Discrete Hamiltonians of the discrete Painlevé equations

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Introduction

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Difference:

The Painlevé differential equation is expressed as a Hamiltonian system, whereas the discrete Painlevé equation does not have such an expression.

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The Painlevé differential equations are written in the form of the canonical systems with the Hamiltonians: $\left[\frac{ds}{dt} = s(s-1)\right]$,

$$H_{\mathrm{VI}}\left(\begin{matrix} a_1, a_2 \\ a_3, a_4 \end{matrix}; t; q, p\right) = q(q-1)(q-s)p^2$$

$$+\left\{(a_1+2a_2)q(q-1)+a_3(s-1)q+a_4 s(q-1)\right\}p + a_2(a_1+a_2)q,$$

$$H_{\mathrm{V}}\left(\begin{matrix} a_1, a_2 \\ a_3 \end{matrix}; t; q, p\right) = p(p+1)q(q+e^t) + a_1 q(p+1) + a_3 pq - a_2 e^t p,$$

$$H_{\mathrm{III}}(D_6)\,(a_1, b_1; t; q, p) = p(p+1)q^2 - a_1 p(q-1) - b_1 pq - e^t q,$$

$$H_{\mathrm{III}}(D_7)\,(a_1; t; q, p) = p^2 q^2 + a_1 qp + e^t p + q,$$

$$H_{\mathrm{III}}(D_8)\,(t; q, p) = p^2 q^2 + qp - q - \frac{e^t}{q},$$

$$H_{\mathrm{IV}}\,(a_1, a_2; t; q, p) = pq(p-q-t) - a_1 p - a_2 q,$$

$$H_{\mathrm{II}}\,(a_1; t; q, p) = p(p-q^2-t) - a_1 q, \qquad H_{\mathrm{I}}\,(t; q, p) = p^2 - q^3 - tq.$$

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If a discrete dynamical system can be described simply using function W on a phase space, we call this W a discrete Hamiltonian, although it is a vague terminology.

Discrete Lagrangian and discrete Hamiltonian

Starting from a function: $L_k(r, s): \; M^n \times M^n \to \mathbb{R}$. (We call it Lagrangian.)

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We consider formal sum $S(\lambda) = \sum_{k\in\mathbb{Z}} L_k(\lambda_k, \lambda_{k+1})$, and $\delta S = 0$:

$$
\begin{aligned}
\delta S(\lambda) &= \sum_{k\in\mathbb{Z}} \delta L_k(\lambda_k, \lambda_{k+1}) \\
&= \sum_{k\in\mathbb{Z}} \{L_k(\lambda_k + \delta\lambda_k, \lambda_{k+1} + \delta\lambda_{k+1}) - L_k(\lambda_k, \lambda_{k+1})\} \\
&= \sum_{k\in\mathbb{Z}} \{\frac{\partial L_k}{\partial r}(\lambda_k, \lambda_{k+1})\delta\lambda_k + \frac{\partial L_k}{\partial s}(\lambda_k, \lambda_{k+1})\delta\lambda_{k+1}\} \\
&= \sum_{k\in\mathbb{Z}} \{\frac{\partial L_k}{\partial r}(\lambda_k, \lambda_{k+1}) + \frac{\partial L_{k-1}}{\partial s}(\lambda_{k-1}, \lambda_k)\}\delta\lambda_k = 0.
\end{aligned}
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&= \sum_{k\in\mathbb{Z}} \{ \frac{\partial L_k}{\partial r}(\lambda_k, \lambda_{k+1})\delta\lambda_k + \frac{\partial L_k}{\partial s}(\lambda_k, \lambda_{k+1})\delta\lambda_{k+1} \} \\
&= \sum_{k\in\mathbb{Z}} \{ \frac{\partial L_k}{\partial r}(\lambda_k, \lambda_{k+1}) + \frac{\partial L_{k-1}}{\partial s}(\lambda_{k-1}, \lambda_k) \} \delta\lambda_k = 0.
\end{aligned}$$

$$\to \qquad \tfrac{\partial L_k}{\partial r}(\lambda_k, \lambda_{k+1}) + \tfrac{\partial L_{k-1}}{\partial s}(\lambda_{k-1}, \lambda_k) = 0 \quad \text{(Euler-Lagrange)}.$$

Discrete Lagrangian and discrete Hamiltonian

Legendre transformation:

We put $\mu_k = \frac{\partial L_k}{\partial r}(\lambda_k, \lambda_{k+1}) = -\frac{\partial L_{k-1}}{\partial s}(\lambda_{k-1}, \lambda_k)$,
and put $H(\lambda, \overline{\mu}) = \overline{\lambda}\overline{\mu} + L(\lambda, \overline{\lambda})$.

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$d\mu \wedge d\lambda$ is an invariant form (symplectic).

We forget the Lagrangian, and only consider the generating function of the canonical transformation.

Content

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Let $D = \sum_{i \in I} m_i D_i$ be an effective divisor on X with irreducible components D_i. We say that D is of canonical type if

$$\mathcal{K}_X \cdot [D_i] = 0 \quad \text{for all } i.$$

- $\dim|-\mathcal{K}_X| = 1 \quad \to$ rational elliptic surface
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Classification by anti-canonical divisor

elliptic	$A_0^{(1)}$
multiplicative	$A_0^{(1)*}$, $A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}, \ldots, A_6^{(1)}$, $A_7^{(1)}$, $A_7^{(1)\prime}$, $A_8^{(1)}$
additive	$A_0^{(1)**}$, $A_1^{(1)*}$, $A_2^{(1)*}$, $D_4^{(1)}$, $D_5^{(1)}$, $D_6^{(1)}$, $D_7^{(1)}$, $D_8^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$

We have non-autonomous differential systems only for $D_k^{(1)}$ and $E_k^{(1)}$.

equations	P_{VI}	P_{V}	$P_{\mathrm{III}}(D_6)$	$P_{\mathrm{III}}(D_7)$	$P_{\mathrm{III}}(D_8)$
geometry	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$
symmetry	$D_4^{(1)}$	$A_3^{(1)}$	$(A_1 + A_1)^{(1)}$	$A_1^{(1)}$	-

P_{IV}	P_{II}	P_{I}
$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
$A_2^{(1)}$	$A_1^{(1)}$	-

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Except $E_8^{(1)}$, the surfaces can be blown down to $\mathbb{P}^1 \times \mathbb{P}^1$. We set the coordinate as $(f_0 : f_1), (g_0 : g_1)$.
We divide them into 4 cases:

1. The imgae of the anti-canonical divisor is ${f_0}^2{g_0}^2 = 0$,
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3. The imgae of the anti-canonical divisor is $f_0 f_1 g_0 g_1 = 0$,
4. The others.

1. ${f_0}^2{g_0}^2=0$: $D_5^{(1)}$, $D_6^{(1)}$, $D_7^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$,
2. $f_0f_1{g_0}^2=0$: $D_4^{(1)}$, $D_5^{(1)}$, $D_6^{(1)}$, $D_7^{(1)}$, $(D_8^{(1)})$,
3. $f_0f_1g_0g_1=0$: $A_3^{(1)}$, $A_4^{(1)}$, $A_5^{(1)}$, $A_6^{(1)}$, $A_7^{(1)}$, $A_7^{(1)\prime}$, $(A_8^{(1)})$,
4. the others: $A_0^{(1)}$, $A_0^{(1)*}$, $A_0^{(1)**}$, $A_1^{(1)}$, $A_1^{(1)*}$, $A_2^{(1)}$, $A_2^{(1)*}$.

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Case1: ${f_0}^2{g_0}^2 = 0$

In this case, the Hamiltonian of the differential system is written by a biquadratic polynomial:

$$H = (g^2, g, 1) \begin{pmatrix} m_{22} & m_{21} & m_{20} \\ m_{12} & m_{11} & m_{10} \\ m_{02} & m_{01} & m_{00} \end{pmatrix} \begin{pmatrix} f^2 \\ f \\ 1 \end{pmatrix},$$

$$\frac{df}{dt} = \frac{\partial H}{\partial g}, \quad \frac{dg}{dt} = -\frac{\partial H}{\partial f}.$$

$$M = M_{D_5} = \begin{pmatrix} 1 & s & 0 \\ 1 & s + a_1 + a_3 & -a_2 s \\ 0 & a_1 & 0 \end{pmatrix}, \quad M_{D_6} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -a_1 - b_1 & -s \\ 0 & -a_1 & 0 \end{pmatrix},$$

$$M_{D_7} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & s \\ 0 & 1 & 0 \end{pmatrix}, \; M_{E_6} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -s & -a_2 \\ 0 & -a_1 & 0 \end{pmatrix}, \; M_{E_7} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & -s \\ 0 & -a_1 & 0 \end{pmatrix}.$$

$s = e^t$ for D-type, $s = t$ for E-type.

Discrete Painlevé equations are written by using these matrices as:

$$g = -\overline{g} - \frac{\hat{m}_{12}f^2 + \hat{m}_{11}f + \hat{m}_{10}}{\hat{m}_{22}f^2 + \hat{m}_{21}f + \hat{m}_{20}}, \qquad \overline{f} = -f - \frac{\overline{m}_{21}\overline{g}^2 + \overline{m}_{11}\overline{g} + \overline{m}_{01}}{\overline{m}_{22}\overline{g}^2 + \overline{m}_{12}\overline{g} + \overline{m}_{02}}.$$

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Hence when we put a generating function W as

$$W(f,\overline{g}) = -f\overline{g} - \int \frac{\hat{m}_{12}f^2 + \hat{m}_{11}f + \hat{m}_{10}}{\hat{m}_{22}f^2 + \hat{m}_{21}f + \hat{m}_{20}} df - \int \frac{\overline{m}_{21}\overline{g}^2 + \overline{m}_{11}\overline{g} + \overline{m}_{01}}{\overline{m}_{22}\overline{g}^2 + \overline{m}_{12}\overline{g} + \overline{m}_{02}} d\overline{g},$$

we can write the discrete equations as

$$g = \frac{\partial W}{\partial f}, \qquad \overline{f} = \frac{\partial W}{\partial \overline{g}}.$$

The explicite formula:

$$
\begin{aligned}
W = W_{D_5} =& - f\overline{g} - f - s\overline{g} - \overline{a}_3 \log(\overline{g}+1) + a_2 \log f \\
& - \overline{a}_1 \log \overline{g} - (a_1 + a_2 + a_3 - 1) \log(f+s), \\
W_{D_6} =& - f\overline{g} - f - \frac{s}{f} + (a_1 + b_1 - 1) \log f \\
& + \overline{a}_1 \log \overline{g} + \overline{b}_1 \log(\overline{g}+1), \\
W_{D_7} =& - f\overline{g} - \frac{s}{f} + \frac{1}{\overline{g}} - (a_1 - 1) \log f - \overline{a}_1 \log \overline{g}, \\
W_{E_6} =& - f\overline{g} + \frac{f^2}{2} + sf + \frac{\overline{g}^2}{2} - s\overline{g} + a_2 \log f - \overline{a}_1 \log \overline{g}, \\
W_{E_7} =& - f\overline{g} + sf + \frac{f^3}{3} - \overline{a}_1 \log \overline{g}.
\end{aligned}
$$

e.g. $E_7^{(1)}$ type:

$$W_{E_7} = -f\overline{g} + sf + \frac{f^3}{3} - \overline{a}_1 \log \overline{g},$$

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$$g = \frac{\partial W_{E_7}}{\partial f} = -\overline{g} + s + f^2,$$
$$\overline{f} = \frac{\partial W_{E_7}}{\partial \overline{g}} = -f - \frac{\overline{a}_1}{\overline{g}}.$$

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Case2: $f_0 f_1 g_0 g_1 = 0$

$$M_{A_7} = \begin{pmatrix} 0 & -a_0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad M_{A_7'} = \begin{pmatrix} 1 & -a_0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$M_{A_6} = \begin{pmatrix} 0 & 1/b & 0 \\ 1 & 0 & -1/b \\ 0 & -a_1 & a_1 \end{pmatrix}, \quad M_{A_5} = \begin{pmatrix} 0 & b_1/a_2 & 0 \\ a_0 & 0 & -b_1/a_2 \\ 1/a_1 & -1-(1/a_1) & 1 \end{pmatrix},$$

$$M_{A_4} = \begin{pmatrix} 0 & 1 & -1 \\ a_0/a_2 & 0 & 1+(1/a_4) \\ -a_0 a_3/a_2 & a_0 a_3 + (1/a_2 a_4) & -1/a_4 \end{pmatrix},$$

$$M_{A_3} = \begin{pmatrix} a_0 a_5 & -1/(a_1 {a_2}^2 a_3) - a_0 a_3 a_5 & 1/(a_1 {a_2}^2) \\ -(1+a_0)a_5 & 0 & -(1+a_1)/a_1 a_2 \\ a_5 & -1-a_5 & 1 \end{pmatrix}.$$

Discrete Painlevé equations are written by using these matrices as:

$$g=\frac{\hat{m}_{02}f^2+\hat{m}_{01}f+\hat{m}_{00}}{\overline{g}(\hat{m}_{22}f^2+\hat{m}_{21}f+\hat{m}_{20})},\qquad \overline{f}=\frac{\overline{m}_{20}\overline{g}^2+\overline{m}_{10}\overline{g}+\overline{m}_{00}}{f(\overline{m}_{22}\overline{g}^2+\overline{m}_{12}\overline{g}+\overline{m}_{02})}$$

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The symplectic form is $\omega = \frac{dg \wedge df}{fg} = d \log g \wedge d \log f$. When we put $F = \log f,\ G = \log g$, we can find a generating function $\widetilde{W}(F, \overline{G})$.
But it is important that the system is a birational mapping, so we want to use the variables f and g.

We put $W(f,\overline{g}) = \widetilde{W}(\log f, \log \overline{g})$, then

$$
\begin{aligned}
W(f,\overline{g}) = &-\log f \log \overline{g} + \int \log\left(\hat{m}_{02} f^2 + \hat{m}_{01} f + \hat{m}_{00}\right) \frac{df}{f} \\
&- \int \log\left(\hat{m}_{22} f^2 + \hat{m}_{21} f + \hat{m}_{20}\right) \frac{df}{f} \\
&+ \int \log\left(\overline{m}_{20} \overline{g}^2 + \overline{m}_{10} \overline{g} + \overline{m}_{00}\right) \frac{d\overline{g}}{\overline{g}} \\
&- \int \log\left(\overline{m}_{22} \overline{g}^2 + \overline{m}_{12} \overline{g} + \overline{m}_{02}\right) \frac{d\overline{g}}{\overline{g}},
\end{aligned}
$$

and we can write the discrete equations as

$$
g = \exp\left(f \frac{\partial W}{\partial f}\right), \qquad \overline{f} = \exp\left(\overline{g} \frac{\partial W}{\partial \overline{g}}\right).
$$

The explicite formula:

$$W_{A_3} = -\log f \log \overline{g} + \mathrm{Li}_2(\overline{g}) + \mathrm{Li}_2(\overline{a}_0\overline{g}) - \mathrm{Li}_2\left(\frac{\overline{g}}{\overline{a}_2}\right) - \mathrm{Li}_2\left(\frac{\overline{g}}{\overline{a}_1\overline{a}_2}\right)$$
$$- \mathrm{Li}_2(f) + \mathrm{Li}_2\left(\frac{f}{a_3}\right) - \mathrm{Li}_2\left(a_5 f\right) + \mathrm{Li}_2\left(\frac{a_0 a_1 {a_2}^2 a_3 a_5 f}{q}\right)$$
$$- \log \overline{a}_3 \log \overline{g} + \log(a_1 {a_2}^2) \log f,$$
$$W_{A_4} = -\log f \log \overline{g} + \mathrm{Li}_2(f) - \mathrm{Li}_2\left(\frac{qf}{a_2}\right) - \mathrm{Li}_2(a_0 a_3 a_4 f) - \mathrm{Li}_2(\overline{g})$$
$$- \mathrm{Li}_2(\overline{a}_4\overline{g}) + \mathrm{Li}_2\left(\frac{\overline{g}}{\overline{a}_3}\right) + \left(\log \frac{\overline{a}_2}{\overline{a}_0\overline{a}_3\overline{a}_4}\right) \log \overline{g} - \log a_4 \log f,$$
$$W_{A_5} = -\log f \log \overline{g} - \mathrm{Li}_2(f) - \mathrm{Li}_2\left(\frac{f}{a_1}\right) - \mathrm{Li}_2\left(\frac{\overline{b}_1\overline{g}}{\overline{a}_2}\right)$$
$$+ \mathrm{Li}_2(-\overline{a}_0\overline{a}_1\overline{g}) - \frac{1}{2}\left(\log \frac{b_1 f}{a_1}\right)^2 + \log \overline{a}_1 \log \overline{g},$$

$$W_{A_6} = -\log f \log \overline{g} - \mathrm{Li}_2(f) - \mathrm{Li}_2\left(\frac{\overline{g}}{\overline{a}_1\overline{b}}\right) + \log \overline{g} \log \overline{a}_1$$
$$- \frac{1}{2}\left(\log \frac{f}{qb}\right)^2 - \frac{1}{2}(\log \overline{g})^2 - \log a_1 \log f,$$
$$W_{A_7'} = -\log f \log \overline{g} - \frac{1}{2}(\log f)^2 - (\log \overline{g})^2 - \mathrm{Li}_2(f) + \mathrm{Li}_2\left(\frac{qf}{a_0}\right)$$
$$- \log \frac{-a_0}{q} \log f,$$
$$W_{A_7} = -\log f \log \overline{g} - \mathrm{Li}_2(f) - \frac{1}{2}\left(\log\left(-q^{-1}a_0 f\right)\right)^2 - \frac{1}{2}(\log(\overline{g}))^2,$$

where $\mathrm{Li}_2(x)$ is the dilogarithmic function:

$$\mathrm{Li}_2(x) = -\int \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty} \frac{x^k}{k^n}.$$

e.g. $A_7^{(1)\prime}$ type:

$$W_{A_7'} = -\log f \log \overline{g} - \frac{1}{2}(\log f)^2 - (\log \overline{g})^2 - \mathrm{Li}_2(f) + \mathrm{Li}_2\left(\frac{qf}{a_0}\right) - \log \frac{-a_0}{q} \log f,$$

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$$\begin{aligned} g =& \exp\left(f\frac{\partial W_{A_7'}}{\partial f}\right) \\ =& \exp\left(-\log\overline{g} - \log f + \log(1-f) - \log\left(1-\frac{qf}{a_0}\right) - \log\frac{-a_0}{q}\right) \\ =& \frac{1-f}{\overline{g}f\left(f-\frac{a_0}{q}\right)}, \\ \overline{f} =& \exp\left(\overline{g}\frac{\partial W_{A_7'}}{\partial \overline{g}}\right) = \exp\left(-\log f - 2\log\overline{g}\right) = \frac{1}{f\overline{g}^2}. \end{aligned}$$

Content

Case3: $f_0 f_1 {g_0}^2 = 0$

$$M_{D_4} = \begin{pmatrix} 1 & -1-s & s \\ a_1+2a_2 & -a_1-2a_2+(s-1)a_3+a_4 & sa_4 \\ a_2(a_1+a_2) & 0 & 0 \end{pmatrix}.$$

Discrete Painlevé equation is written by using the matrix as:

$$g = -\overline{g} - \frac{m_{12}f^2 + m_{11}f + m_{10}}{m_{22}f^2 + m_{21}f + m_{20}}, \qquad \overline{f} = \frac{\overline{m}_{20}\overline{g}^2 + \overline{m}_{10}\overline{g} + \overline{m}_{00}}{f(\overline{m}_{22}\overline{g}^2 + \overline{m}_{12}\overline{g} + \overline{m}_{02})}.$$

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$$M_{D_4} = \begin{pmatrix} 1 & -1-s & s \\ a_1+2a_2 & -a_1-2a_2+(s-1)a_3+a_4 & sa_4 \\ a_2(a_1+a_2) & 0 & 0 \end{pmatrix}.$$

Discrete Painlevé equation is written by using the matrix as:

$$g = -\overline{g} - \frac{m_{12}f^2+m_{11}f+m_{10}}{m_{22}f^2+m_{21}f+m_{20}}, \qquad \overline{f} = \frac{\overline{m}_{20}\overline{g}^2+\overline{m}_{10}\overline{g}+\overline{m}_{00}}{f(\overline{m}_{22}\overline{g}^2+\overline{m}_{12}\overline{g}+\overline{m}_{02})}.$$

Hence when we put a function W as

$$\begin{aligned} W(f,\overline{g}) = &-\overline{g}\log f - \int \frac{m_{12}f^2+m_{11}f+m_{10}}{m_{22}f^2+m_{21}f+m_{20}}\frac{df}{f} \\ &+ \int \log\left(\overline{m}_{20}\overline{g}^2+\overline{m}_{10}\overline{g}+\overline{m}_{00}\right) d\overline{g} \\ &- \int \log\left(\overline{m}_{22}\overline{g}^2+\overline{m}_{12}\overline{g}+\overline{m}_{02}\right) d\overline{g}, \end{aligned}$$

we can write the discrete equations as

$$g = f\frac{\partial W}{\partial f}, \qquad \overline{f} = \exp\left(\frac{\partial W}{\partial \overline{g}}\right).$$

e.g. $D_4^{(1)\prime}$ type:

$$
\begin{aligned}
W_{D_4} = &- \overline{g} \log f + a_4 \log f - a_3 \log(1-f) \\
&- (a_1 + 2a_2 + a_3 + a_4 - 1) \log(1 - f/s) + \overline{g}(\log \overline{g} + \log s) \\
&- (\overline{g} + \overline{a}_1 + \overline{a}_2) \log(\overline{g} + \overline{a}_1 + \overline{a}_2) - (\overline{g} + \overline{a}_2) \log(\overline{g} + \overline{a}_2) \\
&+ (\overline{g} - \overline{a}_4) \log(\overline{g} - \overline{a}_4),
\end{aligned}
$$

e.g. $D_4^{(1)\prime}$ type:

$$\begin{aligned} W_{D_4} =& -\overline{g}\log f + a_4\log f - a_3\log(1-f) \\ & -(a_1+2a_2+a_3+a_4-1)\log(1-f/s) + \overline{g}(\log\overline{g}+\log s) \\ & -(\overline{g}+\overline{a}_1+\overline{a}_2)\log(\overline{g}+\overline{a}_1+\overline{a}_2) - (\overline{g}+\overline{a}_2)\log(\overline{g}+\overline{a}_2) \\ & +(\overline{g}-\overline{a}_4)\log(\overline{g}-\overline{a}_4), \end{aligned}$$

$$\begin{aligned} g =& f\frac{\partial W_{D_4}}{\partial f} = -\overline{g} + a_4 - \frac{a_3 f}{1-f} + \frac{(a_1+2a_2+a_3+a_4-1)f}{s-f} \\ =& -\overline{g} + 1 - a_1 - 2a_2 - \frac{a_3}{1-f} + \frac{a_1+2a_2+a_3+a_4-1}{1-f/s} \\ \overline{f} =& \exp\left(\frac{\partial W_{D_4}}{\partial \overline{g}}\right) \\ =& \exp\left(-\log f + \log\overline{g} + \log(\overline{g}-\overline{a}_4) + \log s - \log(\overline{g}+\overline{a}_1+\overline{a}_2)\right. \\ & \left. -\log(\overline{g}+\overline{a}_2)\right) \\ =& \frac{s\overline{g}(\overline{g}-\overline{a}_4)}{f(\overline{g}+\overline{a}_1+\overline{a}_2)(\overline{g}+\overline{a}_2)}, \end{aligned}$$

Thank you.