Fractional analysis of linear differential equations on the Riemann sphere

Toshio OSHIMA（大島利雄）
Josai University

Web-seminar on Painlevé Equations and related topics
Kobe University (Zoom)
14 January, 2022

Contents

- Generalized Riemann scheme (GRS) and spectral type
- Addition and Middle convolution
- Riemann-Liouville transform
- Connection problem
- Pfaffian form
- Confluence and Versal unfolding
- Extension to several variables and restriction
- Kac-Moody root system and classification of spectral types

Most results are realized in a library [Om] of computer algebra

§ Generalized Riemann scheme

$$\mathcal{M}: \boxed{Pu = 0} \qquad (P = a_n(x)\partial^n + \cdots + a_1(x)\partial + a_0(x))$$

$$P \in W[x] = \mathbb{C}[x, \partial],\ \partial = \tfrac{d}{dx},\ \vartheta = x\partial$$

$c_0 = \infty, c_1, \ldots, c_p$: the singular points of $\mathcal{M}$ $\quad (a_n(c_j) = 0)$

Under the coordinate $c_1 = 0$, we assume $\mathcal{M}$ has (formal) local solutions

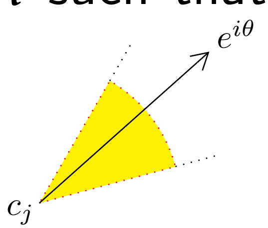
(E) $$\boxed{u_\nu(x) = \phi(x) x^{\lambda^o_{\nu,0}} \exp\Bigl(-\frac{\lambda^o_{\nu,r}}{rx^r} - \frac{\lambda^o_{\nu,r-1}}{(r-1)x^{r-1}} - \cdots - \frac{\lambda^o_{\nu,1}}{x}\Bigr)} \quad (\nu = 1, \ldots, n)$$

$$\phi(x) = 1 + a_1 x + a_2 x^2 + \cdots \in \mathbb{C}[\![x]\!]$$

$$\lambda_\nu(x) := \lambda^o_{\nu,r} x^r + \cdots + \lambda^o_{\nu,1} x + \lambda^o_{\nu,0} \in \mathbb{C}[x], \quad \lambda_\nu \neq \lambda_{\nu'}\ (\Leftarrow 1 \le \nu < \nu' \le n)$$

Theorem [Hu,...] $\theta \in \mathbb{R} \Rightarrow \exists\ R > 0$, θ_1, θ_2 and solutions u_ν to $\mathcal{M}$ such that $\theta \in (\theta_1, \theta_2)$ and u_ν have asymptotic (E) for $x \to c_j = 0$ on

$$V_{c_j, R, (\theta_1, \theta_2)} := \{x \in \mathbb{C} \mid |x - c_j| < R,\ \theta_1 < \arg(x - c_j) < \theta_2\}$$



Characteristic exponents $\lambda_{j,\nu} \in \mathbb{C}[x]$ of $\mathcal{M}$ at $x = c_j$ are defined through $c_j \mapsto 0$

- No logarithmic term ($\Leftarrow$ for simplicity)
- No ramified irregular singularity ($\Leftarrow$ essential)

$\{[\lambda_{j,1}]_{(m_1)},\ldots,[\lambda_{j,n_j}]_{(m_{n_j})}\}$: characteristic exponents of $\mathcal{M}$ at $x=c_j$

$$\lambda_{j,\nu}\in\mathbb{C}[x],\quad n=m_{j,1}+\cdots+m_{j,n_j}\quad (j=0,\ldots,p),$$
$$[\lambda]_{(m)}:=\{\lambda,\lambda+1,\ldots,\lambda+m-1\}\quad (\lambda\in\mathbb{C}[x])$$

Generalized Riemann scheme (GRS)

$$(\bigstar):\left\{\begin{matrix} x=c_j\quad (j=0,\ldots,p)\\ [\lambda_{j,1}]_{(m_{j,1})}\\ \vdots\\ [\lambda_{j,n_j}]_{(m_{j,n_j})}\end{matrix}\right\},\qquad \left\{\begin{matrix} x=0 & x=1 & x=\infty\\ 0 & [0]_{(n-1)} & \alpha_1\\ 1-\beta_1 & & \alpha_2\\ \vdots & & \vdots\\ 1-\beta_{n-1} & -\beta_n & \alpha_n\end{matrix}\right\}$$

$1^n,(n-1)1,1^n$

${}_nF_{n-1}(\alpha;\beta;x)$

$\sum\alpha_\nu=\sum\beta_\nu$

- $\deg\lambda_{j,\nu}=0$ $(\nu=1,\ldots,n_j)$ $\Rightarrow$ c_j is a regular singular point
- Versal unfolding of (GRS) $\hookrightarrow$ (GRS) of a Fuchsian equation (cf. [Ov])

  $\Rightarrow$ spectral type (a tuple of partitions of $n=\operatorname{ord}P$)

Example : spectral type $211|22|22,31$

$Pu = 0$: $\operatorname{ord} P = 4$ with $x = \infty$ (Poincaré rank 2) and 0 (regular singularity)

$$\begin{aligned} u(x) &\approx x^{-a_0}(1+o(x^{-1}))e^{-a_1x-\frac12 a_2x^2},\ x^{-a_0-1}e^{-a_1x-\frac12 a_2x^2} && (x\to+\infty)\\ &\approx x^{-b_0}e^{-b_1x},\ x^{-c_0}e^{-b_1x} && (x\to+\infty)\\ &\approx (1+o(x^2))x^{c_1},\ (x+o(x))^{c_1+1},\ x^{c_1+2},\ x^{c_2} && (x\to+0)\end{aligned}$$

Fuchs-Hukuhara relation : $2a_0+b_0+c_0+2c_1+c_2+c_3=4$

$$\begin{Bmatrix} x=\infty & x=0\\ [a_0+a_1x+a_2x^2]_{(2)} & [c_1]_{(3)}\\ b_0+b_1x & c_2\\ c_0+b_1x & \end{Bmatrix} = \begin{Bmatrix} x=\infty & (1) & (2) & x=0\\ [a_0]_{(2)} & [a_1]_2 & [a_2]_2 & [c_1]_{(3)}\\ b_0 & [b_1]_2 & [0]_2 & c_2\\ c_0 & & & \end{Bmatrix}$$

$\operatorname{idx}(211|22|22,31)=0 \Rightarrow$ 1 $(=1-\frac{\operatorname{idx}}{2})$ accessory parameter $\rightsquigarrow$ versal Heun, P_{IV}

$$\begin{Bmatrix} x=\infty & x=\frac{1}{t_1} & x=\frac{1}{t_2} & x=0\\ [a_0-\frac{a_1}{t_1}+\frac{a_2}{t_1t_2}]_{(2)} & [\frac{a_1}{t_1}+\frac{a_2}{t_1(t_1-t_2)}]_{(2)} & [\frac{a_2}{t_2(t_2-t_1)}]_{(2)} & [c_1]_{(3)}\\ b_0-\frac{b_1}{t_1} & [\frac{b_1}{t_1}]_{(2)} & [0]_{(2)} & c_2\\ c_0-\frac{b_1}{t_1} & & & \end{Bmatrix}$$

§ Index of rigidity and two operations (cf. [Of, Ov])

$$\mathrm{idx}\,\mathbf{m} = 2n^2 - \sum(n^2 - m_{j,\nu}^2) - \sum \deg(\lambda_{j,\nu} - \lambda_{j,\nu'}) \cdot m_{j,\nu} m_{j,\nu'}$$

$$= 2n^2 - \sum\big(n^2 - (m_{j,\nu}^{(r)})^2\big)$$

$$d_1(\mathbf{m}) = 2n - \sum(n - m_{j,1}) - \sum \deg(\lambda_{j,\nu} - \lambda_{j,1}) \cdot m_{j,\nu}$$

$$= 2n - \sum\big(n - m_{j,1}^{(r)}\big)$$

(FC) $\quad \sum m_{j,\nu}\lambda_{j,\nu}(0) - \mathrm{ord}\,\mathbf{m} + \frac{1}{2}\mathrm{idx}\,\mathbf{m} := \big|\{[\lambda_{j,\nu}^{(r)}]_{(m_{j,1}^{(r)})}\}\big| = 0$

Accessory parameters $= 1 - \frac{1}{2}\,\mathrm{idx}\,\mathbf{m}$ (#=0⇔rigid, Pfaffian Eq. $\Rightarrow 2 - \mathrm{idx}\,\mathbf{m}$)

Addition

$$u(x) \mapsto v(x) = \varphi(x)u(x) \;\Rightarrow\; \partial \mapsto \mathrm{Ad}(\varphi)\partial = \varphi \circ \partial \circ \varphi^{-1} = \partial - \frac{\varphi'}{\varphi}$$

$$\mathrm{Ad}\big((x-c)^{\lambda}\big)\partial = \partial - \frac{\lambda}{x-c}, \quad \vartheta \mapsto \mathrm{Ad}\big(x^{\lambda}\big)\vartheta = \vartheta - \lambda,\ \mathrm{Ad}(\varphi)^{-1} = \mathrm{Ad}(\varphi^{-1})$$

Middle convolution : a microlocal operator

$$u(x) \mapsto v(x) = \partial^{-\mu}u = \frac{1}{\Gamma(\mu)}\int_c^x u(t)(x-t)^{\mu-1}dt \quad (c:\ \text{singular point},\ \mu \in \mathbb{C})$$

$$Pu(x) = 0 \Rightarrow \mathrm{mc}_{\mu}(P)v(x) = 0,\quad \mathrm{mc}_{\mu}^{-1} = \mathrm{mc}_{-\mu}\ \ (\mathrm{mc}_{\mu}(\partial) = \partial,\ \mathrm{mc}_{\mu}(\vartheta) = \vartheta - \mu)$$

$$\mathrm{mc}_{\mu}(P) := \partial^{-L}\partial^{-\mu}(\partial^{N}P)\partial^{\mu} \in W[x] \quad (\partial^{N}P \in \mathbb{C}[\vartheta, \partial],\ \text{largest } L \in \mathbb{Z})$$

$$(\bigstar):\left\{\begin{matrix} x=c_0=\infty & x=c_1 & \cdots & x=c_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & \cdots & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{matrix}\right\}$$

$$\mathrm{mc}_\mu : \{[\lambda_{j,\nu}]_{(m_{j,\nu})}\} \mapsto \{[\lambda'_{j,\nu}]_{(m_{j,\nu})}\},\ Pu=0 \mapsto P'v=0 \quad (P'=\mathrm{mc}_\mu(P))$$

Theorem. Suppose $\lambda_{0,1}=1+\mu$ $(\deg\lambda_{0,1}=0)$, $\lambda_{1,1}=\cdots=\lambda_{p,1}=0$ and other exponents are generic ($m_{j,1}$ may be 0). Then ([DR, Of, Hi1])

$$\begin{aligned} m'_{j,\nu} &= m_{j,\nu}-\delta_{\nu,1}\cdot d_1(\mathbf{m}) && (1\le\nu\le n_j,\ 0\le j\le p),\\ \lambda'_{j,1} &= \delta_{j,0}\cdot(1-\mu) && (j=0,\dots,p),\\ \lambda'_{j,\nu} &= \lambda_{j,\nu}+\big(\deg\lambda_{j,\nu}+(-1)^{\delta_{j,0}}\big)\cdot\mu && (2\le\nu\le n_j,\ 0\le j\le p) \end{aligned}$$

and the index of rigidity and the irreducibility are kept.

By suitable (versal) additions and transpositions of the indices, we may assume $(\bigstar)$ with $(\sharp)$ $\sum_r m^{(r)}_{j,1}\ge\sum_r m^{(r)}_{j,\nu}$ $(j=0,\dots,p,\ \nu=2,\dots,n_j)$. Then

Theorem [Kz, Of,...]. If $\mathrm{idx}\,\mathbf{m}>0 \Rightarrow d_1(\mathbf{m})>0 \Rightarrow$ rigid Eq. is reduced to $u'=0$

Theorem [CB, Of, Hi2, ?]. If $d_1(\mathbf{m})\le 0 \ \Rightarrow\ \mathbf{m}$ is irreducibly realizable (basic)

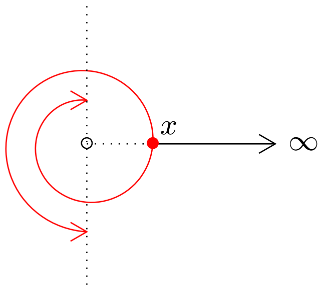
Theorem [Of, HO]. $\#\{\mathbf{m}\mid d_1(\mathbf{m})\le 0,\ \mathrm{idx}\,\mathbf{m}=K\}<\infty$ $(\forall K\in\mathbb{Z})$

$$(\partial^{-\mu}u)(x) = (I_a^\mu u)(x) := \frac{1}{\Gamma(\mu)}\int_a^x u(t)(x-t)^{\mu-1}dt$$

(a : a singular point of $u(x)$)

$$I_0^\mu(x^\lambda) = \frac{1}{\Gamma(\mu)}\int_0^x t^\lambda(x-t)^{\mu-1}dt = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}$$

$$\tilde{I}_\infty^\mu(x^\lambda e^{-x}) \sim x^\lambda e^{-x}{}_2F_0(-\lambda,\mu;-\tfrac{1}{x}) \quad (|x|\to\infty,\ |\arg x| < \tfrac{3\pi}{2})$$



$${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;x) := \sum_{n=0}^\infty \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!}x^n$$

$$e^x = {}_0F_0(x), \quad (1-x)^{-\lambda} = \sum_{n=0}^\infty \frac{(\lambda)_n}{n!}x^n = {}_1F_0(\lambda;x),$$

$$I_0^\mu\big(x^\lambda\cdot{}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\ldots,\beta_q;\pm x)\big)$$
$$= x^{\lambda+\mu}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}\cdot{}_{p+1}F_{q+1}(\alpha_1,\ldots,\alpha_p,\lambda+1;\beta_1,\ldots,\beta_q,\lambda+\mu+1;\pm x)$$

$$I_0^\mu(x^\lambda e^{-x}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}\cdot{}_1F_1(\lambda+1;\lambda+\mu+1;-x) \quad \text{(Kummer)}$$

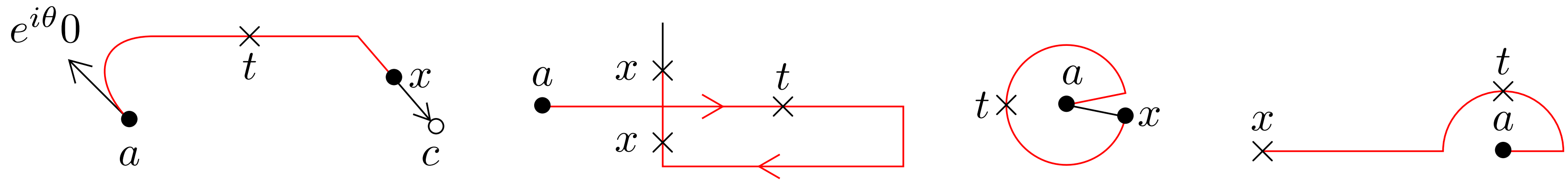
$$I_0^\mu\big(x^\lambda(1-x)^{\lambda'}\big) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}x^{\lambda+\mu}\cdot{}_2F_1(\lambda+1,-\lambda';\lambda+\mu+1;x) \quad \text{(Gauss)}$$

§ Riemann-Liouville transform

$$(I^{\mu}_{a+e^{i\theta}0}u)(x) := \frac{1}{\Gamma(\mu)}\int_{a+e^{i\theta}0}^{x} u(t)(x-t)^{\mu-1}dt = \frac{1}{\Gamma(\mu)}\int_{L} u(t)(x-t)^{\mu-1}dt$$

$$L\ :\ [\alpha,\beta]\ni t\mapsto L(t)\in\mathbb{C},\ L(\alpha)=a,\ L(\beta)=x,\ \theta=\arg L'(\alpha),$$

$$L(s)\neq L(t)\quad \text{for } \alpha\le s<t\le\beta$$



$$\tilde{L}\ :\ [\alpha,\gamma]\ni t\to\tilde{L}(t)\in\mathbb{C},\ a=\tilde{L}(\alpha) \text{ and } c=\tilde{L}(\gamma),$$

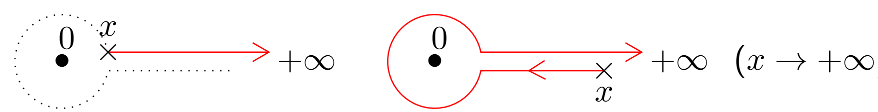
$$\tilde{L}(s)\neq\tilde{L}(t) \text{ for } \alpha\le s<t<\gamma.$$

- $u(x)$ is holomorphic for $x=\tilde{L}(t_0)$ $(t_0\in(\alpha,\gamma))$, $L:=\tilde{L}|_{[\alpha,t_0]}$

Want to know the asymptotics of $(I^{\mu}_{a}u)(x)$ for $t_0\to\alpha$ and $t_0\to\gamma$

in terms of those of $u(x)$.

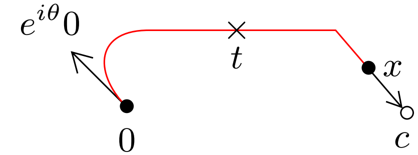
$$(\tilde{I}^{\mu}_{a}u)(x) := \frac{1}{\Gamma(\mu)}\int_{x}^{a} u(t)(t-x)^{\mu-1}dt$$



$$\boxed{(\tilde{I}^{\mu}_{a}u)(x) = x^{\mu-1}\cdot\Big(I^{\mu}_{\frac{1}{a}}x^{-\mu-1}u(\tfrac{1}{x})\Big)(\tfrac{1}{x})}\qquad (a=e^{i\theta}\infty\to\tfrac{1}{a}=e^{-i\theta}0)$$

§ Riemann-Liouville transform (Main Theorem) (cf. [Of, Ov, Or])

$$(I^{\mu}_{e^{i\theta}0}u)(x) = \frac{1}{\Gamma(\mu)}\int_{e^{i\theta}0}^{x} u(t)(x-t)^{\mu-1}dt$$



(1.1) $u(x) \sim \Big(\sum_{n=0}^{\infty} a_n x^n\Big)x^{\lambda} \quad (x \to e^{i\theta}0) \qquad (\phi(x) := a_0 + a_1x + \cdots \in \mathbb{C}[\![x]\!])$

$$\Rightarrow (I^{\mu}_{e^{i\theta}0}u)(x) \sim \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}\Big(\sum_{n=0}^{\infty}\frac{(\lambda+1)_n a_n}{(\lambda+\mu+1)_n}x^n\Big)x^{\lambda+\mu}$$

(1.2) $u(x) \sim \phi(x)x^{\lambda}\exp\big(-\frac{C_0}{x^{m_0}} - \frac{C_1}{x^{m_1}} - \cdots\big) \qquad (x \to e^{i\theta}0)$ and $\operatorname{Re} C_0e^{-im_0\theta} > 0$

$$\Rightarrow (I^{\mu}_{e^{i\theta}0}u)(x) \sim (m_0C_0)^{-\mu}\psi(x)x^{\lambda+(m_0+1)\mu}\exp\big(-\frac{C_0}{x^{m_0}} - \frac{C_1}{x^{m_1}} - \cdots\big)$$

$\phi,\ \psi \in \mathbb{C}[\![x]\!]$, $\phi(0) = \psi(0)$, $m_0 > m_1 > \cdots > m_k > 0$ and $m_j \in \mathbb{Z}_{>0}$

(2.1) $u(x) \approx (c-x)^{\lambda'} \quad (x \to c - e^{i\theta'}0)$ and $\operatorname{Re}(\lambda'+\mu) < 0$

$$\Rightarrow (I^{\mu}_{e^{i\theta}0}u)(x) \approx \frac{\Gamma(-\lambda'-\mu)}{\Gamma(-\lambda')}(c-x)^{\lambda'+\mu} \qquad \Leftarrow (\operatorname{Re}(\lambda-\lambda') > 0 \Leftarrow c = 0,\ (1.1))$$

(2.2) $u(x) \approx (c-x)^{\lambda'}\exp\big(\frac{C'_0}{(c-x)^{m'_0}} + \frac{C'_1}{(c-x)^{m'_1}} + \cdots\big) \quad (x \to c - e^{i\theta'}0)$ and $\operatorname{Re} C'_0e^{-im'_0\theta'} > 0$

$$\Rightarrow (I^{\mu}_{e^{i\theta}0}u)(x) \approx (m'_0C'_0)^{-\mu}(c-x)^{\lambda'+(m'_0+1)\mu}\exp\big(\frac{C'_0}{(c-x)^{m'_0}} + \frac{C'_1}{(c-x)^{m'_1}} + \cdots\big)$$

(2.3) $\lim_{x\to c-e^{i\theta'}0} u(x)(c-x)^{-\operatorname{Re}\mu} = 0 \ \Rightarrow \lim_{x\to c-e^{i\theta'}0} u(x) = (I^{\mu}_{e^{i\theta}0}u)(c-e^{i\theta'}0) \in \mathbb{C}$

§ Example (${}_{n-1}F_{n-1}$: a confluence of ${}_nF_{n-1}$)

$1^n|(n-1)1,1^n$ (n=2 ⇒ Kummer's Eq.)

$$G^{\pm} := \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu}\ (1\le\nu<n) & \lambda_{0,\nu}\ (1\le\nu\le n) \\ \pm x-\lambda_{1,n} & \end{matrix}\right\} \qquad \Big(\sum_{\nu=1}^{n}\lambda_{0,\nu}=\sum_{\nu=1}^{n}\lambda_{1,\nu}\Big)$$

$$\xrightarrow{\mathrm{Ad}(x^{-\lambda_{0,1}})} \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu}+\lambda_{0,1}\ (1\le\nu<n) & 0 \\ \pm x-\lambda_{1,n}+\lambda_{0,1} & \lambda_{0,\nu}-\lambda_{0,1}\ (2\le\nu\le n) \end{matrix}\right\}$$

$$\xrightarrow{\mathrm{mc}_{\lambda_{0,1}-\lambda_{1,1}}} \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu}+\lambda_{1,1}\ (2\le\nu<n) & \lambda_{0,\nu}-\lambda_{1,1}\ (2\le\nu\le n) \\ \pm x-\lambda_{1,n}+\lambda_{0,1} & \end{matrix}\right\}$$

$$\xrightarrow{\mathrm{Ad}(x^{\lambda_{1,1}-\lambda_{0,2}})} \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,\nu}+\lambda_{0,2}\ (3\le\nu<n) & 0 \\ \pm x-\lambda_{1,n}+\lambda_{0,1}+\lambda_{0,2}-\lambda_{1,1} & \lambda_{0,\nu}-\lambda_{0,2}\ (3\le\nu\le n) \end{matrix}\right\}$$

$$\cdots \xrightarrow{\mathrm{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}})}$$

$$\left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0 \\ \pm x-\lambda_{1,n}+\lambda_{0,1}+\sum_{\nu=2}^{n-1}(\lambda_{0,\nu}-\lambda_{1,\nu-1}) & \lambda_{0,n}-\lambda_{0,n-1} \end{matrix}\right\}$$

$$=G_2^{\pm} := \left\{\begin{matrix} x=\infty & x=0 \\ 1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0 \\ \pm x+\lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{0,n-1} \end{matrix}\right\} \qquad \text{(Kummer)}$$

$$G_2^{\pm}=\left\{\begin{matrix} x=\infty & x=0\\ 1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0\\ \pm x+\lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{0,n-1}\end{matrix}\right\}\quad \text{(Kummer)}$$

$$\xrightarrow{\mathrm{mc}_{\lambda_{0,n-1}-\lambda_{1,n-1}}}\left\{\begin{matrix} x=\infty & x=0\\ \pm x+\lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{1,n-1}\end{matrix}\right\}\ni x^{\lambda_{0,n}-\lambda_{1,n-1}}e^{\mp x}$$

$$G_2^{\pm}\xrightarrow{x\mapsto -x}G_2^{\mp}=\left\{\begin{matrix} x=\infty & x=0\\ 1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0\\ \mp x+\lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{0,n-1}\end{matrix}\right\}$$

$$\xrightarrow{\mathrm{Ad}(e^{\mp x})}\left\{\begin{matrix} x=\infty & x=0\\ \pm x+1-\lambda_{1,n-1}+\lambda_{0,n-1} & 0\\ \lambda_{1,n-1}-\lambda_{0,n} & \lambda_{0,n}-\lambda_{0,n-1}\end{matrix}\right\}$$

$$\xrightarrow{\mathrm{mc}_{\lambda_{1,n-1}-\lambda_{0,n-1}}}\left\{\begin{matrix} x=\infty & x=0\\ \pm x+1-\lambda_{1,n-1}+\lambda_{0,n-1} & \lambda_{1,n-1}-\lambda_{0,n-1}-1\end{matrix}\right\}$$

$$G^{\pm}:=\left\{\begin{matrix} x=\infty & x=0\\ 1-\lambda_{1,\nu}\ (1\le\nu<n) & \lambda_{0,\nu}\ (1\le\nu\le n)\\ \pm x-\lambda_{1,n} & \end{matrix}\right\}$$

$$\begin{aligned}
G^{+}:u_{\infty,x-\lambda_{1,n}}(x)&=\mathrm{Ad}(x^{\lambda_{0,1}})\tilde I_\infty^{\lambda_{1,1}-\lambda_{0,1}}\,\mathrm{Ad}(x^{\lambda_{0,2}-\lambda_{1,1}})\tilde I_\infty^{\lambda_{1,2}-\lambda_{0,2}}\cdots\\
&\qquad\cdots\mathrm{Ad}(x^{\lambda_{0,n-1}-\lambda_{1,n-2}})\tilde I_\infty^{\lambda_{1,n-1}-\lambda_{0,n-1}}(x^{\lambda_{0,n}-\lambda_{1,n-1}}e^{-x})\\
&\approx x^{\lambda_{1,n}}e^{-x}\qquad (|x|\to\infty,\ |\arg x|<\tfrac{3\pi}{2}),\\
u^{\pm}_{\infty,1-\lambda_{1,\nu}}(x)&\approx (e^{\pm\pi i}x)^{\lambda_{1,\nu}-1}\qquad (|x|\to\infty,\ |\arg x\pm\pi|<\tfrac{3\pi}{2}),\\
u_{0,\lambda_{0,\nu}}(x)&\approx x^{\lambda_{0,\nu}}\qquad (x\to 0)\qquad (\Leftarrow \tilde I^{\mu}_{\infty}\mapsto I^{\mu}_{0})
\end{aligned}$$

Connection formula: $I_n=\{1,\dots,n\}$, $J_{n-1,m}=\{1,\dots,n-1\}\setminus\{m\}$

$$u^{\pm}_{\infty,1-\lambda_{1,m}}(x)=\sum_{k=1}^{n}c_\Gamma(\infty\!:\!1-\lambda_{1,m}\rightsquigarrow 0\!:\!\lambda_{0,k})e^{\pm\lambda_{0,k}\pi i}u_{0,\lambda_{0,k}}(x)$$

$$\sim(e^{\pm\pi i}x)^{\lambda_{1,m}-1}\cdot{}_nF_{n-2}(\{\lambda_{0,\nu}-\lambda_{1,m}+1\}_{\nu\in I_n};\{\lambda_{1,\nu}-\lambda_{1,m}+1\}_{\nu\in J_{n-1,m}}$$

$$;-\tfrac{1}{e^{\pm\pi i}x})\approx(e^{\pm\pi i}x)^{\lambda_{1,m}-1}\quad(|x|\to\infty,\ |\arg x\pm\pi|<\tfrac{3\pi}{2})$$

$$u_{\infty,x-\lambda_{1,n}}(x)=\sum_{k=1}^{n}c_\Gamma(\infty\!:\!x-\lambda_{1,n}\rightsquigarrow 0\!:\!\lambda_{0,k})u_{0,\lambda_{0,k}}(x)\approx x^{\lambda_{1,n}}e^{-x}\quad(|x|\to\infty,\ |\arg x|<\tfrac{3\pi}{2})$$

$$u_{0,\lambda_{0,k}}(x)=\sum_{m=1}^{n}c_\Gamma(0\!:\!\lambda_{0,k}\rightsquigarrow\infty\!:\!1-\lambda_{1,m})e^{\pm(1-\lambda_{1,m})\pi i}u^{\pm}_{\infty,1-\lambda_{1,m}}(x)$$

$$+\,c_\Gamma(0\!:\!\lambda_{0,k}\rightsquigarrow\infty\!:\!x-\lambda_{1,n})e^{\pm(\lambda_{1,n}-\lambda_{0,k})\pi i}u_{\infty,x-\lambda_{1,n}}(x)$$

$$=x^{\lambda_{0,k}}{}_{n-1}F_{n-1}\big(\{\lambda_{0,k}-\lambda_{1,\nu}+1\}_{\nu\in J_{n-1}};\{\lambda_{0,k}-\lambda_{0,\nu}+1\}_{\nu\in I_{n,k}};-x\big)$$

Stokes relations:

$$e^{\mp(\lambda_{1,m}-1)\pi i}\cdot u^{\pm}_{\infty,1-\lambda_{1,m}}(e^{\mp2\pi i}x)-e^{\pm(\lambda_{1,m}-1)\pi i}\cdot u^{\pm}_{\infty,1-\lambda_{1,m}}(x)$$

$$=\pm2\pi i\cdot c_\Gamma(\infty\!:\!1-\lambda_{1.m}\rightsquigarrow\infty\!:\!x-\lambda_{1,n})\cdot u_{\infty,x-\lambda_{1,n}}(x),$$

$$e^{-\lambda_{1,n}\pi i}\cdot u_{\infty,x-\lambda_{1,n}}(e^{\pi i}x)-e^{\lambda_{1,n}\pi i}\cdot u_{\infty,x-\lambda_{1,n}}(e^{-\pi i}x)$$

$$=-2\pi i\sum_{m=1}^{n-1}c_\Gamma(\infty\!:\!x-\lambda_{1,n}\rightsquigarrow\infty\!:\!1-\lambda_{1,m})\cdot u^{\pm}_{\infty,1-\lambda_{1,m}}(e^{\mp\pi i}x)$$

$$n=2\ \Rightarrow\ {}_2F_0(\alpha,\beta;-\tfrac{1}{e^{\pm\pi i}x})-{}_2F_0(\alpha,\beta;-\tfrac{1}{e^{\mp\pi i}x})=\frac{2\pi i x^{\alpha+\beta-1}e^{-x}}{\Gamma(1-\alpha)\Gamma(1-\beta)}{}_2F_0(1-\alpha,1-\beta;-\tfrac{1}{x})$$

Γ-factors:

$$c_\Gamma(\infty:1-\lambda_{1,m} \rightsquigarrow 0:\lambda_{0,k}) = \frac{\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,\nu}-\lambda_{0,k})\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{1,\nu}-\lambda_{1,m}+1)}{\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,\nu}-\lambda_{1,m}+1)\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{1,\nu}-\lambda_{0,k})}$$

$$c_\Gamma(\infty:x-\lambda_{1,n} \rightsquigarrow 0:\lambda_{0,k}) = \frac{\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,\nu}-\lambda_{0,k})}{\prod_{\nu\in J_{n-1}}\Gamma(\lambda_{1,\nu}-\lambda_{0,k})}$$

$$c_\Gamma(0:\lambda_{0,k} \rightsquigarrow \infty:1-\lambda_{1,m}) = \frac{\prod_{\nu\in I_{n-1,m}}\Gamma(\lambda_{1,m}-\lambda_{1,\nu})\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,k}-\lambda_{0,\nu}+1)}{\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{0,k}-\lambda_{1,\nu}+1)\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{1,m}-\lambda_{0,\nu})}$$

$$c_\Gamma(0:\lambda_{0,k} \rightsquigarrow \infty:x-\lambda_{1,n}) = \frac{\prod_{\nu\in I_{n,k}}\Gamma(\lambda_{0,n}-\lambda_{0,\nu}+1)}{\prod_{\nu\in J_{n-1}}\Gamma(\lambda_{0,k}-\lambda_{1,\nu}+1)}$$

$$c_\Gamma(\infty:x-\lambda_{1,n} \rightsquigarrow \infty:1-\lambda_{1,m}) = \frac{\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{1,\nu}-\lambda_{1,m}+1)}{\prod_{\nu\in I_n}\Gamma(\lambda_{0,\nu}-\lambda_{1,m}+1)}$$

$$c_\Gamma(\infty:1-\lambda_{1,m} \rightsquigarrow \infty:x-\lambda_{1,n}) = \frac{\prod_{\nu\in J_{n-1,m}}\Gamma(\lambda_{1,m}-\lambda_{1,\nu})}{\prod_{\nu\in I_n}\Gamma(\lambda_{1,m}-\lambda_{0,\nu})}$$

Irreducible $\quad\Leftrightarrow\quad \lambda_{0,k}-\lambda_{1,m}\notin\mathbb{Z}\quad(1\le k\le n,\ 1\le m<n)$

Kummer Whittaker

$$\begin{Bmatrix} x=\infty & 0 \\ 1-\lambda_{1,1} & \lambda_{0,1} \\ x-\lambda_{1,2} & \lambda_{0,2}\end{Bmatrix} \xrightarrow{\times x^{-\lambda_{0,1}}e^{x}} \begin{Bmatrix} x=\infty & 0 \\ -x+\gamma-\alpha & 0 \\ \alpha & 1-\gamma\end{Bmatrix} \xrightarrow{\times x^{\frac{\gamma}{2}}e^{-\frac{x}{2}}} \begin{Bmatrix} x=\infty & 0 \\ -\frac{x}{2}-k & \frac{1}{2}-m \\ \frac{x}{2}+k & \frac{1}{2}+m\end{Bmatrix}$$

§ Middle convolution of versal unfolding (Pfaffian form)

$$\frac{du}{dx} = \Big(\sum_{j=1}^{p}\sum_{i=1}^{r_j}\frac{A_{j,i}}{(x-a_j)^i} - \sum_{i=2}^{r_0} A_{0,i}x^{i-2}\Big)u \qquad (A_{j,i}\in M_N(\mathbb{C})) \qquad (*)$$

Definition. Versal unfolding of $(*)$:

$$\frac{d\tilde{u}}{dx} = \Big(\sum_{j=1}^{p}\sum_{i=1}^{r_j}\frac{\tilde{A}_{j,i}(a)}{(x-a_{j,1})(x-a_{j,2})\cdots(x-a_{j,i})} - \sum_{i=2}^{r_0}\frac{\tilde{A}_{0,i}(a)x^{i-2}}{(1-a_{0,2}x)(1-a_{0,3}x)\cdots(1-a_{0,i}x)}\Big)\tilde{u} \qquad (**)$$

$(*) \Leftarrow [a_{j,i}\mapsto a_j \quad (a_0=0)]$

index of rigidity is stable under the parameter (keeps rigidity)

index : $2N^2 - \sum_{j\geq 0}(N^2 - \dim Z_{M_N(\mathbb{C})}C_j)$ for $\dfrac{du}{dx} = \sum_{j\geq 1}\dfrac{C_j}{x-a_j}u,\ C_0 := -\sum_{j\geq 1}C_j$

Assumption. $(*)$ is transformed into Hukuhara-Turrittin-Levelt's normal form by $GL(N,\mathbb{C}[\![x]\!]_{a_j})$ ($\Rightarrow$ without ramified irregular singularities)

Conjecture. $(*)$: irreducible $\Rightarrow \exists!$ versal unfolding

(irreducible $\Leftrightarrow$ no non-trivial subspace V of $\mathbb{C}^N$ with $A_{j,i}V \subset V$)

Theorem [Oc, Ov]. $\exists$ middle convolution of versal unfolding

$\Rightarrow$ Conjecture is stable under middle convolutions

Corollary. $(*)$: rigid $\Rightarrow$ Conjecture is true (True in general by Hiroe)

Example:

$$\frac{du}{dx} = \frac{A_1 u}{x-a_1} + \frac{A_2 u}{(x-a_1)(x-a_2)} + \frac{A_3 u}{(x-a_1)(x-a_2)(x-a_3)}$$

$$\xrightarrow{\mathrm{MC}_\mu} \quad \frac{d\hat{u}}{dx} = \frac{\begin{pmatrix} A_1+\mu & A_2 & A_3 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}}{x-a_1}\hat{u} + \frac{\begin{pmatrix} 0 & 0 & 0 \\ A_1+\mu & A_2+(a_2-a_1)\mu & A_3 \\ 0 & \mu & (a_3-a_1)\mu \end{pmatrix}}{(x-a_1)(x-a_2)}\hat{u}$$

$$+ \frac{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_1+\mu & A_2+(a_3-a_1)\mu & A_3+(a_3-a_1)(a_3-a_2)\mu \end{pmatrix}}{(x-a_1)(x-a_2)(x-a_3)}\hat{u},$$

$$\mathcal{K} := \operatorname{Ker} K, \quad K := \begin{pmatrix} A_1 & A_2 & A_3 \\ A_2 & A_3+(a_2-a_1)A_2 & (a_3-a_1)A_3 \\ A_3 & (a_3-a_1)A_3 & (a_3-a_1)(a_3-a_2)A_3 \end{pmatrix},$$

$$\mathcal{L} := \{{}^t(v,0,0) \in \mathbb{C}^{3N} \mid (A_1+\mu)v = 0\}$$

$$\Longrightarrow \quad \mathrm{mc}_\mu := \mathrm{MC}_\mu\big|_{\mathbb{C}^{3N}/(\mathcal{K}+\mathcal{L})}$$

Theorem [Oc].

$$q_i(x) := \frac{1}{\prod_{\nu=1}^{i}(x-a_\nu)}, \quad \mathbf{q} := \begin{pmatrix} q_1 I_N \\ \vdots \\ q_r I_N \end{pmatrix} \in M(rN, N),$$

$$\frac{du}{dx} = \sum_{i=1}^{r} \frac{A_i}{(x-a_1)\cdots(x-a_i)} u = \sum_{i=1}^{r} A_i q_i u, \quad A_i \in M(N, N; \mathbb{C}(\mathbf{a}))$$

$$\frac{d\hat{u}}{dx} = \mathbf{q}(A_1, \ldots, A_r)\hat{u} + \mu D_r(\mathbf{a})\hat{u} = \sum_{i=1}^{r} \hat{A}_i q_i \hat{u}, \quad \hat{A}_i \in M(rN, rN; \mathbb{C}(\mathbf{a}) + \mu\mathbb{C}(\mathbf{a}))$$

$$D_r(\mathbf{a}) = \sum_{k=1}^{r} D_{r,k}(\mathbf{a}) q_k(x), \quad D_{r,k}(\mathbf{a}) = \big(D_{r,k,i,j}(\mathbf{a}) I_N\big)_{\substack{1\le i\le r,\\ 1\le j\le r}}$$

$$D_{r,k,i,j}(\mathbf{a}) = \begin{cases} 0 & (L<0 \text{ or } i<j \text{ or } i<k), \\ 1 & (L=0), \\ \displaystyle\sum_{0\le\nu_1\le\nu_2\le\cdots\le\nu_L\le i-k} \prod_{p=1}^{L} (a_{i-\nu_p} - a_{\nu_p+p}) & (L>0,\ i\ge j,\ i\ge k) \end{cases}$$

$$L := k + j - i - 1.$$

$$\left(q_i(x), \hat{u} = \begin{pmatrix} I_c^\mu q_1(x) u \\ \vdots \\ I_c^\mu q_r(x) u \end{pmatrix}, D_{r,k,i,j}(\mathbf{a})\right) \leftrightarrow \text{[DR]}: \left(\frac{1}{x-a_i}, \hat{u} = \begin{pmatrix} I_c^\mu \frac{u}{x-x_1} \\ \vdots \\ I_c^\mu \frac{u}{x-x_r} \end{pmatrix}, \delta_{k,i}\delta_{i,j}\right)$$

§ Versal unfolding of systems with several variables

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x - a_j} u \qquad (A_j \in M_N(\mathbb{C})) \qquad (\star)$$

$(\star)$: rigid $\Rightarrow$ $(\star) \to$ extended to KZ (Knizhnik-Zamolodchikov) equation:

$$\frac{\partial u}{\partial x_i} = \sum_{\substack{\nu \neq i \\ 0 \leq \nu \leq p}} \frac{A_{i,\nu}}{x_i - x_\nu} u \quad (x_0 = x,\ x_j = a_j,\ A_{0,\nu} = A_\nu,\ i = 0, \ldots, p) \quad (\star\star)$$

$$A_{i,i} = 0,\ A_{i,j} = A_{j,i},\ [A_{i,j}, A_{k,\ell}] = [A_{i,j}, A_{i,k} + A_{j,k}] = 0 \quad (\#\{i,j,k,\ell\} = 4)$$

$\Leftarrow$ middle convolution mc_μ of $(\star)$ is extended to $(\star\star)$ (Haraoka [Ha])

$$\frac{du}{dx} = \left(\sum_{j=1}^{p} \sum_{i=1}^{r_j} \frac{A_{j,i}}{(x - a_j)^i} - \sum_{i=2}^{r_0} A_{0,i} x^{i-2} \right) u \qquad (x_0 = x,\ x_j = a_j) \qquad (*)$$

Rigid system $(*)$ with unramified irregular singularities

$\Rightarrow$ versal unfolding $(**)$ is rigid Fuchsian system if singular points $a_{j,\nu}$ are generic

Theorem [Oc]. KZ eq. with generic singular points of versal unfolding (Fuchsian)

$\Rightarrow$ **KZ eq. is holomorphically extended to confluent points** $(a_{j,\nu} \to x_j = a_j)$

$\Rightarrow$ **rigid system $(*)$ is extended to confluent KZ Eq.**

middle convolution of $(\star\star)$: $\mathrm{mc}_\mu(A_{i,j}) := \tilde{A}_{i,j}\big|_{\mathbb{C}^{pN}/(\mathcal{K}+\mathcal{L})}$

$$\tilde{A}_{0,j} = j \overset{\displaystyle j}{\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{0,1} & \cdots & A_{0,j}+\mu & \cdots & A_{0,p} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}} \in M(pN,\mathbb{C}) \qquad (1 \le j \le p)$$

$$\mathcal{K} := \mathrm{Ker}\, K,\ K := \mathrm{Diag}(A_{0,1},\ldots,A_{0,p}),$$

$$\mathcal{L} = \mathrm{Ker}\, L,\ L := -(\tilde{A}_{0,1} + \cdots + \tilde{A}_{0,p})$$

by Dettweiler-Reiter [DR],

$$\tilde{A}_{i,j} = \begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} \overset{\displaystyle \qquad i \qquad\qquad\qquad j}{\begin{pmatrix} A_{i,j} & & & & & \\ & \ddots & & & & \\ & & A_{i,j}+A_{0,j} & & -A_{0,j} & \\ & & & \ddots & & \\ & & -A_{0,i} & & A_{i,j}+A_{0,i} & \\ & & & & & \ddots \\ & & & & & & A_{i,j} \end{pmatrix}} \in M(pN,\mathbb{C}) \quad (1 \le i < j \le p)$$

by Haraoka [Ha]

Example (versal unfolding/confluence of KZ Eq. (cf. [Oc]))

$$\begin{cases} \dfrac{\partial u}{\partial x_0} = \Big(\dfrac{A_{01}}{x_0 - x_1} + \dfrac{A_{02}}{(x_0 - x_1)(x_0 - x_1 - e_2)} + \dfrac{A_{03}}{x_0 - x_3}\Big)u, \\ \dfrac{\partial u}{\partial x_1} = \Big(\dfrac{A_{10}}{x_1 - x_0} - \dfrac{A_{20}}{(x_1 - x_0)(x_1 - x_0 + e_2)} \\ \qquad + \dfrac{A_{13}}{x_1 - x_3} - \dfrac{A_{23}}{(x_1 - x_3)(x_1 - x_3 + e_2)}\Big)u, \\ \dfrac{\partial u}{\partial x_3} = \Big(\dfrac{A_{30}}{x_3 - x_0} + \dfrac{A_{31}}{x_3 - x_1} + \dfrac{A_{32}}{(x_3 - x_1)(x_3 - x_1 - e_2)}\Big)u, \end{cases}$$

$$A_{i,j} = A_{j,i} \quad (0 \le i < j \le 3), \qquad (x_0, x_1, x_1 + e_2, x_3) = (x, a_1, a_2, a_3)$$

$$\xrightarrow{\mathrm{MC}_\mu}$$

$$\hat{A}_{01} = \begin{pmatrix} A_{01} + \mu & A_{02} & A_{03} \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{A}_{02} = \begin{pmatrix} 0 & 0 & 0 \\ A_{01} + \mu & A_{02} + e_2\mu & A_{03} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{A}_{03} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{01} & A_{02} & A_{03} + \mu \end{pmatrix}, \quad \hat{A}_{13} = \begin{pmatrix} A_{13} + A_{03} & 0 & -A_{03} \\ 0 & A_{13} + A_{03} & 0 \\ -A_{01} & -A_{02} & A_{13} + A_{01} \end{pmatrix},$$

$$\hat{A}_{23} = \begin{pmatrix} A_{23} & 0 & 0 \\ A_{03} & A_{23} + e_2 A_{03} & -A_{03} \\ -A_{02} & -e_2 A_{02} & A_{23} + A_{02} \end{pmatrix}$$

$$\begin{cases} A_{i,j} \in \mathbb{C} \ \Rightarrow \ \text{Appell's } F_1 \\ 21, 21, 21, 21 \ \to \ 21|21, 21, 21 \end{cases}$$

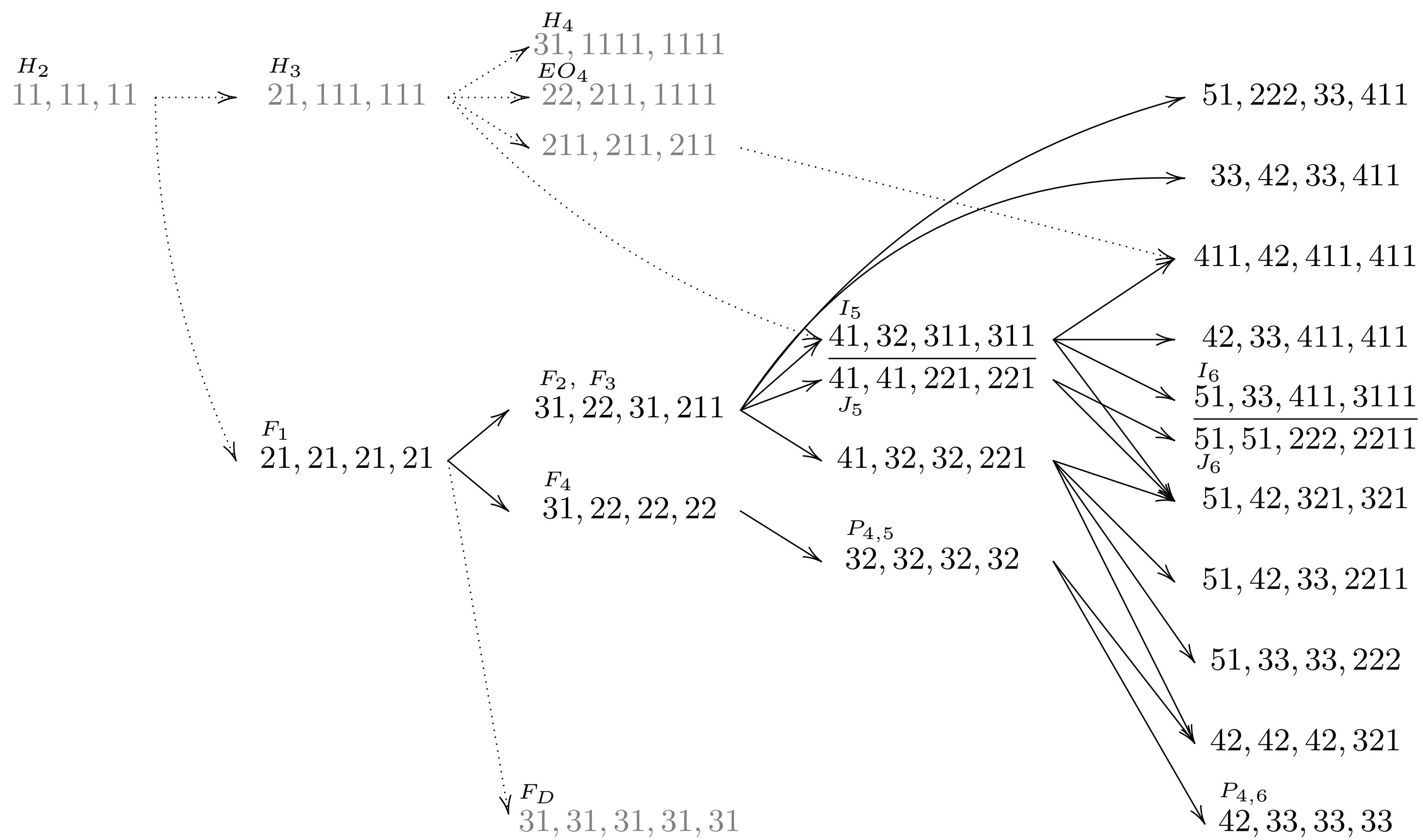
§ Hierarchy of rigid quartets (cf. [Of, Ot])

$\{$Rigid Fuchsian ODE with $p+1$ singular points $\{0,1,\infty,y_1,\dots,y_{p-2}\}\}$

$\hookrightarrow \{$HG Eq. with $(p-1)$-variables $(x,y_1,\dots,y_{p-2})\}$ $\quad (p=3)$

$\hookrightarrow \{$KZ Eq. on $\{(x_0,\dots,x_{p+1}) \mid x_i \in \mathbb{P}^1\}\}$ $\quad (x_0,x_1,\dots,x_{p+1})=(x,y_1,\dots,y_{p-2},0,1,\infty)$

$(m_{0,1}\cdots m_{0,n_0},\cdots,m_{p,1}\cdots m_{p,n_p})$: spectre type ($p+1$ tuples of partitions of $\mathrm{ord}\,\mathbf{m}$)



#parameters $=\sum$(# blocks at sing. point -1)

§ HG with two variables corresponding to KZ Eq's (cf. [Ot])

$$A_{i,p+1} := -\sum_{\nu=0}^{p} A_{i,\nu} \quad \text{(at } \infty\text{)}, \qquad A_{i_1,\dots,i_k} := \sum_{1\le\mu<\nu\le k} A_{i_\mu,i_\nu}$$

$$[A_I, A_J] = 0 \quad (I \subset J \text{ or } I \cap J = \emptyset)$$
$$A_{0,\dots,p+1} = A_K - A_{\{0,\dots,p+1\}\setminus K} \quad (K \subset \{0,\dots,p\})$$
$$= \kappa \ (\text{: scalar} \Leftarrow \text{Irred.})$$

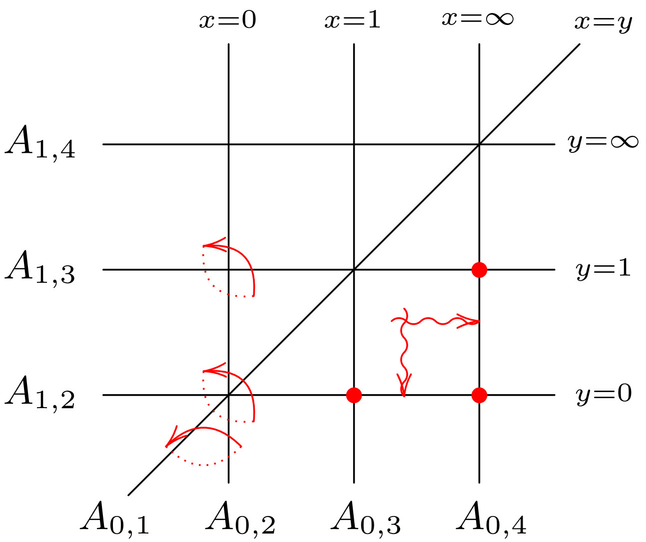
$p = 3 : (x_0, x_1, x_2, x_3, x_4) \leftrightarrow (x, y, 0, 1, \infty)$

$[A_{0,2} + A_{1,3}] \quad \Leftarrow [A_{0,2} : A_{1,3}]$ (conj. class)

$[A_{0,1} + A_{0,2} + A_{1,2}] = [A_{0,1,2}] = [\kappa + A_{3,4}]$

$[A_{0,1} + A_{0,2}] = [A_{0,1,2} - A_{1,2}] = [\kappa + A_{3,4} - A_{1,2}]$

$[A_{0,2}, A_{1,3}] = [A_{3,4}, A_{1,2}] = 0$



In general, the Fuchsian ODE satisfied by x_i $(i \ge 1)$ is not rigid (rigid $\rightsquigarrow$ non-rigid)

$\exists$ Series expansion of (x_i, x_j) around $\{x_i = x_{i'},\ x_j = x_{j'}\}$ $(\#\{i, i', j, j'\} = 4$, cf. [KO])

#Rigid spectral types corresponding to the same KZ Eq. with $p = 3$ and order N

$N \rightarrow$	3	4	5	6	7	8	9	10	11	12	13	14
rigid ODE	1	2	4	11	16	35	58	109	156	299	402	658
same KZ	0	0	1	1	2	4	9	12	18	31	47	68
KZ Eq.	1	2	3	10	14	31	49	97	138	268	355	590

§ Example (Appell's F_4) (cf. [Oh, Ot])

$\mathbf{31, 22, 22, 22}$: rank $= 4$, 4 parameters,

$(1^8 \cdot 2^1) \quad \to \quad F_1 : 21, 12, 12, 12$

$= 10, 01, 01, 01 \oplus 21, 21, 21, 21 \quad (8)$

$= 2(20, 11, 11, 11) \oplus (-1)1, 00, 00, 00 \quad (1)$

$31, 22, 22|22 \quad 31, 22|22|22$ (confluence)

$a + b + c + d + e = 0 (= \frac{3}{2})$ (Fuchs condition)

	t_∞	t_0	t_y	t_1	t_x	idx
t_∞		211	22	211	22	-4
t_0	211		22	211	22	-4
t_y	22	22		22	31	2
t_1	211	211	22		22	-4
t_x	22	22	31	22		2

$$\left\{\begin{array}{cccccccccc} A_{01} & A_{02} & A_{03} & A_{04} & A_{12} & A_{13} & A_{14} & A_{23} & A_{24} & A_{34} \\ x = y & x = 0 & x = 1 & x = \infty & y = 0 & y = 1 & y = \infty & t_0 = t_1 & t_0 = \infty & t_1 = \infty \\ [0]_3 & [0]_2 & [0]_2 & [d]_2 & [0]_2 & [0]_2 & [d]_2 & [-b-c]_2 & [a+c]_2 & [a+b]_2 \\ 2a & [b]_2 & [c]_2 & [e]_2 & [b]_2 & [c]_2 & [e]_2 & 2d & 2c & 2b \\ & & & & & & & 2e & 0 & 0 \end{array}\right\}$$

$(x_0, x_1, x_2, x_3, x_4) = (t_x, t_y, t_0, t_1, t_\infty) = (x, y, 0, 1, \infty)$

$H_1 + DF : [01]$ (Dotsenko-Fateev's Eq. $1^3, 1^3, 1^3$)

$2H_2 : [02], \ [03], \ [04], \ [12], \ [13], \ [14]$

$2H_1 + H_2 : [23], \ [24], \ [34]$

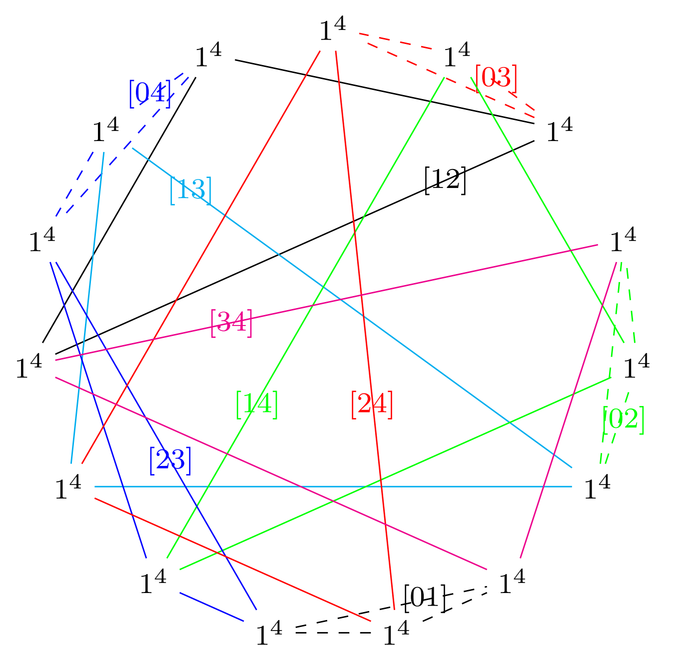
$u(x, y) = F_4\big(\alpha, \beta; \gamma, \gamma'; xy, (1-x)(1-y)\big)$ [Ka],

$b = 1 - \gamma, \ c = -\gamma', \ d = \alpha, \ e = \beta$

$\{d, \ e, \ b + d, \ b + e, c + d, \ c + e, \ b + c + d, \ b + c + e\}$

(8) $\cap \mathbb{Z} = \emptyset$ ($\Leftrightarrow$ irred.)

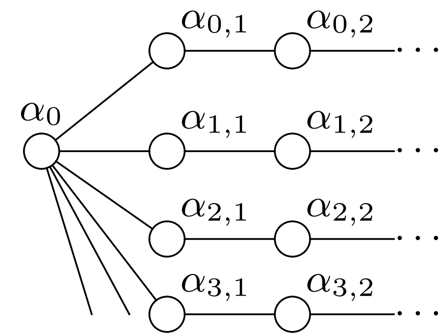
(1) $2a \in \mathbb{Z}$: $u(x, y) = F(\alpha, \beta, \gamma; x) \cdot F(\alpha, \beta, \gamma; y)$



§ A Kac-Moody root system (Fuchsian case) (cf. [CB, Of])

$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi), \quad (\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1},$$

$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \neq j \quad \text{or} \quad |\mu-\nu|>1) \\ -1 & (i=j \quad \text{and} \quad |\mu-\nu|=1) \end{cases}$$



$W := \langle s_\alpha : x \mapsto x - (\alpha|x)\alpha \mid \alpha \in \Pi \rangle$: Weyl group

Δ_+^{re}: positive real roots $\qquad \Delta_+ = \Delta_+^{re} \cup \Delta_+^{im}$

Δ_+^{im}: positive imaginary roots

$$\begin{array}{ccc} \{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, \text{(FC)}\} & \overset{\pi}{\to} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, \text{(FC)}\} \\ \downarrow u \mapsto \begin{cases} \partial^{-\mu} u \\ (x-c_j)^{\lambda_j} u \end{cases} & \circlearrowright & \downarrow \begin{cases} s_\alpha \ (\alpha \in \Pi) : \text{ reflections} \\ +\lambda_j \Lambda^0_{0,j} \end{cases} \\ \{Pu = 0 \text{ with } \{\lambda_{\mathbf{m}}\}, \text{(FC)}\} & \overset{\pi}{\to} & \{(\Lambda(\lambda), \alpha_{\mathbf{m}}) \in \mathfrak{h}^\vee \times \overline{\Delta}_+, \text{(FC)}\} \end{array}$$

$$\{\lambda_{\mathbf{m}}\} = \left\{ \begin{matrix} x = c_j \\ [\lambda_{j,\nu}]_{(m_{j,\nu})} \end{matrix} \right\}$$

$$\{(m_{j,\nu})\} \overset{\sim}{\to} \textstyle\sum_{\alpha \in \Pi} \mathbb{Z}\alpha$$

$\alpha_{\mathbf{m}} := (\operatorname{ord} \mathbf{m})\alpha_0 + \sum_{j \geq 0,\, k \geq 1} \sum_{\nu > k} m_{j,\nu} \alpha_{j,k}$ (Crawley-Boevey [CB])

$\overline{\Delta}_+ := \{\alpha \mid \alpha \in \Delta_+,\ \operatorname{supp} \alpha \ni \alpha_0\} \cup \{k\alpha \mid \alpha \in \Delta_+^{im},\ (\alpha|\alpha) < 0,\ k = 2, 3, \ldots\}$

$\Lambda(\lambda) = -\Lambda_0 - \sum_{j \geq 0} \sum_{\nu \geq 1} \big(\sum_{1 \leq i \leq \nu} \lambda_{j,i}\big) \alpha_{j,\nu} \in \mathfrak{h}^\vee := \prod_{\alpha \in \Pi} \mathbb{C}\alpha / \mathbb{C}\Lambda^0$

$\Lambda_0 := \frac{1}{2}\alpha_0 + \frac{1}{2} \sum_{j \geq 0} \sum_{\nu \geq 1} (1-\nu)\alpha_{j,\nu}$

$\Lambda_{j,\nu} := \sum_{i > \nu} (\nu - i)\alpha_{j,i} \quad (j = 0, \ldots, p,\ \nu = 0, 1, 2, \ldots)$

$\Lambda^0 := 2\Lambda_0 - 2\Lambda_{0,0}, \qquad \Lambda^0_{j,k} := \Lambda_{j,0} - \Lambda_{k,0}$

FC : $|\{\lambda_{\mathbf{m}}\}| = \big(\Lambda(\lambda) + \frac{1}{2}\alpha_{\mathbf{m}} \big| \alpha_{\mathbf{m}}\big) = 0, \quad \operatorname{idx} \mathbf{m} = (\alpha_{\mathbf{m}}|\alpha_{\mathbf{m}})$

Theorem [CB, Of]. $\{\lambda_{\mathbf{m}}\}$ with (FC) is irreducibly realizable $\Leftrightarrow \alpha_{\mathbf{m}} \in \overline{\Delta}_+$

Suppose $\mathbf{m}$ is a rigid Fuchsian spectral type

$\exists ! w_{\mathbf{m}} \in W$ such that $w_{\mathbf{m}} \alpha_{\mathbf{m}} = \alpha_0$ with the minimal length

$\Delta(\mathbf{m}) := \Delta_+^{re} \cap w_{\mathbf{m}}^{-1} \Delta_-^{re}$

$$\left\{ \begin{array}{c} x = c_j \\ [\lambda_{j,\nu}]_{(m_{j,\nu})} \end{array} \right\}$$

Theorem [CB, Of]. $Pu = 0$ is irred. $\Leftrightarrow \sum \lambda_{j,\nu} m'_{j,\nu} \notin \mathbb{Z} \quad (\forall \alpha_{\mathbf{m}'} \in \Delta(\mathbf{m}))$

Theorem [Of]. Suppose $m_{0,n_0} = m_{1,n_1} = 1$, $c_0 = 0$, $c_1 = 1$.

$$c(0 : \lambda_{0,n_0} \rightsquigarrow 1 : \lambda_{1,n_1}) = \frac{\prod_{\nu=1}^{n_0-1} \Gamma(\lambda_{0,n_0} - \lambda_{0,\nu} + 1) \cdot \prod_{\nu=1}^{n_1-1} \Gamma(\lambda_{1,\nu} - \lambda_{1,n_1})}{\prod_{\substack{\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m}) \\ m'_{0,n_0}=1 \\ m'_{1,n_1}=0}} \Gamma(|\{\lambda_{\mathbf{m}'}\}|) \cdot \prod_{\substack{\alpha_{\mathbf{m}'} \in \Delta(\mathbf{m}) \\ m'_{0,n_0}=0 \\ m'_{1,n_1}=1}} \Gamma(1 - |\{\lambda_{\mathbf{m}'}\}|) \cdot \prod_{j=2}^{p} (1 - \frac{1}{c_j})^{-L_j}}$$

Theorem [Of]. Any contiguous relation of $Pu = 0$ is constructed ($\exists$ algorithm).

Basic spectral types with $\mathrm{idx} = 0$ $((\natural), d_1(\mathbf{m}) \le 0, (\alpha_{\mathbf{m}} \mid \alpha_{\mathbf{m}}) = 0)$

$\tilde{D}_4$: $11,11,11,11 \quad 11|11,11,11 \quad 11|11,11|11 \quad 11|11|11,11 \quad 11|11|11|11$

$\tilde{E}_6$: $111,111,111 \quad 111|111,111 \quad 111|111|111$

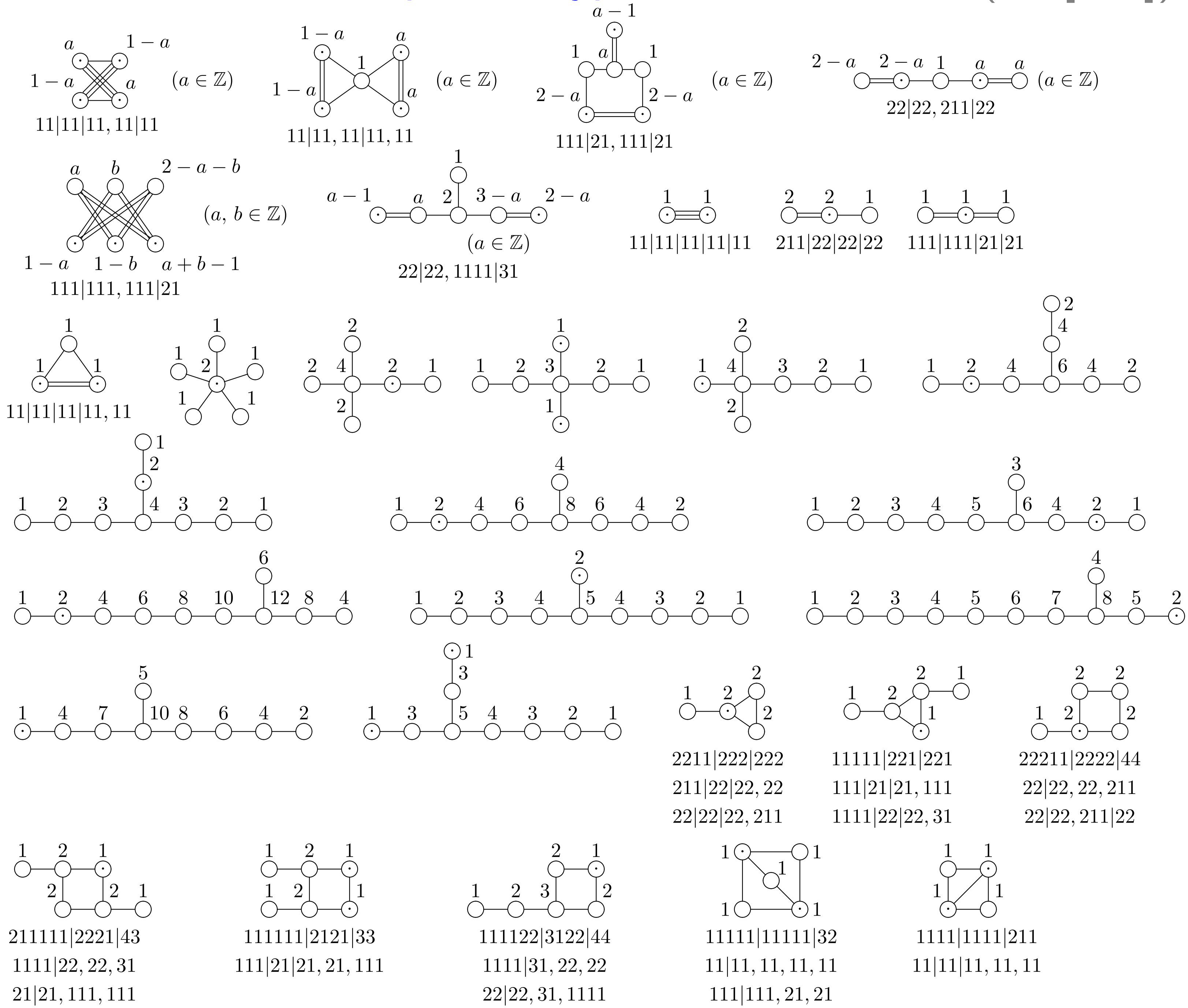
$\tilde{E}_7$: $1111,1111,22 \quad 1111|1111,22 \quad 1111,1111|22 \quad 1111|1111|22$

$\tilde{E}_8$: $111111,222,33 \quad 111111|222,33 \quad 111111|33,222$

Example with $\mathrm{idx} = 6 - 2p$: $11|11|\cdots|11$ $((p+1)$ copies of $11)$ Gauss, Heun,... [Ov]

$$\tilde{P} = \prod_{j=1}^{p} \bigl(1 - t_j x\bigr) \partial^2 + \Bigl(\sum_{j=1}^{p} \lambda_j x^{j-1}\Bigr) \partial + \mu \Bigl(\lambda_p - (-1)^p (\mu + 1) \prod_{j=1}^{p-1} t_j\Bigr) x^{p-2} + \sum_{j=0}^{p-3} r_j x^j$$

List of basic spectral types with $\mathrm{idx} = -2$ (cf. [HO])



References

[Ar] D. Arinkin, *Rigid irregular connections on* $\mathbb{P}^1$, Compos. Math., **146** (2010), 1323–1338

[CB] W. Crawley-Boevey, *On matrices in prescribed conjugacy classes with no common invariant subspaces and sum zero*, Duke Math. J. **118** (2003), 339–352

[DR] M. Dettweiler and S. Reiter, *An algorithm of Katz and its applications to the inverse Galois problems*, J. Symbolic Comput. **30** (2000), 761–798

[Ha] Y. Haraoka, *Middle convolution for completely integrable systems with logarithmic singularities along hyperplane arrangements*, Adv. Studies in Pure Math. **62** (2012), 109–136

[Hi1] K. Hiroe, *Linear differential equations on* $\mathbb{P}^1$ *and root systems*, J. Algebra **382** (2013), 1–38

[Hi2] K. Hiroe, *Linear differential equations on the Riemann sphere and representations of quivers*, Duke Mathematical Journal **166** (2017), 855–935.

[HO] K. Hiroe and T. Oshima, *A classification of roots of symmetric Kac-Moody root systems and its application*, Symmetries, Integrable Systems and Representations, Springer Proceedings in Mathematics and Statistics **40** (2012), 195–241

[Hu] M. Hukuhara, *Sur les points singuliers des équations différentielles linéaires, III*, Mem. Fac. Sci. Kyûsyû Imp. Univ. Ser. A **2** (1941), 125–137

[KO] M. Kashiwara and T. Oshima, *Systems of differential equations with regular singularities and their boundary value problems*, Ann. of Math. **106** (1977), 145—200

[Ka] M. Kato, *Connection formulas for Appell's system F4 and some applications*, Funkcial, Ekvac. **38** (1995), 243–266.

[Kz] N. M. Katz, *Rigid Local Systems*, Annals of Mathematics Studies, vol. 139, Princeton University Press, 1995

[Of] T. Oshima, Fractional calculus of Weyl algebra and Fuchsian differential equations, MSJ Memoirs **58**, Mathematical Society of Japan, 2012

[Oh] T. Oshima, *Reducibility of hypergeometric equations*, Analytic, Algebraic and Geometric Aspects of Differential Equations, Trends in Mathematics, Springer, 2017, 429–453

[Ot] T. Oshima, *Transformations of KZ type equations*, *RIMS Kôkyûroku Bessatsu* **B61**(2017), 141–161

[Os] T. Oshima, *Semilocal monodromy of rigid local systems*, Formal and Analytic Solutions of Diff. Equations, Springer Proceedings in Mathematics and Statistics 256 (2018) 189–199

[Oc] T. Oshima, *Confluence and versal unfolding of Pfaffian systems*, Josai Mathematical Monographs **19** (2019), 1–32

[Ov] T. Oshima, *Versal unfolding of irregular singularities of a linear differential equation on the Riemann sphere*, Publ. RIMS Kyoto Univ. **57** (2021), 893–920

[Or] T. Oshima, *Riemann-Liouville transform and linear differential equations on the Riemann sphere*, preprint, 34pp

[Om] T. Oshima, A library `os_muldif.rr` of Computer Algebra `Risa/Asir`
`https://www.ms.u-tokyo.ac.jp/~oshima/index.html`

Thank you for your attention!