ON KNOTS WITH TRIVIAL ALEXANDER POLYNOMIAL AND THE CYCLIC BRANCHED COVERS

YUKIHIRO TSUTSUMI

Let $K \subset S^3$ be a knot with trivial Alexander polynomial $\Delta_K(t) = 1$. Then, the r-fold cyclic covering space M_K^r over S^3 branched along K is an integral homology sphere. We show that the Casson invariant $\lambda(M_K^r)$ is divisible by $2r$ and there are integers r_0 and k such that $\lambda(M_K^r) = 2rk$ for any $r > r_0$. These restrictions are sharp in the following sense: Given mutually distinct n natural numbers $r_1, r_2, \ldots, r_n$ each greater than one ($r_i > 1$) and $n+1$ integers $k_1, k_2, \ldots, k_n, k$, there are knots K such that $\Delta_K(t) = 1$, $\lambda(M_K^{r_i}) = 2r_ik_i$ and $\lambda(M_K^r) = 2rk$ for any $r(\geq 2) \neq r_i$.

The condition that $\Delta_K(t) = 1$ is essential. It is well-known that the 5-fold cyclic branched cover $M_{3_1}^5$ for the trefoil knot 3_1 is the Poincaré homology sphere $\Sigma(2,3,5)$. Recall that $\Delta_{3_1}(t) = t - 1 + t^{-1}$ and $\lambda(\Sigma(2,3,5)) = 1$.

For the Kinoshita-Terasaka knot KT, $\lambda(M_{KT}^2) = 4$, $\lambda(M_{KT}^{\geq 3}) = 0$. We will also exhibit a sequence of knots K_m such that $K_0 = KT$, $\chi(S^3; (K_m, 0)) = \chi(S^3; (KT, 0))$ (and thus $\Delta_{K_m}(t) = 1$), $\lambda(M_{K_m}^r) = \lambda(M_{KT}^r)$. For the sequence K_m, I suspect the following: $J'_{K_m}(-1) = 48$, $J'''_{K_m}(1) = -72$, $J''''_{K_m}(1) = 720$, $J'''''_{K_m}(1) = -14400m - 9120$, where $J(t)$ denotes the Jones polynomial.

Sophia University