A note on unknotting number, II

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Abstract. In this note, we will give an approach to determine the unknotting number by a surgical view of Alexander matrix.

1. Introduction

This note is a continuation of [8], where we studied the unknotting numbers of knots and related topics, and gave a list of the unknotting numbers of prime knots up to 9 crossings. There were many unknown unknotting numbers. After that, P. Kohn has pointed out a doubt on \(u(929) = 1\), and the author could not detect it. It is still open. At present, several numbers have been determined. And we have a list of unknotting numbers of prime knots up to 10 crossings. (See [2, 3].) In this note, we will give an approach to determine the unknotting number by a surgical view of Alexander matrix, for example \(u(10_{63}) = u(10_{97}) = u(10_{105}) = u(10_{106}) = u(10_{109}) = u(10_{121}) = 2\).

2. Surgical Description of Knot

Let \(k_1, k_2\) be two knots in the 3-sphere \(S^3\). On all diagrams representing \(k_1, k_2\), the minimum number of crossing changes required to deform \(k_1\) to \(k_2\) is called the \textit{Gordian distance} between \(k_1\) and \(k_2\), denote by \(d_G(k_1, k_2)\). And the Gordian distance \(d_G(k, O)\) between \(k\) and the trivial knot \(O\) is called the \textit{unknotting number} of \(k\), denoted by \(u(k)\). A \textit{surgical description} of \(k\) is as follows ([4], [9, 10]):

Let \(k\) be a knot in \(S^3\), there are disjoint solid tori \(T_1, \ldots, T_n\) in \(S^3 - k\) and a homeomorphism \(h : S^3 - (\cup T_1 \cup \cdots \cup T_n) \to S^3\) such that

1. \(h(k)\) is unknotted in \(S^3\),
2. the \(T_i\)’s are unknotted and unlinked,
3. \(\text{lk}(T_i, k) = \text{lk}(T_i, h(k)) = 0\) for all \(i\),
4. \(h(\partial T_i) = \partial T_i\) and \(\text{lk}(h(\mu_i), T_i) = \pm 1\), where \(\mu_i\) is a meridinal curve on \(\partial T_i\).

The minimum number of these solid tori is called the \textit{surgical description number} of \(k\), denoted by \(\text{sd}(k)\). Let \(\widetilde{X}_\infty\) be the infinite cyclic covering space of the exterior of \(k\). \(H_1(\widetilde{X}_\infty)\) is considered as a \(\mathbb{Z}[t, t^{-1}]\)-module, where \(t\) is taken to be represented by the meridian of \(k\). A presentation matrix of \(H_1(\widetilde{X}_\infty)\) as a \(\mathbb{Z}[t, t^{-1}]\)-module is called an \textit{Alexander matrix} of \(k\). The minimum size among all square Alexander matrices of \(k\) is denoted by \(m(k)\), provided that \(m(k) = 0\) if an Alexander matrix is equivalent to (1) as presentation matrices. A surgical view of \(H_1(\widetilde{X}_\infty)\) ([4], [9, 10]) shows the existence of an Alexander matrix with size \(\text{sd}(k)\) as follows:

**Proposition 1.** Let \(k\) be a knot. \(k\) has an Alexander matrix \(M_k(t) = (m_{ij}(t))\) of the following form: (1) \(m_{ij}(t) = m_{ji}(t^{-1})\), (2) \(|m_{ij}(1)| = \delta_{ij}\), where the Kronecker’s
\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

It can be seen that a single crossing change is realized by surgery around the crossing. Therefore, we have the following ([8]):

**Proposition 2.** For a knot \( k \), we have \( 0 \leq m(k) \leq \text{sd}(k) \leq u(k) \).

### 3. Theorems

By a surgical view of Alexander matrix, we have the following Theorems. For a knot \( k \) with \( \text{sd}(k) = 1 \), \( k \) has an Alexander matrix of the form \((\pm \Delta_k(t))\) by a surgical view, where \( \Delta_k(t) \) is the Alexander polynomial of \( k \). Let \( 3_1 \) be a trefoil knot, and \( 4_1 \) a figure-eight knot. Here, the meaning of symbols are as follows: \( N_n \) means the notation by J.W. Alexander and G.B. Briggs in [1] and lately by D. Rolfsen in [10], where \( N \) is the minimum crossing number and \( n \) is the numbering in the \( N \)-crossing prime knots.

**Theorem 3.** There exists no knot \( k \) such that \( d_C(4_1,k) = 1 \) and \( \Delta_k(t) = t - 1 + t^{-1} \). (Therefore, \( d_C(4_1,3_1) \geq 2 \).

**Proof.** If there would exist a knot \( k \) such that \( d_C(4_1,k) = 1 \) and \( \Delta_k(t) = t - 1 + t^{-1} \), then, by a surgical view, \( k \) has an Alexander matrix of the following form:

\[ \begin{pmatrix} \pm (t - 1 + t^{-1}) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}, \]

where each entry is a Laurent polynomial with variable \( t \) such that the following conditions are satisfied: (1) \( m(t) = m(t^{-1}) \), (2) \( |m(1)| = 1, r(1) = 0 \). And the determinant should be \( m(t - 1 + t^{-1}) \).

Put \( t = -1 \), and we have the determinant

\[ \begin{vmatrix} \pm 5 & r(-1) \\ r(-1) & m(-1) \end{vmatrix} = \pm 3. \]

From the equation modulo 5, \( r(-1)^2 \equiv \pm 3 \mod 5 \). It is a contradiction. \( \Box \)

**Remark.** H. Murakami [7] showed Theorem 3 and that there exists no knot \( k \) such that \( d_C(3_1,k) = 1 \) and \( \Delta_k(t) = t - 3 + t^{-1} \), by observation on Seifert matrices. It is still unknown whether there exists a pair of knots \( k_1, k_2 \) such that \( d_C(k_1,k_2) = 1, \Delta_{k_1}(t) = t - 1 + t^{-1} \), and \( \Delta_{k_2}(t) = t - 3 + t^{-1} \).

**Theorem 4.** Let \( k \) be the knot \( 5_1 \) (or \( 7_4, 10_{83}, 10_{106}, 10_{109}, 10_{121}, \) respectively). Then, we have \( \text{sd}(k) = u(k) = 2 \).

**Proof.** Let \( k \) be the knot \( 5_1 \) (or \( 7_4, 10_{83}, 10_{106}, 10_{109}, 10_{121}, \) respectively). It is easily seen that a single crossing change deforms \( k \) to \( 3_1 \). (In the above Fig. 1, consider the crossing with the asterisk \(*\).) We would suppose that \( \text{sd}(k) = 1 \). Therefore, by a surgical view, \( 3_1 \) has an Alexander matrix of the following form:

\[ \begin{pmatrix} \pm \Delta_k(t) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix}, \]

where each entry is a Laurent polynomial with variable \( t \) such that the following conditions are satisfied: (1) \( m(t) = m(t^{-1}) \), (2) \( |m(1)| = 1, r(1) = 0 \). And the determinant should be \( m(t - 1 + t^{-1}) \).

Put \( t = -1 \), and we have the determinant

\[ \begin{vmatrix} \pm \Delta_k(-1) & r(-1) \\ r(-1) & m(-1) \end{vmatrix} = \pm 3. \]

From the equation modulo 5, \( \Delta_k(-1) \equiv 0 \mod 5 \), and we have \( r(-1)^2 \equiv \pm 3 \).
mod 5. It is a contradiction. Hence, we have $sd(k) \geq 2$ and $u(k) \geq 2$. It is easily seen that $u(k) \leq 2$. Hence, we have $sd(k) = u(k) = 2$.  

**Remarks.** $u(5_1) = 2$ is shown by his observation of K. Murasugi on the signature [6], and $u(7_4) = 2$ is shown by his observation of W.B.R. Lickorish on the linking form of the double branched cover [5]. We remark that the diagram of $10_{83}$ should be exchanged to that of $10_{86}$.

**Theorem 5.** Let $k$ be the knot $10_{97}$. Then, we have $sd(k) = u(k) = 2$.

**Proof.** Let $k$ be the knot $10_{97}$. It is easily seen that a single crossing change deforms $k$ to $3_1 \# 4_1$. (In the above Fig 2, consider the crossing with the asterisk *) We would suppose that $sd(k) = 1$. By the parallel argument to the proof of Theorem 4, we have the determinant

\[
\begin{vmatrix}
\pm \Delta_k(-1) & r(-1) \\
r(-1) & m(-1)
\end{vmatrix}
= \pm 15.
\]
From the equation modulo 29, $\Delta_k(-1) \equiv 0 \mod 29$, and we have $r(-1)^2 \equiv \pm 15 \mod 29$. It is a contradiction. Hence, we have $\text{sd}(k) \geq 2$ and $u(k) \geq 2$. It is easily seen that $u(k) \leq 2$. Hence, we have $\text{sd}(k) = u(k) = 2$. □

**Theorem 6.** Let $k$ be the knot 10_{105}. Then, we have $\text{sd}(k) = u(k) = 2$. 

![Fig. 3](image)

**Proof.** Let $k$ be the knot 10_{105}. It is easily seen that a single crossing change deforms $k$ to 41. (In the above Fig. 3, consider the crossing with the asterisk *) We would suppose that $\text{sd}(k) = 1$. By the parallel argument to the proof of Theorem 4, we have the determinant

$$
\begin{vmatrix}
\pm \Delta_k(-1) & r(-1) \\
r(-1) & m(-1)
\end{vmatrix} = \pm 5.
$$

From the equation modulo 13, $\Delta_k(-1) \equiv 0 \mod 13$, and we have $r(-1)^2 \equiv \pm 5 \mod 13$. It is a contradiction. Hence, we have $\text{sd}(k) \geq 2$ and $u(k) \geq 2$. It is easily seen that $u(k) \leq 2$. Hence, we have $\text{sd}(k) = u(k) = 2$. □

**Theorem 7.** Let $k$ be the knot 10_{117}. Then, we have $\text{sd}(k) = u(k) = 2$. 

![Fig. 4](image)

**Proof.** Let $k$ be the knot 10_{117}. It is easily seen that a single crossing change deforms $k$ to 31. (In the above Fig. 4, consider the crossing with the asterisk *) We would suppose that $\text{sd}(k) = 1$. By the parallel argument to the proof of Theorem 4, we have the determinant

$$
\begin{vmatrix}
\pm \Delta_k(-1) & r(-1) \\
r(-1) & m(-1)
\end{vmatrix} = \pm 3.
$$

From the equation modulo 101, $\Delta_k(-1) \equiv 0 \mod 101$, and we have $r(-1)^2 \equiv \pm 3 \mod 101$. It is a contradiction. Hence, we have $\text{sd}(k) \geq 2$ and $u(k) \geq 2$. It is
easily seen that \( u(k) \leq 2 \). Hence, we have \( \text{sd}(k) = u(k) = 2 \). \( \square \)

3. Example.

For an arbitrary non-negative integer \( n \), there is a knot \( k \) with \( u(k) - \text{sd}(k) = n \).

**Theorem 3.** For an arbitrary positive integer \( n \), there exists a knot \( k \) with \( \text{sd}(k) = 1, u(k) = n \).

![Fig. 5](image)

**Proof.** We consider the knot with \( n \) clasps as illustrated in Fig. 5, where the case \( n = 4 \) is shown, has the signature \( |\sigma(k)| = 2n \). By the theorem in [6]:

\[
0 \leq |\sigma(k)|/2 \leq u(k),
\]

we have \( u(k) \geq n \). It is easily seen that \( u(k) \leq n \). Then we have \( u(k) = n \). On the other hand, one right-hand-full-twist can be cancelled by surgery around a bunch of \( 2n \) strings. Therefore, we have \( \text{sd}(k) = 1 \). \( \square \)

**References**