A note on unknotting number, II

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Abstract. In this note, we will give an approach to determine the unknotting number by a surgical view of Alexander matrix.

1. Introduction

This note is a continuation of [8], where we studied the unknotting numbers of knots and related topics, and gave a list of the unknotting numbers of prime knots up to 9 crossings. There were many unknown unknotting numbers. After that, P. Kohn has pointed out a doubt on $\mathrm{u}(9_{29}) = 1$, and the author could not detect it. It is still open. At present, several numbers have been determined. And we have a list of unknotting numbers of prime knots up to 10 crossings. (See [2, 3].) In this note, we will give an approach to determine the unknotting number by a surgical view of Alexander matrix, for example $\mathrm{u}(10_{83}) = \mathrm{u}(10_{97}) = \mathrm{u}(10_{105}) = \mathrm{u}(10_{106}) = \mathrm{u}(10_{109}) = \mathrm{u}(10_{121}) = 2$.

2. Surgical Description of Knot

Let k_1, k_2 be two knots in the 3-sphere S^3. On all diagrams representing k_1, k_2, the minimum number of crossing changes required to deform k_1 to k_2 is called the *Gordian distance* between k_1 and k_2, denote by $d_G(k_1, k_2)$. And the Gordian distance $d_G(k, O)$ between k and the trivial knot O is called the *unknotting number* of k, denoted by $\mathrm{u}(k)$. A *surgical description* of k is as follows ([4], [9, 10]):

Let k be a knot in S^3, there are disjoint solid tori $T_1, \dots, T_n$ in $S^3 - k$ and a homeomorphism $h : S^3 - (°T_1 \cup \cdots \cup °T_n)$ such that

(1) $h(k)$ is unknotted in S^3,
(2) the T_i's are unknotted and unlinked,
(3) $\mathrm{lk}(T_i, k) = \mathrm{lk}(T_i, h(k)) = 0$ for all i,
(4) $h(\partial T_i) = \partial T_i$ and $\mathrm{lk}(h(\mu_i), T_i) = \pm 1$, where μ_i is a meridinal curve on ∂T_i.

The minimum number of these solid tori is called the *surgical description number* of k, denoted by $\mathrm{sd}(k)$. Let $\widetilde{X_\infty}$ be the infinite cyclic covering space of the exterior of k. $H_1(\widetilde{X_\infty})$ is considered as a $\mathbf{Z}[t, t^{-1}]$-module, where t is taken to be represented by the meridian of k. A presentation matrix of $H_1(\widetilde{X_\infty})$ as a $\mathbf{Z}[t, t^{-1}]$-module is called an *Alexander matrix* of k. The minimum size among all square Alexander matrices of k is denoted by $\mathrm{m}(k)$, provided that $\mathrm{m}(k) = 0$ if an Alexander matrix is equivalent to (1) as presentation matrices. A surgical view of $H_1(\widetilde{X_\infty})$ ([4], [9, 10]) shows the existence of an Alexander matrix with size $\mathrm{sd}(k)$ as follows:

Proposition 1. *Let k be a knot. k has an Alexander matrix $M_k(t) = (m_{ij}(t))$ of the following form:* (1) $m_{ij}(t) = m_{ji}(t^{-1})$, (2) $|m_{ij}(1)| = \delta_{ij}$, *where the Kronecker's* *delta* $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$

It can be seen that a single crossing change is realized by surgery around the crossing. Therefore, we have the following ([8]):

Proposition 2. *For a knot k, we have* $0 \leq \mathrm{m}(k) \leq \mathrm{sd}(k) \leq \mathrm{u}(k)$.

3. Theorems

By a surgical view of Alexander matrix, we have the following Theorems. For a knot k with $\mathrm{sd}(k) = 1$, k has an Alexander matrix of the form $(\pm\Delta_k(t))$ by a surgical view, where $\Delta_k(t)$ is the Alexander polynomial of k. Let 3_1 be a trefoil knot, and 4_1 a figure-eight knot. Here, the meaning of symbols are as follows: N_n means the notation by J.W. Alexander and G.B. Briggs in [1] and lately by D. Rolfsen in [10], where N is the minimum crossing number and n is the numbering in the N-crossing prime knots.

Theorem 3. *There exists no knot k such that $d_G(4_1, k) = 1$ and $\Delta_k(t) = t - 1 + t^{-1}$. (Therefore, $d_G(4_1, 3_1) \geq 2$).*

Proof. If there would exist a knot k such that $d_G(4_1, k) = 1$ and $\Delta_k(t) = t - 1 + t^{-1}$, then, by a surgical view, k has an Alexander matrix of the following form:

$$\begin{pmatrix} \pm(t - 3 + t^{-1}) & r(t^{-1}) \\ r(t) & \mathrm{m}(t) \end{pmatrix},$$

where each entry is a Laurent polynomial with variable t such that the following conditions are satisfied: (1) $\mathrm{m}(t) = \mathrm{m}(t^{-1})$, (2) $|\mathrm{m}(1)| = 1, r(1) = 0$. And the determinant should be $\mathrm{m}(t - 1 + t^{-1})$.

Put $t = -1$, and we have the determinant $\begin{vmatrix} \pm 5 & r(-1) \\ r(-1) & \mathrm{m}(-1) \end{vmatrix} = \pm 3.$

From the equation modulo 5, $r(-1)^2 \equiv \pm 3 \mod 5$. It is a contradiction. □

Remark. H. Murakami [7] showed Theorem 3 and that there exists no knot k such that $d_G(3_1, k) = 1$ and $\Delta_k(t) = t - 3 + t^{-1}$, by observation on Seifert matrices. It is still unknown whether there exists a pair of knots k_1, k_2 such that $d_G(k_1, k_2) = 1$, $\Delta_{k_1}(t) = t - 1 + t^{-1}$, and $\Delta_{k_2}(t) = t - 3 + t^{-1}$.

Theorem 4. *Let k be the knot 5_1 (or 7_4, 10_{83}, 10_{106}, 10_{109}, 10_{121}, respectively). Then, we have* $\mathrm{sd}(k) = \mathrm{u}(k) = 2$.

Proof. Let k be the knot 5_1 (or 7_4, 10_{83}, 10_{106}, 10_{109}, 10_{121}, respectively). It is easily seen that a single crossing change deforms k to 3_1. (In the above Fig. 1, consider the crossing with the asterisk $*$.) We would suppose that $\mathrm{sd}(k) = 1$. Therefore, by a surgical view, 3_1 has an Alexander matrix of the following form:

$$\begin{pmatrix} \pm\Delta_k(t) & r(t^{-1}) \\ r(t) & \mathrm{m}(t) \end{pmatrix},$$

where each entry is a Laurent polynomial with variable t such that the following conditions are satisfied: (1) $\mathrm{m}(t) = \mathrm{m}(t^{-1})$, (2) $|\mathrm{m}(1)| = 1, r(1) = 0$. And the determinant should be $\mathrm{m}(t - 1 + t^{-1})$.

Put $t = -1$, and and we have the determinant $\begin{vmatrix} \pm\Delta_k(-1) & r(-1) \\ r(-1) & \mathrm{m}(-1) \end{vmatrix} = \pm 3.$

From the equation modulo 5, $\Delta_k(-1) \equiv 0 \mod 5$, and we have $r(-1)^2 \equiv \pm 3$

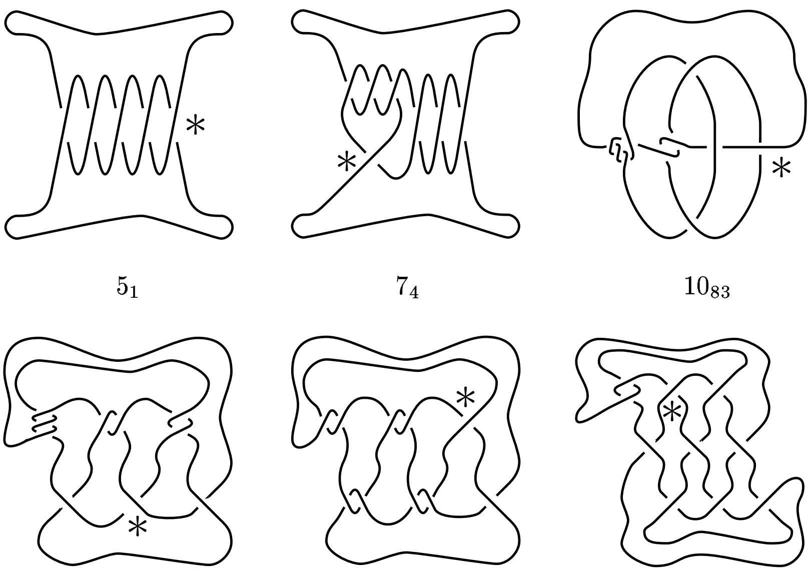
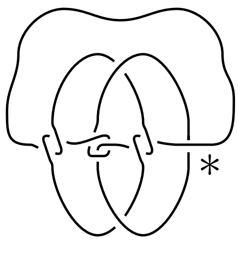


Fig. 1

mod 5. It is a contradiction. Hence, we have $\mathrm{sd}(k) \geq 2$ and $\mathrm{u}(k) \geq 2$. It is easily seen that $\mathrm{u}(k) \leq 2$. Hence, we have $\mathrm{sd}(k) = \mathrm{u}(k) = 2$. □

Remarks. $\mathrm{u}(5_1) = 2$ is shown by his observation of K. Murasugi on the signature [6], and $\mathrm{u}(7_4) = 2$ is shown by his observation of W.B.R. Lickorish on the linking form of the double branched cover [5]. We remark that the diagram of 10_{83} should be exchanged to that of 10_{86}.

Theorem 5. *Let k be the knot 10_{97}. Then, we have* $\mathrm{sd}(k) = \mathrm{u}(k) = 2$. .



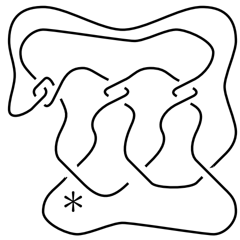
10_{97}

Fig. 2

Proof. Let k be the knot 10_{97}. It is easily seen that a single crossing change deforms k to $3_1\#4_1$. (In the above Fig. 2, consider the crossing with the asterisk $*$.) We would suppose that $\mathrm{sd}(k) = 1$. By the parallel argument to the proof of Theorem 4, we have the determinant $\begin{vmatrix} \pm\Delta_k(-1) & r(-1) \\ r(-1) & m(-1) \end{vmatrix} = \pm 15$.

From the equation modulo 29, $\Delta_k(-1) \equiv 0 \mod 29$, and we have $r(-1)^2 \equiv \pm 15 \mod 29$. It is a contradiction. Hence, we have $\mathrm{sd}(k) \geq 2$ and $\mathrm{u}(k) \geq 2$. It is easily seen that $\mathrm{u}(k) \leq 2$. Hence, we have $\mathrm{sd}(k) = \mathrm{u}(k) = 2$. □

Theorem 6. *Let k be the knot 10_{105}. Then, we have* $\mathrm{sd}(k) = \mathrm{u}(k) = 2$. .



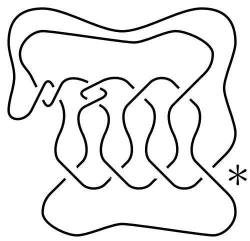
10_{105}

Fig. 3

Proof. Let k be the knot 10_{105}. It is easily seen that a single crossing change deforms k to 4_1. (In the above Fig. 3, consider the crossing with the asterisk $*$.) We would suppose that $\mathrm{sd}(k) = 1$. By the parallel argument to the proof of Theorem 4, we have the determinant $\begin{vmatrix} \pm\Delta_k(-1) & r(-1) \\ r(-1) & \mathrm{m}(-1) \end{vmatrix} = \pm 5$.

From the equation modulo 13, $\Delta_k(-1) \equiv 0 \mod 13$, and we have $r(-1)^2 \equiv \pm 5 \mod 13$. It is a contradiction. Hence, we have $\mathrm{sd}(k) \geq 2$ and $\mathrm{u}(k) \geq 2$. It is easily seen that $\mathrm{u}(k) \leq 2$. Hence, we have $\mathrm{sd}(k) = \mathrm{u}(k) = 2$. □

Theorem 7. *Let k be the knot 10_{117}. Then, we have* $\mathrm{sd}(k) = \mathrm{u}(k) = 2$. .



10_{117}

Fig. 4

Proof. Let k be the knot 10_{117}. It is easily seen that a single crossing change deforms k to 3_1. (In the above Fig. 4, consider the crossing with the asterisk $*$.) We would suppose that $\mathrm{sd}(k) = 1$. By the parallel argument to the proof of Theorem 4, we have the determinant $\begin{vmatrix} \pm\Delta_k(-1) & r(-1) \\ r(-1) & \mathrm{m}(-1) \end{vmatrix} = \pm 3$.

From the equation modulo 101, $\Delta_k(-1) \equiv 0 \mod 101$, and we have $r(-1)^2 \equiv \pm 3 \mod 101$. It is a contradiction. Hence, we have $\mathrm{sd}(k) \geq 2$ and $\mathrm{u}(k) \geq 2$. It is easily seen that $\mathrm{u}(k) \leq 2$. Hence, we have $\mathrm{sd}(k) = \mathrm{u}(k) = 2$. □

3. Example.

For an arbitrary non-negative integer n, there is a knot k with $\mathrm{u}(k) - \mathrm{sd}(k) = n$.

Theorem 3. *For an arbitrary positive integer n, there exists a knot k with $\mathrm{sd}(k) = 1, \mathrm{u}(k) = n$.*

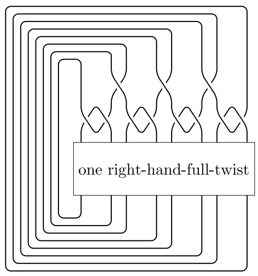


Fig. 5

Proof. We consider the knot with n clasps as illustrated in Fig. 5, where the case $n = 4$ is shown, has the signature $|\sigma(k)| = 2n$. By the theorem in [6]: $0 \leq |\sigma(k)|/2 \leq \mathrm{u}(k)$, we have $\mathrm{u}(k) \geq n$. It is easily seen that $\mathrm{u}(k) \leq n$. Then we have $\mathrm{u}(k) = n$. On the other hand, one right-hand-full-twist can be cancelled by surgery around a bunch of $2n$ strings. Therefore, we have $\mathrm{sd}(k) = 1$. □

References

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