(Considerations on) twisted Pieri formulas for Macdonald polynomials

Genki Shibukawa (Kobe university)
野海 65

Department of Mathematics, Graduate School of Science, Kobe University,
1-1, Rokkodai, Nada-ku, Kobe, 657-8501, JAPAN
E-mail: g-shibukawa@math.kobe-u.ac.jp

本研究は科研費 **(課題番号: 18J00233)** の助成を受けたものである.

Abstract

結論:予告した結果は得られませんでした.

『
なんの成果も ···!! 得られませんでした ···!!
私が無能なばかりに ···!! ただ いたずらに時間を浪費し ···!!
ヤツら (twisted Pieri for M) の正体を ···!!
突きとめることができませんでした!!
』

しかたがないので,

前半: 今回の研究の元 (動機) になった Jack 版 (rational version). 特に補間 Jack 多項式の差分方程式の証明.

後半: 今回の調査 (研究) 報告 (失敗)

の話を行う.

Contents

1. Jack version (motivation)

Notations

Let $r \in \mathbb{Z}_{\geq 1}$, $d \in \mathbb{C}$ and

$$\mathcal{P} := \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0\},$$

$$\delta := (r-1, r-2, \ldots, 2, 1, 0) \in \mathcal{P},$$

$$\epsilon_i := (0, \ldots, 0, \overset{\overset{i}{\vee}}{1}, 0, \ldots, 0),$$

$$e_k(\mathbf{z}) := \sum_{1 \leq i_1 < \cdots < i_k \leq r} z_{i_1} \cdots z_{i_k} \quad (k \in \mathbb{Z}_{\geq 0}),$$

$$E_k(\mathbf{z}) := \sum_{j=1}^{r} z_j^k \partial_{z_j} \quad (k \in \mathbb{Z}_{\geq 0}),$$

$$D_k^{(d)}(\mathbf{z}) := \sum_{j=1}^{r} z_j^k \partial_{z_j}^2 + d \sum_{1 \leq j \neq l \leq r} \frac{z_j^k}{z_j - z_l} \partial_{z_j} \quad (k \in \mathbb{Z}_{\geq 0}).$$

Jack and interpolation Jack

For any partition $\mathbf{m} = (m_1, \ldots, m_r) \in \mathcal{P}$ and $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$, Jack polynomials $P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ are defined by the following two conditions.

(1) $D_2(\mathbf{z}) P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) \sum_{j=1}^{r} m_j \left(m_j - 1 - d(r-j)\right)$,

(2) $P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = m_{\mathbf{m}}(\mathbf{z}) + \sum_{\mathbf{k}<\mathbf{m}} c_{\mathbf{mk}} m_{\mathbf{k}}(\mathbf{z})$.

Similarly, interpolation (or shifted) Jack polynomials $P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ are defined by the following two conditions.

$(1)^{\mathrm{ip}}$ $P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right) = 0$, unless $\mathbf{k} \subset \mathbf{m} \in \mathcal{P}$

$(2)^{\mathrm{ip}}$ $P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ + (lower terms).

Further, we put

$$\Phi^{(d)}_{\mathbf{m}}(\mathbf{z}) := \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P_{\mathbf{m}}\left(\mathbf{1}; \frac{d}{2}\right)}, \quad \Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) := \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)},$$

$$\binom{\mathbf{z}}{\mathbf{k}}^{(d)} := \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{z} + \frac{d}{2}\delta; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k} + \frac{d}{2}\delta; \frac{d}{2}\right)}$$ (generalized (or Jack) binomial coefficients).

$\underline{r = 1}$

$$\Phi^{(d)}_{m}(z) := z^m, \quad \Psi^{(d)}_{m}(z) := \frac{z^m}{m!}, \quad \binom{z}{k}^{(d)} := \frac{z(z-1)\cdots(z-k+1)}{k!}.$$

Special values

$$P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)=\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i+1)\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)\right)_{m_i-m_j}},$$

$$P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)=\prod_{j=1}^{r}\left(\frac{d}{2}(r-j)+1\right)_{m_j}\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i-1)+1\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)+1\right)_{m_i-m_j}}.$$

Therefore,

$$\Phi^{(d)}_{\mathbf{m}}(\mathbf{z})=\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i)\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i+1)\right)_{m_i-m_j}}P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right),$$

$$\Psi^{(d)}_{\mathbf{m}}(\mathbf{z})=\prod_{j=1}^{r}\frac{1}{\left(\frac{d}{2}(r-j)+1\right)_{m_j}}\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i)+1\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i-1)+1\right)_{m_i-m_j}}P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right).$$

1. Sekiguchi operators (Sekiguchi, Debiard, Macdonald, . . .)

Let

$$H^{(d)}_{r,p}(\mathbf{z}) := \sum_{l=0}^{p} \left(\frac{2}{d}\right)^{p-l} \sum_{\substack{I\subset[r],\\ |I|=l}} \frac{1}{\Delta(\mathbf{z})} \left(\prod_{i\in I} z_i\partial_{z_i}\right) \Delta(\mathbf{z}) \sum_{\substack{J\subset[r]\setminus I,\\ |J|=p-l}} \left(\prod_{j\in J} z_j\partial_{z_j}\right),$$

$$S^{(d)}_r(u,\mathbf{z}) := \sum_{p=0}^{r} H^{(d)}_{r,p}(\mathbf{z}) u^{r-p},$$

where $\Delta(\mathbf{z}) := \prod_{1\le i<j\le r}(z_i - z_j)$. We have

$$S^{(d)}_r(u,\mathbf{z})P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right) I^{(d)}_r(u,\mathbf{m}), \tag{1}$$

$$H^{(d)}_{r,p}(\mathbf{z})P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right) e_{r,k}\left(\mathbf{m} + \frac{d}{2}\delta\right), \tag{2}$$

where

$$I^{(d)}_r(u,\mathbf{m}) := \prod_{k=1}^{r}\left(u + r - k + \frac{2}{d}m_k\right) = \left(\frac{2}{d}\right)^r \prod_{k=1}^{r}\left(m_k + \frac{d}{2}(u + r - k)\right).$$

2. Pieri type formulas for Jack polynomials (Lassalle, . . .)

$$E_0(\mathbf{z})\Phi^{(d)}_{\mathbf{x}}(\mathbf{z}) = \sum_{i=1}^{r} \Phi^{(d)}_{\mathbf{x}-\epsilon_i}(\mathbf{z}) \left(x_i + \frac{d}{2}(r-i) \right) A^{(d)}_{-,i}(\mathbf{x}), \tag{3}$$

$$E_0(\mathbf{z})\Psi^{(d)}_{\mathbf{x}}(\mathbf{z}) = \sum_{\substack{1\le i\le r,\\ \mathbf{x}-\epsilon_i\in\mathcal{P}}} \Psi^{(d)}_{\mathbf{x}-\epsilon_i}(\mathbf{z}) A^{(d)}_{+,i}(\mathbf{x}-\epsilon_i), \tag{4}$$

$$|\mathbf{z}|\Phi^{(d)}_{\mathbf{x}}(\mathbf{z}) = \sum_{i=1}^{r} \Phi^{(d)}_{\mathbf{x}+\epsilon_i}(\mathbf{z}) A^{(d)}_{+,i}(\mathbf{x}), \tag{5}$$

$$|\mathbf{z}|\Psi^{(d)}_{\mathbf{x}}(\mathbf{z}) = \sum_{\substack{1\le i\le r,\\ \mathbf{x}+\epsilon_i\in\mathcal{P}}} \Psi^{(d)}_{\mathbf{x}+\epsilon_i}(\mathbf{z}) \left(x_i + 1 + \frac{d}{2}(r-i) \right) A^{(d)}_{-,i}(\mathbf{x}+\epsilon_i), \tag{6}$$

where

$$|\mathbf{z}| := \sum_{j=1}^{r} z_j, \;\; E_0(\mathbf{z}) := \sum_{j=1}^{r} \partial_{z_j}, \;\; A^{(d)}_{\pm,i}(\mathbf{x}) := \prod_{1\le k\ne i\le r} \frac{x_i - x_k - \frac{d}{2}(i-k) \pm \frac{d}{2}}{x_i - x_k - \frac{d}{2}(i-k)}.$$

3. Binomial formulas (Knop-Sahi, Okounkov-Olshanski, . . .)

For any partition $\mathbf{x}$, $\mathbf{k}$

$$\Phi_{\mathbf{x}}^{(d)}(\mathbf{1}+\mathbf{z}) = \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}}^{(d)} \Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) = \sum_{\mathbf{k}\subset\mathbf{x}} \frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}\left(\mathbf{1};\frac{d}{2}\right)} \Psi_{\mathbf{k}}^{(d)}(\mathbf{z}) \qquad (7)$$

or

$$e^{|\mathbf{z}|}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z}) = \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}}^{(d)} \Psi_{\mathbf{x}}^{(d)}(\mathbf{z}) = \sum_{\mathbf{k}\subset\mathbf{x}} \frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)} \Psi_{\mathbf{x}}^{(d)}(\mathbf{z}). \qquad (8)$$

4. Mysterious summation

For any $I \subset [r]$ and $\mathbf{x} = (x_1, \ldots, x_r) \in \mathbb{C}^r$, we have

$$\sum_{i\in I}\left(x_i + 1 + \frac{d}{2}(r-i)\right) A^{(d)}_{-,i,I\setminus i}(\mathbf{x}+\epsilon_i) A^{(d)}_{+,i,I\setminus i}(\mathbf{x}) - \sum_{i\in I}\left(x_i + \frac{d}{2}(r-i)\right) A^{(d)}_{+,i,I\setminus i}(\mathbf{x}-\epsilon_i) A^{(d)}_{-,i,I\setminus i}(\mathbf{x}) = |I|, \tag{9}$$

where

$$A^{(d)}_{\pm,i,I\setminus i}(\mathbf{x}) := \prod_{j\in I\setminus i} \frac{x_i - x_j - \frac{d}{2}(i-j) \pm \frac{d}{2}}{x_i - x_j - \frac{d}{2}(i-j)}.$$

The proof of the mysterious summation

We recall

$$[E_0(\mathbf{z}), |\mathbf{z}|]\Phi^{(d)}_{\mathbf{x}}(\mathbf{z}) = \Phi^{(d)}_{\mathbf{x}}(\mathbf{z})r$$

and Pieri type formulas for the Jack polynomials

$$\begin{aligned}
&[E_0(\mathbf{z}), |\mathbf{z}|]\Phi^{(d)}_{\mathbf{x}}(\mathbf{z}) \\
&\quad = \sum_{i=1}^{r} E_0(\mathbf{z})\Phi^{(d)}_{\mathbf{x}+\epsilon_i}(\mathbf{z})A^{(d)}_{+,i}(\mathbf{x}) - \sum_{j=1}^{r} |\mathbf{z}|\Phi^{(d)}_{\mathbf{x}-\epsilon_j}(\mathbf{z})\left(x_j + \frac{d}{2}(r-j)\right)A^{(d)}_{-,j}(\mathbf{x}) \\
&\quad = \Phi^{(d)}_{\mathbf{x}}(\mathbf{z}) \\
&\quad \cdot \sum_{i=1}^{r}\left(\left(x_i + 1 + \frac{d}{2}(r-i)\right)A^{(d)}_{-,i}(\mathbf{x}+\epsilon_i)A^{(d)}_{+,i}(\mathbf{x}) \right.\\
&\qquad\qquad \left. -\left(x_i + \frac{d}{2}(r-i)\right)A^{(d)}_{-,i}(\mathbf{x})A^{(d)}_{+,i}(\mathbf{x}-\epsilon_i)\right).
\end{aligned}$$

5. Twisted Pieri formulas for Jack polynomials (difficult)

For $l = 0, 1, \ldots, r$, we have

$$\left[\frac{(-\mathrm{ad}\,|\mathbf{z}|)^l}{l!} S_r^{(d)}(u;\mathbf{z})\right]\Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) = \sum_{\substack{J\subset[r]\\|J|=l}} \Phi_{\mathbf{k}+\epsilon_J}^{(d)}(\mathbf{z}) I_{J^c}^{(d)}(u,\mathbf{k}) A_{+,J}^{(d)}(\mathbf{k}), \tag{10}$$

$$\left[\frac{(-\mathrm{ad}\,|\mathbf{z}|)^l}{l!} S_r^{(d)}(u,\mathbf{z})\right]\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})$$

$$= \sum_{\substack{J\subset[r],|J|=l\\ \mathbf{k}+\epsilon_J\in\mathcal{P}}} \Psi_{\mathbf{k}+\epsilon_I}^{(d)}(\mathbf{z}) I_{J^c}^{(d)}(u,\mathbf{k}) A_{-,J}^{(d)}(\mathbf{k}+\epsilon_I) \prod_{j\in J}\left(k_j + 1 + \frac{d}{2}(r-j)\right). \tag{11}$$

$$A_{\pm,J}^{(d)}(\mathbf{k}) := \prod_{j\in J, l\in J^c} \frac{k_j - k_l - \frac{d}{2}(j-l) \pm \frac{d}{2}}{k_j - k_l - \frac{d}{2}(j-l)},$$

$$I_{J^c}^{(d)}(u,\mathbf{k}) := \left(\frac{2}{d}\right)^r \prod_{l\in J^c}\left(k_l + \frac{d}{2}(u+r-l)\right).$$

Sketch of the proof of the twisted Pieri formulas

These formulas are proved by induction on p.
The case of $p = 0$ is

$$S_r^{(d)}(u, \mathbf{z})\Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) = \Phi_{\mathbf{k}}^{(d)}(\mathbf{z})I_r^{(d)}(u, \mathbf{m}), \tag{12}$$

$$S_r^{(d)}(u, \mathbf{z})\Psi_{\mathbf{k}}^{(d)}(\mathbf{z}) = \Psi_{\mathbf{k}}^{(d)}(\mathbf{z})I_r^{(d)}(u, \mathbf{m}). \tag{13}$$

This is (1)

$$S_r^{(d)}(u, \mathbf{z})P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) I_r^{(d)}(u, \mathbf{m})$$

exactly.

If $p = 1$, then

$$
\begin{aligned}
&[\mathrm{ad}\,(|\mathbf{z}|) S_r^{(d)}(u,\mathbf{z})]\Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) \\
&\quad = |\mathbf{z}|\Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) I_r^{(d)}(u,\mathbf{k}) - S_r^{(d)}(u,\mathbf{z}) \sum_{i=1}^{r} \Phi_{\mathbf{k}+\epsilon_i}^{(d)}(\mathbf{z}) A_{+,i}^{(d)}(\mathbf{k}) \\
&\quad = \sum_{i=1}^{r} \Phi_{\mathbf{k}+\epsilon_i}^{(d)}(\mathbf{z}) A_{+,i}^{(d)}(\mathbf{k}) (I_r^{(d)}(u,\mathbf{k}) - I_r^{(d)}(u,\mathbf{k}+\epsilon_i)) \\
&\quad = \left(-\frac{2}{d}\right) \sum_{i=1}^{r} \Phi_{\mathbf{k}+\epsilon_i}^{(d)}(\mathbf{z}) A_{+,i}^{(d)}(\mathbf{k}) \prod_{j\neq i} \left(u + r - j + \frac{2}{d} k_j\right) \\
&\quad = -\sum_{i=1}^{r} \Phi_{\mathbf{k}+\epsilon_i}^{(d)}(\mathbf{z}) A_{+,i}^{(d)}(\mathbf{k}) I_{[r]\setminus\{i\}}^{(d)}(u,\mathbf{k}).
\end{aligned}
$$

Assume the $n = l$ case holds. Hence,

$$\begin{aligned}
&\left[\frac{(\mathrm{ad}\,(|\mathbf{z}|))^{l+1}}{(l+1)!} S_r^{(d)}(u,\mathbf{z})\right] \Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) \\
&\quad = \frac{1}{l+1}|\mathbf{z}| \left[\frac{(\mathrm{ad}\,(|\mathbf{z}|))^{l}}{l!} S_r^{(d)}(u,\mathbf{z})\right] \Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) \\
&\qquad - \left[\frac{(\mathrm{ad}\,(|\mathbf{z}|))^{l}}{l!} S_r^{(d)}(u,\mathbf{z})\right] \frac{1}{l+1}|\mathbf{z}|\Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) \\
&\quad = \frac{1}{l+1}(-1)^l \sum_{\substack{J\subset[r] \\ |J|=l}} \sum_{\nu=1}^{r} \Phi_{\mathbf{k}+\epsilon_J+\epsilon_\nu}^{(d)}(\mathbf{z}) \\
&\qquad \left(A_{+,\nu}^{(d)}(\mathbf{k}+\epsilon_J) I_{J^c}^{(d)}(u,\mathbf{k}) A_{+,J}^{(d)}(\mathbf{k}) - I_{J^c}^{(d)}(u,\mathbf{k}+\epsilon_\nu) A_{-,J}^{(d)}(\mathbf{k}+\epsilon_\nu) A_{-,\nu}^{(d)}(\mathbf{k})\right).
\end{aligned}$$

Put

$$s_j := k_j + \frac{d}{2}(r-j).$$

$$
\begin{aligned}
&\left[\frac{(\mathrm{ad}\,(|\mathbf{z}|))^{l+1}}{(l+1)!} S_r^{(d)}(u,\mathbf{z})\right]\Phi_{\mathbf{k}}^{(d)}(\mathbf{z}) \\
&\quad = (-1)^l \sum_{\substack{I\subset[r] \\ |I|=l+1}} \frac{1}{l+1}\Phi_{\mathbf{k}+\epsilon_I}^{(d)}(\mathbf{z}) I_{I^c}^{(d)}(u,\mathbf{k}) A_{+,I}^{(d)}(\mathbf{k}) \\
&\qquad \cdot (-1)\sum_{i\in I}\left\{\left(s_i + 1 + \frac{d}{2}u\right) A_{-,\{i\},I\setminus\{i\}}^{(d)}(\mathbf{k}+\epsilon_i) A_{+,\{i\},I\setminus\{i\}}^{(d)}(\mathbf{k})\right. \\
&\qquad\qquad \left. - \left(s_i + \frac{d}{2}u\right) A_{-,\{i\},I\setminus\{i\}}^{(d)}(\mathbf{k}) A_{+,\{i\},I\setminus\{i\}}^{(d)}(\mathbf{k}-\epsilon_i)\right\} \\
&\quad = (-1)^{l+1} \sum_{\substack{I\subset[r] \\ |I|=l+1}} \Phi_{\mathbf{k}+\epsilon_I}^{(d)}(\mathbf{z}) I_{I^c}^{(d)}(u,\mathbf{k}) A_{+,I}^{(d)}(\mathbf{k}).
\end{aligned}
$$

0. Difference equations (Knop-Sahi)

Let

$$T_{x_j} f(\mathbf{x}) := f(\mathbf{x} - \epsilon_j), \quad T_{\mathbf{x}}^J := \prod_{j \in J} T_{x_j},$$

$$A_{\pm,J}^{(d)}(\mathbf{x}) := \prod_{j \in J, k \in J^c} \frac{x_j - x_k - \frac{d}{2}(j-k) \pm \frac{d}{2}}{x_j - x_k - \frac{d}{2}(j-k)},$$

$$I_{J^c}^{(d)}(u; \mathbf{x}) := \left(\frac{2}{d}\right)^r \prod_{k \in J^c} \left(x_k + \frac{d}{2}(u + r - k)\right).$$

Put

$$D_r^{(d)\,\mathrm{ip}}(u; \mathbf{x}) := \sum_{J \subset [r]} (-1)^{|J|} I_{J^c}^{(d)}(u; \mathbf{x}) h_{-,J}^{(d)}(\mathbf{x}) \prod_{j \in J} \left(x_j + \frac{d}{2}(r - j)\right) T_{\mathbf{x}}^J.$$

We have

$$D_r^{(d)\,\mathrm{ip}}(u; \mathbf{x}) P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x} + \frac{d}{2}\delta; \frac{d}{2}\right) = P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x} + \frac{d}{2}\delta; \frac{d}{2}\right) I_r^{(d)}(u; \mathbf{k}).$$

The proof of the difference equations for interpolation Jack

From the twisted Pieri (10) and the binomial (8),

$$
\begin{aligned}
&[e^{\mathrm{ad}\,(|\mathbf{z}|)} S_r^{(d)}(u,\mathbf{z})] e^{|\mathbf{z}|} \Psi_{\mathbf{k}}^{(d)}(\mathbf{z}) \\
&= \sum_{\mathbf{k}\subset\mathbf{x}} \frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)} [e^{\mathrm{ad}\,(|\mathbf{z}|)} S_r^{(d)}(u;\mathbf{z})] \Psi_{\mathbf{x}}^{(d)}(\mathbf{z}) \\
&= \sum_{\mathbf{k}\subset\mathbf{x}} \frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)} \left[\sum_{l=0}^{r} \frac{\mathrm{ad}\,(|\mathbf{z}|)^l}{l!} S_r^{(d)}(u;\mathbf{z})\right] \Psi_{\mathbf{x}}^{(d)}(\mathbf{z}) \\
&= \sum_{\mathbf{k}\subset\mathbf{x}} \frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)} \\
&\cdot \sum_{l=0}^{r} (-1)^l \sum_{\substack{J\subset[r],|J|=l \\ \mathbf{x}+\epsilon_J\in\mathcal{P}}} \Psi_{\mathbf{x}+\epsilon_J}^{(d)}(\mathbf{z}) I_{J^c}^{(d)}(u,\mathbf{x}) A_{-,J}^{(d)}(\mathbf{x}+\epsilon_J) \prod_{j\in J}\left(x_j+1+\frac{d}{2}(r-j)\right)
\end{aligned}
$$

$$= \sum_{\mathbf{k}\subset\mathbf{x}} \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{x} - \epsilon_J + \frac{d}{2}\delta; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k} + \frac{d}{2}\delta; \frac{d}{2}\right)}$$

$$\cdot \sum_{l=0}^{r} (-1)^l \sum_{\substack{J\subset[r],|J|=l \\ \mathbf{x}\in\mathcal{P}}} \Psi^{(d)}_{\mathbf{x}}(\mathbf{z}) I^{(d)}_{J^c}(u,\mathbf{x}) A^{(d)}_{-,J}(\mathbf{x}) \prod_{j\in J}\left(x_j + \frac{d}{2}(r-j)\right)$$

$$= \sum_{\mathbf{k}\subset\mathbf{x}} \Psi^{(d)}_{\mathbf{x}}(\mathbf{z})$$

$$\cdot \sum_{J\subset[r]} (-1)^{|J|} I^{(d)}_{J^c}(u;\mathbf{x}) A^{(d)}_{-,J}(\mathbf{x}) \prod_{j\in J}\left(x_j + \frac{d}{2}(r-j)\right) \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{x} - \epsilon_J + \frac{d}{2}\delta; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k} + \frac{d}{2}\delta; \frac{d}{2}\right)}.$$

On the other hand, from the binomial formula (8) and (1),

$$[e^{\mathrm{ad}\,(|\mathbf{z}|)}S_r^{(d)}(u,\mathbf{z})]e^{|\mathbf{z}|}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z}) = e^{|\mathbf{z}|}S_r^{(d)}(u,\mathbf{z})e^{-|\mathbf{z}|}e^{|\mathbf{z}|}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})$$
$$= e^{|\mathbf{z}|}S_r^{(d)}(u,\mathbf{z})\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})$$
$$= e^{|\mathbf{z}|}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})I_r^{(d)}(u,\mathbf{k})$$
$$= \sum_{\mathbf{k}\subset\mathbf{x}}\Psi_{\mathbf{x}}^{(d)}(\mathbf{z})\frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)}I_r^{(d)}(u;\mathbf{k}).$$

Then we have

$$\sum_{J\subset[r]}(-1)^{|J|}I_{J^c}^{(d)}(u;\mathbf{x})A_{-,J}^{(d)}(\mathbf{x})\prod_{j\in J}\left(x_j+\frac{d}{2}(r-j)\right)\frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x}-\epsilon_J+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)}$$
$$= \frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)}I_r^{(d)}(u;\mathbf{k}).$$

注

半年前の RIMS で似たような話をしたが, その時の証明と似て非なるものであることに注意する. 実際, 前回は

$$S_r^{(d)}(u,\mathbf{z})\Psi_{\mathbf{x}}^{(d)}(\mathbf{1}+\mathbf{z})$$

を binomial formula (7) と twisted Pieri で 2 通りの方法で計算して, 補間 Jack の差分方程式を導出したが, 今回は

$$[e^{\mathrm{ad}\,(|\mathbf{z}|)}S_r^{(d)}(u,\mathbf{z})]e^{|\mathbf{z}|}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})$$

を binomial formula (8) と twisted Pieri で計算している.

違い: 今回の方法は q 類似がある可能性がある.

今回のお題: q, q and q (ククク $\cdots$)

今回の動機: 上述の結果, 論法, 特に 5. twisted Pieri (ヤツら) の q 類似を導出したい.

結果: なんの成果も $\cdots$!! 得られませんでした $\cdots$

しかたがないので何が (どこが) ダメだったかを報告する.

2. Considerations on $\cdots$

q 類似早見表 (結論)

先に述べた

1. Sekiguchi operators
2. Pieri type formulas for Jack polynomials
3. Binomial formulas
4. Mysterious summation
5. Twisted Pieri formulas for Jack polynomials
0. Difference equations for interpolation Jack

の q 類似の有無については以下の通り.

	1.	2.	3.	4.	5.	0.
$r = 1$	○	○	○	○	○	○
$r > 1$	○	△?	△?	○	×	○

今回は $r > 1$ の場合の 5. twisted Pieri の q 類似について話すつもりだった.
しかし, よりにもよってここが唯一どうにもならない場所だった!!

$r = 1$

$$\Phi_m(z;q,t) := z^m,$$

$$\Psi_m(z;q,t) := \frac{z^m}{(q;q)_m},$$

$$(a;q)_m := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}) & (n \neq 0) \\ 1 & (n = 0) \end{cases},$$

$$P^{\mathrm{ip}}_m(z;q,t) := z^m(z^{-1};q)_m = \begin{cases} (z-1)(z-q)\cdots(z-q^{m-1}) & (m \neq 0) \\ 1 & (m = 0) \end{cases},$$

$$\binom{m}{k}_{q,t} := \frac{P^{\mathrm{ip}}_k(q^m;q,t)}{P^{\mathrm{ip}}_k(q^k;q,t)} = (-1)^k q^{-\frac{k(k-1)}{2}} \frac{P^{\mathrm{ip}}_k(q^m;q,t)}{(q;q)_k} = \frac{(q;q)_m}{(q;q)_k(q;q)_{m-k}},$$

$$e_q(z) := \frac{1}{(z;q)_\infty} = \sum_{n\geq 0} \frac{z^n}{(q;q)_n}.$$

1. Macdonald operators

$$T_{q,z}\Phi_m(z;q,t) = \Phi_m(z;q,t)q^m,$$
$$T_{q,z}\Psi_m(z;q,t) = \Psi_m(z;q,t)q^m.$$

ただし,

$$T_{q,z}f(z) := f(qz).$$

2. Pieri formulas

$$E_0(z)\Phi_m(z;q,t) = \Phi_{m-1}(z;q,t)[m]_q,$$
$$E_0(z)\Psi_m(z;q,t) = \Psi_{m-1}(z;q,t),$$
$$z\Phi_m(z;q,t) = \Phi_{m+1}(z;q,t),$$
$$z\Psi_m(z;q,t) = \Psi_{m+1}(z;q,t)[m+1]_q.$$

ただし

$$E_0(z) := \frac{T_{q,z}-1}{qz-q}.$$

3. Binomial formulas

$$\prod_{j=1}^{x}(1+q^{j-1}z)=\sum_{k\geq 0}\binom{x}{k}_{q,t}\Phi_k(z;q,t),$$

$$e_q(z)\Psi_k(z;q,t)=\sum_{x\geq 0}\binom{x}{k}_{q,t}\Psi_x(z;q,t)=\sum_{x\geq 0}\frac{P_k^{\mathrm{ip}}(q^x)}{P_k(1)}\Psi_x(z;q,t).$$

4. Mysterious summation

$$[x+1]_q-[x]_q=q^x.$$

5. Twisted Pieri formulas

$$\left[\frac{(\mathrm{ad}\,z)^1}{[1]_q}(1+T_{q,z}u)\right]\Phi_k(z;q,t)=(1-q)q^k u\Phi_{k+1}(z;q,t),$$

$$\left[\frac{(\mathrm{ad}\,z)^1}{[1]_q}(1+T_{q,z}u)\right]\Psi_k(z;q,t)=(1-q^{k+1})q^k u\Psi_{k+1}(z;q,t).$$

今回はこの 5. を多変数化しようと思った.

1. Macdonald operators

言わずと知れた Macdonald 作用素 (M, M and M)!!

$$D_k(\mathbf{z}) := \sum_{I\subseteq[r],\ |I|=k} t^{\binom{k}{2}} A_I(\mathbf{z};t) T_q^I.$$

ただし $T_{q,i}$ は i 変数に関しての q-shift 作用素で

$$A_I(\mathbf{z};t) := t^{\binom{|I|}{2}} \prod_{i\in I,\, j\notin I} \frac{tz_i - z_j}{z_i - z_j}, \quad T_q^I := \prod_{i\in I} T_{q,i}.$$

$P_\lambda(\mathbf{z};q,t)$ を Macdonald 多項式 (いつもの) とすると

$$D_k(\mathbf{z})P_\lambda(\mathbf{z};q,t) = P_\lambda(\mathbf{z};q,t)e_k(t^\delta q^\lambda).$$

母函数形

$$D(u,\mathbf{z}) := \sum_{k=0}^{r} D_k(\mathbf{z})u^k, \quad I(u,\lambda;q,t) := \prod_{j=1}^{r}(1 + t^{r-j}q^{\lambda_j}u)$$

を用いれば

$$D(u,\mathbf{z})P_\lambda(\mathbf{z};q,t) = P_\lambda(\mathbf{z};q,t)I(u,\lambda).$$

2. Pieri formulas

$\Phi_\lambda^{(d)}(\mathbf{z})$ の q 類似に関しては, Macdonald 多項式の主特殊化による正規化

$$\Phi_\lambda(\mathbf{z};q,t) := \frac{P_\lambda(\mathbf{z};q,t)}{P_\lambda(t^\delta q^\lambda;q,t)}$$

を用いれば

$$|\mathbf{z}|\Phi_\lambda(\mathbf{z};q,t) = \sum_{i=1}^{r} \Phi_{\lambda+\epsilon_i}(\mathbf{z};q,t)A_i(t^\delta q^\lambda;t), \tag{14}$$

$$E_0(\mathbf{z})\Phi_\lambda(\mathbf{z};q,t) = \sum_{i=1}^{r} \Phi_{\lambda-\epsilon_i}(\mathbf{z};q,t)t^{1-r}\frac{1-q^{\lambda_i}t^{r-i}}{1-q}A_i(t^\delta q^\lambda;t) \tag{15}$$

が得られる. ただし,

$$E_0(\mathbf{z};q,t) := \frac{1}{q-1}\left(\sum_{i=1}^{r}\frac{1}{z_i}(A_i(\mathbf{z};q,t)T_{q,i}-1)\right).$$

しかし $\Psi_\lambda^{(d)}(\mathbf{z})$ の q 類似に関しては定義もまだ決めかめている?

3. Binomial formulas

どうも

$$\Phi^{(d)}_{\mathbf{x}}(\mathbf{1}+\mathbf{z}) = \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}}^{(d)} \Phi^{(d)}_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{k}\subset\mathbf{x}} \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}\left(\mathbf{1};\frac{d}{2}\right)} \Psi^{(d)}_{\mathbf{k}}(\mathbf{z})$$

の方の q 類似はない?(ないこともないが色々微妙?).
ただし

$$e^{|\mathbf{z}|}\Psi^{(d)}_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{k}\subset\mathbf{x}} \binom{\mathbf{x}}{\mathbf{k}}^{(d)} \Psi^{(d)}_{\mathbf{x}}(\mathbf{z}) = \sum_{\mathbf{k}\subset\mathbf{x}} \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)} \Psi^{(d)}_{\mathbf{x}}(\mathbf{z})$$

の方の二項定理は

$$\prod_{j=1}^{r} e_q(z_j)\Psi_{\mathbf{k}}(\mathbf{z};q,t) = \prod_{j=1}^{r} \frac{1}{(z_j;q)_\infty}\Psi_{\mathbf{k}}(\mathbf{z};q,t)$$

を考えるとワンチャンありうる (Lassalle)?

4. Mysterious summation

これは (Weyl 関係式の q 類似の多変数版)

$$[E_0(\mathbf{z};q,t), e_1(\mathbf{z})] = D_1(\mathbf{z})$$

と先程の $\Phi_\lambda(\mathbf{z};q,t)$ についての Pieri (14), (15) を用いれば, Jack のときと同様にして得られる.

$$\begin{aligned}\sum_{i=1}^{r} \Big(& (1-q^{\lambda_i+1}t^{r-i}) A_i(t^\delta q^\lambda;t) A_i(t^\delta q^{\lambda+\epsilon_i};t) \\ & -(1-q^{\lambda_i}t^{r-i}) A_i(t^\delta q^{\lambda-\epsilon_i};t) A_i(t^\delta q^\lambda;t) \Big) \\ = (1-q)t^{r-1} e_1(t^\delta q^\lambda).\end{aligned}$$

5. Twisted Pieri formulas for Macdonald

$[(\mathrm{ad}\,(|\mathbf{z}|))^l D(u,\mathbf{z})]\Phi_\lambda(\mathbf{z};q,t)$ の $l=1$ の場合は

$$[|\mathbf{z}|, D(u,\mathbf{z})]\Phi_\lambda(\mathbf{z};q,t)$$
$$= (1-q)u\sum_{i=1}^{r}\Phi_{\lambda+\epsilon_i}(\mathbf{z};q,t)A_i(t^\delta q^\lambda;t)t^{r-i}q^{\lambda_i}\prod_{j\in[r]\setminus\{i\}}(1+t^{r-j}q^{\lambda_j}u)$$

のようには計算できる (だから 1 変数の場合はうまくいっている) が, 一般の $l>1$ については Jack の時とは似ても似つかぬ大変複雑な形になってしまう (この q^{λ_i} が悪さをするらしい) !!

実は (この形の)Macdonald の twisted Pieri はないのでは?
実際, 補間 Jack の差分方程式の証明の q 類似としてもスジが悪いような気がする (rational の場合は共役が Lie theoretic に ad へと還元されたが, q の場合はそうなっていない).
なので Macdonald 作用素を適当な函数の共役をとった作用素を Macdonald 多項式に当てることは意味がありそうだが, そもそも Macdonald 作用素を ad で捻ることには余り意味がないのかも?

0. Difference equations (Okounkov)

$$D^{\mathrm{ip}}(u,\mathbf{z};q,t) := \prod_{i=1}^{r}\frac{1}{z_i}\sum_{I\subset[r]}(-1)^{|I|}t^{\binom{r-|I|}{2}}\prod_{i\in I}(1-z_it^{r-i})\prod_{j\notin I}(1-uz_it^{r-i})$$
$$\cdot\, t^{-|I|(r-|I|)}A_I(t^{\delta}\mathbf{z};t^{-1})T_q^{-I}$$

とすると補間 Macdonald 多項式 $P^{\mathrm{ip}}_{\lambda}(\mathbf{z};q,t)$ について

$$D^{\mathrm{ip}}(u,\mathbf{z};q,t)P^{\mathrm{ip}}_{\lambda}(\mathbf{z};q,t) = P^{\mathrm{ip}}_{\lambda}(\mathbf{z};q,t)\prod_{i=1}^{r}(q^{-\lambda_i}t^{i-1}-ut^{r-1}).$$

q の場合, 5. twisted Pieri よりも 0. 補間 Macdonald の差分方程式の方がやさしい?

逆にこちらを使うことで, twisted Pieri みたいなものが導出できないか? たとえば Macdonald 作用素を Cauchy 核で共役を取った

$$\left[\prod_{j=1}^{r}\frac{(z_j;q)_\infty}{(az_j;q)_\infty}D(u,\mathbf{z};q,t)\prod_{j=1}^{r}\frac{(az_j;q)_\infty}{(z_j;q)_\infty}\right]$$

$$=\sum_{I\subset[r]}u^{|I|}\prod_{i\in I}\frac{1-z_i}{1-az_i}t^{\binom{|I|}{2}}A_I(\mathbf{z};t)T_q^I$$

$$=\prod_{k=1}^{r}(1-az_k)\sum_{I\subset[r]}u^{|I|}\prod_{i\in I}(1-z_i)\prod_{j\notin I}(1-az_j)t^{\binom{|I|}{2}}A_I(\mathbf{z};t)T_q^I$$

のような作用素 (先の $D^{\mathrm{ip}}(u,\mathbf{z};q,t)$ に似ている?) を Macdonald 多項式にあてた Pieri(こちらが正しい twisted Pieri の q 版?)

$$\left[\prod_{j=1}^{r}\frac{(z_j;q)_\infty}{(az_j;q)_\infty}D(u,\mathbf{z};q,t)\prod_{j=1}^{r}\frac{(az_j;q)_\infty}{(z_j;q)_\infty}\right]P_\lambda(\mathbf{z};q,t)$$

はもとまるか?

q 類似早見表 (再録)

	1.	2.	3.	4.	5.	0.
$r = 1$	○	○	○	○	○	○
$r > 1$	○	$\triangle$?	$\triangle$?	○	×	○

1. Sekiguchi operators
2. Pieri type formulas for Jack polynomials
3. Binomial formulas
4. Mysterious summation
5. Twisted Pieri formulas for Jack polynomials
0. Difference equations for interpolation Jack

ご清聴ありがとうございました.