Birational geometry of moduli spaces of rank 2 logarithmic connections

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June. 23, 2021

Web-seminar on Painlevé Equations and related topics

Introduction (1)

Rationality of moduli spaces

C: a smooth projective curve of genus $g \geq 2$
L: a line bundle on C of degree d
$\mathbf{t} = (t_1, \ldots, t_n)$: a set of $n(\geq 1)$-distinct points on C

- Let $M(r, L)$ be the moduli space of stable vector bundles on C of rank r and with the determinat L. When r and d are coprime, $M(r, L)$ is rational. (King-Schofield, 1999)
- Let $P^{\boldsymbol{\alpha}}(r, L)$ be the moduli space of $\boldsymbol{\alpha}$-semistable quasi-parabolic bundles over $(C, \mathbf{t})$ of rank r and with the determinat L. $P^{\boldsymbol{\alpha}}(r, L)$ is rational in the case of certain flag structures including full flag structures. (Boden-Yokogawa, 1999)

Introduction (2)

In the case of parabolic connections (logarithmic case)

Theorem (Loray-Saito, 2015)

In the case $C = \mathbb{P}^1$, assume $n, d, \boldsymbol{\nu}, \boldsymbol{\alpha}$ satisfy appropriate conditions. Then the rational map

$$\mathrm{App} \times \mathrm{Bun} \colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, 2, d) \cdots \to |\mathcal{O}_C(n-3)| \times P^{\boldsymbol{\alpha}}(2, d)$$

is birational. In particular, $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, 2, d)$ is a rational variety.

Problem Is $\mathrm{App} \times \mathrm{Bun}$ birational for any rank and any genus?

In the case of fixing the determinant and the trace connection;

- $r = 2, g = 0$: Loray-Saito
- $r = 2, g = 1$: Fassarella-Loray-Muniz (arXiv: 2008.11767)
- $r = 2, g \geq 2$: M (arXiv: 2105.06892) $\longleftarrow$ Today's talk
- $r \geq 3$: in progress

Setting

- $\mathbb{C}$: the field of complex numbers
- C: a smooth projective curve over $\mathbb{C}$ of genus $g \geq 0$
- $\mathbf{t} = (t_1, \ldots, t_n)$: a set of $n(\geq 1)$-distinct points on C
- $D = t_1 + \cdots + t_n$: the effective divisor on C

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Quasi-parabolic bundles

Definition

$(E, l_* = \{l_*^{(i)}\}_{1 \leq i \leq n})$: a quasi-parabolic bundle over $(C, \mathbf{t})$

- E: an algeblic vector bundle on C of rank r and of degree d
- $l_*^{(i)}$: $E|_{t_i} = l_0^{(i)} \supsetneq l_1^{(i)} \supsetneq \cdots \supsetneq l_r^{(i)} = 0$: a filtration of $E|_{t_i}$ such that for each i, $1 \leq i \leq n$, $\dim l_j^{(i)} = r - j$

Here, $E|_{t_i} = E \otimes \mathcal{O}_C/\mathcal{O}_C(-t_i)$.

$l_*^{(i)}$ is called a parabolic structure (or a flag structure) of E at t_i.

A weight $\boldsymbol{\alpha} = (\alpha_j^{(i)})_{1 \leq j \leq r}^{1 \leq n \leq n}$ is a collection of rational numbers such that for each i, $0 < \alpha_1^{(i)} < \cdots < \alpha_r^{(i)} < 1$

Parabolic degree

(E, l_*): a quasi-parabolic bundle

$F \subset E$: a subbundle

$\boldsymbol{\alpha}$: a weight

Definition

$$\operatorname{par}\deg_{\boldsymbol{\alpha}} F := \deg F + \sum_{i=1}^{n}\sum_{j=1}^{r} \alpha_j^{(i)} \dim(F|_{t_i} \cap l_{j-1}^{(i)} / F|_{t_i} \cap l_j^{(i)}).$$

In the case $F = E$,

$$\operatorname{par}\deg_{\boldsymbol{\alpha}} E = \deg E + \sum_{i=1}^{n}\sum_{j=1}^{r} \alpha_j^{(i)}.$$

In the case $\operatorname{rank} E = 2$, for any sub line bundle F,

$$\operatorname{par}\deg_{\boldsymbol{\alpha}} F = \deg F + \sum_{F|_{t_i} \neq l_1^{(i)}} \alpha_1^{(i)} + \sum_{F|_{t_i} = l_1^{(i)}} \alpha_2^{(i)}.$$

The stability condition of quasi-parabolic bundles

Definition

A quasi-parabolic bundle (E, l_*) is $\boldsymbol{\alpha}$-semistable ($\boldsymbol{\alpha}$-stable) if for any nonzero proper subbundle $F \subsetneq E$, the inequality

$$\frac{\operatorname{par}\deg_{\boldsymbol{\alpha}} F}{\operatorname{rank} F} \underset{(<)}{\leq} \frac{\operatorname{par}\deg_{\boldsymbol{\alpha}} E}{\operatorname{rank} E}$$

holds.

In the case $\operatorname{rank} E = 2$, the above inequality is equivalent to the following inequality

$$\deg E - 2\deg F + \sum_{F|_{t_i} \neq l_1^{(i)}} (\alpha_2^{(i)} - \alpha_1^{(i)}) - \sum_{F|_{t_i} = l_1^{(i)}} (\alpha_2^{(i)} - \alpha_1^{(i)}) \underset{(>)}{\geq} 0.$$

Moduli spaces of quasi-parabolic bundles

Suppose $g \geq 2$.

$P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,d)$: the moduli space of $\boldsymbol{\alpha}$-semistable quasi-parabolic bundles of rank r and of degree d

Theorem (Mehta, Seshadri)

$P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,d)$ is an irreducible normal projective variety of dimension

$$N = r^2(g-1) + \frac{nr(r-1)}{2} + 1$$

L: a line bundle on C of degree d

$$P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,L) := \{(E,l_*) \in P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,d) \mid \det E \simeq L\}$$

Proposition

$P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(r,L)$ is an irreducible normal projective variety of dimension

$$(r^2-1)(g-1) + \frac{nr(r-1)}{2} = N - g$$

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Logarithmic connections

Let us fix a complex number $\lambda \in \mathbb{C}$.

Definition

(E, ∇): a logarithmic λ-connection over $(C, \mathbf{t})$

- E: an algebraic vector bundle on C of rank r and of degree d
- $\nabla \colon E \to E \otimes \Omega^1_C(D)$
  $\forall a \in \mathcal{O}_C, \forall \sigma \in E, \nabla(a\sigma) = \lambda\sigma \otimes da + a\nabla(\sigma)$

Fuchs relation

(E, ∇): a logarithmic λ-connection

$\mathrm{res}_{t_i}(\nabla) \in \mathrm{End}(E|_{t_i})$: the residue homomorphism

$\{\nu_0^{(i)}, \ldots, \nu_{r-1}^{(i)}\}$: the set of ordered eigenbalues of $\mathrm{res}_{t_i}(\nabla)$ (local exponents of ∇ at t_i)

Proposition (Fuchs relation)

$$\lambda \deg E + \sum_{i=1}^{n} \sum_{j=0}^{r-1} \nu_j^{(i)} = 0$$

$$\mathcal{N}^{(n)}(\lambda, d) := \left\{ (\nu_j^{(i)})^{1 \leq i \leq n}_{0 \leq j \leq r-1} \in \mathbb{C}^{nr} \;\middle|\; \lambda d + \sum_{i=1}^{n} \sum_{j=0}^{r-1} \nu_j^{(i)} = 0 \right\}$$

$\boldsymbol{\nu}$-parabolic connections

Let us fix a complex number $\lambda \in \mathbb{C}$ and $\boldsymbol{\nu} = (\nu_j^{(i)})_{0 \le j \le r-1}^{1 \le i \le n} \in \mathcal{N}^{(n)}(\lambda, d)$.

Definition

$(E, \nabla, l_* = \{l_*^{(i)}\}_{1 \le i \le n})$: a $\boldsymbol{\nu}$-parabolic λ-connection over $(C, \mathbf{t})$.

- (E, ∇): a logarithmic λ-connection over $(C, \mathbf{t})$ of rank r
- $l_*^{(i)}$: $E|_{t_i} = l_0^{(i)} \supsetneq l_1^{(i)} \supsetneq \cdots \supsetneq l_r^{(i)} = 0$: a parabolic structure of $E|_{t_i}$ for each $i, 1 \le i \le n$ such that for any $j, 0 \le j \le r-1$ $(\mathrm{res}_{t_i}(\nabla) - \nu_j^{(i)} \mathrm{id}_{E|_{t_i}})(l_j^{(i)}) \subset l_{j+1}^{(i)}$

- $\lambda = 1$: $\boldsymbol{\nu}$-parabolic connection
- $\lambda = 0$: $\boldsymbol{\nu}$-parabolic Higgs bundle
- $\lambda = 0, \boldsymbol{\nu} = 0$: parabolic Higgs bundle

The stability condition of parabolic connections

$\boldsymbol{\alpha} = (\alpha_j^{(i)})_{1 \leq j \leq r}^{1 \leq i \leq n}$: a weight

Definition

(1) A $\boldsymbol{\nu}$-parabolic λ-connection (E, ∇, l_*) is $\boldsymbol{\alpha}$-stable if for any nonzero subbundle $F \subsetneq E$ satisfying $\nabla(F) \subset F \otimes \Omega_C^1(D)$, the inequality

$$\frac{\operatorname{par} \deg_{\boldsymbol{\alpha}} F}{\operatorname{rank} F} < \frac{\operatorname{par} \deg_{\boldsymbol{\alpha}} E}{\operatorname{rank} E}$$

holds.

(2) A $\boldsymbol{\nu}$-parabolic λ-connection (E, ∇, l_*) is irreducible if for any nonzero subbundle $F \subsetneq E$, $\nabla(F) \not\subset F \otimes \Omega_C^1(D)$

For a $\boldsymbol{\nu}$-parabolic λ-connection (E, ∇, l_*),

$$(E, \nabla, l_*)\text{: irreducible} \Longrightarrow (E, \nabla, l_*)\text{: } \boldsymbol{\alpha}\text{-stable}$$

Moduli spaces of parabolic connections (1)

$M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},\lambda,r,d)$: the moduli space of $\boldsymbol{\alpha}$-stable $\boldsymbol{\nu}$-parabolic λ-connections
In the case $\lambda \neq 0$, the morphism

$$M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},\lambda,r,d) \longrightarrow M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}/\lambda,1,r,d)$$

$$(E,\nabla,l_*) \longmapsto (E,\tfrac{1}{\lambda}\nabla,l_*)$$

is an isomorphism.

$$M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d) := M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},1,r,d)$$

Theorem (Inaba-Iwasaki-Saito, 2006, Inaba, 2013)

$M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,d)$ is an irreducible smooth quasi-projective variety of dimension

$$2r^2(g-1)+nr(r-1)+2=2N$$

Moduli spaces of parabolic connections (2)

$\mathrm{tr}(\boldsymbol{\nu}) := (\nu_0^{(i)} + \cdots + \nu_{r-1}^{(i)})_{1 \leq i \leq n}$

$$\begin{aligned}\det \colon M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d) &\longrightarrow M_{(C,\mathbf{t})}(\mathrm{tr}(\boldsymbol{\nu}), 1, d) \\ (E, \nabla, l_*) &\longmapsto (\det E, \mathrm{tr}(\nabla))\end{aligned}$$

(L, ∇_L): a $\mathrm{tr}(\boldsymbol{\nu})$-parabolic connection of degree d

$$M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, (L, \nabla_L)) := \det^{-1}((L, \nabla_L))$$

Theorem (Inaba, 2013)

Assume

$$g = 0, rn - 2(r+1) > 0 \text{ or } g = 1, n \geq 2 \text{ or } g \geq 2, n \geq 1.$$

$M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, (L, \nabla_L))$ is an irreducible smooth quasi-projective variety of dimension

$$2r^2(g-1) + nr(r-1) + 2 - 2g = 2(N - g)$$

Elementary transformations (1)

For each i, $1 \leq i \leq n$, we can define the lower transformation.

$$\begin{aligned} \mathrm{elm}^-_{t_i} \colon M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, 2, d) &\longrightarrow M^{\boldsymbol{\alpha}'}_{(C,\mathbf{t})}(\boldsymbol{\nu}', 2, d-1) \\ (E, \nabla, l_*) &\longmapsto (E', \nabla', l'_*) \end{aligned}$$

- E' is defined by the exact sequence of sheaves

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E|_{t_i}/l^{(i)}_1 \longrightarrow 0.$$

- $\nabla' \colon E' \to E' \otimes \Omega^1_C(D)$ is the restriction of $\nabla \colon E \to E \otimes \Omega^1_C(D)$.
- $l'^{(i)}_1$ is given by the exact sequence of vector spaces

$$0 \longrightarrow l'^{(i)}_1 \longrightarrow E'|_{t_i} \longrightarrow l^{(i)}_1 \longrightarrow 0.$$

Here, $\boldsymbol{\nu}' = (\nu'^{(i)}_j)^{1 \leq i \leq n}_{1 \leq j \leq 2}$ is given by

$$(\nu'^{(i)}_0, \nu'^{(i)}_1) = (\nu^{(i)}_1, \nu^{(i)}_0 + 1)$$

and $\boldsymbol{\alpha}'$ is a weight satisfying

$$\alpha'^{(i)}_2 - \alpha'^{(i)}_1 = 1 - (\alpha^{(i)}_2 - \alpha^{(i)}_1).$$

Elementary transformations (2)

For each i, $\mathbf{b}_i((E, \nabla, l_*))$ is defined by

$$\mathbf{b}_i((E, \nabla, l_*)) = (E, \nabla, l_*) \otimes \mathcal{O}_C(t_i).$$

Then, we define the upper transformation

$$\mathrm{elm}^+_{t_i} \colon M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, 2, d) \longrightarrow M^{\boldsymbol{\alpha}'}_{(C,\mathbf{t})}(\boldsymbol{\nu}'', 2, d+1)$$

by the composition

$$\mathrm{elm}^+_{t_i} := \mathrm{elm}^-_{t_i} \circ \mathbf{b}_i.$$

Here, $\boldsymbol{\nu}'' = (\nu''^{(i)}_j)^{1 \le i \le n}_{1 \le j \le 2}$ is defined by

$$(\nu'^{(i)}_0, \nu'^{(i)}_1) = (\nu^{(i)}_1 - 1, \nu^{(i)}_0).$$

Proposition

Elementary transformations $\mathrm{elm}^{\pm}_{t_i}$ give an isomorphism between two moduli spaces of stable parabolic connections. In particular, we can change degree of parabolic connections freely.

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Apparent singularities (1)

(E, ∇, l_*): an irreducible $\boldsymbol{\nu}$-parabolic connection of rank 2
$\sigma \in H^0(C, E) \setminus \{0\}$
$E/\mathcal{O}_C = \mathrm{Coker}(\sigma \colon \mathcal{O}_C \to E)$: torsion free
$\varphi_{(\nabla,\sigma)} \in H^0(C, E/\mathcal{O}_C \otimes \Omega^1_C(D))$: the section given by the composition

$$\mathcal{O}_C \xrightarrow{\sigma} E \xrightarrow{\nabla} E \otimes \Omega^1_C(D) \to E/\mathcal{O}_C \otimes \Omega^1_C(D)$$

Proposition

$\varphi_{(\nabla,\sigma)}$ induces the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E/\mathcal{O}_C \otimes \Omega^1_C(D) \longrightarrow T_{(\nabla,\sigma)} \longrightarrow 0.$$

Then,

$$\mathrm{length}\ T_{(\nabla,\sigma)} = d - 2(g-1) + r^2(g-1) + \frac{nr(r-1)}{2}.$$

In the case $d = 2(g-1) + 1 = 2g - 1$,

$$\mathrm{length}\ T_{(\nabla,\sigma)} = N.$$

Apparent singularities (2)

$d = 2g - 1$.
$S^N(C)$: the N-fold symmetric product

Definition (Saito-Szabó)

We call the effective divisor

$$\mathrm{App}((\nabla, \sigma)) := \mathrm{Supp}\, T_{(\nabla,\sigma)} = \sum_{i=1}^{k} n_i q_i \in |E/\mathcal{O}_C \otimes \Omega_C^1(D)| \subset S^N(C)$$

the apparent singularity divisor of (E, ∇, l_*, σ).

Remark: $\mathrm{App}((\nabla, \sigma))$ does't depend on the scalar multiplication of σ, that is, for any $a \in \mathbb{C}^\times$,

$$\mathrm{App}((\nabla, \sigma)) = \mathrm{App}((\nabla, a\sigma))$$

Apparent map

Suppose $d = 2g - 1$.

$$\dim H^0(C,E) - \dim H^1(C,E) = 2g - 1 + 2(1-g) = 1$$

$$M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},2,d)^{0,irr} := \{(E,\nabla,l_*) \mid \text{irreducible}, \dim H^0(C,E) = 1\}/\sim$$

$M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},2,d)^{0,irr}$ is a Zariski open subset of $M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},2,d)$.

Definition (Saito-Szabó)

We define the apparent map

$$\mathrm{App}\colon M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},2,d)^{0,irr} \longrightarrow S^N(C)$$

by $\mathrm{App}((E,\nabla,l_*)) = \mathrm{App}((E,\nabla,l_*,\sigma))$, $\sigma \in H^0(C,E)\setminus\{0\}$. We can extend this map to the rational map

$$\mathrm{App}\colon M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},2,d) \cdots \to S^N(C).$$

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Birational structure of moduli spaces (0)

Suppose $d = 2g - 1$. Let (L, ∇_L) be a $\mathrm{tr}(\boldsymbol{\nu})$-parabolic connection.

$$M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) := M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, 2, (L, \nabla_L)),\ \ P^{\boldsymbol{\alpha}}(L) := P^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(2, L)$$

To show that $\mathrm{App} \times \mathrm{Bun}$ is birational, we take two steps.

Step 1

To find a good open subset V_0 of $P^{\boldsymbol{\alpha}}(L)$. "good" means that for any $(E, l_*) \in V_0$, $\mathrm{App}\colon \mathrm{Bun}^{-1}((E, l_*)) \longrightarrow |L \otimes \Omega^1_C(D)|$ is injective.

Step 2

To show the injectivity of App.

$$\begin{array}{ccccc}
(E, \nabla, l_*) & \in & M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L)) & \overset{\mathrm{App}}{\dashrightarrow} & |L \otimes \Omega^1_C(D)| \\
\big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Bun}} & & \\
(E, l_*) & \in & P^{\boldsymbol{\alpha}}(L) & &
\end{array}$$

Birational structure of moduli spaces (1)

Suppose $\sum_{i=1}^{n}(\alpha_2^{(i)} - \alpha_1^{(i)}) < 1$ and $g \geq 1$

Proposition

Let $V_0 \subset P^{\boldsymbol{\alpha}}(L)$ be the subset consiting of all elements (E, l_*) satisfying the following conditions:

- (E, l_*) is an extension of (L, D) by $(\mathcal{O}_C, \emptyset)$, that is, (E, l_*) fits into an exact sequence of quasi-parabolic bundles.

$$0 \longrightarrow (\mathcal{O}_C, \emptyset) \longrightarrow (E, l_*) \longrightarrow (L, D) \longrightarrow 0.$$

- $\dim H^0(C, E) = 1.$

Then V_0 is a nonempty Zariski open subset of $P^{\boldsymbol{\alpha}}(L)$. Moreover, there is an open immersion

$$V_0 \hookrightarrow \mathbb{P}\operatorname{Ext}^1((L, D), (\mathcal{O}_C, \emptyset)) = \mathbb{P}H^1(C, L^{-1}(-D)).$$

Birational structure of moduli spaces (2)

Let

$$\langle\ ,\ \rangle\colon H^0(C, L \otimes \Omega^1_C(D)) \times H^1(C, L^{-1}(-D)) \longrightarrow H^1(C, \Omega^1_C)$$

be the natural cup-product. Let us define the subvariety

$$\Sigma \subset \mathbb{P}H^0(C, L \otimes \Omega^1_C(D)) \times \mathbb{P}H^1(C, L^{-1}(-D))$$

by

$$\Sigma := \{([s], [b]) \mid \langle s, b \rangle = 0\}.$$

Birational structure of moduli spaces (3)

$$M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))^0 := \{(E,\nabla,l_*) \in M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L)) \mid (E,l_*) \in V_0\}$$

Theorem (M)

Assume $g \geq 1$, $d = 2g-1$, $\sum_{i=1}^n \nu_0^{(i)} \neq 0$ and $\sum_{i=1}^n(\alpha_2^{(i)} - \alpha_1^{(i)}) < 1$. Then tha map

$$\mathrm{App} \times \mathrm{Bun} \colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))^0 \longrightarrow (\mathbb{P}H^0(C, L \otimes \Omega_C^1(D)) \times V_0) \setminus \Sigma$$

is an isomorphism. Therefore, the rational map

$$\mathrm{App} \times \mathrm{Bun} \colon M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L)) \dashrightarrow |L \otimes \Omega_C^1(D)| \times P^{\boldsymbol{\alpha}}(L)$$

is birational. In particular, $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))$ is a rational variety.

Outline of the proof (1)

$(E, l_*) \in V_0$
$b \in H^1(C, L^{-1}(-D))$
$[b] = (E, l_*)$

$$\begin{Bmatrix} \lambda\boldsymbol{\nu}\text{-parabolic} \quad \lambda\text{-connections} \quad \text{over} \quad (E, l_*) \\ \text{with the trace connection } \lambda\nabla_L \ (\forall\lambda \in \mathbb{C}) \end{Bmatrix} \overset{1:1}{\longleftrightarrow} H^0(C, L \otimes \Omega^1_C(D))$$

$$\begin{Bmatrix} \text{parabolic} \quad \text{Higgs} \quad \text{bundles} \quad \text{over} \\ (E, l_*) \text{ with the trace } 0 \end{Bmatrix} \overset{1:1}{\longleftrightarrow} \{s \in H^0(C, L \otimes \Omega^1_C(D)) \mid \langle s, b \rangle = 0\}$$

The correpondence above is given by the following composition

$$\mathcal{O}_C \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega^1_C(D) \to E/\mathcal{O}_C \otimes \Omega^1_C(D) \simeq L \otimes \Omega^1_C.$$

This composite map is denoted by φ_∇. We have

$$\mathrm{App}(E, \nabla, l_*) = [\varphi_\nabla].$$

Outline of the proof (2)

$\{U_i\}_i$: an open covering of C

A λ-connection ∇ is given in U_i by $\lambda d + A_i$

$$A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \in M_2(\Omega^1_C(D)(U_i)).$$

∇_L is given in U_i by $d + \omega_i$, $\omega_i \in \Omega^1_C(D)$.

We show that there exist $\lambda, \alpha_i, \beta_i, \gamma_i, \delta_i$ satisfying the following four conditions uniquely.

- the condition of the apparent singularities

$$\varphi_\nabla = \gamma \in H^0(C, L \otimes \Omega^1_C)$$

- the condition of the trace connection

$$\mathrm{tr}(\nabla) = \lambda \nabla_L \Longleftrightarrow \alpha_i + \delta_i = \lambda \omega_i$$

Outline of the proof (3)

- the compatibility condition

$$\lambda dM_{ij} + A_i M_{ij} = M_{ij} A_j.$$

Here, M_{ij} is a transition matrix of (E, l_*) on $U_i \cap U_j$. This condition is equivalent to the following conditions

$$\begin{cases} \dfrac{\gamma_i}{c_i} - \dfrac{\gamma_j}{c_j} = 0 \\ \alpha_i - \alpha_j = b_{ij}\gamma_j \\ \delta_i - \delta_j = -b_{ij}\gamma_j - \lambda \dfrac{dc_{ij}}{c_{ij}} \\ c_i\beta_i - c_j\beta_j = -(\lambda c_j db_{ij} + (b_{ij}c_j)(\alpha_i - \delta_j)). \end{cases}$$

- the residual condition

$$\mathrm{res}_{t_k}(A_i) = \begin{pmatrix} \lambda\nu_0^{(k)} & 0 \\ * & \lambda\nu_1^{(k)} \end{pmatrix}$$

Outline of another proof

For $(E, l_*) \in V_0$, we define a sheaf $\mathcal{E}$ by

$$\mathcal{E} := \{s \in \mathcal{E}nd(E) \otimes \Omega^1_C(D) \mid \mathrm{tr}(s) = 0, \mathrm{res}_{t_i}(s)(l^{(i)}_j) \subset l^{(i)}_{j+1} \text{ for all } i, j\}.$$

$\Phi \in H^0(C, \mathcal{E})$ is a parabolic Higgs field over (E, l_*) such that $\mathrm{tr}(\Phi) = 0$.
For $(E, \nabla, l_*), (E, \nabla', l_*) \in M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0$, we have

$$\nabla' - \nabla \in H^0(C, \mathcal{E}).$$

Therefore, we obtain

$$\mathrm{Bun}^{-1}(E, l_*) = \nabla + H^0(C, \mathcal{E})$$

$\mathrm{App}\colon \mathrm{Bun}^{-1}(E, l_*) \longrightarrow \mathbb{P}H^0(C, L, \otimes\Omega^1_C)$ is injective.

$\Longleftrightarrow$ the map

$$H^0(C, \mathcal{E}) \longrightarrow H^0(C, L \otimes \Omega^1_C(D)),\ \Phi \longmapsto \varphi_\Phi$$

is injective.

Birational structure of moduli spaces (4)

Proposition

Suppose $\sum_{i=1}^{n} \nu_0^{(i)} = 0$. Then for each $(E, l_*) \in V_0$, there is a unique $\boldsymbol{\nu}$-parabolic connection ∇_0 such that

$$\nabla_0(\mathcal{O}_C) \subset \mathcal{O}_C \otimes \Omega_C^1(D).$$

Moreover, we can take a section $s_0 \colon V_0 \longrightarrow M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0$ such that for each $(E, l_*) \in V_0$, $s_0((E, l_*)) = (E, \nabla_0, l_*)$.

$(E, l_*) \in V_0$
$\mathcal{E}$: as above
There is the natural isomorphism

$$T^*_{(E,l_*)} V_0 \simeq H^0(C, \mathcal{E}).$$

Birational structure of moduli spaces (5)

Proposition

Suppose $\sum_{i=1}^{n} \nu_0^{(i)} = 0$. By the section s_0, we can identify $M^{\boldsymbol{\alpha}}(\boldsymbol{\nu}, (L, \nabla_L))^0$ with the total space of the cotangent bundle T^*V_0.

$$\begin{array}{ccc} M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))^0 & \xrightarrow{\ \sim\ } & T^*V_0 \\ & {\scriptstyle \mathrm{Bun}} \searrow \quad \swarrow {\scriptstyle \mathrm{projection}} & \\ & V_0 & \end{array} \qquad \begin{array}{ccc} M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))^0 & \xrightarrow{\ \sim\ } & T^*V_0 \\ & {\scriptstyle s_0} \nwarrow \quad \nearrow {\scriptstyle 0} & \\ & V_0 & \end{array}$$

$$\begin{array}{ccc} M^{\boldsymbol{\alpha}}(\boldsymbol{\nu},(L,\nabla_L))^0 & \overset{\mathrm{App}\times\mathrm{Bun}}{\dashrightarrow} & \Sigma \\ \downarrow \wr & & \downarrow \wr \\ T^*V_0 & \overset{\text{projectivization}}{\dashrightarrow} & \mathbb{P}T^*V_0 \end{array}$$