GEOMETRY OF KNOTS AND LINKS

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We consider knots and links as a singular set of hyperbolic cone–manifolds with the three-sphere as underlying space. Then, in many important cases, the lengths of singular geodesics and cone angles are related with each other by number of relations similar to the classical Sine, Cosine and Tangent rules. In particular, the following results take a place

Theorem 1. *Let $B(\alpha,\beta,\gamma)$, $0<\alpha,\beta,\gamma<\pi$ be a hyperbolic Borromean rings cone–manifold. Denote by l_α, l_β, l_γ the lengths of singular geodesics of $B(\alpha,\beta,\gamma)$ with cone angles α, β, γ, respectively. Then we have*

$$\frac{\tan\frac{\alpha}{2}}{\tanh\frac{l_\alpha}{4}}=\frac{\tan\frac{\beta}{2}}{\tanh\frac{l_\beta}{4}}=\frac{\tan\frac{\gamma}{2}}{\tanh\frac{l_\gamma}{4}}\ \text{(The Tangent Rule)}$$

and

$$\frac{\sin\frac{\alpha}{2}}{\sinh\frac{l_\alpha}{4}}\,\frac{\sin\frac{\beta}{2}}{\sinh\frac{l_\beta}{4}}\,\frac{\cos\frac{\gamma}{2}}{\cosh\frac{l_\gamma}{4}}=1\ \text{(The Sine}-\text{Cosine Rule).}$$

Theorem 2. *Let $W(\alpha,\beta)$ be a hyperbolic Whitehead cone–manifold. Denote by γ_α, γ_β the complex lengths of singular geodesics of $W(\alpha,\beta)$ with cone angles α, β, respectively. Then we obtain*

$$\frac{\tan\frac{\alpha}{2}}{\tanh\frac{\gamma_\alpha}{4}}=\frac{\tan\frac{\beta}{2}}{\tanh\frac{\gamma_\beta}{4}}\ \text{(The Tangent Rule).}$$

Similar results as well the Sine and Cosine rules are obtained in [3] for the the twist link cone–manifolds $\frac{4k+4}{2k+1}(\alpha,\,\beta)$, $k=1,2,3,\ldots$. Follow Rolfsen denote by 6_2^2 the two bridge link with a slope 10/3.

Theorem 3. *Let $6_2^2(\alpha,\beta)$ be a hyperbolic cone–manifold. Denote by l_α, l_β the lengths of singular geodesics of $6_2^2(\alpha,\,\beta)$ with cone angles α, β, respectively. Then we have*

$$\frac{\sin\frac{\alpha}{2}}{\sinh\frac{l_\alpha}{2}}=\frac{\sin\frac{\beta}{2}}{\sinh\frac{l_\beta}{2}}\ \text{(The Sine Rule)}$$

and

$$\frac{\cos\frac{\alpha}{2}\cosh\frac{l_\beta}{2}-\cos\frac{\beta}{2}\cosh\frac{l_\alpha}{2}}{\cos\alpha-\cos\beta}=1\ \text{(The Cosine Rule).}$$

Similar results are expected to be true for any two bridge cone– manifold

$$(2k+1+\frac{1}{2k+1})(\alpha,\,\beta),\;k=1,2,3,\ldots.$$

References

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