A C^∞ closing lemma for Hamiltonian diffeos on closed surfaces

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§1. Closing lemma

M : closed mfd

$f : M \to M$ homeo / diffeo

$n \in \mathbb{Z}$

$$f^n := \begin{cases} \mathrm{Id}_X & (n=0) \\ \underbrace{f \circ \cdots \circ f}_{n\text{-times}} & (n \geq 1) \\ \underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{|n|\text{-times}} & (n \leq -1) \end{cases}$$

iterations of f

Goal: Understand orbits $(f^n(x))_{n \in \mathbb{Z}}$

$\mathrm{Fix}(f) := \{x \in X \mid f(x) = x\}$ fixed pt. set

$\mathrm{Per}(f) := \{x \in X \mid {}^{\exists} n \geq 1 \text{ s.t. } f^n(x) = x\}$

periodic pt. set

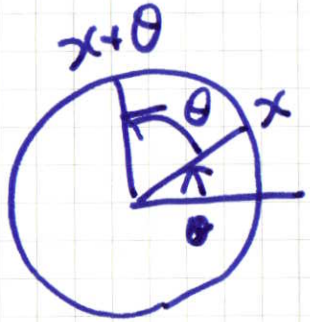
Ex. (rigid rotation)

$S^1 = \mathbb{R}/\mathbb{Z}$ $\quad \theta \in \mathbb{R}$

$R_\theta : S^1 \longrightarrow S^1 \; ; \; x \longmapsto x + \theta$

$\theta = p/q \in \mathbb{Q} \Rightarrow \mathrm{Per}(R_\theta) = \mathrm{Fix}(R_\theta^q) = S^1$

$\theta \notin \mathbb{Q} \Rightarrow \mathrm{Per}(R_\theta) = \phi.$



Importance of periodic pts

(1) Lefschetz Fixed Pt. Thm

local (fixed pt. index) $\leftrightarrow$ global ($f_* : H_*(X) \circlearrowleft$)

(2) hyperbolicity of per. pts.

$p \in \mathrm{Fix}(f^n)$ is hyperbolic

$\Leftrightarrow \forall \lambda$: e.v. of $Df^n_p : T_pM \circlearrowleft$, $|\lambda| \neq 1$

- local linearization (Hartman-Grobman, Sternberg)
- the stable/unstable mfds.
- $^\exists$non-hyp. per pt $\Rightarrow$ not structurally stable.

Closing Problem

$f \in \mathrm{Diff}^r(M) \quad x \in M$

$\overset{?}{\Rightarrow} \ {}^\exists f_n \xrightarrow{C^r} f \quad {}^\exists x_n \to x$ in M

s.t. $x_n \in \mathrm{Per}(f_n)$

Prop ("Non-recurrence $\Rightarrow$ Non-closable")

$K \subset M$: cpt $\quad f(K) \subset \mathrm{Int}\, K$

$x \in \mathrm{Int}\, K \setminus f(K)$

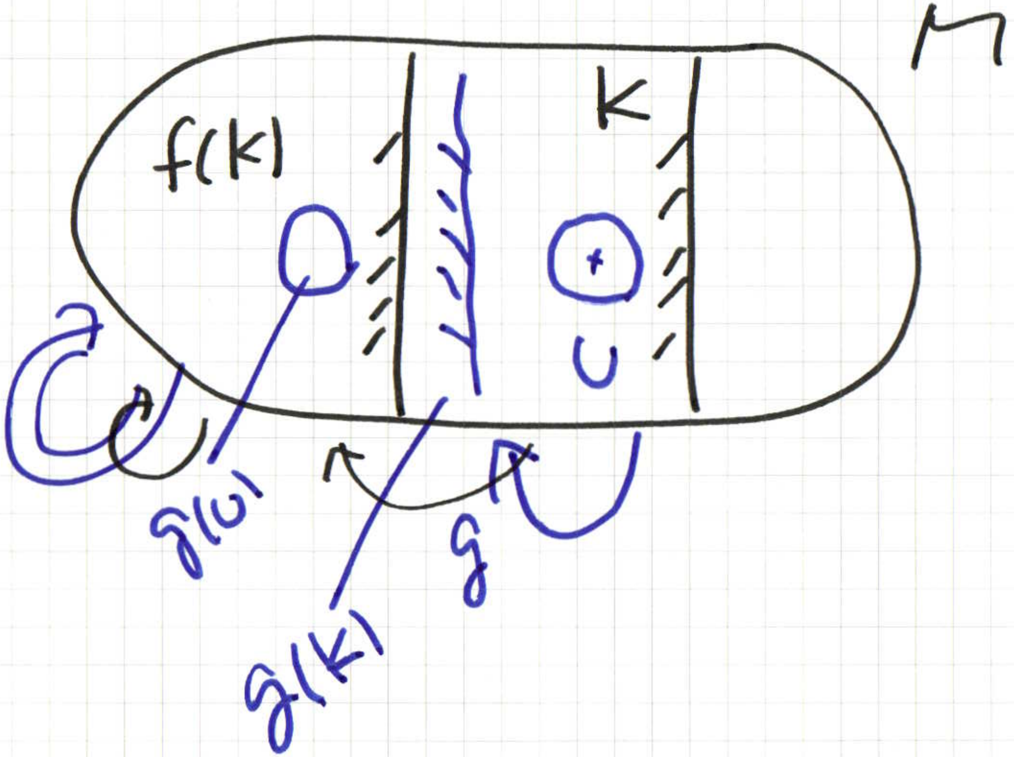
$\Rightarrow {}^\exists \mathcal{U} \subset \mathrm{Homeo}(M)$ C^0-nbd $\ {}^\exists U \subset M$: nbd of x

s.t. ${}^\forall g \in \mathcal{U}$ ${}^\forall n \geq 1$.

$g^n(U) \cap U = \phi \quad (\Rightarrow \mathrm{Per}(g) \cap U = \phi)$

(Proof) Choose $\begin{cases} U \\ \mathcal{U} \end{cases}$ s.t. $\begin{cases} \overline{U} \subset \mathrm{Int}\,K \setminus f(K) \\ g \in \mathcal{U} \Rightarrow g(K) \subset \mathrm{Int}\,K \setminus \overline{U} \end{cases}$ ⑤

$n \geq 1 \Rightarrow g^n(U) \subset g^n(K) \subset g(K) \subset \mathrm{Int}\,K \setminus U$ //



Recurrence : non-wandering set

$x \in M$ is **non-wandering**

$\Longleftrightarrow {}^\forall U$: nbd of x

${}^\exists n \geq 1$ s.t. $f^n(U) \cap U \neq \phi$.

$(\Longrightarrow {}^\forall U,\ \#\{n \geq 1 \mid f^n(U) \cap U \neq \phi\} = \infty)$

$\Omega(f) := \{\text{non-wandering pts}\}$

the non-wandering set

Ex. $\Omega(R_\theta) = S^1$ for ${}^\forall \theta \in \mathbb{R}$

※ (a) $\Omega(f)$: f-inv. closed subset of M ⑪

(b) M : cpt $\Rightarrow \Omega(f) \neq \phi$

(c) μ : f-inv prob. measure on M

s.t. $\underbrace{\mathrm{supp}\,\mu}_{} = M$

$= \{x \in M \mid {}^\forall U : \text{open nbd of } x,\ \mu(U) > 0\}$

$\Rightarrow \Omega(f) = M$

$\because)$ $U \subset M$. open, $\neq \phi$ $(\Rightarrow \mu(U) > 0)$

$U, f(U), \cdots, f^N(U)$ disjoint

$$\Rightarrow 1 \geq \mu\left(\bigcup_{n=0}^{N} f^n(U)\right) = \sum_{n=0}^{N} \mu(f^n(U))$$

$$= (N+1)\mu(U)$$

$\therefore N < 1/\mu(U)$

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C^0 closing lemma

$f \in \mathrm{Homeo}(M)$ $\quad x \in \Omega(f)$

$\Rightarrow \exists f_n \xrightarrow{C^0} f \quad \exists x_n \longrightarrow x$

s.t. $x_n \in \mathrm{Per}(f_n)$

Cor (C^0 General Density Thm)

$\{ f \in \mathrm{Homeo}(M) \mid \overline{\mathrm{Per}(f)} = \Omega(f) \}$ is residual in $\mathrm{Homeo}(M)$ w.r.t. C^0-top

$\left(\begin{array}{l} X : \text{top sp.} \quad Y \subset X \text{ is } \underline{\text{residual}} \\ \Leftrightarrow \exists X_1, X_2, \cdots \subset X \text{ open dense subsets} \\ \qquad \text{s.t. } Y \supset \bigcap_{n=1}^{\infty} X_n \end{array} \right)$

(Proof of C^0 closing lemma) ⑨

$V \subset M$: small nbd of x

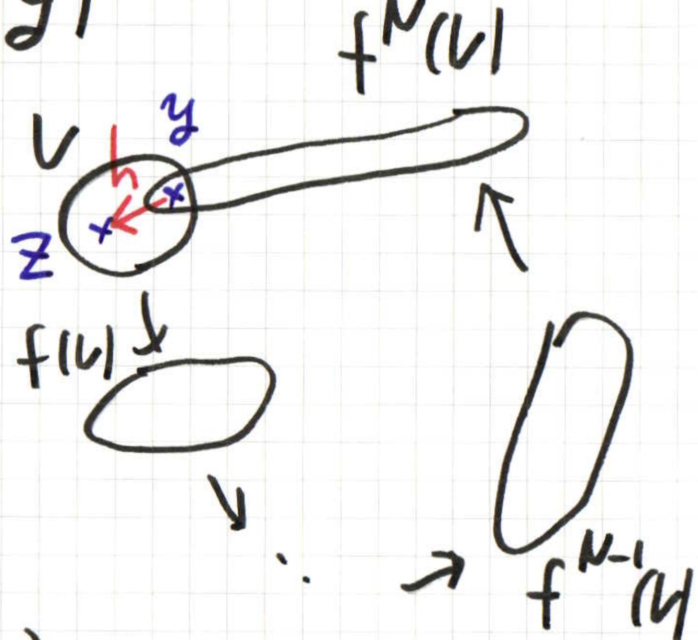
$\exists N \geq 1$ s.t. $f^n(V) \cap V = \phi$ $(n = 1, 2, \cdots N-1)$

$$f^N(V) \cap V \neq \phi$$

Take $y \in V \cap f^N(V)$. $z := f^{-N}(y)$

$\exists h \in \mathrm{Homeo}(M)$

s.t. $h(y) = z$ $\quad \underbrace{\mathrm{supp}(h)}_{\{x \in M \mid f(x) \neq x\}} \subset V$



Then,

- $(f \circ h)^N(y) = f^N(z) = y$
- $d_{C^0}(f, f \circ h) \leq \mathrm{diam}(V) + \mathrm{diam}(f(V))$

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$\left(d_{C^0}(f, g) = \sup_{x \in M} \left(d(f(x), g(x)) + d(f^{-1}(x), g^{-1}(x)) \right) \right)$

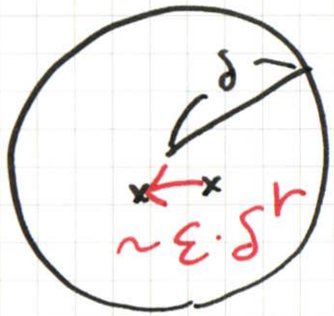
Difficulty in C^r closing lemma for $r \geq 1$

Ex "$M = \mathbb{R}$"

$h : \mathbb{R} \to \mathbb{R}$ C^r-diffeo

s.t $\|(h - Id)^{(r)}\|_{sup} < \varepsilon$ $\quad$ $\operatorname{diam}(\operatorname{supp}(h)) < \delta$

$\Longrightarrow \|h - Id\|_{sup} < \frac{1}{r!}\varepsilon \cdot \delta^r$



"push"

n-times "pushs" with disjoint supports $\rightsquigarrow$ move $\sim n \cdot \varepsilon\delta^r$

$r = 1$: We can move a pt with distance comparable with $\operatorname{diam}(\operatorname{supp}(h_i))$

by ($1/\varepsilon$)-times pushs $\qquad \underset{\delta}{\wr}$

C^1 Closing Lemma (Pugh. 1967)

$f \in \mathrm{Diff}^1(M)$ $\quad x \in \Omega(f)$

$\Rightarrow {}^\exists f_n \xrightarrow{C^1} f \quad {}^\exists x_n \to x$

s.t. $x_n \in \mathrm{Per}(f_n)$

Cor (C^1 General Density Thm)

$\{f \in \mathrm{Diff}^1(M) \mid \overline{\mathrm{Per}(f)} = \Omega(f)\}$ is residual

in $\mathrm{Diff}^1(M)$

$r \geq 2$

$\exists$ Counterexamples in several categories:

Guitierrez (1980's):

2-dim flows with a fixed stationary pts

Herman (1991)

Hamiltonian flows on $\mathbb{T}^3 \times [0,1]$

※ Smale's "Mathematical problems for the next (=21st) centuries" (~~200~~ 1998)

Problem 10 = C^r closing lemma.

Almost 50 years have passed after Pugh ....

§2 Irie's C^∞ closing lemma for 3-dim Reeb flows

M : closed mfd

$\Phi = \{\Phi^t\}_{t\in\mathbb{R}}$ flow on M

$\mathrm{Per}(\Phi) := \{x \in M \mid {}^\exists T > 0 \text{ s.t. } \Phi^T(x) = x\}$

$x \in M$ is non-wandering

$\Leftrightarrow {}^\forall U$: nbd of x

${}^\exists T \geq 1$ s.t. $\Phi^T(U) \cap U \neq \emptyset$

$\Omega(\Phi) = \{\text{non-wandering pts}\}$

the non-wandering set.

Reeb flows

M : 3-dim. closed. oriented mfd

α : 1-form on M. $\alpha \wedge d\alpha > 0$

(contact form)

$\exists^1 X_\alpha$: vector field on M

s.t. $\iota_{X_\alpha} \alpha \equiv 1$, $\iota_{X_\alpha}(d\alpha) \equiv 0$

(Reeb vector field of α)

$\rightsquigarrow$ flow $\Phi_\alpha = (\Phi_\alpha^t)_{t \in \mathbb{R}}$

(Reeb flow of α)

※ (a) $f : C^\infty, > 0$ $\quad \alpha$: contact form

$\Rightarrow f \cdot \alpha$: contact.

(b) Φ_α preserves the volume form $\alpha \wedge d\alpha$

$(\Rightarrow \Omega(\Phi_\alpha) = M)$

Thm (Irie)

M: closed, 3-dim mfd. $\quad \alpha$: contact form on M

$f : M \to \mathbb{R}$ C^∞, ≥ 0, $\mathrm{supp}(f) \neq \phi$.

$s \in [0.1]$ $\quad \Phi_s$: Reeb flow of $(1 + s \cdot f) \cdot \alpha$

$\Rightarrow {}^\exists s \in [0.1]$

s.t. $\quad \mathrm{Per}(\Phi_s) \cap \mathrm{supp}(f) \neq \phi$.

$\mathcal{U}$ $\qquad$ U

$\mathcal{U} \subset \{\text{Reeb flow}\}$

Φ_α a nbd

$U \subset M$: x a nbd

Cor $x \in M$

$\Rightarrow \exists \alpha_n \xrightarrow{C^\infty} \alpha$ contact forms

$\exists x_n \longrightarrow x$

s.t $x_n \in \mathrm{Per}(\Phi_{\alpha_n})$

(Take f s.t $\|f\|_{C^r}$: small ($r \gg 1$), $\mathrm{supp}\, f$: small)

Cor (General Density Thm)

$\{\alpha$: contact form $\mid \overline{\mathrm{Per}(\Phi_\alpha)} = M\}$ is residual in $\{$contact forms$\}$ w.r.t. C^∞ top.

Application : "Closing lemma for geodesics of closed surfaces"

(Idea of Proof: ECH spectral inv.)

$\mathcal{A}(\alpha)_+ = \{ \sum_{i=1}^{n} (\text{period of } x_i) \mid n \geq 1, x_1, \dots x_n \in \mathrm{Per}(\Phi_\alpha) \} \cup \{0\}$

(M, α), $\Gamma \in H_1(M, \mathbb{Z})$

$\rightsquigarrow$ $ECH(M, \alpha, \Gamma)$ "Embedded Contact Homology"

$c : ECH(M, \alpha, \Gamma) \to \mathcal{A}(\alpha)_+$

ECH spectral inv.

※ - $f \mapsto c(\sigma ; f\alpha)$ is continuous

- $\exists (\sigma_n \in ECH(M, \alpha, \Gamma))_{n \geq 1}$

s.t. $\lim_{n \to \infty} c(\sigma_n)^2 / |\sigma_n| = \int_M \alpha \wedge d\alpha$

grade in $ECH(M, \alpha, \Gamma)$

(Cristofaro-Gardiner - Hutchings - Ramos)

Suppose $^\forall s \in [0,1]$ $\mathrm{Per}(\Phi_s) \cap \mathrm{supp}(f) = \phi$

$\Longrightarrow \mathcal{A}(\alpha_s)_+ = \mathcal{A}(\alpha)_+ \quad (\alpha_s := (1+s\cdot f)\alpha)$

nowhere measure zero set

$(\mathbb{R} \setminus \mathcal{A}(\alpha)_+ : \text{dense in } \mathbb{R})$

$\Longrightarrow c(\sigma, \alpha_s) = c(\sigma, \alpha)$

$\underset{\text{CHR}}{\Longrightarrow} \int_M \alpha_s \wedge d\alpha_s = \int_M \alpha \wedge d\alpha.$

But, $\int_M \alpha_s \wedge d\alpha_s = \int_M (1+s\cdot f)^2 \alpha \wedge d\alpha > \int_M \alpha \wedge d\alpha$ //

$f \geq 0$, $\mathrm{supp}(f) \neq \phi$.

§3. C^∞ closing lemma for Hamiltonian diffeos on closed surfaces

Σ : cpt ori'. surface (possibly $\partial\Sigma \neq \phi$)

ω : symp. form on Σ (= volume form)

$H : \Sigma \longrightarrow \mathbb{R}$ C^∞

$^\exists X_H$: vector field on Σ

s.t $dH = -\iota_{X_H}\omega$

(Hamiltonian vector field of H)

$f : \Sigma \to \Sigma$ is Hamiltonian

$\Longleftrightarrow$ $^\exists \bar{H} : \Sigma \times [0,1] \to \mathbb{R}$ C^∞

$^\exists (\varphi_s)_{s \in [0,1]}$ smooth family of diffeos of Σ

s.t $\frac{d}{ds} \varphi_s = X_{\bar{H}(\cdot, s)}$ $(^\forall s \in [0,1])$

$\varphi_0 = \mathrm{Id}_\Sigma$, $\varphi_1 = f$

$\mathrm{Ham}(\Sigma) := \{ \text{Ham. diffeos} \}$

※ $f \in \mathrm{Ham}(\Sigma)$ preserves ω.

$(\Rightarrow \Omega(f) = \Sigma)$

Thm (A.-Irie)

$f \in Ham(\Sigma)$ $x \in \Sigma$

$\Rightarrow {}^\exists f_n \to f$ in $Ham(\Sigma)$ ${}^\exists x_n \to x$

s.t. $x_n \in Per(f_n)$

(Idea of Proof)

Lefschetz Fixed Pt Thm or Solution of Arnold's Conj

$\Rightarrow {}^\exists p \in Fix(f)$

(with a small perturbation), we may assume

$\left(\begin{array}{l} p \text{ is "non-degenerate".} \\ f \text{ is } C^\omega \text{ on a nbd of } p. \end{array} \right.$

We may assume $x \neq p$.

(Step 1)

By KAM theory / normal form theory, we can "blow up f at p":

$\exists V \subset \Sigma$: open, f-inv. $\ni x$

$\exists \Sigma'$: cpt surface, $\partial\Sigma' \neq \phi$

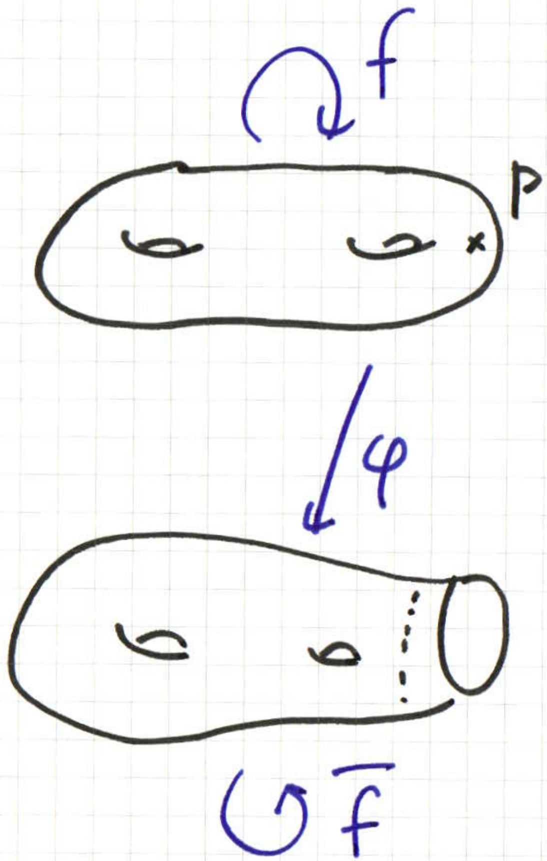
$\exists \omega'$: symp. form on Σ'

$\exists \varphi : V \hookrightarrow \Sigma'$ C^∞ embedding

$\exists \bar{f} \in Ham(\Sigma')$

s.t. $\varphi^* \omega' = \omega$, $\varphi \circ f = \bar{f} \circ \varphi$

$\bar{f} = Id$ on a nbd of $\partial\Sigma'$



(Step 2)

$$M_{\bar f} := \Sigma' \times \mathbb{R} / (x, s+1) \sim (\bar f(x), s)$$

$\frac{\partial}{\partial s}$: vector field on $M_{\bar f}$.

Lemma (Hutchings)

$^\exists M$: 3-dim closed mfd $^\exists \alpha$: contact form on M
$^\exists \psi : M_{\bar f} \hookrightarrow M$ C^∞ embed. $^\exists h : M \to \mathbb{R}$ C^∞, >0
s.t. $X_\alpha = h \cdot \psi_* \left(\frac{\partial}{\partial s}\right)$ on $\psi(M_{\bar f})$

C^∞ closing lemma for Φ_α

$\Rightarrow$ "C^∞ closing lemma for $\bar f$"

$\Rightarrow$ C^∞ closing lemma for f

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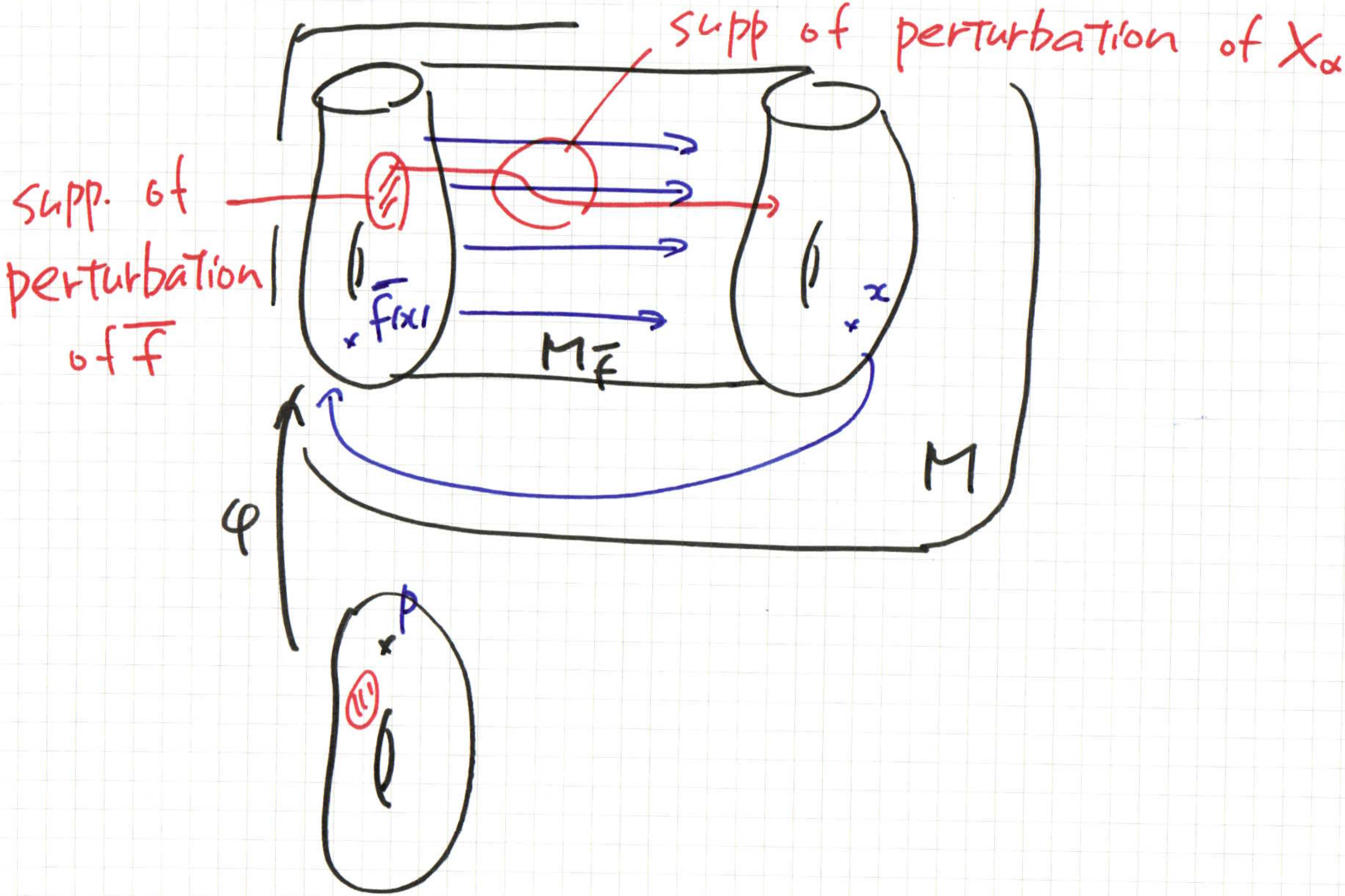
supp of perturbation of X_α
supp. of perturbation of $\overline{f}$
$\overline{f}(x)$
x
$M_{\overline{f}}$
M
φ
p

What is next?

- Non-Hamiltonian case for closed surfaces:
  area-pres / non-area pres
- Ergodic closing lemma.
  Approximating an ergodic inv. probability by the unif. meas. on per. pts
- More dynamical proof.