An elementary proof of walk dimension being greater than two for Brownian motion on Sierpiński carpets

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Abstract

We give an elementary self-contained proof of the fact that the walk dimension of the Brownian motion on an arbitrary generalized Sierpiński carpet is greater than two, no complete proof of which had been available in the literature. Our proof is based solely on the self-similarity and hypercubic symmetry of the associated Dirichlet form and on several very basic pieces of the theory of regular symmetric Dirichlet forms. We also present an application of this fact to the singularity of the energy measures with respect to the symmetric measure in this case, proved first by M. Hino in [Probab. Theory Related Fields 132 (2005), no. 2, 265–290].

1 Introduction

It is an established result in the field of analysis on fractals that, on a large class of typical fractal spaces, there exists a nice diffusion process \( \{X_t\}_{t \in [0, \infty)} \) which is symmetric with respect to some canonical measure \( \mu \) and exhibits strong sub-diffusive behavior in the sense that its transition density (heat kernel) \( p_t(x, y) \) satisfies the following sub-Gaussian estimates:

\[
\frac{c_1}{\mu(B(x, t^{1/d_w}))} \exp \left( -c_2 \left( \frac{\rho(x, y)^{d_w}}{t} \right)^{\frac{1}{d_u-1}} \right) \leq p_t(x, y) \\
\leq \frac{c_3}{\mu(B(x, t^{1/d_w}))} \exp \left( -c_4 \left( \frac{\rho(x, y)^{d_w}}{t} \right)^{\frac{1}{d_u-1}} \right)
\]

(1.1)

for any points \( x, y \) and any \( t \in (0, \infty) \), where \( c_1, c_2, c_3, c_4 \in (0, \infty) \) are some constants, \( \rho \) is a natural metric on the space comparable to a complete geodesic metric, \( B(x, r) \) denotes

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the open ball of radius $r$ centered at $x$, and $d_w \in [2, \infty)$ is a characteristic of the diffusion called the walk dimension. This result was obtained first for the Sierpiński gasket in [8], then for nested fractals in [23], for affine nested fractals in [12] and for Sierpiński carpets in [1, 3, 4] (see also [2, 26, 6, 7]), and it is believed for essentially all the known examples, and has been verified for many of them, that the walk dimension $d_w$ is strictly greater than 2. Therefore (1.1) implies in particular that a typical distance the diffusion travels by time $t$ is of order $t^{1/d_w}$, which is in sharp contrast with the order $t^{1/2}$ of such a distance for the Brownian motion and uniformly elliptic diffusions on Euclidean spaces and Riemannian manifolds, where (1.1) with $d_w = 2$, the usual Gaussian estimates, are known to hold widely; see, e.g., [28, 29, 27, 14] and references therein. The sub-Gaussian estimates, (1.1) with $d_w > 2$, are also known to imply a number of other anomalous features of the diffusion, one of the most important among which is the singularity of the associated energy measures with respect to the symmetric measure $\mu$, proved recently in [21, Theorem 2.13-(a)]; see also [24, 25, 9, 15, 17] for earlier results on singularity of energy measures for diffusions on fractals.

The main concern of this paper is the verification of the strict inequality $d_w > 2$ for the Brownian motion on an arbitrary generalized Sierpiński carpet (see Figure 1 below), which constitutes the most typical examples of infinitely ramified self-similar fractals and has been intensively studied, e.g., in [1, 2, 3, 26, 4, 7, 16]. In fact, the existing proof of $d_w > 2$ for this case due to Barlow and Bass in [4, Proof of Proposition 5.1-(a)] requires a certain extra geometric assumption on the generalized Sierpiński carpet (see Remark 2.10 below), and there is no proof of it in the literature that is applicable to any generalized Sierpiński carpet although they claimed to have one in [4, Remarks 5.4-1.]. The purpose of the present paper is to give such a proof as is also elementary, self-contained and based solely on the self-similarity and hypercubic symmetry of the associated Dirichlet form (see Theorem 2.6 below) and on several very basic pieces of the theory of regular symmetric Dirichlet forms in [13, Sections 1.3, 1.4, 2.1, 2.2, 2.3, 3.1 and 4.4]. As an important consequence of $d_w > 2$, we also see that [21, Theorem 2.13-(a)] applies and yields the singularity of the energy measures with respect to the symmetric measure $\mu$ in this case.

This paper is organized as follows. In Section 2, we first introduce the framework of a generalized Sierpiński carpet and the canonical Dirichlet form on it, then give the precise statement of our main theorem on the strict inequality $d_w > 2$ (Theorem 2.9) and deduce the singularity of the energy measures (Corollary 2.12). Finally, we give our elementary self-contained proof of Theorem 2.9 in Section 3.

**Notation.** Throughout this paper, we use the following notation and conventions.

1. The symbols $\subset$ and $\supset$ for set inclusion allow the case of the equality.
2. $\mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$, i.e., $0 \not\in \mathbb{N}$.
3. The cardinality (the number of elements) of a set $A$ is denoted by $\#A$.
4. We set $a \lor b := \max\{a, b\}$, $a \land b := \min\{a, b\}$ and $a^+ := a \lor 0$ for $a, b \in [-\infty, \infty]$, and we use the same notation also for $[-\infty, \infty]$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty, \infty]$-valued.
5. Let $K$ be a non-empty set. We define $1_A = 1^K_A \in \mathbb{R}^K$ for $A \subset K$ by $1_A(x) := 1^K_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \not\in A \end{cases}$, and set $\|u\|_{\sup} := \|u\|_{\sup, K} := \sup_{x \in K} |u(x)|$ for $u : K \to [-\infty, \infty]$. 

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(6) Let $K$ be a topological space. The interior and closure of $A \subset K$ in $K$ are denoted by $\text{int}_K A$ and $\overline{A}$, respectively. We set $C(K) := \{u \mid u : K \to \mathbb{R}, u \text{ is continuous}\}$ and $\text{supp}_K[u] := K \setminus u^{-1}(0)$ for $u \in C(K)$. The Borel $\sigma$-field of $K$ is denoted by $\mathcal{B}(K)$.

(7) For $d \in \mathbb{N}$, we equip $\mathbb{R}^d$ with the Euclidean norm denoted by $\| \cdot \|$ and set $0_d := (0)^d_{k=1}$.

### 2 Framework, the main theorem and an application

We fix the following setting throughout this and the next sections.

**Framework 2.1.** Let $d, l \in \mathbb{N}$, $d \geq 2$, $l \geq 3$ and set $Q_0 := [0, 1]^d$. Let $S \subset \{0, 1, \ldots, l-1\}^d$ be non-empty, define $f_i : \mathbb{R}^d \to \mathbb{R}^d$ by $f_i(x) := l^{-1}i + l^{-1}x$ for each $i \in S$ and set $Q_1 := \bigcup_{i \in S} f_i(Q_0)$, so that $Q_1 \subsetneq Q_0$. Let $K$ be the self-similar set associated with $\{f_i\}_{i \in S}$, i.e., the unique non-empty compact subset of $\mathbb{R}^d$ such that $K = \bigcup_{i \in S} f_i(K)$, which exists and satisfies $K \subsetneq Q_0$ thanks to $Q_1 \subsetneq Q_0$ by [22, Theorem 1.1.4], and set $F_i := f_i|_K$ for each $i \in S$ and $\text{GSC}(d, l, S) := (K, S, \{F_i\}_{i \in S})$. Let $\rho : K \times K \to [0, \infty)$ be the Euclidean metric on $K$ given by $\rho(x, y) := |x - y|$, set $d_l := \log_l \#S$, and let $\mu$ be the self-similar measure on $\text{GSC}(d, l, S)$ with weight $(1/\#S)$, i.e., the unique Borel probability measure on $K$ such that $\mu = (\#S)\mu \circ F_i$ (as Borel measures on $K$) for any $i \in S$, which exists by [22, Propositions 1.5.8, 1.4.3, 1.4.4 and Corollary 1.4.8].

Recall that $d_l$ is the Hausdorff dimension of $(K, \rho)$ and that $\mu$ is a constant multiple of the $d_l$-dimensional Hausdorff measure on $(K, \rho)$; see, e.g., [22, Proposition 1.5.8 and Theorem 1.5.7]. Note that $d_l < d$ by $S \subset \{0, 1, \ldots, l-1\}^d$.

The following definition is essentially due to Barlow and Bass [4, Section 2].

**Definition 2.2** (Generalized Sierpiński carpet, [7, Subsection 2.2]). $\text{GSC}(d, l, S)$ is called a **generalized Sierpiński carpet** if and only if the following four conditions are satisfied:

(GSC1) **(Symmetry)** $f(Q_1) = Q_1$ for any isometry $f$ of $\mathbb{R}^d$ with $f(Q_0) = Q_0$.

(GSC2) **(Connectedness)** $Q_1$ is connected.

(GSC3) **(Non-diagonality)** $\text{int}_{\mathbb{R}^d}(Q_1 \cap \prod_{k=1}^d [i_k - \varepsilon_k, i_k + 1 - \varepsilon_k])$ is either empty or connected for any $(i_k)_{k=1}^d \in \mathbb{Z}^d$ and any $(\varepsilon_k)_{k=1}^d \in \{0, 1\}^d$.

(GSC4) **(Borders included)** $[0, 1] \times \{0\}^{d-1} \subset Q_1$.

![Figure 1: Sierpiński carpet, some other generalized Sierpiński carpets with $d = 2$ and Menger sponge](image)
As special cases of Definition 2.2, GSC(2, 3, SC) and GSC(3, 3, MS) are called the Sierpiński carpet and the Menger sponge, respectively, where \( S_{SC} := \{0, 1, 2\}^2 \setminus \{(1, 1)\} \) and \( S_{MS} := \{(i_1, i_2, i_3) \in \{0, 1, 2\}^3 \mid \sum_{k=1}^3 1_{\{i_k\}}(i_k) \leq 1\} \) (see Figure 1 above).

See [4, Remark 2.2] for a description of the meaning of each of the four conditions (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 2.2. We remark that there are several equivalent ways of stating the non-diagonality condition, as in the following proposition.

**Proposition 2.3** ([19, §2]). Set \( |x| := \sum_{k=1}^d |x_k| \) for \( x = (x_k)_{k=1}^d \in \mathbb{R}^d \). Then (GSC3) is equivalent to any one of the following three conditions:

1. \((\text{ND})_N\) \( \text{int}_{\mathbb{R}^d}(Q_1 \cap \prod_{k=1}^d [(i_k - 1)l^{-m}, (i_k + 1)l^{-m}]) \) is either empty or connected for any \( m \in \mathbb{N} \) and any \( (i_k)_{k=1}^d \in \{1, 2, \ldots, l^m - 1\}^d \).
2. \((\text{ND})_2\) \( \text{int}_{\mathbb{R}^d}(Q_1 \cap \prod_{k=1}^d [(i_k - 1)l^{-2}, (i_k + 1)l^{-2}]) \) is either empty or connected for any \( (i_k)_{k=1}^d \in \{1, 2, \ldots, l^2 - 1\}^d \).
3. \((\text{NDF})\) For any \( i, j \in S \) with \( f_i(Q_0) \cap f_j(Q_0) \neq \emptyset \), there exists \( n(k)|_{k=0}^i \subseteq S \) such that \( n(0) = i, n(i - j|_1) = j \) and \( n(k) - n(k + 1)|_{k=1}^i = 1 \) for any \( k \in \{0, \ldots, |i - j| - 1\} \).

**Remark 2.4.** (1) Only the case of \( m = 1 \) of \((\text{ND})_N\) had been assumed in the original definition of generalized Sierpiński carpets in [4, Section 2], but Barlow, Bass, Kuma-gai and Teplyaev [7] later realized that it had been too weak for [4, Proof of Theorem 3.19] and had to be replaced by \((\text{ND})_N\) (or equivalently, by (GSC3)).

(2) In fact, [7, Subsection 2.2] assumes instead of (GSC2) the seemingly stronger condition that \( \text{int}_{\mathbb{R}^d} Q_1 \) is connected, but it is implied by (GSC2) and (GSC3) in view of (NDF) in Proposition 2.3 and is thus equivalent to (GSC2) under the assumption of (GSC3).

Throughout the rest of this paper, we assume that GSC\((d, l, S) = (K, S, \{F_i\}_{i \in S})\) is a generalized Sierpiński carpet.

We next recall some basics of the canonical (regular symmetric) Dirichlet form on GSC\((d, l, S)\); we refer to [13, 10] for details of the theory of regular symmetric Dirichlet forms. There are two established ways of constructing a non-degenerate \( \mu \)-symmetric diffusion on \( K \), or equivalently, a non-zero conservative local regular symmetric Dirichlet form on \( L^2(K, \mu) \), one by Barlow and Bass [1, 4] using the reflecting Brownian motions on the domains approximating \( K \), and the other by Kusuoka and Zhou [26] based on graph approximations. It had been a long-standing open problem to prove that the constructions in [1, 4] and in [26] give rise to the same diffusion on \( K \), which Barlow, Bass, Kumagai and Teplyaev [7] have finally solved by proving the uniqueness of a non-zero conservative regular symmetric Dirichlet form on \( L^2(K, \mu) \) possessing certain local symmetry. As a consequence of the results in [7], after some additional arguments in [16, 20] we have the following unique existence of a canonical self-similar Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \( L^2(K, \mu) \).

**Definition 2.5.** We define

\[
\mathcal{G}_0 := \{ f|_K \mid f \text{ is an isometry of } \mathbb{R}^d, f(Q_0) = Q_0 \},
\]

which forms a finite subgroup of the group of homeomorphisms of \( K \) by virtue of (GSC1).
Theorem 2.6 ([7, Theorems 1.2 and 4.32], [16, Proposition 5.1], [20, Proposition 5.9]). There exists a unique (up to constant multiples of $E$) conservative regular symmetric Dirichlet form $(E, F)$ on $L^2(K, \mu)$ satisfying $E(u, u) > 0$ for some $u \in F$ and the following:

(GSCDF1) If $u \in F \cap C(K)$ and $g \in G_0$ then $u \circ g \in F$ and $E(u \circ g, u \circ g) = E(u, u)$.

(GSCDF2) $F \cap C(K) = \{u \in C(K) \mid u \circ f_i \in F$ for any $i \in S\}.

(GSCDF3) There exists $r \in (0, \infty)$ such that for any $u \in F \cap C(K)$,

$$E(u, u) = \sum_{i \in S} \frac{1}{r} E(u \circ F_i, u \circ F_i).$$

(2.2)

Definition 2.7. The regular symmetric Dirichlet form $(E, F)$ on $L^2(K, \mu)$ as in Theorem 2.6 is called the canonical Dirichlet form on GSC$(d, l, S)$, and the walk dimension $d_w$ of $(E, F)$ (or of GSC$(d, l, S)$) is defined by $d_w := \log_2(\#S/r)$.

Remark 2.8. The walk dimension $d_w$ as defined in Definition 2.7 coincides with the exponent $d_w$ in (1.1) for the Dirichlet space $(K, \mu, E, F)$ equipped with the Euclidean metric $\rho$; see the proof of Corollary 2.12 below and the references therein for details.

The main result of this paper is an elementary self-contained proof of the following theorem based solely on our standing assumption that $S \neq \{0, 1, \ldots, l-1\}^d$, on the properties of $(E, F)$ stated in Theorem 2.6 except its uniqueness and on several very basic pieces of the theory of regular symmetric Dirichlet forms in [13, Sections 1.3, 1.4, 2.1, 2.2, 2.3, 3.1 and 4.4]. To keep the proof as elementary and self-contained as possible, we refrain from using any known properties of $(E, F)$ other than those in Theorem 2.6.

Theorem 2.9. $d_w > 2$.

Remark 2.10. The existing proof of Theorem 2.9 due to Barlow and Bass in [4, Proof of Proposition 5.1-(a)] requires the extra assumption on GSC$(d, l, S)$ that

$$\#\{(i_k)_{k=1}^d \in S \mid i_1 = j\} \neq \#\{(i_k)_{k=1}^d \in S \mid i_1 = 0\} \quad \text{for some } j \in \{1, \ldots, l-1\},$$

(2.3)

which holds for any generalized Sierpiński carpet with $d = 2$ but does fail for infinitely many examples of generalized Sierpiński carpets with fixed $d$ for each $d \geq 3$; indeed, for each $d, l \in \mathbb{N}$ with $d \geq 3$ and $l \geq 2$, it is not difficult to see that GSC$(d, 2ld, S_{dl})$ with

$$S_{dl} := \left\{ i \mid \begin{array}{l} i = (i_k)_{k=1}^d \in \{0, 1, \ldots, 2ld-1\}^d, \text{and for any } j \in \{1, 3, \ldots, 2l-1\}, \\
\{2i_k - 2ld + 1 \mid k \in \{1, 2, \ldots, d\}\} \neq \{j, j+2l, \ldots, j+2l(d-1)\} \end{array} \right\}$$

(2.4)

satisfies (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 2.2 but not (2.3).

The proof of Theorem 2.9 is given in the next section. We conclude this section by presenting an application of Theorem 2.9 to the singularity with respect to $\mu$ of the energy measures associated with $(K, \mu, E, F)$, which was proved first by Hino in [15, Subsection 5.2] via $d_w > 2$ and is obtained here by combining [21, Theorem 2.13-(a)] with $d_w > 2$. 

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**Definition 2.11** (Cf. [13, (3.2.13), (3.2.14) and (3.2.15)]). The energy measure $\mu(u)$ of $u \in \mathcal{F}$ associated with $(\mathcal{K}, \mu, \mathcal{E}, \mathcal{F})$ is defined, first for $u \in \mathcal{F} \cap L^\infty(\mathcal{K}, \mu)$ as the unique $([0, \infty]-valued)$ Borel measure on $\mathcal{K}$ such that
\[
\int_{\mathcal{K}} v \, d\mu(u) = \mathcal{E}(uv, u) - \frac{1}{2} \mathcal{E}(v, v^2)
\]
and then by $\mu(u)(A) := \lim_{n \to \infty} \mu((-n) \vee (u \wedge n))(A)$ for each $A \in \mathcal{B}(\mathcal{K})$ for general $u \in \mathcal{F}$; note that $uv \in \mathcal{F}$ for any $u, v \in \mathcal{F} \cap L^\infty(\mathcal{K}, \mu)$ by [13, Theorem 1.4.2-(ii)] and that $\{(n) \vee (u \wedge n)\}^\infty_{n=1} \subset \mathcal{F}$ and $\lim_{n \to \infty} \mathcal{E}(u - (n) \vee (u \wedge n), u - (n) \vee (u \wedge n)) = 0$ by [13, Theorem 1.4.2-(iii)].

**Corollary 2.12.** $\mu(u)$ is singular with respect to $\mu$ for any $u \in \mathcal{F}$.

**Proof.** $(\mathcal{E}, \mathcal{F})$ is local by [18, Lemma 3.4], whose proof is based only on (GSCDF2), (GSCDF3) and [13, Exercise 1.4.1 and Theorem 3.1.2], and is therefore strongly local by its conservativeness (see also Lemma 3.4 below). We easily see that $c_5 r^{dc} \leq \mu(B(x, r)) \leq c_6 r^{dc}$ for any $(x, r) \in \mathcal{K} \times (0, d]$ for some $c_5, c_6 \in (0, \infty)$, where $B(x, r) := \{y \in \mathcal{K} \mid \rho(x, y) < r\}$. It is also immediate that $(\mathcal{K}, \rho)$ satisfies the chain condition as defined in [21, Definition 2.10-(a)], in view of the fact that by (GSC4), (GSC1) and (GSC2) there exists $c_7 \in (0, \infty)$ such that for any $x, y \in \mathcal{K}$ there exists a continuous map $\gamma : [0, 1] \to \mathcal{K}$ with $\gamma(0) = x$ and $\gamma(1) = y$ whose Euclidean length is at most $c_7 \rho(x, y)$. Finally, by [7, Theorem 4.30 and Remark 4.33] (see also [4, Theorem 1.3]) the heat kernel $p_t(x, y)$ of $(\mathcal{K}, \mu, \mathcal{E}, \mathcal{F})$ exists and there exist $\beta_0 \in (1, \infty)$ and $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that (1.1) with $\beta_0$ in place of $d_w$ holds for $\mu$-a.e. $x, y \in \mathcal{K}$ for each $t \in (0, \infty)$, but then necessarily $\beta_0 = \log_4(\#S/r) = d_w$ as proved in [20, Proposition 5.9], whence $\beta_0 = d_w = 2$ by Theorem 2.9. Thus $(\mathcal{K}, \rho, \mu, \mathcal{E}, \mathcal{F})$ satisfies all the assumptions of [21, Theorem 2.13-(a)], which implies the desired claim. \(\square\)

## 3 The elementary proof of the main theorem

This section is devoted to giving our elementary self-contained proof of the main theorem (Theorem 2.9), which is an adaptation of, and has been inspired by, an elementary proof of the counterpart of Theorem 2.9 for Sierpiński gaskets presented in [21, Proof of Proposition 5.3, Second paragraph]. We start with standard definitions and some simple lemmas.

**Definition 3.1.** We set $W_m := S^m = \{w_1 \ldots w_m \mid w_i \in S \text{ for } i \in \{1, \ldots, m\}\}$ for $m \in \mathbb{N}$ and $W_* := \bigcup_{m=1}^\infty W_m$. For each $w = w_1 \ldots w_m \in W_*$, the unique $m \in \mathbb{N}$ with $w \in W_m$ is denoted by $|w|$ and we set $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$, $K_w := F_w(\mathcal{K})$ and $q^w := (q^w_k)_{k=1}^\infty := F_w(0_d)$.

**Lemma 3.2.** Let $w, v \in W_*$ satisfy $|w| = |v|$ and $w \neq v$. Then $\mu(K_w \cap K_v) = 0$.

**Proof.** This follows easily from the fact that $K_w \cap K_v = F_w(\mathcal{K} \setminus (0, 1)^d) \cap F_v(\mathcal{K} \setminus (0, 1)^d)$. \(\square\)

**Lemma 3.3.** Let $w \in W_*$. Then $\int_{\mathcal{K}} |u \circ F_w| \, d\mu = (\#S)^{|w|} \int_{K_w} |u| \, d\mu$ and $\int_{K_w} |u \circ F^{-1}_w| \, d\mu = (\#S)^{-|w|} \int_{\mathcal{K}} |u| \, d\mu$ for any Borel measurable function $u : \mathcal{K} \to [-\infty, \infty]$. In particular, bounded linear operators $F_w^* : L^2(\mathcal{K}, \mu) \to L^2(\mathcal{K}, \mu)$ can be defined by setting $F_w^* u := u \circ F_w$ and $(F_w)_* u := \begin{cases} u \circ F^{-1}_w & \text{on } K_w, \\ 0 & \text{on } K \setminus K_w \end{cases}$.
for each \( u \in L^2(K, \mu) \). Moreover, \( u \circ F_w \in \mathcal{F} \) and (2.2) holds for any \( u \in \mathcal{F} \).

**Proof.** The former assertions are immediate from \( \mu = (\#S)^{|w|} \mu \circ F_w \). For the latter, let \( u \in \mathcal{F} \). Since \( \mathcal{F} \cap \mathcal{C}(K) \) is dense in the Hilbert space \((\mathcal{F}, \mathcal{E}_1) := \mathcal{E} + (\cdot, \cdot)_{L^2(K, \mu)}\) by the regularity of \((\mathcal{E}, \mathcal{F})\), we can choose \( \{u_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap \mathcal{C}(K) \) so that \( \lim_{n \to \infty} \mathcal{E}_1(u - u_n, u - u_n) = 0 \), and then \( \{u_n \circ F_w\}_{n=1}^{\infty} \) is a Cauchy sequence in \((\mathcal{F}, \mathcal{E}_1)\) with \( \lim_{n \to \infty} \|u \circ F_w - u_n \circ F_w\|_{L^2(K, \mu)} = 0 \) by (GSCDF2) and (GSCDF3) and therefore has to converge to \( u \circ F_w \) in norm in \((\mathcal{F}, \mathcal{E}_1)\). Thus \( u \circ F_w \in \mathcal{F} \), and (2.2) for \( u \) follows by letting \( n \to \infty \) in (2.2) for \( u_n \in \mathcal{F} \cap \mathcal{C}(K) \).

**Lemma 3.4.** \( \mathbb{1}_K \in \mathcal{F} \) and \( \mathcal{E}(\mathbb{1}_K, v) = 0 \) for any \( v \in \mathcal{F} \).

**Proof.** This is immediate from the conservativeness of \((\mathcal{E}, \mathcal{F})\), \( \mathbb{1}_K \in L^2(K, \mu) \) and [13, Lemma 1.3.4-(i)]. \( \square \)

**Definition 3.5.** Let \( U \) be a non-empty open subset of \( K \). We define \( \mu|_U := \mu|_{\mathcal{B}(U)} \),

\[
\mathcal{C}_U := \{ u \in \mathcal{F} \cap \mathcal{C}(K) \mid \text{supp}_K[u] \subset U \}, \quad \mathcal{F}_U := \overline{\mathcal{C}_U}^\mathcal{F} \quad \text{and} \quad \mathcal{E}^U := \mathcal{E}|_{\mathcal{F}_U \times \mathcal{F}_U}, \tag{3.2}
\]

where \( \mathcal{F} \) is equipped with the inner product \( \mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_{L^2(K, \mu)} \), and call \((\mathcal{E}^U, \mathcal{F}_U)\) the part of \((\mathcal{E}, \mathcal{F})\) on \( U \). Since \( u = 0 \) \( \mu \)-a.e. on \( K \setminus U \) for any \( u \in \mathcal{F}_U \), we can consider \( \mathcal{F}_U \) as a linear subspace of \( L^2(U, \mu|_U) \) through the linear injection \( \mathcal{F}_U \ni u \mapsto u|_U \in L^2(U, \mu|_U) \), and then \((\mathcal{E}^U, \mathcal{F}_U)\) is a regular symmetric Dirichlet form on \( L^2(U, \mu|_U) \) by [13, Lemma 1.4.2-(ii) and Theorem 3.1.1]. Moreover, by [13, Corollary 2.3.1] we also have

\[
\mathcal{F}_U = \{ u \in \mathcal{F} \mid \widetilde{u} = 0 \ \mathcal{E}\text{-q.e. on } K \setminus U \}, \tag{3.3}
\]

where “\( \mathcal{E}\text{-q.e.} \)” means “outside a set of capacity zero with respect to \((K, \mu, \mathcal{E}, \mathcal{F})\)” and \( \widetilde{u} \) denotes any \( \mu \)-version of \( u \in \mathcal{F} \) quasi-continuous with respect to \((K, \mu, \mathcal{E}, \mathcal{F})\), which exists by [13, Theorem 2.1.3] and is unique \( \mathcal{E}\text{-q.e.} \) by [13, Lemma 2.1.4]; see [13, Section 2.1] for the definitions of these notions with respect to a regular symmetric Dirichlet space.

**Definition 3.6.** Let \( U \) be a non-empty open subset of \( K \). Then \( h \in \mathcal{F} \) is said to be \( \mathcal{E}\text{-harmonic} \) on \( K \setminus U \) if and only if either of the following two conditions, which are easily seen from (3.3) to be equivalent to each other, holds:

\[
\mathcal{E}(h, h) = \inf \{ \mathcal{E}(u, u) \mid u \in \mathcal{F}, \widetilde{u} = \tilde{h} \ \mathcal{E}\text{-q.e. on } K \setminus U \}, \tag{3.4}
\]

\[
\mathcal{E}(h, v) = 0 \quad \text{for any } v \in \mathcal{C}_U, \quad \text{or equivalently, for any } v \in \mathcal{F}_U. \tag{3.5}
\]

**Definition 3.7.** (1) We set \( V^d_{\mathbb{0}} := K \cap \{ \varepsilon \} \times \mathbb{R}^{d-1} \) for each \( \varepsilon \in \{0, 1\} \).

(2) We define \( g_{\varepsilon} \in \mathcal{G}_0 \) by \( g_{\varepsilon} := \tau_{\varepsilon}|_K \) for each \( \varepsilon = (\varepsilon_k)_{k=1}^d \in \{ 0, 1 \}^d \), where \( \tau_{\varepsilon} : \mathbb{R}^d \to \mathbb{R}^d \) is given by \( \tau_{\varepsilon}(x) := (\varepsilon_k + (1 - 2\varepsilon_k)x_k)_{k=1}^d \), and define a subgroup \( \mathcal{G}_1 \) of \( \mathcal{G}_0 \) by

\[
\mathcal{G}_1 := \{ g_{\varepsilon} \mid \varepsilon \in \{ 0 \} \times \{ 0, 1 \}^{d-1} \}. \tag{3.6}
\]

Now we proceed to the core part of the proof of Theorem 2.9. It is divided into three propositions, proving respectively the existence of a good sequence \( \{u_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap \mathcal{C}(K) \) converging in norm in \((\mathcal{F}, \mathcal{E}_1)\) to an \( \mathcal{E}\text{-harmonic function } h_0 \in \mathcal{F} \) on \( K \setminus (V^d_{\mathbb{0}} \cup V^d_{\mathbb{1}}) \) with
\( \tilde{h}_0 = \mathbb{1}_{V_0^1} \mathcal{E}\text{-q.e. on } V_0^0 \cup V_0^1 \) (Proposition 3.8), \( \mathcal{E}(h_0, h_0) > 0 \) (Proposition 3.11) and the non-\( \mathcal{E}\)-harmonicity on \( K \setminus (V_0^0 \cup V_0^1) \) of \( h_2 := \sum_{v \in V_2} (F_v)_* (l^{-2} h_0 + q_0^v \mathbb{1}_K) \) (Proposition 3.12). Then Theorem 2.9 will follow from \( \mathcal{E}(h_0, h_0) < \mathcal{E} (h_2, h_2) \) and (2.2) for \( u \in \mathcal{F} \). While the existence of such \( h_0 \) is implied by [13, Exercise 1.4.1, Theorems 4.6.5, A.2.6-(i), 4.2.1-(ii) and 1.5.2-(iii)], that of \( \{u_n\}_{n=1}^\infty \subset \mathcal{F} \cap C(K) \) as in the following proposition cannot be obtained directly from the theory of regular symmetric Dirichlet forms in [13, 10].

**Proposition 3.8.** There exist \( h_0 \in \mathcal{F} \) and \( \{u_n\}_{n=1}^\infty \subset \mathcal{F} \cap C(K) \) satisfying the following:

1. \( h_0 \) is \( \mathcal{E}\)-harmonic on \( K \setminus (V_0^0 \cup V_0^1) \) and \( \tilde{h}_0 = \mathbb{1}_{V_0^1} \mathcal{E}\text{-q.e. on } V_0^0 \cup V_0^1 \).
2. For each \( n \in \mathbb{N} \), \( u_n \circ g = u_n \) for any \( g \in \mathcal{G}_1 \) and \( u_n|_{V_0^0 \cup V_0^1} = \mathbb{1}_{V_0^1} \).
3. \( \lim_{n \to \infty} \mathcal{E}_1(h_0 - u_n, h_0 - u_n) = 0 \).

**Proof.** Noting that \( \{u \in \mathcal{F} \cap C(K) \mid u|_{V_0^0 \cup V_0^1} = \mathbb{1}_{V_0^1} \} \neq \emptyset \) by [13, Exercise 1.4.1], we set

\[
a_\alpha := \inf \{ \mathcal{E}(u, u) + \alpha \| u \|^2_{L^2(K, \mu)} \mid u \in \mathcal{F}, \; u = \mathbb{1}_{V_0^1} \mathcal{E}\text{-q.e. on } V_0^0 \cup V_0^1, \; \alpha \in [0, \infty) \}. \tag{3.7}
\]

Then for each \( \alpha \in [0, \infty) \), for any \( u \in \mathcal{F} \) with \( \tilde{u} = \mathbb{1}_{V_0^1} \mathcal{E}\text{-q.e. on } V_0^0 \cup V_0^1 \) we have

\[
\mathcal{E}(u, u) \geq \mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \geq \mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) + \alpha \| u^+ \wedge 1 \|^2_{L^2(K, \mu)} - \alpha \geq a_\alpha - \alpha
\]

and hence \( a_0 \geq a_\alpha - \alpha \). Also since \( a_\alpha^{-1} a_\alpha \) is the capacity of \( V_0^1 \) with respect to the regular symmetric Dirichlet space \( (K \setminus V_0^0, \mu|_{K \setminus V_0^0}, a_\alpha^{-1} \mathcal{E}|_{K \setminus V_0^0}, \mathcal{F}|_{K \setminus V_0^0}) \) for any \( \alpha \in (0, \infty) \) by [13, Theorem 4.4.3-(ii) and Theorem 2.1.5-(i)], it follows from [13, Exercise 1.4.1 and Lemma 2.2.7-(ii)] that for each \( n \in \mathbb{N} \) we can choose \( v_n \in \mathcal{F} \cap C(K) \) with \( v_n|_{V_0^0 \cup V_0^1} = \mathbb{1}_{V_0^1} \) so that

\[
\mathcal{E}(v_n^+ \wedge 1, v_n^+ \wedge 1) \leq \mathcal{E}(v_n, v_n) + n^{-1} \| v_n \|^2_{L^2(K, \mu)} < a_{n-1} + n^{-1} \leq a_0 + 2n^{-1}. \tag{3.8}
\]

Recalling (GSCDF1), now for each \( n \in \mathbb{N} \) we can define \( u_n \in \mathcal{F} \cap C(K) \) with the properties in (2) by \( u_n := (\# \mathcal{G}_1)^{-1} \sum_{g \in \mathcal{G}_1} (v_n^+ \wedge 1) \circ g \) and see from the triangle inequality for \( \mathcal{F} \ni u \mapsto \mathcal{E}(u, u)^{1/2}, \mathcal{E}((v_n^+ \wedge 1) \circ g, (v_n^+ \wedge 1) \circ g) = \mathcal{E}(v_n^+ \wedge 1, v_n^+ \wedge 1) \) for \( g \in \mathcal{G}_1 \) and (3.8) that

\[
\mathcal{E}(u_n, u_n) \leq \mathcal{E}(v_n^+ \wedge 1, v_n^+ \wedge 1) < a_0 + 2n^{-1}. \tag{3.9}
\]

Finally, since \( \| u_n \|^2_{L^2(K, \mu)} \leq 1 \) by \( 0 \leq u_n \leq 1 \) for any \( n \in \mathbb{N} \), the Banach–Saks theorem [10, Theorem A.4.1-(i)] yields \( h_0 \in L^2(K, \mu) \) and a strictly increasing sequence \( \{j_k\}_{k=1}^\infty \subset \mathbb{N} \) such that the Cesàro mean sequence \( \{\overline{u}_n\}_{n=1}^\infty \subset \mathcal{F} \cap C(K) \) given by \( \overline{u}_n := n^{-1} \sum_{k=1}^n u_{j_k} \) satisfies \( \lim_{n \to \infty} \| h_0 - \overline{u}_n \|^2_{L^2(K, \mu)} = 0 \). Then (2) obviously holds for \( \{\overline{u}_n\}_{n=1}^\infty \), and it follows from (3.7) and (3.9) that for any \( n, k \in \mathbb{N} \),

\[
\mathcal{E}(\overline{u}_n - \overline{u}_k, \overline{u}_n - \overline{u}_k) = 2\mathcal{E}(\overline{u}_n, \overline{u}_n) + 2\mathcal{E}(\overline{u}_k, \overline{u}_k) - 4\mathcal{E}(\overline{u}_n + \overline{u}_k)/2, (\overline{u}_n + \overline{u}_k)/2) \leq 2\mathcal{E}(\overline{u}_n, \overline{u}_n) + 2\mathcal{E}(\overline{u}_k, \overline{u}_k) - 4a_0 \to 0, \quad n, k \to \infty,
\]

which together with \( \lim_{n \to \infty} \| h_0 - \overline{u}_n \|^2_{L^2(K, \mu)} = 0 \) and the completeness of \( (\mathcal{F}, \mathcal{E}_1) \) implies that \( h_0 \in \mathcal{F} \) and \( \lim_{n \to \infty} \mathcal{E}_1(h_0 - \overline{u}_n, h_0 - \overline{u}_n) = 0 \). Thus \( \mathcal{E}(h_0, h_0) = \lim_{n \to \infty} \mathcal{E}(\overline{u}_n, \overline{u}_n) = a_0, \quad h_0 = \mathbb{1}_{V_0^1} \mathcal{E}\text{-q.e. on } V_0^0 \cup V_0^1 \) by [13, Theorem 2.1.4-(i)], and therefore \( h_0 \) is \( \mathcal{E}\)-harmonic on \( K \setminus (V_0^0 \cup V_0^1) \) in view of (3.7) and (3.4), completing the proof. \( \square \)
We need the following two lemmas for the remaining two propositions and their proofs.

**Lemma 3.9.** Let \( h_0 \in \mathcal{F} \) be as in Proposition 3.8, \( m \in \mathbb{N} \) and define \( h_m \in L^2(K, \mu) \) by

\[
h_m := \sum_{w \in W_m} (F_w)_+ (l^{-m} h_0 + q^w_1 \mathbf{1}_K). \tag{3.10}
\]

Then \( h_m \in \mathcal{F} \) and \( \tilde{h}_m = \mathbf{1}_{V_0^1} \mathcal{E} \)-q.e. on \( V_0^0 \cup V_0^1 \).

**Proof.** Let \( \{u_n\}_{n=1}^\infty \subset \mathcal{F} \cap C(K) \) be as in Proposition 3.8. For each \( n \in \mathbb{N} \), by \( u_n \circ g = u_n \) for \( g \in G_1 \) and \( u_n|_{V_0^0 \cup V_0^1} = \mathbf{1}_{V_0^1} \) from Lemma 3.2-(2) we can define \( u_{m,n} \in C(K) \) by setting \( u_{m,n}|_K := (l^{-m}u_n + q^w_1 \mathbf{1}_K) \circ F_w^{-1} \) for each \( w \in W_m \), so that \( u_{m,n} \circ F_w = l^{-m}u_n + q^w_1 \mathbf{1}_K \) \( \in \mathcal{F} \) by Lemma 3.4 and hence \( u_{m,n} \in \mathcal{F} \) by (GSCDF2). Then we see from (GSCDF3), Lemmas 3.2, 3.3 and Proposition 3.8-(3) that \( \{u_{m,n}\}_{n=1}^\infty \) is a Cauchy sequence in the Hilbert space \((\mathcal{F}, \mathcal{E}_1)\) with \( \lim_{n \to \infty} h_m - u_{m,n} \|_{L^2(K, \mu)} = 0 \) and therefore has to converge to \( h_m \) in norm in \((\mathcal{F}, \mathcal{E}_1)\), whence \( h_m \in \mathcal{F} \) and \( \tilde{h}_m = \mathbf{1}_{V_0^1} \mathcal{E} \)-q.e. on \( V_0^0 \cup V_0^1 \) by [13, Theorem 2.1.4-(i)]. \( \square \)

**Lemma 3.10.** Let \( k \in \{1, 2, \ldots, d\} \) and define \( f_k \in C(\mathbb{R}^d) \) by \( f_k((x_j)_{j=1}^d) := x_k \). Then either \( f_k|_K \in \mathcal{F} \) and \( \mathcal{E}(f_k|_K, f_k|_K) = 0 \) or \( f_k|_K \notin \mathcal{F} \).

**Proof.** Suppose to the contrary that \( f_k|_K \in \mathcal{F} \) and \( \mathcal{E}(f_k|_K, f_k|_K) = 0 \). Then \( f|_K \in \mathcal{F} \) and \( \mathcal{E}(f|_K, f|_K) = 0 \) for any polynomial \( f \in \{1_{\mathbb{R}^d}, f_1, f_2, \ldots, f_d\} \) by Lemma 3.4 and (GSCDF1), and hence also for any polynomial \( f \in C(\mathbb{R}^d) \) since for any \( u, v \in \mathcal{F} \cap L^\infty(K, \mu) \) we have \( u \in \mathcal{F} \) and \( \mathcal{E}(uv, uv)^{1/2} \leq \|u\|_{L^\infty(K, \mu)} \mathcal{E}(u, v)^{1/2} + \|v\|_{L^\infty(K, \mu)} \mathcal{E}(u, v)^{1/2} \) by [13, Theorem 1.4.2-(ii)]. On the other hand, \( \mathcal{E}(u, u) > 0 \) for some \( u \in \mathcal{F} \cap C(K) \) by the existence of such \( u \in \mathcal{F} \) and the denseness of \( \mathcal{F} \cap C(K) \) in \((\mathcal{F}, \mathcal{E}_1)\), the Stone–Weierstrass theorem [11, Theorem 2.4.11] implies that \( \lim_{n \to \infty} \|u - f_n|_K\|_{\sup} = 0 \) for some sequence \( \{f_n\}_{n=1}^\infty \subset C(\mathbb{R}^d) \) of polynomials, then \( \lim_{n \to \infty} \|u - f_n|_K\|_{L^2(K, \mu)} = 0 \) and \( \lim_{n \to \infty} \mathcal{E}_1(f_n|_K - f_k|_K, f_n|_K - f_k|_K) = 0 \). Thus \( \lim_{n \to \infty} \mathcal{E}_1(f_n|_K - f_k|_K, u - f_n|_K) = 0 \) by the completeness of \((\mathcal{F}, \mathcal{E}_1)\) and therefore \( \mathcal{E}(u, u) = \lim_{n \to \infty} \mathcal{E}_1(f_n|_K, f_n|_K) = 0 \), which is a contradiction and completes the proof. \( \square \)

**Proposition 3.11.** Let \( h_0 \in \mathcal{F} \) be as in Proposition 3.8. Then \( \mathcal{E}(h_0, h_0) > 0 \).

**Proof.** Let \( f_1 \in C(\mathbb{R}^d) \) be as in Lemma 3.10 with \( k = 1 \) and for each \( m \in \mathbb{N} \) let \( h_m \in \mathcal{F} \) be as in Lemma 3.9, so that by (3.10), Lemmas 3.2 and 3.3 we have \( \|f_1|_K - h_m\|_{L^2(K, \mu)} \leq l^{-m}(1 + \|h_0\|_{L^2(K, \mu)}) \) and \( h_m \circ F_w = l^{-m} h_0 + q^w_1 \mathbf{1}_K \) \( \mu \)-a.e. for any \( w \in W_m \) and hence (2.2) for \( u \in \mathcal{F} \) from Lemma 3.3 and Lemma 3.4 together yield

\[
\mathcal{E}(h_m, h_m) = \sum_{w \in W_m} \frac{1}{p^m} \mathcal{E}(l^{-m} h_0 + q^w_1 \mathbf{1}_K, l^{-m} h_0 + q^w_1 \mathbf{1}_K) = \left( \frac{\# S}{p} l^{-2} \right)^m \mathcal{E}(h_0, h_0). \tag{3.11}
\]

Now if \( \mathcal{E}(h_0, h_0) = 0 \), then \( \mathcal{E}(h_m, h_m) = 0 \) by (3.11) for any \( m \in \mathbb{N} \), thus \( \{h_m\}_{m=1}^\infty \) would be a Cauchy sequence in the Hilbert space \((\mathcal{F}, \mathcal{E}_1)\) with \( \lim_{m \to \infty} \mathcal{E}(f_1|_K - h_m, f_1|_K) = 0 \) and therefore convergent to \( f_1|_K \) in norm in \((\mathcal{F}, \mathcal{E}_1)\), hence \( f_1|_K \in \mathcal{F} \) and \( \mathcal{E}(f_1|_K, f_1|_K) = \lim_{m \to \infty} \mathcal{E}(h_m, h_m) = 0 \), which contradicts Lemma 3.10 and completes the proof. \( \square \)

It is the proof of the following proposition that requires our standing assumption that \( S \neq \{0, 1, \ldots, l-1\}^d \), which excludes the case of \( K = [0, 1]^d \) from the present framework.
Proposition 3.12. Let $h_2 \in \mathcal{F}$ be as in Lemma 3.9 with $m = 2$. Then $h_2$ is not $\mathcal{E}$-harmonic on $K \setminus (V_0^0 \cup V_0^1)$.

**Proof.** Suppose to the contrary that $h_2$ were $\mathcal{E}$-harmonic on $K \setminus (V_0^0 \cup V_0^1)$. We claim that then $h_0 \in \mathcal{F}$ as in Proposition 3.8 would turn out to be $\mathcal{E}$-harmonic on $K \setminus V_0^0$, which together with $\tilde{h}_0 = 0$ $\mathcal{E}$-q.e. on $V_0^0$ from Proposition 3.8-(1) would imply $\mathcal{E}(h_0, h_0) = 0$ by (3.3) and (3.5), a contradiction to Proposition 3.11 and will thereby complete the proof.

For each $\varepsilon = (\varepsilon_k)_{k=1}^d \in \{1\} \times \{0, 1\}^{d-1}$, set $U^\varepsilon := K \cap \prod_{k=1}^d (\varepsilon_k - 1, \varepsilon_k + 1/2)$ and choose $\varphi_{\varepsilon} \in \mathcal{C}_U$, so that $\varphi_{\varepsilon}|_{K^\varepsilon} = \mathbf{1}_{K^\varepsilon}$, such $\varphi_{\varepsilon}$ exists by [13, Exercise 1.4.1]. Let $v \in \mathcal{C}_{K \setminus V_0^0}$ and, taking an enumeration $\{\varepsilon(k)\}_{k=1}^{d-1}$ of $\{1\} \times \{0, 1\}^{d-1}$ and recalling that $u_1, u_2 \in \mathcal{F}$ for any $u_1, u_2 \in \mathcal{F} \cap L^\infty(K, \mu)$ by [13, Theorem 1.4.2-(ii)], define $v_{\varepsilon} \in \mathcal{C}_{U^\varepsilon}$ for $\varepsilon \in \{1\} \times \{0, 1\}^{d-1}$ by $v_{\varepsilon} := v_{\varphi_{\varepsilon}}(i)$ and $v_{\varepsilon} := v_{\varphi_{\varepsilon}}(i) \prod_{j=1}^{d-1}(\mathbf{1}_K - \varphi_{\varepsilon(j)})$ for $k \in \{2, \ldots, 2^{d-1}\}$. Then $v - \sum_{\varepsilon \in \{1\} \times \{0, 1\}^{d-1}} v_{\varepsilon} = v \prod_{\varepsilon \in \{1\} \times \{0, 1\}^{d-1}} (1 - \varphi_{\varepsilon}) \in \mathcal{C}_{K \setminus V_0^0 \cup V_0^1}$, hence $\mathcal{E}(h_0, v) = \sum_{\varepsilon \in \{1\} \times \{0, 1\}^{d-1}} \mathcal{E}(h_0, v_{\varepsilon})$ by Proposition 3.8-(1) and (3.5), and therefore the desired $\mathcal{E}$-harmonic $h_0$ on $K \setminus V_0^0$, i.e., (3.5) with $h = h_0$ and $U = K \setminus V_0^0$, would be obtained by deducing that $\mathcal{E}(h_0, v_{\varepsilon}) = 0$ for any $\varepsilon \in \{1\} \times \{0, 1\}^{d-1}$.

To this end, set $\varepsilon^{(1)} := (1_{\{1\}}(k))_{k=1}^d$ and choose $i = (i_k)_{k=1}^d \in S$ satisfying $i_1 < l - 1$ and $i + \varepsilon^{(1)} \not\in S$; such $i$ exists by $S \neq \{0, 1, \ldots, l - 1\}^d$ and (GSC1). Let $\varepsilon = (\varepsilon_k)_{k=1}^d \in \{1\} \times \{0, 1\}^{d-1}$ and set $\varepsilon^{(i)} := (i - 1)_{\varepsilon_k} + (2 - 2 \varepsilon_k)_{\varepsilon_k} = (\eta_k)_{k=1}^d$ for each $\eta = (\eta_k)_{k=1}^d \in \{0, 1\} \times \{0, 1\}^{d-1}$ and $I^{(i)} := \{\eta \in \{0\} \times \{0, 1\}^{d-1} \mid \varepsilon^{(i)} \eta \in S\}$, so that $\varepsilon^{(i)}\eta \in S$ by (GSC4) and (GSC1) and hence $0_d \in I^{(i)}$. Thanks to $v_{\varepsilon} \in \mathcal{C}_{U^\varepsilon}$ and $i + \varepsilon^{(1)} \not\in S$ we can define $v_{\varepsilon, 2} \in \mathcal{C}(K)$ by setting

$$v_{\varepsilon, 2}|_{K_w} := \begin{cases} 0 & \text{if } \eta \in I^{(i)} \text{ and } w = \varepsilon^{(i)} \eta \\ v_{\varepsilon} \circ g_{1 - \varepsilon^{(i)}(i)} \circ \eta \circ F_{\varepsilon}^{-1} & \text{if } w \not\in \{\varepsilon^{(i)} \eta \mid \eta \in I^{(i)}\} \end{cases}$$

for each $w \in W_2$, (3.12)

where $1 := (1)_{k=1}^d$, then $\text{supp}[v_{\varepsilon, 2}] \subset K \setminus V_0^0 \subset K \setminus (V_0^0 \cup V_0^1)$ by (3.12) and $i_1 < l - 1$, $v_{\varepsilon, 2} \circ F_{\varepsilon} \in \mathcal{F}$ for any $w \in W_2$ by (3.12), $v_{\varepsilon, 2} \in \mathcal{F}$ by (GSCD1) and therefore $v_{\varepsilon, 2} \in \mathcal{C}_{K \setminus (V_0^0 \cup V_0^1)}$. On the other hand, recalling that $h_2 \circ F_w = r^2h_0 + q^2_{\varepsilon}r^2 I_K \mu$-a.e. for any $w \in W_2$ by (3.10), Lemmas 3.2 and 3.3 and taking $\{u_n\}_{n=1}^\infty \subset \mathcal{F} \cap \mathcal{C}(K)$ as in Proposition 3.8, we see from (2.2) for $u \in \mathcal{F}$ in Lemma 3.3, (3.12), Lemma 3.4, Proposition 3.8-(3), (GSCD1) and Proposition 3.8-(2) that

$$\mathcal{E}(h_2, v_{\varepsilon, 2}) = \frac{1}{r^2} \mathcal{E}(h_0, v_{\varepsilon} \circ g_{1 - \varepsilon^{(i)}(i)} \circ \eta) = \lim_{n \to \infty} \sum_{\eta \in I^{(i)}} \frac{1}{r^2} \mathcal{E}(u_n, v_{\varepsilon} \circ g_{1 - \varepsilon^{(i)}(i)} \circ \eta)$$

$$= \lim_{n \to \infty} \sum_{\eta \in I^{(i)}} \frac{1}{r^2} \mathcal{E}(u_n \circ g_0 \circ g_{1 - \varepsilon^{(i)}(i)}, v_{\varepsilon, 2}) = \lim_{n \to \infty} \frac{\#I^{(i)}}{r^2} \mathcal{E}(u_n, v_{\varepsilon, 2}) = \frac{\#I^{(i)}}{r^2} \mathcal{E}(h_0, v_{\varepsilon, 2}).$$

(3.13)

Now our supposition that $h_2$ is $\mathcal{E}$-harmonic on $K \setminus (V_0^0 \cup V_0^1)$, in combination with (3.13), $\#I^{(i)} > 0$ and $v_{\varepsilon, 2} \in \mathcal{C}_{K \setminus (V_0^0 \cup V_0^1)}$, would yield $\mathcal{E}(h_0, v_{\varepsilon, 2}) = r^2 \mathcal{E}(\#I^{(i)})^{-1} \mathcal{E}(h_2, v_{\varepsilon, 2}) = 0$, which would imply a contradiction as explained above and thus completes the proof. \qed

**Proof of Theorem 2.9.** Let $h_0 \in \mathcal{F}$ be as in Proposition 3.8 and $h_2 \in \mathcal{F}$ as in Lemma 3.9 with $m = 2$, so that $h_0$ is $\mathcal{E}$-harmonic on $K \setminus (V_0^0 \cup V_0^1)$, $\tilde{h}_0 = \mathbf{1}_{V_0^1} = h_2$ $\mathcal{E}$-q.e. on $V_0^0 \cup V_0^1$,
$h_2$ is not $\mathcal{E}$-harmonic on $K \setminus (V_0^0 \cup V_0^1)$ by Proposition 3.12 and hence $\mathcal{E}(h_0, h_0) < \mathcal{E}(h_1, h_1)$ in view of (3.4). This strict inequality combined with (3.11) shows that

$$\mathcal{E}(h_0, h_0) < \mathcal{E}(h_2, h_2) = \left( \frac{\#S}{r} l^{-2} \right)^2 \mathcal{E}(h_0, h_0),$$

whence $l^2 < \#S/r$, namely $d_w = \log_2(\#S/r) > 2$. \hfill \Box

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**References**


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