* $\boldsymbol{l}=\sum_{i=1}^{3}l_i\epsilon_i\in\mathbb{Z}^3/(\epsilon_1+\epsilon_2+\epsilon_3)$.



The $(1, 1, 1)$-reduction causes the reduction from the C_3 root lattice:

$$\Omega = \left\{ \sum_{i=1}^{3} l_i \epsilon_i \;\middle|\; l_i \in \mathbb{Z},\ l_1 + l_2 + l_3 \in 2\mathbb{Z} \right\},$$

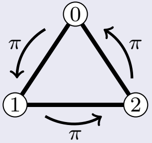
to the A_2 root lattice (triangle lattice):

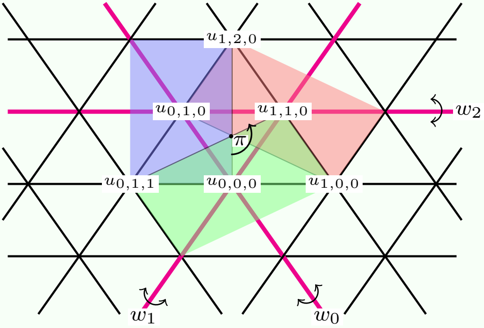
$$\mathbb{Z}^3/(\epsilon_1 + \epsilon_2 + \epsilon_3) = \left\{ \sum_{i=1}^{3} l_i \epsilon_i \;\middle|\; l_i \in \mathbb{Z},\ l_1 + l_2 + l_3 = 0 \right\}.$$

Lemma (Joshi-Nakazono)

The reduced system has the extended affine Weyl group symmetry of type $A_2^{(1)}$:

$$\widetilde{W}(A_2^{(1)}) = \langle w_0, w_1, w_2 \rangle \rtimes \langle \pi \rangle.$$

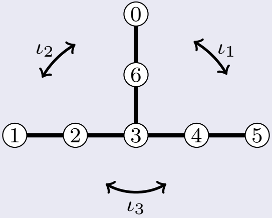




Lemma (Joshi-Nakazono)

The symmetry group $\widetilde{W}(A_2^{(1)})$ is a subgroup of the symmetry group for δ-$P(E_6^{(1)})$. Indeed, the birational action of $\widetilde{W}(A_2^{(1)})$ can be reconstructed from

$$\widetilde{W}(E_6^{(1)}) = \langle s_0, \ldots, s_6 \rangle \rtimes \langle \iota_1, \iota_2, \iota_3 \rangle,$$



as the following:

$$w_0 = s_2 s_1 s_3 s_2, \quad w_1 = s_4 s_5 s_3 s_4, \quad w_2 = s_6 s_0 s_3 s_6, \quad \pi = \iota_3 \iota_1.$$

Moreover, the u-variables are given by the ratios of the τ-functions of δ-$P(E_6^{(1)})$.

Theorem (Joshi-Nakazono)

The birational action of the square of shortest translation on the triangle lattice gives δ-$P(E_6^{(1)})$:

$$(\underline{g}+f)(f+g)=\frac{\left(f-\dfrac{\alpha_{23}+c-\delta_1+\delta_2-\delta_3}{4}\right)\left(f+\dfrac{\alpha_{23}+c+\delta_1+\delta_2+\delta_3}{4}\right)}{f+\dfrac{\alpha_{23}+c+\delta_1-\delta_2-\delta_3-2}{4}+\dfrac{\alpha_{12}}{2}}\frac{\left(f-\dfrac{\alpha_{23}-c+\delta_1-\delta_2+\delta_3}{4}\right)\left(f+\dfrac{\alpha_{23}-c-\delta_1-\delta_2-\delta_3}{4}\right)}{f+\dfrac{\alpha_{23}-c-\delta_1+\delta_2+\delta_3-2}{4}+\dfrac{\alpha_{12}}{2}},$$

$$(\overline{f}+g)(f+g)=\frac{\left(g-\dfrac{\alpha_{23}+c+\delta_1+\delta_2+\delta_3}{4}\right)\left(g+\dfrac{\alpha_{23}+c-\delta_1+\delta_2-\delta_3}{4}\right)}{g+\dfrac{\alpha_{23}+c-\delta_1-\delta_2+\delta_3}{4}+\dfrac{\alpha_{12}}{2}}\frac{\left(g-\dfrac{\alpha_{23}-c-\delta_1-\delta_2-\delta_3}{4}\right)\left(g+\dfrac{\alpha_{23}-c+\delta_1-\delta_2+\delta_3}{4}\right)}{g+\dfrac{\alpha_{23}-c+\delta_1+\delta_2-\delta_3}{4}+\dfrac{\alpha_{12}}{2}},$$

where

$$\overline{\alpha_{12}}=\alpha_{12}+2.$$

The f, g*-variables are given by the rational functions of the* u*-variables.*

Note that $\overline{u}=u_{\overline{11}}$, $\underline{u}=u_{\underline{11}}$.

Concluding remarks

Summary

We gave definitions of CACO and square properties and presented a system of PΔEs which has such properties. Moreover, we showed the reduction from the system of PΔEs to δ-$P(E_6^{(1)})$.

Future works

- Construction of a Lax pair of PΔEs which have the CACO property.
- Extend the idea of consistency around a cuboctahedron to polytopes in higher dimensions.