曲面と円柱の接触と曲面の輪郭線

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福井敏純氏と中川幸一氏（埼玉大学）との共同研究および，

M. Salarinoghabi 氏（サンパウロ大学）との共同研究に基づく

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Contact (1)

Definition 1. Let $X_i, Y_i \subset \mathbb{R}^n$ $(i = 1, 2)$ be submanifolds with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$, and let $y_i \in X_i \cap Y_i$. We say that the contact of X_1 and Y_1 at y_1 is same type as the contact of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, x_1) \to (\mathbb{R}^n, x_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case, we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$.

Contact (2)

Definition 2. Two function germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are $\mathcal{K}$-equivalent if there are a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a function germ $A : (\mathbb{R}^n, 0) \to \mathbb{R}$ with $A(0) \neq 0$ such that $g \circ \phi(x) = A(x) f(x)$.

Theorem 3 (Montaldi, 1986)**.** Let $g_i : (X_i, x_i) \to (\mathbb{R}^n, y_i)$ $(i = 1, 2)$ be immersion germs and $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be submersion germs with $Y_i = f_i^{-1}(0)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

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Example 4. Let $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a regular curve, and set

$$F(x, y) = (x - a)^2 + (y - b)^2 - r^2.$$

Then γ has a circle with k-point contact at $\gamma(t_0)$ iff

$$f(t_0) = F(\gamma(t_0)) = 0, \quad f^{(i)}(t_0) = 0 \quad (1 \le i \le k - 1) \quad \text{and} \quad f^{(k)}(t_0) \ne 0.$$

The curve γ has k-th order vertex at $\gamma(t_0)$ if

$$\kappa(t_0) \ne 0, \quad \kappa^{(i)}(t_0) = 0 \quad (1 \le i \le k) \quad \text{and} \quad \kappa^{(k+1)}(t_0) \ne 0.$$

The curve γ has a circle with k-point contact ($k \ge 4$) at $\gamma(t_0)$ iff γ has a $(k - 3)$-th order vertex at $\gamma(t_0)$.

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A_k and D_k-singularities

In this talk, we consider the following contact types corresponding to $\mathcal{K}$-equivalence classes of $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ represented by

$$A_k^{\pm} : x^2 \pm y^{k+1}, \qquad D_k^{\pm} : x^2 y \pm y^{k-1}\, (k \geq 4).$$

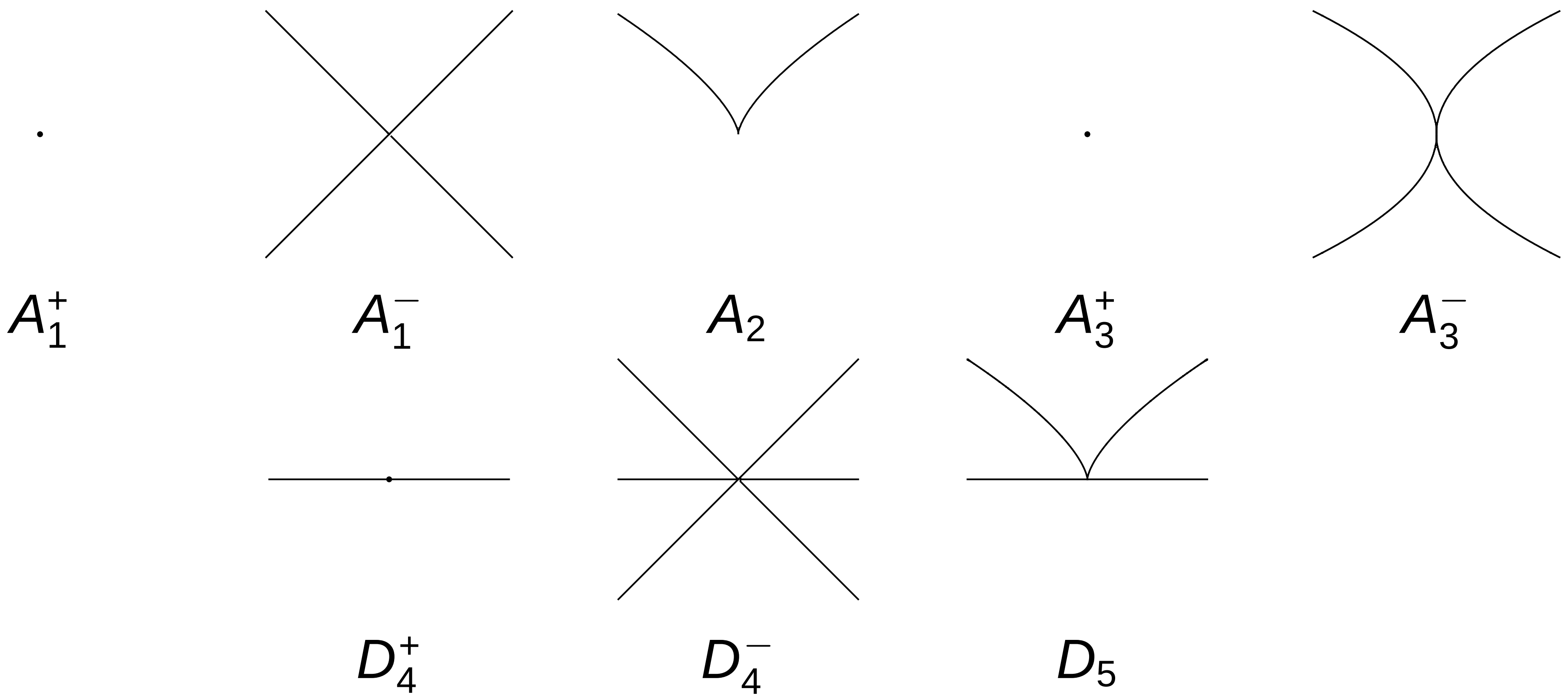

Figure 1: Zero sets of A_k and D_k-singularities.

Contact with planes

Let $S \subset \mathbb{R}^3$ be a regular surface, and let $\pi_{\boldsymbol{v},d}$ be a plane defined by

$$\langle \boldsymbol{x}, \boldsymbol{v} \rangle = d \qquad (\boldsymbol{x} = (x, y, z),\ \boldsymbol{v} \in S^2,\ d \in \mathbb{R}).$$

The moduli space of planes is three dimensional.

The contact of S with $\pi_{\boldsymbol{v},d}$ at $\boldsymbol{p} \in S \cap \pi_{\boldsymbol{v}}$ is as follows:

- $\star$ $A_{\geq 1}$-contact $\Longleftrightarrow$ $\boldsymbol{v} = \pm \boldsymbol{n}(\boldsymbol{p})$
- $\star$ $A_{\geq 2}$-contact $\Longleftrightarrow$ $\boldsymbol{p}$ is parabolic (i.e., $\kappa_i(\boldsymbol{p}) = 0$, $\kappa_j(\boldsymbol{p}) \neq 0$ $(i \neq j)$)
- $\star$ $A_{\geq 3}$-contact $\Longleftrightarrow$ $\boldsymbol{p}$ is a ridge point w.r.t. $\boldsymbol{v}_i$

Definition 5. A non-umbilic point $\boldsymbol{p}$ is called a ridge point w.r.t. the principal direction $\boldsymbol{v}_i$ if $\boldsymbol{v}_i\kappa_i(\boldsymbol{p}) = 0$. Moreover, $\boldsymbol{p}$ is called a k-th order ridge point w.r.t. $\boldsymbol{v}_i$ if $\boldsymbol{v}_i^{(n)}\kappa_i(\boldsymbol{p}) = 0$ $(1 \leq n \leq k)$ and $\boldsymbol{v}_i^{(k+1)}\kappa_i(\boldsymbol{p}) \neq 0$.

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The kernel filed of A_1^--contact with planes defines asymptotic directions in the hyperbolic region.

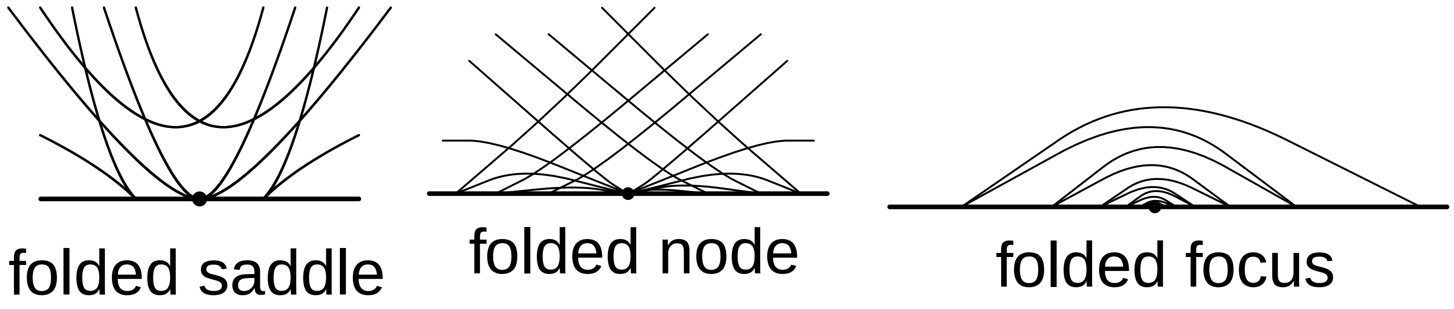

Figure 2: Singularities of asymptotic curves.

Contact with spheres

Let $S \subset \mathbb{R}^3$ be a regular surface, and let $S_{\boldsymbol{a},r}$ be the sphere defined by

$$|\boldsymbol{x} - \boldsymbol{a}|^2 = r^2 \qquad (\boldsymbol{x} = (x, y, z),\ \boldsymbol{a} \in \mathbb{R}^3,\ r > 0).$$

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The kernel filed of A_2-contact with spheres defines principal directions except at umbilic points.

Classification of singularity types of the lines of curvature at generic umbilics are known as Darboux classification.

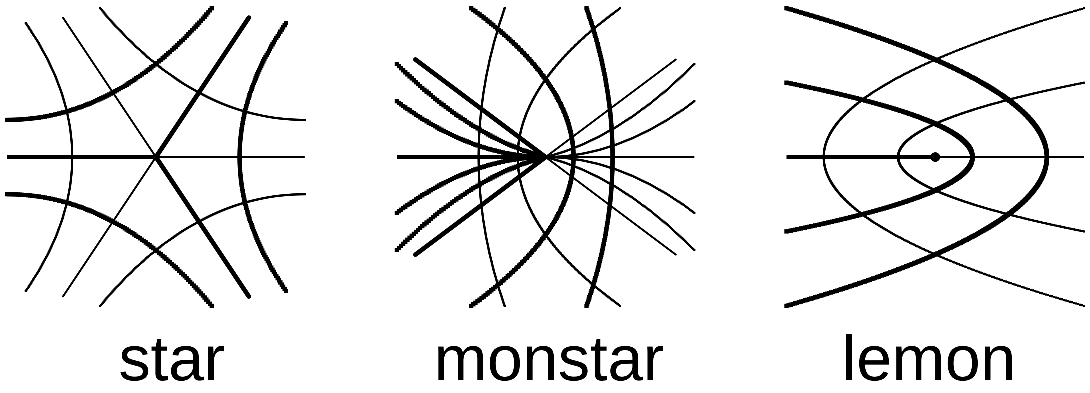

Figure 3: Singularities of lines of curvature near umbilic points.

Definition 6. A homogenous surface in $\mathbb{R}^3$ is an orbit of a certain subgroup of the Euclidean motion group $G = O(3) \rtimes \mathbb{R}^3$.

Theorem 7 (Takahashi, 1970)**.** Homogenous surfaces in $\mathbb{R}^3$ are planes, spheres and cylinders.

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Let $S \subset \mathbb{R}^3$ be a regular surface, and let denote $C_{\boldsymbol{v},\boldsymbol{a},r}$ the cylinder defined by

$$|\boldsymbol{x} - \langle \boldsymbol{x}, \boldsymbol{v} \rangle - \boldsymbol{a}|^2 = r^2$$

$$(\boldsymbol{x} = (x, y, z), (\boldsymbol{v}, \boldsymbol{a}) \in \{(\boldsymbol{v}, \boldsymbol{a}) \in S^2 \times \mathbb{R}^3 | \langle \boldsymbol{v}, \boldsymbol{a} \rangle = 0\}, r > 0).$$

The moduli space of cylinders is five dimensional, and thus we expect that there are generically A_1, A_2, A_3, A_4, A_5, D_4 and D_5-contact cylinders for a regular surface $S \subset \mathbb{R}^3$.

Cylinders

Let $S \subset \mathbb{R}^3$ be a regular surface, and let denote $C_{\boldsymbol{v},\boldsymbol{a},r}$ the cylinder defined by

$$|\boldsymbol{x} - \langle \boldsymbol{x}, \boldsymbol{v} \rangle - \boldsymbol{a}|^2 = r^2$$

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The moduli space of cylinders is five dimensional, and thus we expect that there are generically A_1, A_2, A_3, A_4, A_5, D_4 and D_5-contact cylinders for a regular surface $S \subset \mathbb{R}^3$.

Contact with cylinders (1)

Theorem 8. The contact of S with $C_{\boldsymbol{v},\boldsymbol{a},r}$ at $\boldsymbol{p} \in S \cap C_{\boldsymbol{v},\boldsymbol{a},r}$ is as follows:

- $\star$ $A_{\geq 1}$-contact $\Longleftrightarrow$ $\boldsymbol{v} \in T_{\boldsymbol{p}}S$ and $\boldsymbol{a} = \boldsymbol{p} - \langle \boldsymbol{p}, \boldsymbol{v} \rangle \boldsymbol{v} \pm r\, \boldsymbol{n}(\boldsymbol{p})$
- $\star$ $A_{\geq 2}$-contact $\Longleftrightarrow$ one of the following conditions holds:
  - $\boldsymbol{p}$ is not a parabolic point, $\boldsymbol{v}$ is not asymptotic at $\boldsymbol{p}$ and $\boldsymbol{a} = \boldsymbol{p} - \langle \boldsymbol{p}, \boldsymbol{v} \rangle \boldsymbol{v} + \boldsymbol{n}(\boldsymbol{p})/\kappa_c$,
  - $\boldsymbol{p}$ is a parabolic but not umbilic point (i.e., $\kappa_i(\boldsymbol{p}) = 0$, $\kappa_j(\boldsymbol{p}) \neq 0$), $\boldsymbol{v}$ is asymptotic at p and $\lambda \neq 1/\kappa_j(\boldsymbol{p})$,
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- ⋆ $A_{\geq 1}$-contact $\iff$ $\boldsymbol{v} \in T_{\boldsymbol{p}}S$ and $\boldsymbol{a} = \boldsymbol{p} - \langle \boldsymbol{p}, \boldsymbol{v} \rangle \boldsymbol{v} \pm r\,\boldsymbol{n}(\boldsymbol{p})$
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Theorem 9. Assume that $\boldsymbol{p}$ is not a parabolic point, $\boldsymbol{v}$ is not asymptotic at $\boldsymbol{p}$ and $r = 1/\kappa_c$. Then

A_k-contact

$\Longleftrightarrow$ the apparent contour of S along $\boldsymbol{v}$ has a circle with $(k+1)$-contact at the corresponding point to $\boldsymbol{p}$,

$\Longleftrightarrow$ the apparent contour of S along $\boldsymbol{v}$ has a $(k-2)$-th order vatex at the corresponding point to $\boldsymbol{p}$.

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Contact with cylinders (2)

Theorem 9. Assume that $\boldsymbol{p}$ is not a parabolic point, $\boldsymbol{v}$ is not asymptotic at $\boldsymbol{p}$ and $r = 1/\kappa_c$. Then
A_k-contact
$\Longleftrightarrow$ the apparent contour of S along $\boldsymbol{v}$ has a circle with $(k+1)$-contact at the corresponding point to $\boldsymbol{p}$,
$\Longleftrightarrow$ the apparent contour of S along $\boldsymbol{v}$ has a $(k-2)$-th order vatex at the corresponding point to $\boldsymbol{p}$.

Monge form of S is given by the following expression:

(1)

$$\begin{cases} (x,y) \mapsto (x,y,f(x,y)), \\ f(x,y) = \sum_{s=2}^{k} f_s(x,y) + o(x,y)^k, \quad f_s = \sum_{i+j=s} \frac{a_{ij}}{i!j!} x^i y^j \end{cases}$$

In case of $k = 3$, f is given by

$$\begin{aligned} f(x,y) = &\frac{1}{2}(a_{20}x^2 + 2a_{11}xy + a_{02}y^2) \\ &+ \frac{1}{6}(a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3) + o(u,v)^3 \end{aligned}$$

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Definition 10. We say that (dx_1, dy_2) and (dx_2, dy_2) are conjugate at $\boldsymbol{p}$ if

$$Ldx_1dx_2 + M(dx_1dy_2 + dx_2dy_1) + Ndy_1dy_2 = 0$$

at $\boldsymbol{p}$, where L, M, N are the coefficients of the second fundamental form at $\boldsymbol{p}$.

Assume that S is given in Monge form (1).

Directions (dx, dy) and $(a_{11}dx + a_{02}dy, -a_{20}dx - a_{11}dy)$ are conjugate.

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Directions (dx, dy) and $(a_{11}dx + a_{02}dy, -a_{20}dx - a_{11}dy)$ are conjugate.

Cylindrical directions

Assume that the origin be not a parabolic point.

If S has $A_{\geq 3}$-contact with $C_{\boldsymbol{v},\boldsymbol{a},r}$ at the origin, then $(a_{11}dx + a_{02}dy, -a_{20}dx - a_{11}dy)$ is a root of f_3.

Definition 11. We say that (dx, dy) is the cylindrical direction at the origin if

$$f_3(a_{11}dx + a_{02}dy, -a_{20}dx - a_{11}dy) = 0.$$

The kernel filed of A_3-contact with cylinders defines cylindrical directions.

The principal and asymptotic directions are defined by quadratic Binary Differential Equations (BEDs), but the cylindrical direction is defined by a cubic BDE.

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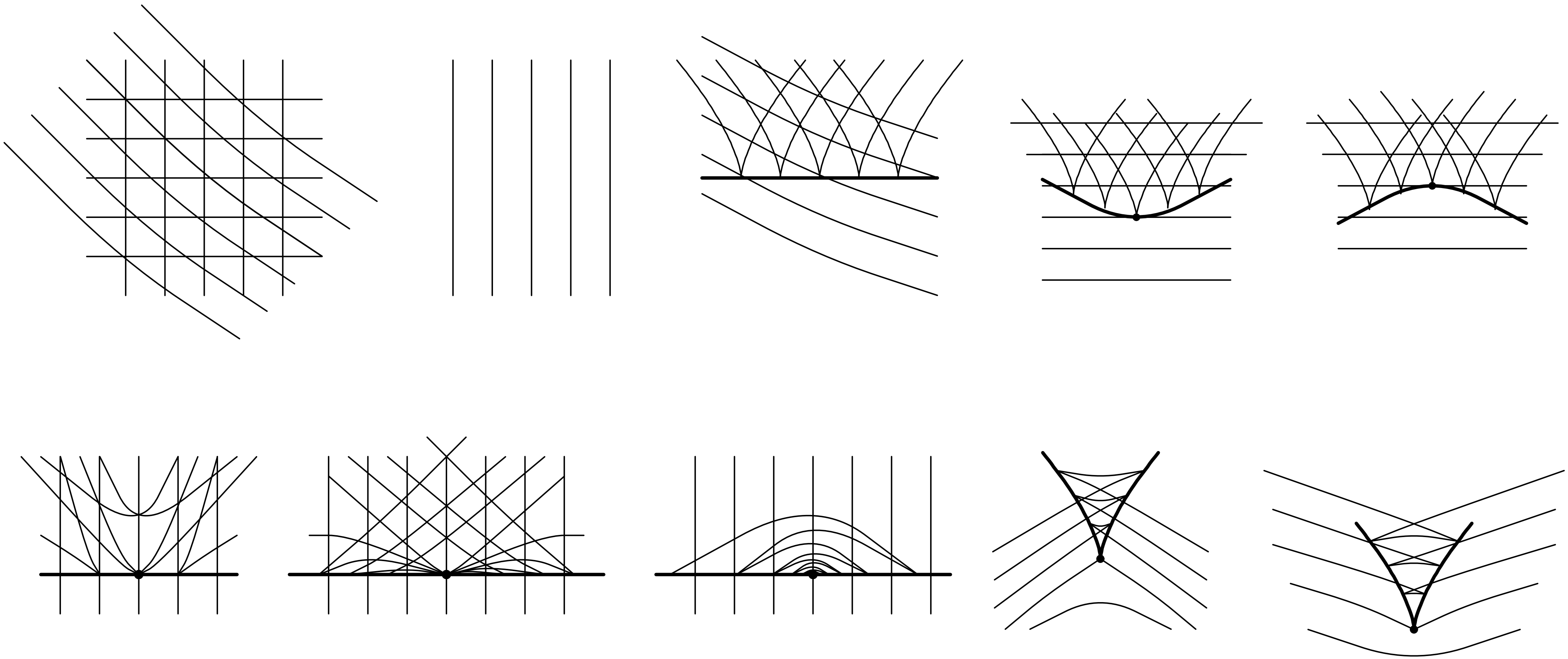
Singularities of integral curves of cylindrical directions



The thick lines represent the locus of points where the discriminant of f_3 is zero.

Definition 12. Two map-germs $f, g : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism-germs $\sigma : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ and $\tau : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\tau \circ f \circ \sigma = g$ holds.

Table 1: Generic $\mathcal{A}$-singularities of orthogonal projections of regular surfaces.

Name	Normal form
Fold	(x, y^2)
Cusp	$(x, xy + y^3)$
Lips/Beaks	$(x, y^3 \pm x^2 y)$
Goose	$(x, y^3 + x^3 y)$
Swallowtail	$(x, y + y^4)$
Butterfly	$(x, xy + y^5 \pm y^7)$
Gulls	$(x, xy^2 + y^4 + y^5)$

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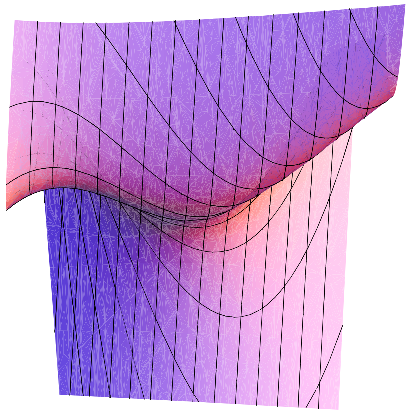

Figure 4:

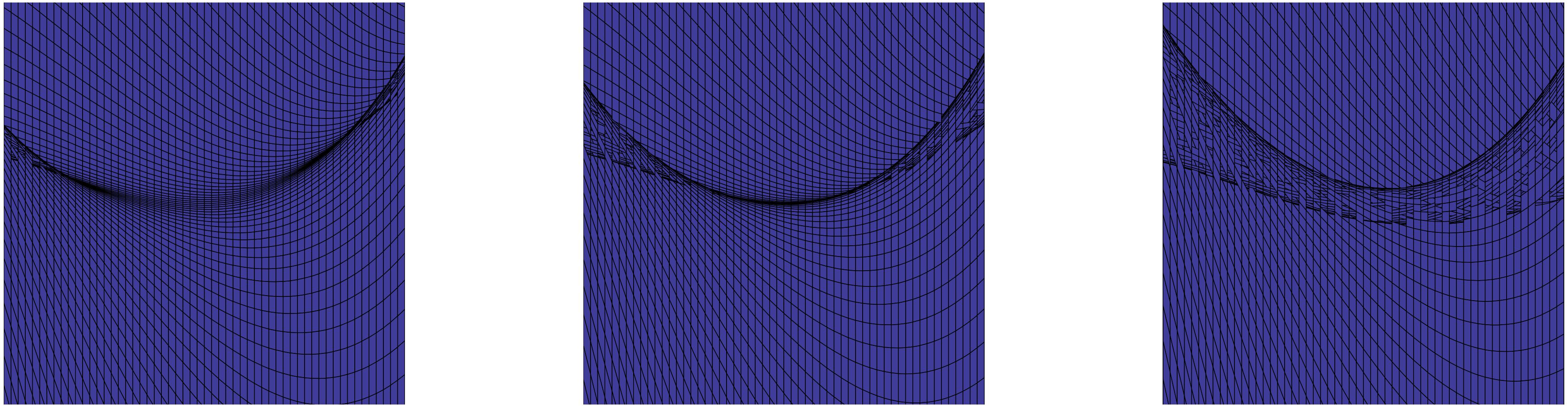

Figure 5:

Apparent contours

Let $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parameterization of a regular surface S.

The family of orthogonal projections $P : U \times S^2 \to TS^2$ of S is given by

$$P(x, y, \boldsymbol{v}) = (\boldsymbol{v}, P_{\boldsymbol{v}}), \quad P_{\boldsymbol{v}} = \langle \phi(x, y), \boldsymbol{v} \rangle \boldsymbol{v}$$

We consider $P_{\boldsymbol{v}}$ as the orthogonal projection of M along fixed direction $\boldsymbol{v}$.

We denote $\Sigma(P_{\boldsymbol{v}})$ the set of critical points of $P_{\boldsymbol{v}}$.

The image of $\Sigma(P_{\boldsymbol{v}})$ by $P_{\boldsymbol{v}}$ is the apparent contour of S along $\boldsymbol{v}$.

Apparent contours

Let $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parameterization of a regular surface S.
The family of orthogonal projections $P : U \times S^2 \to TS^2$ of S is given by

$$P(x, y, \boldsymbol{v}) = (\boldsymbol{v}, P_{\boldsymbol{v}}), \quad P_{\boldsymbol{v}} = \langle \phi(x, y), \boldsymbol{v} \rangle \boldsymbol{v}$$

We consider P_v as the orthogonal projection of M along fixed direction $\boldsymbol{v}$.
We denote $\Sigma(P_v)$ the set of critical points of P_v.
The image of $\Sigma(P_v)$ by P_v is the apparent contour of S along $\boldsymbol{v}$.

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We take $\boldsymbol{v}_0 \in T_{\boldsymbol{p}}S$.

We can rotate the coordinate axes if necessary and set $\boldsymbol{v}_0 = (0, 1, 0)$.

We then take the surface S in Monge form

$$\phi(x, y) = (x, y, f(x, y)), \quad f(x, y) = \sum_{i+j=2}^{k} \frac{a_{ij}}{i!j!} x^i y^j + o(x, y)^k$$

at a point $\boldsymbol{p}$ considered to be the origin in $\mathbb{R}^3$

With above setting, we have

$$P_{\boldsymbol{v}_0} = (x, f(x, y)).$$

We parametrize the directions near $\boldsymbol{v}_0$ by

$$\boldsymbol{v} = (-\cos v \sin u, \cos u \cos v, -\sin v).$$

The modified family of orthogonal projections (1)

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$$\boldsymbol{v} = (-\cos v \sin u, \cos u \cos v, -\sin v).$$

We modify P by the rotation

$$R = \begin{pmatrix} \cos u & \sin u & 0 \\ -\cos v \sin u & \cos u \cos v & -\sin v \\ -\sin u \sin v & \cos u \sin v & \cos v \end{pmatrix},$$

and we obtain the family of (germs of) projections $\widetilde{P} : (\mathbb{R}^2 \times \mathbb{R}^2, (0,0)) \to (\mathbb{R}^2, 0)$ of S given by

$$\widetilde{P}(x, y, u, v) = \begin{pmatrix} x \cos u + y \sin v \\ -x \sin u \sin v + y \cos u \sin v + f(x, y) \cos v \end{pmatrix}.$$

Remark that

$$P_{v_0} = \widetilde{P}(x, y, 0, 0) = (x, f(x, y)).$$

The modified family of orthogonal projections (2)

We modify P by the rotation

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Remark that

$$P_{\mathbf{v}_0} = \widetilde{P}(x, y, 0, 0) = (x, f(x, y)).$$

The modified family of orthogonal projections (3)

Assume that $P_{\boldsymbol{v}_0}$ has a beaks singularity, that is, $P_{\boldsymbol{v}_0}$ is $\mathcal{A}$-equivalent to $(x, y^3 - x^2y)$.

Then $\Sigma(P_{v_0})$ is locally a pair of intersecting smooth curves, and the deformation of $\Sigma(\widetilde{P}_v)$ is as shown in Figure 6.

Assume that $P_{\boldsymbol{v}_0}$ has a beaks singularity, that is, $P_{\boldsymbol{v}_0}$ is $\mathcal{A}$-equivalent to $(x, y^3 - x^2y)$.
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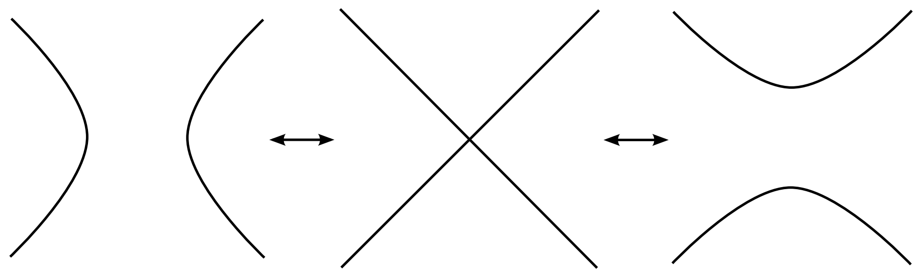

Figure 6: The modele of the deformation of $\Sigma(\widetilde{P}_{\boldsymbol{v}})$.

The modified family of orthogonal projections (4)

The beaks singularity is stable under perturbation of u, and thus we consider

$$F(x, y, v) = \widetilde{P}(x, y, 0, v).$$

To investigate inflections, cusps and vertices on the deformation in the apparent contour, we consider intersections between $\Sigma(F)$ and sets haveing some properties in the parameter space, where

$$\Sigma(F) = \{(x, y) | F_x = F_y = 0\}.$$

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Koenderink's theorem shows that if $\kappa_n(\boldsymbol{v}) \neq 0$, then

$$K = \kappa_n(\boldsymbol{v})\kappa_c$$

holds.

Therefore, $K = 0$ iff $\kappa_c = 0$ when $\kappa_n(\boldsymbol{v}) \neq 0$.

Set

$$I(x,y) = L(x,y)N(x,y) - M(x,y)^2.$$

Intersections between $\Sigma(F)$ and $I(x,y) = 0$ correspond to points where $\kappa_c = 0$, that is, inflections on the apparent contour along $\boldsymbol{v}$.

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Cusps on $\Sigma(F)$

If $\boldsymbol{v}$ is asymptotic at $\boldsymbol{p}$, that is, $\kappa_n(\boldsymbol{v}) = 0$ then the apparent contour along $\boldsymbol{v}$ has a cusp at point corresponding to $\boldsymbol{p}$.

For a tangent vector $\boldsymbol{w} = a\phi_x + b\phi_y$, set

$$C_w(x,y) = a^2L(x,y) + 2abM(x,y) + b^2N(x,y).$$

Since $C_w(x,y) = 0$ iff $\kappa_n(\boldsymbol{w}) = 0$, $C_w(x,y) = 0$ is the locus of points where $\boldsymbol{w}$ is asymptotic.

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To consider the locus of points where a tangen vector $\boldsymbol{w} = a\phi_x + b\phi_y$ is cylinderical, we consider Monge form of a surface S given by Monge form

$$\phi(x, y) = (x, y, f(x, y)), \quad f(x, y) = \sum_{i+j=2}^{k} \frac{a_{ij}}{i!j!} x^i y^j + o(x, y)^k$$

at a point $(x, y, f(x, y))$ considered to be the origin.

By the translation, change of basis matrix and change of coordinates, we obtain new Monge form of the surface S at $(x, y, f(x, y))$:

$$\tilde{\phi}(s, t) = (s, t, \tilde{f}(s, t)), \quad \tilde{f}(s, t) = \sum_{i+j=2}^{k} \frac{A_{ij}(x, y)}{i!j!} s^i t^j + o(s, t)^k,$$

where, for example,

$$A_{20}(x, y) = a_{20} + a_{30}x + a_{21}y + o(x, y).$$

Vertices on $\Sigma(F)$ (1)

To consider the locus of points where a tangen vector $\boldsymbol{w} = a\phi_x + b\phi_y$ is cylinderical, we consider Monge form of a surface S given by Monge form

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At the orign $\widetilde{\phi}(0,0)$, that is, the point $(x, y, f(x, y))$, the conjugate direction of $\boldsymbol{w} = a\widetilde{\phi}_s + b\widetilde{\phi}_t$ is

$$(aA_{11}(x,y) + bA_{20}(x,y))\widetilde{\phi}_s + (-aA_{20}(x,y) - bA_{11}(x,y))\widetilde{\phi}_t.$$

Hence, a tangent vector $\boldsymbol{w} = a\widetilde{\phi}_s + b\widetilde{\phi}_t$ is cylinderical iff

$$V_{\boldsymbol{w}}(x,y) = \widetilde{f}_3(aA_{11}(x,y) + bA_{20}(x,y), -aA_{20}(x,y) - bA_{11}(x,y)) = 0,$$

where $\widetilde{f}_3$ denotes the cubic part of $\widetilde{f}$.

Therefore, intersections between $\Sigma(F)$ and $V_{\boldsymbol{w}}(x,y) = 0$ correspond to vertices on the apparent contour along $\boldsymbol{w}$.

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Therefore, intersections between $\Sigma(F)$ and $V_w(x,y) = 0$ correspond to vertices on the apparent contour along $\boldsymbol{w}$.

Vertices on $\Sigma(F)$ (2)

At the orign $\widetilde{\phi}(0,0)$, that is, the point $(x, y, f(x, y))$, the conjugate direction of $\boldsymbol{w} = a\widetilde{\phi}_s + b\widetilde{\phi}_t$ is

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Hence, a tangent vector $\boldsymbol{w} = a\widetilde{\phi}_s + b\widetilde{\phi}_t$ is cylinderical iff

$$V_{\boldsymbol{w}}(x,y) = \widetilde{f}_3(aA_{11}(x,y) + bA_{20}(x,y), -aA_{20}(x,y) - bA_{11}(x,y)) = 0,$$

where $\widetilde{f}_3$ denotes the cubic part of $\widetilde{f}$.

Therefore, intersections between $\Sigma(F)$ and $V_{\boldsymbol{w}}(x,y) = 0$ correspond to vertices on the apparent contour along $\boldsymbol{w}$.

Now we consider $\boldsymbol{v} = (0, \cos v, -\sin v)$.

We then show that $I(x,y) = 0$ and $C_{\boldsymbol{v}}(x,y) = 0$ are tangent at $(0,0)$, and that $V_{\boldsymbol{v}}(x,y)$ has generically a $D_4^{\pm}$-singularity, that is $V_{\boldsymbol{v}}(x,y) = 0$ is locally one smooth curve or three intersecting smooth curves near $(0,0)$.

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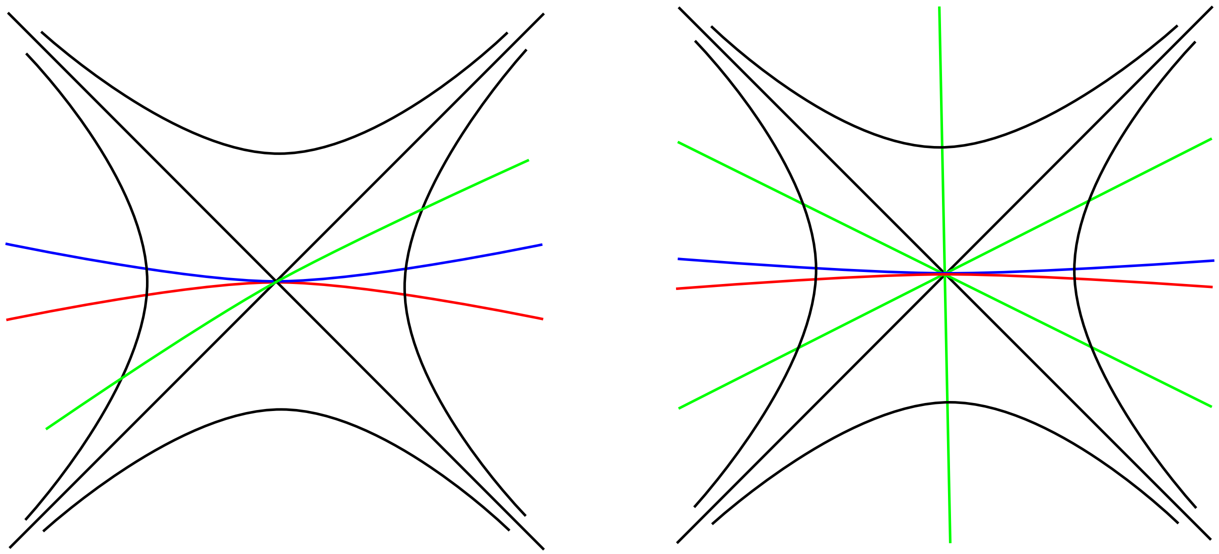

Figure 7: The arrangement of $\Sigma(F)$, $I(x, y) = 0$, $C_{\boldsymbol{v}}(x, y) = 0$ and $V_{\boldsymbol{v}}(x, y) = 0$: $V_{\boldsymbol{v}}$ has a D_4^+ (left), $V_{\boldsymbol{v}}$ has a D_4^- (right).

Inflections, cusps and vertices on $\Sigma(F)$

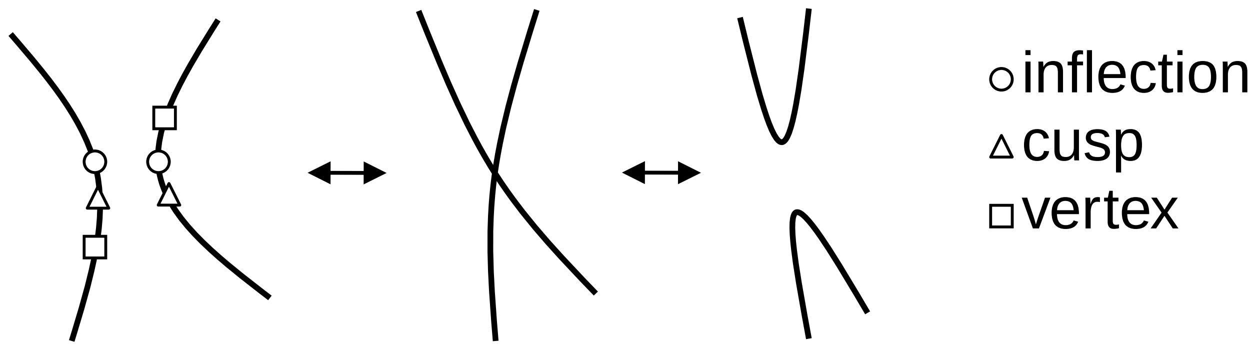

Figure 8: Inflections, cusps and vertices on $\Sigma(F)$ when $V_{\boldsymbol{v}}$ has a D_4^+-singularity.

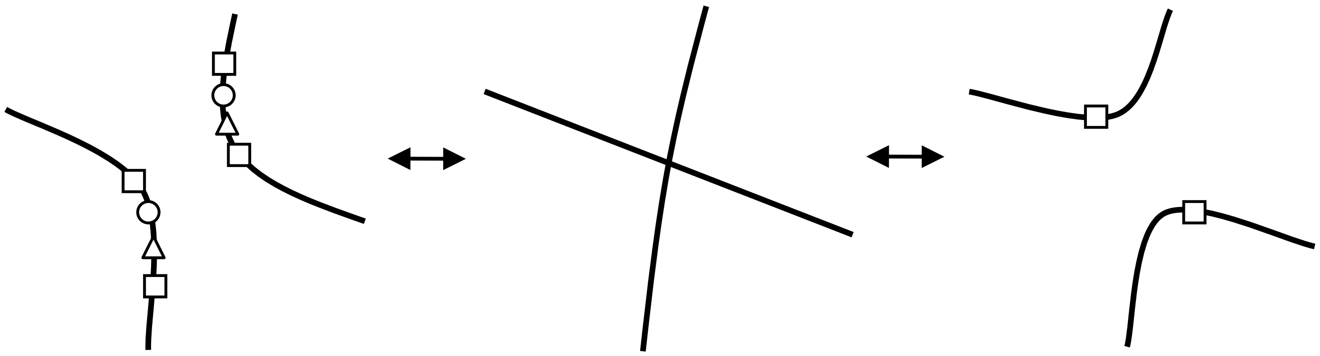

Figure 9: Inflections, cusps and vertices on $\Sigma(F)$ when $V_{\boldsymbol{v}}$ has a D_4^--singularity.

Theorem 13. There are generically two beaks type singularity of orthogonal projections of S along $\boldsymbol{v}$, one labeled D_4^+-beaks where $V_{\boldsymbol{v}}$ has a D_4^+-singularity, and the other labeled D_4^--beaks where $V_{\boldsymbol{v}}$ has a D_4^--singularity. The deformation of the apparent coutour of D_4^+ and D_4^--beaks are shown as in Figure 10 and 11, respectively.

The deformation of the apparent contour of $D_4^{\pm}$-beaks

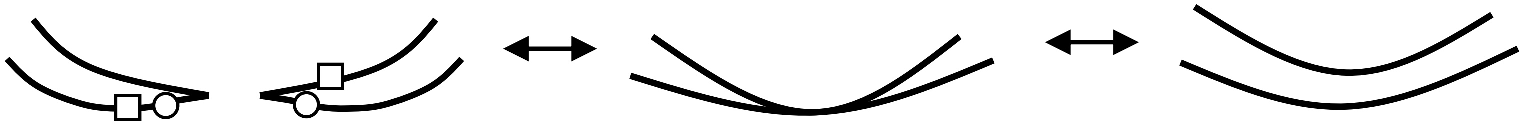

Figure 10: The deformation of the apparent contour of D_4^{+}-beaks.

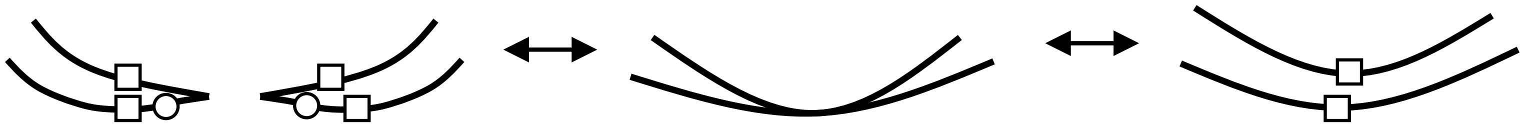

Figure 11: The deformation of the apparent contour of D_4^{-}-beaks.