Intersection numbers of twisted cohomology groups associated with Selberg-type integrals

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1 Introduction

Let $h_1, \ldots, h_n$ be linear forms in $\mathbb{C}[x_1, \ldots, x_r]$. We call the set of linear forms $\{h_1, \ldots, h_n\}$ the hyperplane arrangement. Put $X = \{(x_1, \ldots, x_r) \in \mathbb{C}^r \mid \prod_{k=1}^n h_k \neq 0\}$ and $\omega = \sum_{k=1}^n \lambda_k d \log h_k$ where λ_k is a given constant. Define $\nabla_+ \tau = d\tau + \omega \wedge \tau$ and $\nabla_- \tau = d\tau - \omega \wedge \tau$ for given differential form τ. The de Rham cohomology group on X with the differential map $\nabla_\pm$ is denoted by $H^r(X, \nabla_\pm)$. Among hyperplane arrangements and cohomology groups on the complement of the arrangement, we are concerned with the Selberg-type arrangement, which is defined by the linear forms $x_i - x_j$ $(1 \leq i < j \leq r)$ and $x_i - t_k$ $(1 \leq i \leq r,\ 1 \leq k \leq m)$. Here $t_1, \ldots, t_m$ are distinct constants. In this paper, we study intersection numbers

$$I(\phi_i, \psi_j) = \int_X \phi_i \wedge \psi_j$$

for suitable bases of twisted cocycles $\phi_i \in H^r(X, \nabla_+)$ and $\psi_j \in H_c^r(X, \nabla_-)$ for the Selberg-type arrangement where the support of φ_j is compact.

K. Matsumoto [6] gave a formula of intersection numbers for suitable bases of $H^r(X, \nabla_+)$ in the case that the hyperplane arrangement $\{h_1, \ldots, h_n\}$ is in a general position. However, in the case of Selberg-type arrangements, his formula and his method to evaluate intersection numbers are hard to be applied because the Selberg-type arrangement is degenerate.

In this paper, we introduce a new method to evaluate intersection numbers. Let us briefly illustrate our method. We assume that we can take a one dimensional space Y and a projection $f : X \ni (x_1, \ldots, x_r) \mapsto x_r \in Y$. If an r-form ϕ of $x_1, \ldots, x_r$ has a decomposition $\sigma \wedge \varphi$ such that σ is an $(r-1)$-form on the fibre $f^{-1}(x_r)$ and φ is a one form on Y, then the multiple integral $\displaystyle\int_X \phi \wedge \psi$ can be written by a successive integral as follows:

$$I(\phi, \psi) = \int_X \phi \wedge \psi = \int_Y \left\{ \int_{f^{-1}(x_r)} \sigma \wedge \nu \right\} \varphi \wedge \mu,$$

where $\psi = \nu \wedge \mu$. The integral $\displaystyle\int_{f^{-1}(x_r)} \sigma \wedge \nu$ is the intersection number of σ and ν on the fibre $f^{-1}(x_r)$. Thus, the intersection number is expressed in terms of the intersection numbers on the fibre $f^{-1}(x_r)$. Note that this method can be appliable to any arrangements. A general description of our method will be given in Section 2. We apply this method to Selberg-type arrangements and get a recursion formula of intersection numbers of the symmetric part of twisted cohomology groups, introduced by Aomoto [1], on the dimension of X in Section 3.

2 Twisted cohomology groups

Let X be an r-dimensional complex manifold. We denote by $\mathcal{E}^p$ the sheaf of smooth p-forms on the $2r$-dimensional real manifold X and by $\mathcal{E}^p_c$ the sheaf of smooth p-forms on the $2r$-dimensional real manifold X with compact supports. Let ω^+ be an $m \times m$-matrix valued holomorphic one forms on X. We suppose that a covariant derivation

$$\nabla_+ : \Gamma(U, \mathcal{E}^p) \otimes \mathbb{C}^m \ni \tau \longmapsto d\tau + \omega^+ \wedge \tau \in \Gamma(U, \mathcal{E}^{p+1}) \otimes \mathbb{C}^m$$

is integrable, that is, $\nabla_+\nabla_+ = 0$. We define p-th cohomology groups $H^p(X, \nabla_+)$ and $H^p_c(X, \nabla_+)$ as follows:

$$\begin{aligned} H^p(X, \nabla_+) &:= \frac{\mathrm{Ker}(\nabla_+ : \Gamma(X, \mathcal{E}^p) \otimes \mathbb{C}^m \to \Gamma(X, \mathcal{E}^{p+1}) \otimes \mathbb{C}^m)}{\mathrm{Im}(\nabla_+ : \Gamma(X, \mathcal{E}^{p-1}) \otimes \mathbb{C}^m \to \Gamma(X, \mathcal{E}^p) \otimes \mathbb{C}^m)}, \\ H^p_c(X, \nabla_+) &:= \frac{\mathrm{Ker}(\nabla_+ : \Gamma(X, \mathcal{E}^p_c) \otimes \mathbb{C}^m \to \Gamma(X, \mathcal{E}^{p+1}_c) \otimes \mathbb{C}^m)}{\mathrm{Im}(\nabla_+ : \Gamma(X, \mathcal{E}^{p-1}_c) \otimes \mathbb{C}^m \to \Gamma(X, \mathcal{E}^p_c) \otimes \mathbb{C}^m)}. \end{aligned}$$

In the sequel, we assume that X is the complement of hyperplanes in $\mathbb{C}^r$. Let $h_1, \dots, h_N$ be linear forms in $\mathbb{C}[x_1, \dots, x_r]$ and set

$$\begin{aligned} X &= \left\{ (x_1, \dots, x_r) \in \mathbb{C}^r \;\middle|\; \prod_{i=1}^N h_i \neq 0 \right\}, \\ \omega^+_X &= -\omega^-_X = \lambda_1 \frac{dh_1}{h_1} + \cdots + \lambda_N \frac{dh_N}{h_N}, \\ \nabla_+ &= d + \omega^+_X, \qquad \nabla_- = d + \omega^-_X, \end{aligned}$$

where $\lambda_1, \dots, \lambda_N$ are sufficiently generic complex numbers so that all k-th twisted cohomology groups appearing in this section vanish, where k is not equal to the dimension of the base space X, and so that $H^k_c(X, \nabla_\pm)$ and $H^k(X, \nabla_\pm)$ agree.

We assume that we can choose linear forms $h'_1, \dots, h'_n$ in $\mathbb{C}[y]$ such that there exists the projection

$$f : X \ni (x_1, \dots, x_r) \longmapsto x_r \in Y = \left\{ y \in \mathbb{C} \;\middle|\; \prod_{i=1}^n h'_i \neq 0 \right\}.$$

Then we have the natural inclusion $\iota : f^{-1}(y) \hookrightarrow X$, where $f^{-1}(y)$ is the fibre of y. We denote by ∇'_+ and ∇'_- covarinat derivations $d + (\iota^*\omega^+_X)$ and

$d + (\iota^* \omega_X^-)$ on $f^{-1}(y)$. Let $\{\sigma_1^+, \ldots, \sigma_m^+\}$ be a basis of $H^{r-1}(f^{-1}(y), \nabla'_+)$ and $\{\sigma_1^-, \ldots, \sigma_m^-\}$ a basis of $H_c^{r-1}(f^{-1}(y), \nabla'_-)$. Put $S = (S_{ij})_{i,j=1,\ldots,m}$ as

$$S_{ij} = \int_{f^{-1}(y)} \sigma_i^+ \wedge \sigma_j^-,$$

that is, it is an intersection matrix of twisted $(r-1)$-cocycles σ_i^+ and σ_j^-. We suppose that we can take the bases $\{\sigma_i^+\}$ and $\{\sigma_i^-\}$ so that the matrix S does not depend on y. We take a suitable integral domain $\triangle$ and set

$$u^+(\triangle) = {}^t\left(\int_\triangle \prod_{k=1}^{N} h_k^{\lambda_k} \sigma_1^+, \ldots, \int_\triangle \prod_{k=1}^{N} h_k^{\lambda_k} \sigma_m^+ \right),$$

$$u^-(\triangle) = {}^t\left(\int_\triangle \prod_{k=1}^{N} h_k^{-\lambda_k} \sigma_1^-, \ldots, \int_\triangle \prod_{k=1}^{N} h_k^{-\lambda_k} \sigma_m^- \right).$$

The functions $u^+(\triangle)$ (resp. $u^-(\triangle)$) satisfies a system of first order differntial equation; let Ω^+, Ω^- be $m \times m$-matrix valued one forms which satisfy the following equations:

$$du^+ - {}^t\Omega^+ u^+ = 0, \qquad du^- - {}^t\Omega^- u^- = 0.$$

Here,the complete integrable condition $d\Omega^+ = \Omega^+ \wedge \Omega^+$ and $d\Omega^- = \Omega^- \wedge \Omega^-$ are satisfied. In the sequel, we assume the condition

$$ {}^t\Omega^+ S + S\Omega^- = 0. \tag{2.1}$$

If matrices S, Ω^+, Ω^- satisfy this relation (2.1), then the locally constant sheaf $\mathrm{Ker}(d + \Omega^+)$ is the dual sheaf of $\mathrm{Ker}(d + \Omega^-)$. In Section 5, we will prove (2.1) for the Selberg-type arrrangement using a recursive method.

Put $\nabla''_+ = d + \Omega^+$ and $\nabla''_- = d + \Omega^-$. We consider cohomology groups $H^1(Y, \nabla''_+)$ and $H_c^1(Y, \nabla''_-)$. We suppose that we can choose bases $\{\tau^\pm\}$ of $H^r(X, \nabla_+)$ and $H_c^r(X, \nabla_-)$ such that $\tau^\pm$ can be expressed as

$$\tau^\pm = \sum_{i=1}^{m} \sigma_i^\pm \wedge \varphi_i^\pm, \qquad {}^t(\varphi_1^\pm, \ldots, \varphi_m^\pm) \in H^1(Y, \nabla''_\pm).$$

A small example of this decomposition is given in (2.6). For these bases $\{\tau^{\pm}\}$, we define the following $\mathbb{C}$-linear maps

$$R^{+}: H^{r}(X, \nabla_{+}) \ni \tau^{+} = \sigma_1^{+} \wedge \varphi_1^{+} + \cdots + \sigma_m^{+} \wedge \varphi_m^{+} \mapsto \begin{pmatrix} \varphi_1^{+} \\ \vdots \\ \varphi_m^{+} \end{pmatrix} \in H^{1}(Y, \nabla''_{+}), \tag{2.2}$$

$$R^{-}: H_c^{r}(X, \nabla_{-}) \ni \tau^{-} = \sigma_1^{-} \wedge \varphi_1^{-} + \cdots + \sigma_m^{-} \wedge \varphi_m^{-} \mapsto \begin{pmatrix} \varphi_1^{-} \\ \vdots \\ \varphi_m^{-} \end{pmatrix} \in H_c^{1}(Y, \nabla''_{-}). \tag{2.3}$$

We define a bilinear form

$$S: (\Gamma(U, \mathcal{E}^{p}) \otimes \mathbb{C}^{m}) \times (\Gamma(U, \mathcal{E}^{q}) \otimes \mathbb{C}^{m}) \longrightarrow \Gamma(U, \mathcal{E}^{p+q})$$

by

$$S(\varphi^{+}, \varphi^{-}) = \sum_{i=1}^{m} \sum_{j=1}^{m} s_{ij} \varphi_i^{+} \wedge \varphi_j^{-},$$

where $\varphi^{+} = {}^{t}(\varphi_1^{+}, \ldots, \varphi_m^{+})$ and $\varphi^{-} = {}^{t}(\varphi_1^{-}, \ldots, \varphi_m^{-})$.

Let us see that this bilinear form induces the bilinear form on $H^{1}(Y, \nabla''_{+}) \times H_c^{1}(Y, \nabla''_{-})$ under the condition (2.1). For $\xi \in \Gamma(U, \mathcal{E}^{q}) \otimes \mathbb{C}^{m}$ and $\tau \in \Gamma(U, \mathcal{E}^{p}) \otimes \mathbb{C}^{m}$, we have

$$\begin{aligned} S(\xi, \nabla_{-}\tau) &= S(\xi, d\tau + \omega^{-}\tau) \\ &= S(\xi, d\tau) + S(\xi, \omega^{-}\tau) \\ &= dS(\xi, \tau) + (-1)^{p} S(d\xi, \tau) + (-1)^{p} S(\omega^{+}\xi, \tau) \\ &= dS(\xi, \tau) + (-1)^{p} S(\nabla_{+}\xi, \tau). \end{aligned}$$

Hence, if $\nabla_+\xi = 0$ and $\tau \in \Gamma(X, \mathcal{E}^p) \otimes \mathbb{C}^m$, then we have

$$\begin{aligned} \int_Y S(\xi, \nabla_- \tau) &= \int_Y dS(\xi, \tau) + (-1)^p \int_Y S(\nabla_+ \xi, \tau) \\ &= 0. \end{aligned}$$

Therefore the value of the integral $\displaystyle\int_Y S(\psi, \tau)$ depends only on the cohomology classes of cocycles ψ and τ. Thus, we introduce the following definition.

Definition 1. The intersection number of cocycles φ^+ and φ^- in $H^1(Y, \nabla''_+) \times H^1_c(Y, \nabla''_-)$ is

$$I_Y(\varphi^+, \varphi^-) = \int_Y S(\varphi^+, \varphi^-).$$

As we have seen, I_Y is a bilinear form. Rewriting a multiple integral to a successive integral, we get the following lemma.

Lemma 1. *The following diagram is commutative:*

$$\begin{array}{ccc} H^r(X, \nabla_+) \times H^r_c(X, \nabla_-) & \xrightarrow{I_X} & \mathbb{C} \\ {\scriptstyle R^+ \times R^-}\big\downarrow & & \big\| \\ H^1(Y, \nabla''_+) \times H^1_c(Y, \nabla''_-) & \xrightarrow[I_Y]{} & \mathbb{C} \end{array}$$

Proof. Set

$$\begin{aligned} \tau^+ &= \sigma_1^+ \wedge \varphi_1^+ + \cdots + \sigma_m^+ \wedge \varphi_m^+, \\ \tau^- &= \sigma_1^- \wedge \varphi_1^- + \cdots + \sigma_m^- \wedge \varphi_m^-. \end{aligned}$$

Then, we have

$$\begin{aligned}
I_X(\tau^+, \tau^-) &= \int_X \tau^+ \wedge \tau^- \\
&= \int_X \left(\sum_{i=1}^m \sigma_i^+ \wedge \varphi_i^+ \right) \wedge \left(\sum_{j=1}^m \sigma_j^- \wedge \varphi_j^- \right) \\
&= \sum_{i=1}^m \sum_{j=1}^m \int_X \sigma_i^+ \wedge \varphi_i^+ \wedge \sigma_j^- \wedge \varphi_j^- \\
&= \sum_{i=1}^m \sum_{j=1}^m \int_Y \left(\int_{f^{-1}(y)} \sigma_i^+ \wedge \sigma_j^- \right) \varphi_i^+ \wedge \varphi_j^- \\
&= \sum_{i=1}^m \sum_{j=1}^m \int_Y s_{ij} \varphi_i^+ \wedge \varphi_j^- \\
&= \int_Y S(\varphi^+, \varphi^-) \\
&= I_Y(R^+\tau^+, R^-\tau^-).
\end{aligned}$$

□

We next explain a method to evaluate intersection numbers on $H^1(Y, \nabla''_\pm)$. Put

$$\begin{aligned}
\Omega^+ &= -\Omega^- = \frac{dh'_1}{h'_1} A_1 + \cdots + \frac{dh'_n}{h'_n} A_n, \\
A_\infty &= -\sum_{k=1}^n A_k
\end{aligned}$$

where $A_1, \ldots, A_n \in M(m, \mathbb{C})$, and recall that

$$Y = \left\{ y \in \mathbb{C} \;\middle|\; \prod_{i=1}^n h'_i \neq 0 \right\}.$$

Here $h'_1, \ldots, h'_n$ are linear forms in $\mathbb{C}[y]$.

Theorem 2. *Suppose that there exists $A_1, \ldots, A_n, A_\infty \in M(m, \mathbb{C})$ of which eigenvalues are not integers and so that $A_1 + \cdots + A_n + A_\infty = 0$. In addition, if the connections $\nabla_+ = d + \Omega$ and $\nabla_- = d - \Omega$ on Y are written as*

$$\Omega = \frac{dh'_1}{h'_1} A_1 + \cdots + \frac{dh'_n}{h'_n} A_n,$$

then the intersection number of cocycles $[\frac{dt}{t-x_i} v^+] \in H^1(Y, \nabla''_+)$ and $[\frac{dt}{t-x_j} v^-] \in H^1(Y, \nabla''_-)$ is

$$I_Y\left(\frac{dt}{t-x_i} v^+, \frac{dt}{t-x_j} v^-\right) = 2\pi\sqrt{-1} \times \left\{\delta_{ij}\ S(A_i^{-1} v^+,\ v^-) + S(A_\infty^{-1} v^+,\ v^-)\right\},$$

where $v^\pm \in \mathbb{C}^m$.

Theorem 2 is a generalization of Theorem 2.1 ($k = 1$) in Matsumoto [6] to twisted cohomology groups with the coefficient sheaf of which rank is more than one. We omit the proof because it can be shown in a similar method with that explained in Section 4 of [6]. In Section 3, Theorem 2 and Lemma 1 are used to derive a recursion formula of intersection numbers for a basis of symmetric subspaces of twisted cohomology groups associated with Selberg-type integrals.

Remark 1. Let $b_1, \ldots, b_m$ be the eigenvalues of the matrix A_i. Then the eigenvalues of the local monodromy of the local constant sheaf $\operatorname{Ker} \nabla''_+$ at the point t_i are $\exp(2\pi\sqrt{-1} b_1), \ldots, \exp(2\pi\sqrt{-1} b_m)$. If none of $b_1, \ldots, b_m$ is an integer, then none of the eigenvalues is one. Therefore, if the hypothesis of Theorem 2 holds, then there exists the isomorphism $H^1(Y, \operatorname{Ker} \nabla''_+) \simeq H^1_c(Y, \operatorname{Ker} \nabla''_+)$. ([4, Lemma (1.6)])

Example 1. We consider twisted cohomology groups associated with the B-function

$$\int_0^1 x^a (1-x)^b \frac{dx}{x(1-x)}.$$

Set

$$
\begin{aligned}
X &= \{x \in \mathbb{C} \mid x(1-x) \neq 0\},\\
\omega &= a\frac{dx}{x} + b\frac{dx}{x-1},\\
\nabla_+ &= d + \omega, \qquad \nabla_- = d - \omega
\end{aligned}
$$

and assume that $a, b, a+b \notin \mathbb{Z}$. Then the dimension of $H^1(X, \nabla_-)$ is one and an intersection number on $H^1(X, \nabla_\pm)$ is given by

$$
I_X\left(\frac{dx}{x(1-x)}, \frac{dx}{x(1-x)}\right) = 2\pi\sqrt{-1}\frac{a+b}{ab}.
$$

Example 2. Let us illustrate our method of the succesive integration by a small example. Set

$$
\begin{aligned}
X &= \{(x,y) \in \mathbb{C}^2 \mid xy(1-x-y) \neq 0\},\\
Y &= \{y \in \mathbb{C} \mid y(y-1) \neq 0\},\\
\omega^+ &= -\omega^- = \lambda_1\frac{dx}{x} + \lambda_2\frac{dy}{y} + \lambda_3\frac{dx+dy}{x+y-1}.
\end{aligned}
$$

We consider the map $f : X \ni (x,y) \mapsto y \in Y$. Then

$$
\begin{aligned}
f^{-1}(y) &= \{x \in \mathbb{C} \mid x(1-x-y) \neq 0\},\\
\iota^*\omega^+ &= -\iota^*\omega^- = \lambda_1\frac{dx}{x} + \lambda_3\frac{dx}{x+y-1}.
\end{aligned}
$$

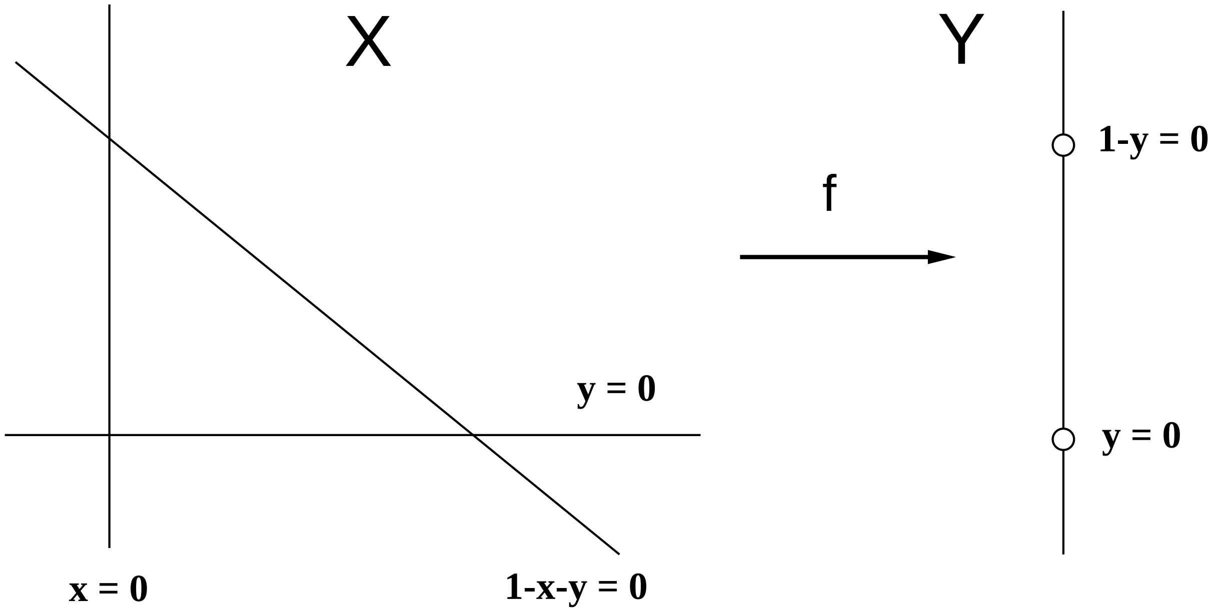
If the non-integral condition

$$
\lambda_1, \lambda_3, \lambda_1 + \lambda_3 \notin \mathbb{Z} \tag{2.4}
$$

holds, then the cohomology group $H^1(f^{-1}(y), \nabla'_-)$ is one dimensional $\mathbb{C}$-vector space and the cocycle $\frac{(1-y)dx}{x(1-x-y)}$ spans $H^1(f^{-1}(y), \nabla'_-)$.

We take a domain of integration $\{x \in \mathbb{R} \mid 0 \leq x \leq 1-y\}$ and put

$$
u^-(y) = \int_0^{1-y} x^{-\lambda_1} y^{-\lambda_2} (1-x-y)^{-\lambda_3} \frac{(1-y)dx}{x(1-x-y)}.
$$



The function $u^-(y)$ satisfies the equation

$$du^- + \left\{(\lambda_1 + \lambda_3)\frac{dy}{y-1} + \lambda_2\frac{dy}{y}\right\} u^- = 0.$$

It is proved in Example 1 that the self intersection number of the cocycle $\frac{(1-y)dx}{x(1-x-y)} \in H^1(f^{-1}(y), \nabla'_-)$ on the fibre $f^{-1}(y)$ is $S = 2\pi\sqrt{-1}\frac{\lambda_1+\lambda_3}{\lambda_1\lambda_3}$.

Set

$$\Omega = (\lambda_1 + \lambda_3)\frac{dy}{y-1} + \lambda_2\frac{dy}{y},$$
$$\nabla''_+ = d + \Omega, \qquad \nabla''_- = d - \Omega.$$

In addition to (2.4), we suppose

$$\lambda_1 + \lambda_3, \lambda_2, \lambda_1 + \lambda_2 + \lambda_3 \notin \mathbb{Z}. \tag{2.5}$$

Then the $\mathbb{C}$-vector space $H^1(Y, \nabla''_-)$ is one dimensional and $\{\frac{dy}{y(1-y)}\}$ is a basis of $H^1(Y, \nabla''_-)$.

Since

$$\frac{dx \wedge dy}{xy(1-x-y)} = \frac{(1-y)dx}{x(1-x-y)} \wedge \frac{dy}{y(1-y)}, \tag{2.6}$$

the $\mathbb{C}$-linear map (2.2) explicitly written as

$$H^2(X, \nabla_-) \ni \frac{dx \wedge dy}{xy(1-x-y)} \longmapsto \frac{dy}{y(1-y)} \in H^1(Y, \nabla''_-),$$

that is, the basis $\{\frac{dy}{y(1-y)}\}$ corresponds to the basis $\{\frac{dx\wedge dy}{xy(1-x-y)}\}$ of $H^2(X, \nabla_-)$.

From Theorem 2, the intersection pairing on $H^1(Y, \nabla''_\pm)$ is given by

$$\begin{aligned} I_Y\left(\frac{dy}{y}, \frac{dy}{y}\right) &= 2\pi\sqrt{-1}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2+\lambda_3}\right) S, \\ I_Y\left(\frac{dy}{y-1}, \frac{dy}{y}\right) = I_Y\left(\frac{dy}{y}, \frac{dy}{y-1}\right) &= 2\pi\sqrt{-1}\left(-\frac{1}{\lambda_1+\lambda_2+\lambda_3}\right) S, \\ I_Y\left(\frac{dy}{y-1}, \frac{dy}{y-1}\right) &= 2\pi\sqrt{-1}\left(\frac{1}{\lambda_1+\lambda_3} - \frac{1}{\lambda_1+\lambda_2+\lambda_3}\right) S. \end{aligned}$$

Since I_Y is bilinear,

$$\begin{aligned} I_Y\left(\frac{dy}{y(1-y)}, \frac{dy}{y(1-y)}\right) &= I_Y\left(\frac{dy}{y} - \frac{dy}{y-1}, \frac{dy}{y} - \frac{dy}{y-1}\right) \\ &= I_Y\left(\frac{dy}{y}, \frac{dy}{y}\right) - I_Y\left(\frac{dy}{y}, \frac{dy}{y-1}\right) - I_Y\left(\frac{dy}{y-1}, \frac{dy}{y}\right) + I_Y\left(\frac{dy}{y-1}, \frac{dy}{y-1}\right) \\ &= 2\pi\sqrt{-1}\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1+\lambda_3}\right) S \\ &= -4\pi^2\frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1\lambda_2\lambda_3}, \end{aligned}$$

that is, the self intersection number of $\frac{dx\wedge dy}{xy(1-x-y)}$ on $H^2(X, \nabla_\pm)$ is

$$I_X\left(\frac{dx \wedge dy}{xy(1-x-y)}, \frac{dx \wedge dy}{xy(1-x-y)}\right) = (2\pi\sqrt{-1})^2\frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1\lambda_2\lambda_3}.$$

We have the following theorem for the next section.

Theorem 3 ([2], [8]). *Let $A_1, \ldots, A_n$ be $m \times m$ matrices which eigenvalues are not integers. Set*

$$\begin{aligned} X &= \mathbb{C} \setminus \{t_1, \ldots, t_n\}, \\ \Omega &= \frac{dx}{x-t_1} A_1 + \cdots + \frac{dx}{x-t_n} A_n, \\ \nabla_\pm &= d \pm \Omega, \qquad \nabla^*_\pm = d \mp {}^t\Omega. \end{aligned}$$

Let $\{\varphi_i^\pm\}$ be a basis of $H^1(T, \nabla_\pm)$ and $\tau^\pm$ a basis of $H_1(T, \nabla^*_\pm)$. We put

$$P_\pm = \left(\int_{\tau_i^\pm} \varphi_j^\pm \right)_{ij}, \qquad I_{ch} = ([\varphi_i^+] \cdot [\varphi_j^-])_{ij}, \qquad I_h = ([\tau_i^+] \cdot [\tau_j^-])_{ij}.$$

Then the matrices $P_\pm$, I_h and I_{ch} satisfy the relation

$${}^tP_+ {}^tI_{ch}^{-1} P_- = I_h$$

Theorem 3 is proved in [2] at first. As to the intersection theory of twisted homology groups of which rank is more than one, see [8] for detail.

3 Recursion formulae of intersection numbers of symmetric parts of cohomology groups asscoiated with the Selberg-type integral.

In this section, using the method explained in previous sections, we study the intersection matrix of cohomology groups associated with the Selberg-type integral:

$$\int_C \prod_{1 \le i < j \le n} (x_i - x_j)^\nu \prod_{i=1}^n \prod_{k=1}^m (x_i - t_k)^{\lambda_k} dx_1 \cdots dx_n.$$

Set

$$
\begin{aligned}
X = X(n,m) &= \left\{ (x_1,\ldots,x_n) \in \mathbb{C}^n \;\middle|\; \prod_{1\le i<j\le n} (x_i - x_j) \prod_{i=1}^{n}\prod_{k=1}^{m} (x_i - t_k) \neq 0 \right\}, \\
\Phi(n,m) &= \prod_{1\le i<j\le n} (x_i - x_j)^{\nu} \prod_{i=1}^{n}\prod_{k=1}^{m} (x_i - t_k)^{\lambda_k}, \\
\omega &= \sum_{1\le i<j\le n} \nu \frac{dx_i - dx_j}{x_i - x_j} + \sum_{i=1}^{n}\sum_{k=1}^{m} \lambda_k \frac{dx_i}{x_i - t_k}, \\
\nabla_{\pm}\tau &= d\tau \pm \omega \wedge \tau.
\end{aligned}
$$

The cohomology group $H^n(X, \nabla_+)$ admits the natural action of $\mathfrak{S}_n$ by the change of indices of $x_1, \ldots, x_n$. The symmetric subspace $H^n(X, \nabla_+)^{\mathfrak{S}_n}$ was studied in Aomoto [1] and Mimachi [7]. Our purpose is to derive recurence relations of intersection numbers for a basis of the symmetric subspace. Our intersection matrix is expressed in terms of $n, m, \nu, \lambda_1, \ldots, \lambda_m$. We will derive a recursion formula of intersection numbers with respect to n and m. We do not have explicit expressions of intersection numbers in general, but we can obtain the explicit formula of intersection numbers for small n and m using computer algebra systems.

In order to describe a basis of the symmetric subspace, we define some notations as follows. Put

$$
\varphi_{(a_1 a_2 \cdots a_n)} = \left\{ \sum_{\sigma\in\mathfrak{S}_n} \prod_{i=1}^{n} \frac{1}{x_{\sigma(i)} - t_{a_i}} \right\} dx_1 \wedge \cdots \wedge dx_n.
$$

This index $(a_1 a_2 \cdots a_n)$ is abbreviated as

$$
(1^{k_1} 2^{k_2} \cdots m^{k_m}) := (\underbrace{1\cdots 1}_{k_1}\underbrace{2\cdots 2}_{k_2}\cdots\underbrace{m\cdots m}_{k_m}).
$$

We define the following finite set of indices:

$$
\Xi_{n,m} = \{(1^{k_1} \cdots (m-1)^{k_{m-1}}) \mid k_1 + k_2 + \cdots + k_{m-1} = n\}.
$$

The cardinal number of the set $\Xi_{n,m}$ is $\binom{n+m-2}{m-2}$. We regard $\Xi_{n,m}$ as a subset of $\Xi_{n,m+1}$ by $(1^{k_1}\cdots(m-1)^{k_{m-1}}) = (1^{k_1}\cdots(m-1)^{k_{m-1}}m^0)$.

We define the following maps:

$$\Xi_{n,m} \ni \xi = (1^{k_1}\cdots(m-1)^{k_{m-1}}) \mapsto \xi_j := (1^{k_1}\cdots j^{k_j-1}\cdots(m-1)^{k_{m-1}}) \in \Xi_{n-1,m}, \tag{3.1}$$

$$\Xi_{n-1,m} \ni \xi = (1^{k_1}\cdots(m-1)^{k_{m-1}}) \mapsto \xi^r := (1^{k_1}\cdots r^{k_r+1}\cdots(m-1)^{k_{m-1}}) \in \Xi_{n,m}, \tag{3.2}$$

$$\ell_j : \xi = (1^{k_1}\cdots(m-1)^{k_{m-1}}) \mapsto \ell_j(\xi) = k_j. \tag{3.3}$$

Then we get the following relation between n-forms φ_ξ ($\xi \in \Xi_{n,m}$) and $(n-1)$-forms φ_{ξ_j} ($\xi_j \in \Xi_{n-1,m+1}$):

$$\varphi_\xi = \sum_{j=1}^{m-1} \varphi_{\xi_j} \wedge \ell_j(\xi)\frac{dx_n}{x_n - t_j}. \tag{3.4}$$

Let $\lambda_{m+1} = \lambda_{m+2} = \cdots = \lambda_{m+n-1} = \nu$. For any i such that $0 \le i < n$ we put

$$\begin{aligned}\Lambda(n-i, m+i) = &\left\{ k\lambda_j + \frac{k(k-1)}{2}\nu \;\middle|\; 1 \le j \le m+i, 1 \le k \le n-i \right\} \\ &\cup \left\{ -k\sum_{j=1}^{m+i}\lambda_j - \frac{k(k-1)}{2}\nu \;\middle|\; 1 \le k \le n-i \right\}.\end{aligned}$$

Note that, if $\Lambda(n,m) \cap \mathbb{Z} = \emptyset$, then $\prod_{k=0}^{n-1}(\lambda_m + \frac{k}{2}\nu) \neq 0$.

Theorem 4 (Aomoto [1]). *Suppose the following conditions:*

1. $\Lambda(n,m) \cap \mathbb{Z}_{>0} = \emptyset$,
2. $\prod_{k=0}^{n-1}(\lambda_m + \frac{k}{2}\nu) \neq 0$.

Then the set $\{\varphi_\xi\}_{\xi\in\Xi_{n,m}}$ *is a basis of* $H^n(X(n,m), \nabla_+)^{\mathfrak{S}_n}$.

In order to evaluate intersection numbers for φ_ξ ($\xi \in \Xi_{n,m}$) by using the method explained in the previous section, we use the projection

$$f : X(n,m) \ni (x_1, \ldots, x_n) \mapsto x_n \in Y = \left\{ y \in \mathbb{C} \;\middle|\; \prod_{k=1}^{m} (y - t_k) \neq 0 \right\}.$$

The fibre of $y \in Y$ is

$$f^{-1}(y) = \left\{ (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1} \;\middle|\; \prod_{1 \le i < j \le n-1} (x_i - x_j) \prod_{i=1}^{n-1} \prod_{k=1}^{m+1} (x_i - t_k) \neq 0 \right\} = X(n-1, m+1),$$

where $t_{m+1} = y$.

Since our purpose is to derive a recursion formula of intersection numbers, we assume that

$$\bigcup_{i=1}^{n} \Lambda(i,\ m+n-i) \cap \mathbb{Z} = \emptyset. \tag{3.5}$$

By virtue of Theorem 4, the set $\{\varphi_\xi \mid \xi \in \Xi_{n-1,m+1}\}$ is a basis of $H^{n-1}(f^{-1}(y), \nabla'_{-})^{\mathfrak{S}_{n-1}}$ under the condition (3.5). For any $\xi \in \Xi_{n-1,m+1}$ we put $u_\xi = \displaystyle\int_\Gamma \Phi(n,m)\varphi_\xi$ where we assume that the integral domain Γ is invariant by the action of $\mathfrak{S}_n$ ([1]). The function u_ξ of y satisfies the ordinary differential equation:

$$\frac{d}{dy} u_\xi = \sum_{s=1}^{m} \sum_{r=1}^{m} \frac{\ell_s(\xi)(\lambda_r + \frac{\nu}{2}\ell_r(\xi))}{y - t_s} u_{\xi_s^r} + \sum_{s=1}^{m} \frac{\lambda_s + \frac{\nu}{2}\ell_s(\xi)}{y - t_s} u_\xi.$$

(see [7], Prop. 2.1.) Namely the $\binom{n+m-2}{m-1}$-dimensional vector valued function $U = (u_\xi)_{\xi \in \Xi_{n-1,m+1}}$ of y satisfies the system:

$$\frac{d}{dy} U = \left(\frac{1}{y - t_1} {}^t A_1 + \cdots + \frac{1}{y - t_m} {}^t A_m \right) U,$$

where $A_1, \dots, A_m$ are square matrices of size $\binom{n+m-2}{m-1}$ and all elements of $A_1, \dots, A_m$ are linear forms of $\lambda_1, \dots, \lambda_m, \nu$. We put

$$\Omega = \frac{dy}{y-t_1}A_1 + \cdots + \frac{dy}{y-t_m}A_m,$$
$$\nabla''_+ = d + \Omega, \qquad \nabla''_- = d - \Omega.$$

By Aomoto [1] Lemma 1.6, we can see that none of eigenvalues of matrices $A_1, \dots, A_m, A_\infty$ is a non-positive integer under the condition $\Lambda \cap \mathbb{Z}_{\leq 0} = \emptyset$.

Remark 2. Under a suitable total order in $\Xi_{n-1,m+1}$, one of A_i can be expressed as a tridiagonal matrix. For example, A_1 is expressed as a lower tridiagonal matrix with respect to the lexicographic order in $\Xi_{n-1,m+1}$.

Example 3. In the case $n = 2$, $m = 3$, the coefficient matrices $A_1, \dots, A_3$ are written as

$$A_1 = \begin{pmatrix} 2\lambda_1+\nu & & \\ \lambda_2 & \lambda_1 & \\ \lambda_3 & & \lambda_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \lambda_2 & \lambda_1 & \\ & 2\lambda_2+\nu & \\ & \lambda_3 & \lambda_2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} \lambda_3 & & \lambda_1 \\ & \lambda_3 & \lambda_2 \\ & & 2\lambda_3+\nu \end{pmatrix}.$$

We prove the relation (2.1) for the Selberg-type arrangement.

Theorem 5. *Let S be the intersection matrix for the basis $\{\varphi_\xi\}_{\xi\in\Xi_{n,m}}$ of $H^n(X(n,m), \nabla_\pm)$. If the condition* (3.5) *holds, then S satisfies the relation* (2.1)*;*

$$S^t\Omega - S\Omega = 0.$$

Proof. We use an induction on the dimension n of X. If $n = 1$ then the theorem is clear. Assume $n > 1$. We indentify S with an intersection matrix on $H^1(Y, \nabla''_\pm)$. From the condition (3.5) none of eigenvalues of each A_i are intergers. Then we can apply Theorem 3;

$$I_h = {}^tP_+{}^tS^{-1}P_-.$$

From Theorem 6 for $n-1$ the matrix S does not depend on $y \in Y$ and, from the intersection theory of twisted homology groups, the intersection matrix I_h is also constant (cf. [5, Theorem 1.3]). Therefore

$$\begin{aligned} 0 = d({}^tP_+{}^tS^{-1}P_-) &= (d^tP_+)^tS^{-1}P_- + {}^tP_+{}^tS^{-1}(dP_-) \\ &= (-{}^tP_+\Omega)^tS^{-1}P_- + {}^tP_+{}^tS^{-1}({}^t\Omega P_-) \\ &= -{}^tP_+(\Omega^tS^{-1} - {}^tS^{-1t}\Omega)P_-. \end{aligned}$$

Hence $\Omega^tS^{-1} - {}^tS^{-1t}\Omega = 0$, that is, $S^t\Omega - S\Omega = 0$. □

In order to apply Theorem 2 to evaluate intersection numbers for the basis $\{\varphi_\xi\}$, we suppose that inverse matrices of $A_1, \cdots, A_m, A_\infty$ are expressed as

$$A_1^{-1} = (h^1_{\xi\xi'})_{\xi,\xi'\in\Xi_{n-1,m+1}}, \ldots, A_m^{-1} = (h^m_{\xi\xi'})_{\xi,\xi'\in\Xi_{n-1,m+1}},\ A_\infty^{-1} = (h^\infty_{\xi\xi'})_{\xi,\xi'\in\Xi_{n-1,m+1}}.$$

Note that we can get $h^i_{\xi\xi'}$ from A_i for given small n and m using computer algebra systems.

The following theorem gives a recursion formula in which the intersection form $S_{\xi\xi'}$ on $H^n(X(n,m),\nabla_\pm)$ are expressed in terms of the intersection form $S_{\pi\xi'_j}$ on $H^{n-1}(X(n-1,m+1),\nabla'_\pm)$.

Theorem 6. *Put $S_{\xi\xi'} = I(\varphi_\xi,\varphi_{\xi'})$. Suppose the condition* (3.5). *Then, for any $\xi,\xi'\in\Xi_{n,m}$, we have*

$$S_{\xi\xi'} = 2\pi\sqrt{-1}\sum_{\pi\in\Xi_{n-1,m+1}}\left\{\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\ell_i(\xi)\ell_j(\xi')h^\infty_{\pi\xi_i}S_{\pi\xi'_j} + \sum_{i=1}^{m-1}\ell_i(\xi)\ell_i(\xi')h^i_{\pi\xi_i}S_{\pi\xi'_i}\right\}. \tag{3.6}$$

Proof. We use an induction on n.

In the case $n=1$, since an intersection number $S_{\pi\xi'_j}$ in the right hand side of (3.6) is Kronecker's delta $\delta_{\pi\xi'_j}$, the theorem holds.

Next we assume $n > 2$. From the formula (3.4), we have the $\mathbb{C}$-linear map

$$H^n(X, \nabla_+)^{\mathfrak{S}_n} \ni \varphi_\xi \longmapsto \psi_\xi = \sum_{j=1}^{m-1} \frac{dy}{y - t_j} \ell_j(\xi) e_{\xi_j} \in H^1(Y, \nabla''_+), \tag{3.7}$$

where e_ξ is the vector whose ξ-th element is 1 and the other elements are 0.

Since this $\mathbb{C}$-linear map preserves intersection numbers, i.e. $I_X(\varphi_\xi, \varphi_{\xi'}) = I_Y(\psi_\xi, \psi_{\xi'})$, we may prove the theorem with respect to the right hand side.

By (3.5), none of eigenvalues of matrices $A_1, \dots, A_m, A_\infty$ is a non-positive integer, that is, it holds the hypothesis of Theorem 2. From the hypothesis of the induction, the intersection matrix for the basis $\{\varphi_\xi\}_{\xi \in \Xi_{n-1,m+1}}$ of $H^{n-1}(f^{-1}(y), \nabla'_-)^{\mathfrak{S}_{n-1}}$ is $S = (S_{\xi\xi'})_{\xi,\xi' \in \Xi_{n-1,m+1}}$. Since

$$A_i^{-1} e_\xi = \sum_{\pi \in \Xi_{n-1,m+1}} h^i_{\pi\xi} e_\pi,$$

we have

$$\begin{aligned} S(A_i^{-1} e_\xi,\ e_{\xi'}) &= \sum_{\pi \in \Xi_{n-1,m+1}} h^i_{\pi\xi} S(e_\pi, e_{\xi'}) \\ &= \sum_{\pi \in \Xi_{n-1,m+1}} h^i_{\pi\xi} S_{\pi\xi'}. \end{aligned}$$

From Theorem 2, for any $\xi, \xi' \in \Xi_{n-1,m+1}$, we have

$$I_Y\left(\frac{dy}{y - t_i} e_\xi, \frac{dy}{y - t_j} e_{\xi'}\right) = 2\pi\sqrt{-1} \sum_{\pi \in \Xi_{n-1,m+1}} (\delta_{ij} h^i_{\pi\xi} + h^\infty_{\pi\xi}) S_{\pi\xi'}. \tag{3.8}$$

From the formulae (3.7) and (3.8), the intersection numbers of $\{\psi_\xi\}_{\xi \in \Xi_n}$ are

$$\begin{aligned} S_{\xi\Xi'} &= I(\psi_\xi, \psi_{\xi'}) \\ &= 2\pi\sqrt{-1} \sum_{\pi \in \Xi_{n-1,m+1}} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} \ell_i(\xi) \ell_j(\xi') h^\infty_{\pi\xi_i} S_{\pi\xi'_j} + \sum_{i=1}^{m} \ell_i(\xi) \ell_i(\xi') h^i_{\pi\xi_i} S_{\pi\xi'_i} \right\}. \end{aligned}$$

$\square$

4 Example: the case $n = 2$, $m = 3$.

Using Theorem 6, we evaluate intersection numbers in the case $n = 2$, $m = 3$. Set

$$
\begin{aligned}
&X = \{(x,y) \in \mathbb{C}^2 \mid xy(x-1)(y-1)(x-t)(y-t)(x-y) \neq 0\},\\
&Y = \{y \in \mathbb{C} \mid y(y-1)(y-t) \neq 0\},\\
&\Phi = (xy)^{\lambda_1}((x-1)(y-1))^{\lambda_2}((x-t)(y-t))^{\lambda_3}(x-y)^{\nu},\\
&\omega = \lambda_1\left(\frac{dx}{x} + \frac{dy}{y}\right) + \lambda_2\left(\frac{dx}{x-1} + \frac{dy}{y-1}\right) + \lambda_3\left(\frac{dx}{x-t} + \frac{dy}{y-t}\right) + \nu\frac{dx-dy}{x-y}.
\end{aligned}
$$

Then

$$\Xi_{1,4} = \{(1),(2),(3)\}.$$

Note that

$$
\begin{aligned}
\Lambda(2,3) \cup \Lambda(1,4) = \{&\lambda_1, \lambda_2, \lambda_3, \nu, 2\lambda_1+\nu, 2\lambda_2+\nu, 2\lambda_3+\nu,\\
&-(\lambda_1+\lambda_2+\lambda_3), -(\lambda_1+\lambda_2+\lambda_3+\nu), -(2\lambda_1+2\lambda_2+2\lambda_3+\nu)\}.
\end{aligned}
$$

We asssume that $(\Lambda(2,3) \cup \Lambda(1,4)) \cap \mathbb{Z} = \emptyset$.

The fibre of the projection $f : X \ni (x,y) \mapsto y \in Y$ is $f^{-1}(y) = \{x \in \mathbb{C} \mid x(x-1)(x-t)(x-y) \neq 0\}$.

We take a basis of $H^1(f^{-1}(y), \nabla'_+)$ as

$$\varphi_{(1)} = \frac{1}{x}, \quad \varphi_{(2)} = \frac{1}{x-1}, \quad \varphi_{(3)} = \frac{1}{x-t}.$$

We put $b = \lambda_1 + \lambda_2 + \lambda_3 + \nu$.

The intersection matrix of the basis $\{\varphi_{(1)}, \varphi_{(2)}, \varphi_{(3)}\}$ is

$$
\begin{pmatrix} S_{(1),(1)} & S_{(1),(2)} & S_{(1),(3)} \\ S_{(2),(1)} & S_{(2),(2)} & S_{(2),(3)} \\ S_{(3),(1)} & S_{(3),(2)} & S_{(3),(3)} \end{pmatrix} = 2\pi\sqrt{-1}\left\{\begin{pmatrix} 1/\lambda_1 & & \\ & 1/\lambda_2 & \\ & & 1/\lambda_3 \end{pmatrix} - \frac{1}{b}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right\}
$$

(e.g. Example 1).

We already gave A_1, A_2, A_3 in Example 3. Put

$$A_\infty = -(A_1 + A_2 + A_3) = -\begin{pmatrix} \lambda_1 + b & \lambda_1 & \lambda_1 \\ \lambda_2 & \lambda_2 + b & \lambda_2 \\ \lambda_3 & \lambda_3 & \lambda_3 + b \end{pmatrix}.$$

Then

$$A_1^{-1} = \begin{pmatrix} \frac{1}{2\lambda_1+\nu} & 0 & 0 \\ -\frac{\lambda_2}{(2\lambda_1+\nu)\lambda_1} & \frac{1}{\lambda_1} & 0 \\ -\frac{\lambda_3}{(2\lambda_1+\nu)\lambda_1} & 0 & \frac{1}{\lambda_1} \end{pmatrix}, \quad A_2^{-1} = \begin{pmatrix} \frac{1}{\lambda_2} & -\frac{\lambda_1}{(2\lambda_2+\nu)\lambda_2} & 0 \\ 0 & \frac{1}{2\lambda_2+\nu} & 0 \\ 0 & -\frac{\lambda_3}{(2\lambda_2+\nu)\lambda_2} & \frac{1}{\lambda_2} \end{pmatrix}, \quad A_3^{-1} = \begin{pmatrix} \frac{1}{\lambda_3} & 0 & -\frac{\lambda_1}{(2\lambda_3+\nu)\lambda_3} \\ 0 & \frac{1}{\lambda_3} & -\frac{\lambda_2}{(2\lambda_3+\nu)\lambda_3} \\ 0 & 0 & -\frac{1}{2\lambda_3+\nu} \end{pmatrix}$$

and

$$A_\infty^{-1} = \frac{1}{bc}\begin{pmatrix} \lambda_1 - c & \lambda_1 & \lambda_1 \\ \lambda_2 & \lambda_2 - c & \lambda_2 \\ \lambda_3 & \lambda_3 & \lambda_3 - c \end{pmatrix},$$

where $c = 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + \nu$.

From Theorem 6, we have

$$S_{(11),(11)} = 2\pi\sqrt{-1}\left\{4\sum_{\pi=1}^{3}(h^\infty_{\pi1} + h^1_{\pi1})S_{\pi1}\right\},$$

$$S_{(12),(22)} = 2\pi\sqrt{-1}\left\{2\sum_{\pi=1}^{3}(h^\infty_{\pi1} + h^\infty_{\pi2})S_{\pi2} + 2\frac{1}{\lambda_2}S_{12}\right\},$$

$$S_{(12),(11)} = 2\pi\sqrt{-1}\left\{2\sum_{\pi=1}^{3}(h^\infty_{\pi1} + h^\infty_{\pi2})S_{\pi1} + 2\frac{1}{\lambda_1}S_{21}\right\},$$

$$S_{(12),(12)} = 2\pi\sqrt{-1}\left\{\frac{1}{\lambda_2}S_{11} + \frac{1}{\lambda_1}S_{22} + \sum_{\pi=1}^{3}\{(h^\infty_{\pi1} + h^\infty_{\pi2})(S_{\pi1} + S_{\pi2})\}\right\},$$

$$S_{(22),(11)} = 2\pi\sqrt{-1}\left\{4\sum_{\pi=1}^{3}h^\infty_{\pi2}S_{\pi1}\right\},$$

$$S_{(22),(22)} = 2\pi\sqrt{-1}\left\{4\sum_{\pi=1}^{3}(h^\infty_{\pi2} + h^2_{\pi2})S_{\pi2}\right\}.$$

Therefore, we have

$$
\begin{aligned}
I(\varphi_{(11)},\varphi_{(11)}) &= -4\pi^2\frac{8(\lambda_2+\lambda_3)(\nu+2\lambda_2+2\lambda_3)}{\lambda_1(\nu+2\lambda_1)(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
I(\varphi_{(11)},\varphi_{(12)}) &= 4\pi^2\frac{4(\nu+2\lambda_2+2\lambda_3)}{\lambda_1(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
I(\varphi_{(11)},\varphi_{(22)}) &= -4\pi^2\frac{8}{(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
I(\varphi_{(12)},\varphi_{(22)}) &= 4\pi^2\frac{4(\nu+2\lambda_1+2\lambda_3)}{\lambda_2(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
I(\varphi_{(22)},\varphi_{(22)}) &= -4\pi^2\frac{8(\lambda_1+\lambda_3)(\nu+2\lambda_1+2\lambda_3)}{\lambda_2(\nu+2\lambda_2)(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)},\\
I(\varphi_{(12)},\varphi_{(12)}) &= -4\pi^2\frac{2(\nu^2+(2\lambda_1+2\lambda_2+3\lambda_3)\nu+4\lambda_1\lambda_2+2\lambda_1\lambda_3+2\lambda_2\lambda_3+2\lambda_3^2)}{\lambda_1\lambda_2(\nu+\lambda_1+\lambda_2+\lambda_3)(\nu+2\lambda_1+2\lambda_2+2\lambda_3)}.
\end{aligned}
$$

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