グラフの Hom 複体と彩色数について

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- The Hom complex $\mathrm{Hom}(T, G)$ is a poset associated to a pair of graphs (T, G).
- A group action on T induces a group action on $\mathrm{Hom}(T, G)$.
- The equivariant homotopy type of $\mathrm{Hom}(T, G)$ is closely related to the chromatic number $\chi(G)$ (graph coloring problem).

Graphs

A graph is a pair $G = (V(G), E(G))$ consisting of a set $V(G)$ together with a symmetric subset $E(G)$ of $V(G) \times V(G)$, i.e.

$$(x, y) \in E(G) \Rightarrow (y, x) \in E(G).$$

- Namely, our graphs are undirected, have no multiple edges, may have loops.
- A graph is *simple* if a graph has no looped vertices.

Graph coloring problem

Definition 1 (Chromatic number)

An *n-coloring of G* is a map $c : V(G) \to \{1, \cdots, n\}$ such that $(x, y) \in E(G) \Rightarrow c(x) \neq c(y)$. The *chromatic number $\chi(G)$ of G* is the smallest integer n such that G has an n-coloring.

Problem 2 (Graph coloring problem)

Compute the chromatic number of a given graph G.

Lovász applied algebraic topology to this subject, in the proof of Kneser's conjecture.

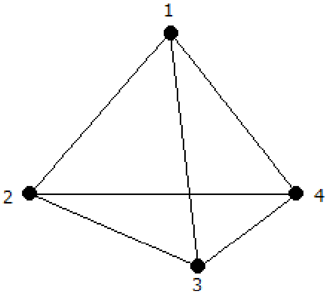
Graph homomorphism

A graph homomorphism is a map $f : V(G) \to V(H)$ such that $(f \times f)(E(G)) \subset E(H)$, i.e. $(x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(H)$. For $n \geq 0$, define the graph K_n by $V(K_n) = \{1, \cdots, n\}$ and $E(K_n) = \{(x, y) \mid x \neq y\}$.

- Then an n-coloring of G is identified with a graph homomorphism from G to K_n. Thus

$$\chi(G) = \inf\{n \geq 0 \mid \exists G \to K_n\}$$

K_4



K_4

Multi-homomorphism

A multi-homomorphism from G to H is a map $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ such that $(v, w) \in E(G) \Rightarrow \eta(v) \times \eta(w) \subset E(H)$. We write $\eta \leq \eta'$ if $\eta(v) \subset \eta'(v)$ for every $v \in V(G)$.

- A graph homomorphism $f : G \to H$ is regarded as a multi-homomorphism $v \mapsto \{f(v)\}$.

Definition 3

The *Hom complex* $\mathrm{Hom}(G, H)$ of graphs G and H is the poset of multi-homomorphisms from G to H.

In particular, we write $B(G)$ instead of $\mathrm{Hom}(K_2, G)$, and call it the *box complex of* G.

Functorial property

Let $\mathcal{G}$ denote the category of graphs and $\mathcal{P}$ the category of posets. The Hom complex gives a functor

$$\mathcal{G}^{\mathrm{op}} \times \mathcal{G} \to \mathcal{P},\ (T, G) \mapsto \mathrm{Hom}(T, G).$$

Let Γ be a finite group.

- Let T be a (right) Γ-graph. Then $\mathrm{Hom}(T, G)$ is a (left) Γ-poset. In particular, $B(G) = \mathrm{Hom}(K_2, G)$ as a $\mathbb{Z}_2$-poset.
- A graph homomorphism $f : G \to H$ induces a Γ-poset map $\mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$.

Conversely, if there is no Γ-equivariant map from $\mathrm{Hom}(T, G)$ to $\mathrm{Hom}(T, H)$, then there is no graph homomorphism $G \to H$.

Box complex and neighborhood complex

In general, it is difficult to describe $\mathrm{Hom}(T, G)$ even if $T = K_2$.

Definition 4 (Lovász '79)

Let G be a graph. The *neighborhood complex* $N(G)$ *of* G is the simplicial complex

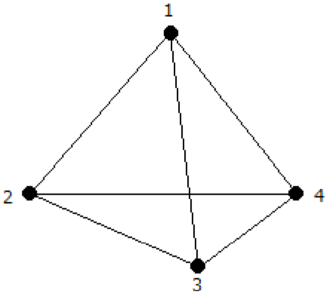
$$N(G) = \{\sigma \subset V(G) \mid \#\sigma < +\infty, \exists v \in V(G) \text{ s.t. } \sigma \subset N(v)\}.$$

Here we denote by $N(v)$ the set of vertices adjacent to v.

Proposition 5 (Babson-Kozlov '07)

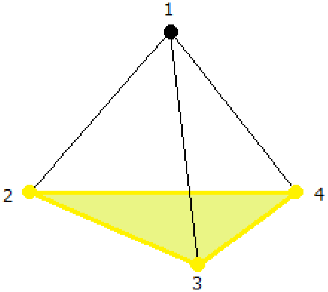
Let G be a graph. Then $N(G)$ and $B(G)$ are homotopy equivalent.

The neighborhood complex of K_n



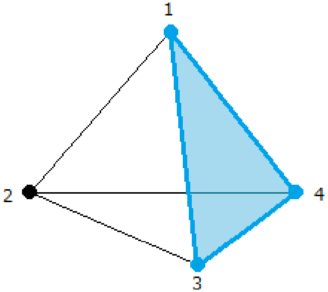
K_4

The neighborhood complex of K_n



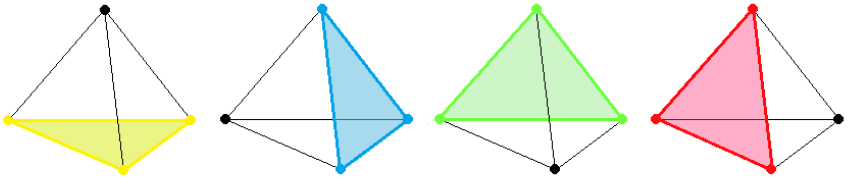
$$N(1) = \{2, 3, 4\}$$

The neighborhood complex of K_n



$$N(2) = \{1, 3, 4\}$$

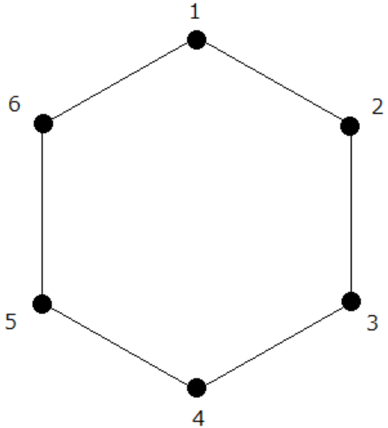
The neighborhood complex of K_n



$$N(K_4) \approx \partial\Delta^3 \approx S^2$$

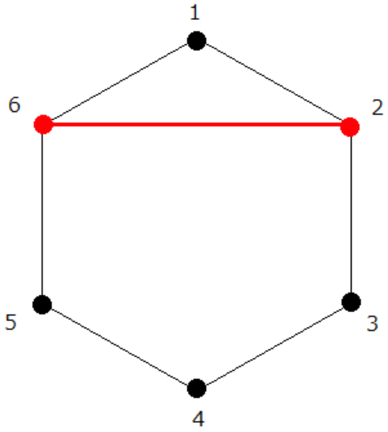
In general, $N(K_n) \approx \partial\Delta^{n-1} \approx S^{n-2}$ $(n \geq 1)$.

The neighborhood complex of C_{2n}



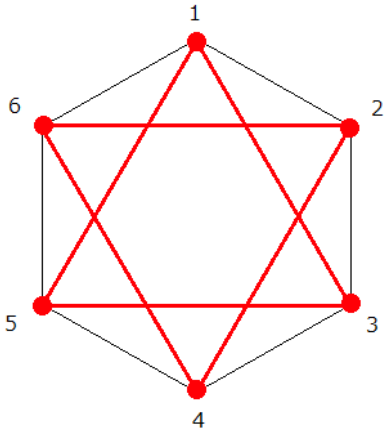
C_6

The neighborhood complex of C_{2n}



$$N(1) = \{6, 2\}$$

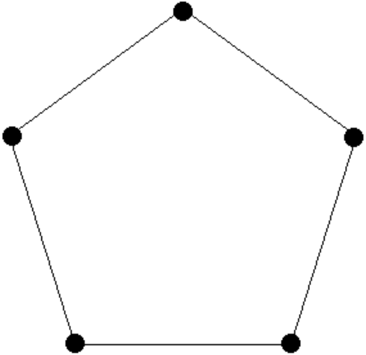
The neighborhood complex of C_{2n}



$$N(C_6) \approx S^1 \sqcup S^1$$

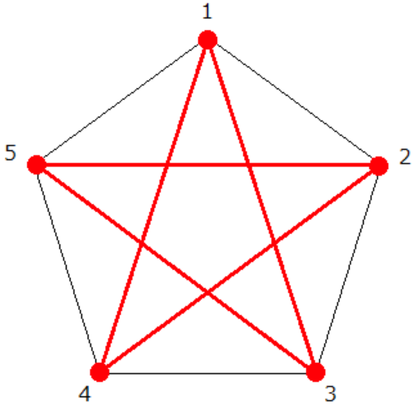
In general, $B(C_{2n}) \simeq N(C_{2n}) \approx S^1 \sqcup S^1$ $(n \geq 3)$. If $n = 1$, one can show $N(C_4) \simeq S^0$.

The neighborhood complex of C_{2n+1}



C_5

The neighborhood complex of C_{2n+1}



$N(C_5) \approx S^1$. In general $N(C_{2n+1}) \approx S^1$. $(n \geq 1)$.

Theorem 6 (Babson-Kozlov '06)

For $n \geq 1$, $B(K_n) \approx_{\mathbb{Z}_2} S^{n-2}$.

Corollary 7 (Lovász '79)

If $N(G)(\simeq B(G))$ *is* n*-connected, then* $\chi(G) \geq n+3$.

Proof.

Suppose $\chi(G) = m$ and let $f : G \to K_m$. Then $f_* : B(G) \to B(K_m) \approx_{\mathbb{Z}_2} S^{m-2}$. If $B(G)$ is n-connected, there is a $\mathbb{Z}_2$-map $S^{n+1} \to B(G)$. Thus there is a $\mathbb{Z}_2$-map from S^{n+1} to S^{m-2}. By the Borsuk-Ulam theorem, we have $n+1 \leq m-2$. □

Kneser's conjecture

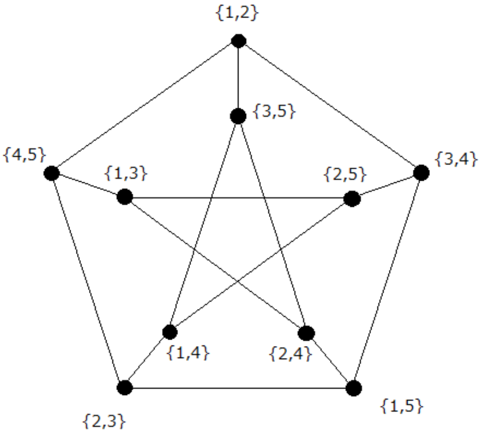
Let n and k be positive integers with $n \geq 2k$. Define the Kneser graph $KG_{n,k}$ as follows:

$$V(KG_{n,k}) = \{\sigma \subset \{1, \cdots, n\} \mid \#\sigma = k\},$$

$$E(KG_{n,k}) = \{(\sigma, \tau) \mid \sigma \cap \tau = \emptyset\}.$$

- $KG_{n,1} = K_n$
- $KG_{2k,k}$ is a disjoint union of K_2.

Kneser's conjecture

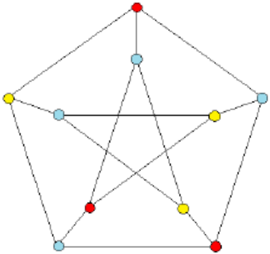


$KG_{5,2}$: Peterson graph

Kneser's conjecture

Problem 8 (Kneser's conjecture, '55)

$\chi(KG_{n,k}) = n - 2k + 2$



$\chi(KG_{5,2}) = 5 - 2 \cdot 2 + 2 = 3$

It is easy to see $\chi(KG_{n,k}) \leq n - 2k + 2$. Lovász showed that $N(KG_{n,k})$ ($\simeq B(KG_{n,k})$) is $(n-2k-1)$-connected and $\chi(KG_{n,k}) = n - 2k + 2$.

In this talk, we investigate the relationship between the topology of Hom complexes and the chromatic number.

- A conjecture by Kozlov about "homotopy test graphs"
- The homotopy types of $\mathrm{Hom}(T, G)$ do not determine $\chi(G)$. Namely, for a finite graph T and a graph G with $\chi(G) > 2$, there is a graph H such that $\mathrm{Hom}(T, G) \simeq \mathrm{Hom}(T, H)$ but $\chi(H)$ is much greater than $\chi(G)$.
- $B(G) \cong B(H)$ as $\mathbb{Z}_2$-posets, then $G \cong H$ up to isolated vertices. Moreover, we have $\chi(G) = \chi(H)$.

Homotopy test graphs

Definition 9 (Kozlov '07)

A graph T is a homotopy test graph if the following inequality holds for every graph G:

$$\chi(G) > \mathrm{conn}(\mathrm{Hom}(T, G)) + \chi(T)$$

For a space X, $\mathrm{conn}(X)$ is the maximal integer n such that X is n-connected.

Example 10

Since $\mathrm{Hom}(K_2, G) = B(G) \simeq N(G)$, Lovász's result implies that

$$\begin{aligned} \chi(G) &\geq \mathrm{conn}(N(G)) + 3 \\ &= \mathrm{conn}(\mathrm{Hom}(K_2, G)) + \chi(K_2) + 1. \end{aligned}$$

Thus K_2 is a homotopy test graph.

Homotopy test graphs

- K_n $(n \geq 3)$ is a homotopy test graph (Babson-Kozlov).
- An odd cycle C_{2r+1} $(r \geq 3)$ is a homotopy test graph (Babson-Kozlov).
- Lovász conjectured that every graph with at least one edge is a homotopy test graph. But there is a graph which is not a homotopy test graph (Hoory-Linial).

Kozlov (2007) conjectured that if $\chi(T) = 2$, then T is a homotopy test graph. I solved this conjecture affirmatively.

Theorem 11 (M.)

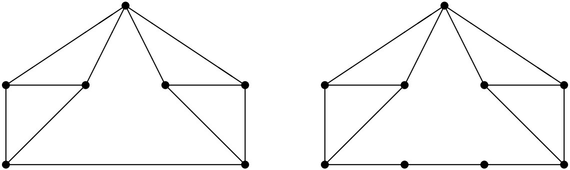
$\chi(T) = 2$ *implies that* T *is a homotopy test graph.*

Here we give a proof of the theorem.
If $\chi(T) = 2$, then K_2 is a retract of T. Namely, there are graph homomorphisms $i : K_2 \to T$ and $r : T \to K_2$ such that $ri = \mathrm{id}_{K_2}$.
Suppose that $\mathrm{conn}(\mathrm{Hom}(T, G)) = n$. Then $\mathrm{Hom}(K_2, G)$ is n-connected since $\mathrm{Hom}(K_2, G)$ is a retract of $\mathrm{Hom}(T, G)$. Since K_2 is a homotopy test graph, we have

$$\chi(G) > n + 2 = \mathrm{conn}(\mathrm{Hom}(T, G)) + \chi(T).$$

□

Thus one can expect that $\mathrm{Hom}(T, G)$ determines $\chi(G)$ for some homotopy test graph T. Walker considered that the $\mathbb{Z}_2$-homotopy type of $\mathrm{Hom}(K_2, G)$ does not determine $\chi(G)$.



G_1 G_2

Walker (1983) showed that $B(G_1) \simeq_{\mathbb{Z}_2} B(G_2)$ but $\chi(G_1) \neq \chi(G_2)$.

Theorem 12 (M.)

Let T be a finite graph, G a graph with $\chi(G) > 2$, and n an integer. Then there is a graph H which contains G as a subgraph and satisfies the following properties:

(1) *The inclusion $\mathrm{Hom}(T, G) \hookrightarrow \mathrm{Hom}(T, H)$ is a homotopy equivalence.*

(2) $\chi(H) > n$

In case $T = K_2$, then we can take H so that $\mathrm{Hom}(T, G) \hookrightarrow \mathrm{Hom}(T, H)$ is a $\mathbb{Z}_2$-homotopy equivalence.

Thus the homotopy type of $\mathrm{Hom}(T, G)$ or the $\mathbb{Z}_2$-homotopy type of $B(G)$ does not determine $\chi(G)$. Moreover, there is no homotopy invariant of $\mathrm{Hom}(T, G)$ which gives an upper bound for $\chi(G)$.

A sketch of the proof

We give a sketch of the proof in case $T = K_2$.

Definition 13

The *girth* $g(G)$ *of a graph* G is the minimum integer n such that the n-cycle C_n embeds into G.

Intuitively, the girth of a graph G is large implies that G is "locally tree". Thus the following theorem implies that the chromatic number is not a local invariant of graphs.

Theorem 14 (Erdős)

Let n be an integer. Then there is a finite graph G such that $\chi(G) > n$ and $g(G) > n$.

Sketch of the proof

It is known that if the girth of G is greater than 4, then $B(G)$ is $\mathbb{Z}_2$-homotopy equivalent to a 1-dimensional $\mathbb{Z}_2$-complex.

Proposition 15 (M.)

Let G and H be finite graphs and $\varphi : |B(G)| \to |B(H)|$ a $\mathbb{Z}_2$-map. Then there are a graph G' and graph homomorphisms $\varepsilon : G' \to G$ and $f : G' \to H$ which satisfies the following:

- *$\varepsilon_* : B(G') \to B(G)$ is a $\mathbb{Z}_2$-homotopy equivalence.*
- *The diagram*

$$\begin{array}{ccc} B(G) & \xrightarrow{\varphi} & B(H) \\ {\scriptstyle\varepsilon}\uparrow & & \| \\ B(G') & \xrightarrow{f_*} & B(H) \end{array}$$

  is commutative up to $\mathbb{Z}_2$-homotopy.

Sketch of the proof

We turn to the proof of the theorem. Let G be a graph with $\chi(G) > 2$ and n an positive integer. We want to construct a graph H containing G such that $B(G) \hookrightarrow B(H)$ is a $\mathbb{Z}_2$-homotopy equivalence and $\chi(H) > n$.

Since $\chi(G) > 2$, there is a graph homomorphism from C_{2m+1} to G for some m. Thus there is a $\mathbb{Z}_2$-map from $S^1 (\approx_{\mathbb{Z}_2}) B(C_{2m+1})$ to $B(G)$.

Let X be a finite graph such that $\chi(X) > n$ and $g(X) > 4$. Since $g(X) > 4$, the box complex $B(X)$ is $\mathbb{Z}_2$-homotopy equivalent to a 1-dimensional $\mathbb{Z}_2$-complex. Hence there is a $\mathbb{Z}_2$-map from $B(X)$ to $B(G)$.

Let X' be a finite graph and graph homomorphisms $\varepsilon : X' \to X$ and $f : X' \to G$ such that $\varphi \circ \varepsilon_* \simeq_{\mathbb{Z}_2} f$.

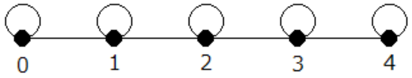
Sketch of the proof

Let k be a positive integer with $k \geq 2$. Define the graph I_n as follows:

$$V(I_n) = \{0, 1, \cdots, n\}$$

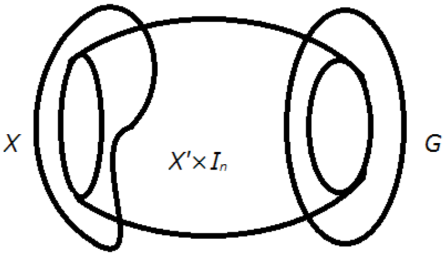
$$E(I_n) = \{(x, y) \mid |x - y| \leq 1\}.$$

Consider "the cylinder" $X' \times I_n$ of X'. The graph H is obtained by attaching two ends of $X' \times I_n$ by $\varepsilon : X' \to X$ and $f : X' \to G$.



I_4

Sketch of the proof



※ H のイメージ

Since $X \subset H$, we have that $\chi(H) \geq \chi(X) > n$. Moreover, one can show that $B(H)$ is a homotopy pushout of $\varepsilon_* : B(X') \to B(X)$ and $f_* : B(X') \to B(G)$. Since ε_* is a $\mathbb{Z}_2$-homotopy equivalence, we have that $B(G) \hookrightarrow B(H)$ is a $\mathbb{Z}_2$-homotopy equivalence. □

Poset structure of $B(G)$

Thus to determine the chromatic number, we must investigate more rigid structures of the Hom complex $\mathrm{Hom}(T, G)$. However, in case $T = K_2$, the following theorem holds:

Theorem 16 (M.)

Let G and H be graphs having no isolated vertices. Then the following hold:

(1) $K_2 \times G \cong K_2 \times H$ *iff* $B(G) \cong B(H)$ *as posets.*

(2) $G \cong H$ *iff* $B(G) \cong B(H)$ *as* $\mathbb{Z}_2$*-posets.*

(3) *If* $K_2 \times G \cong K_2 \times H$, *then* $N(G) \cong N(H)$. *On the other hand, if* G *and* H *are stiff and* $N(G) \cong N(H)$, *then* $K_2 \times G \cong K_2 \times H$.

The graph $K_2 \times G$ is called the *Kronecker double covering of* G.

Poset structure of $B(G)$

- In the previous theorem, we have that $B(G) \cong B(H)$ as $\mathbb{Z}_2$-posets, then $G \cong H$ and hence $\chi(G) = \chi(H)$.
- On the other hand, we use (1) to construct graphs G and H whose neighborhood complexes and box complexes are isomorphic (as posets), but whose chromatic numbers are different.

Kronecker double covering

Let G and H be graphs. Define the (Kronecker) product $G \times H$ as follows:

$$V(G \times H) = V(G) \times V(H),$$

$$E(G \times H) = \{((x,y),(x',y')) \mid (x,x') \in E(G), (y,y') \in E(H)\}.$$

In general, it is difficult to describe the product of graphs. However, the Kronecker double covering $K_2 \times G$ has a simple geometric description.

Kronecker double coverings

A graph G is *bipartite* if $\chi(G) \leq 2$. Suppose that X is a connected bipartite graph. Then an involution α of X is *odd* if a length of a path joining v to $\alpha(v)$ is odd for some (or any) vertex of v. For an odd involution α of X, the quotient by the $\mathbb{Z}_2$-action is denoted by X/α. The Kronecker double covering $K_2 \times G$ is bipartite. The involution

$$\alpha : K_2 \times G \to K_2 \times G,\ (1, v) \leftrightarrow (2, v)$$

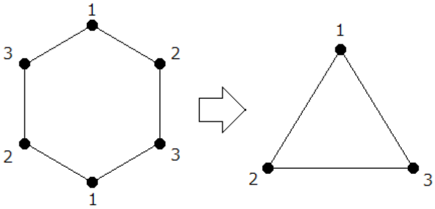
is an odd involution of $K_2 \times G$.

Proposition 17

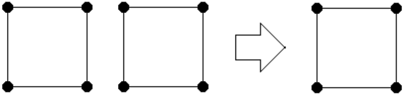
Let α be an odd involution of a bipartite graph X. Then there is an isomorphism

$$K_2 \times (X/\alpha) \xrightarrow{\cong} X.$$

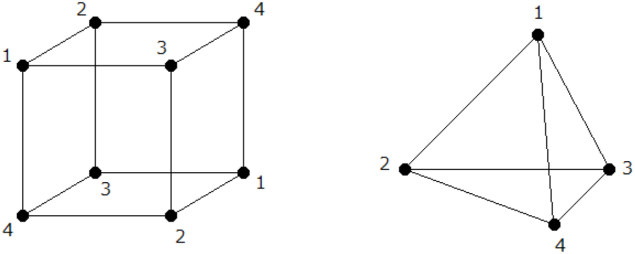
Examples of Kronecker double coverings



$$C_6 \cong K_2 \times C_3$$



$$C_4 \sqcup C_4 \cong K_2 \times C_4$$



$$K_2 \times K_4$$

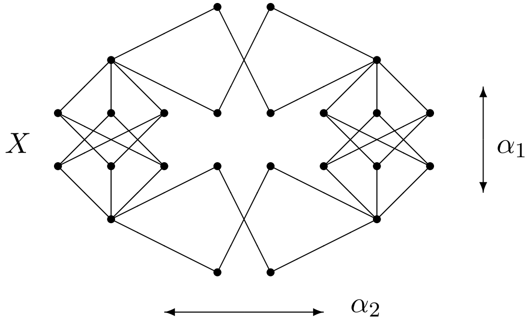
Proposition 18 (M.)

Let m and n be integers greater than 1. Then there are graphs G and H such that $\chi(G) = m$, $\chi(H) = n$, and $K_2 \times G \cong K_2 \times H$.

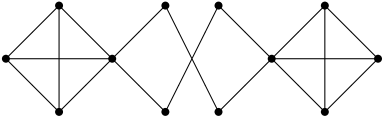
Corollary 19 (M.)

Let m and n be integers greater than 1. Then there are graphs G and H such that $\chi(G) = m$, $\chi(H) = n$, but $B(G) \cong B(H)$ as posets and $N(G) \cong N(H)$.

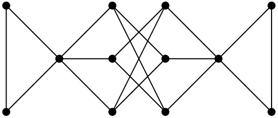
Lovász (1979) asked if there is a topological invariant of the neighborhood complex $N(G)$ which is equivalent to the chromatic number. The above corollary gives a negative answer to his question.



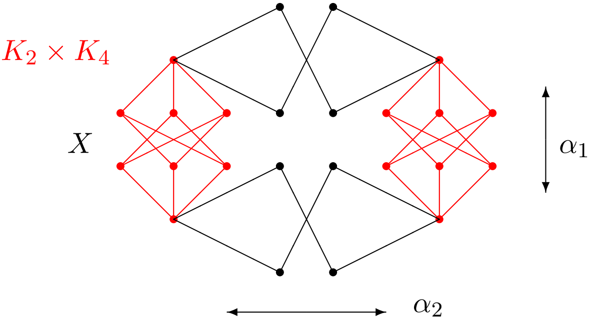
$$X \cong K_2 \times G \cong K_2 \times H$$



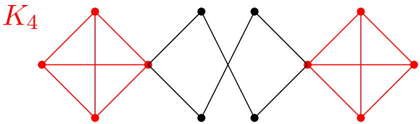
$G = X/\alpha_1, \ \chi(G) = 4$

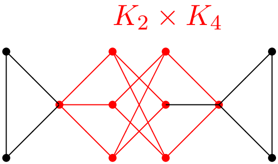


$H = X/\alpha_2, \ \chi(H) = 3$

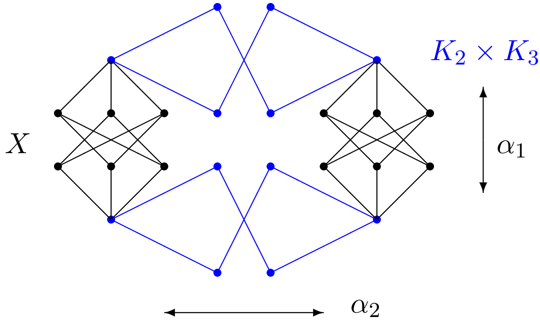


$$X \cong K_2 \times G \cong K_2 \times H$$

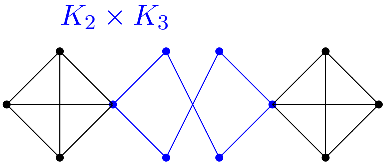




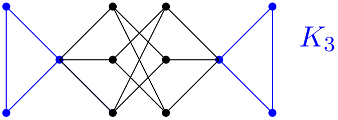
$G = X/\alpha_1, \ \chi(G) = 4$ $\qquad$ $H = X/\alpha_2, \ \chi(H) = 3$



$$X \cong K_2 \times G \cong K_2 \times H$$



$G = X/\alpha_1, \ \chi(G) = 4$



$H = X/\alpha_2, \ \chi(H) = 3$

Summary

- The homotopy type of $\mathrm{Hom}(T, G)$ does not determine $\chi(G)$.
- On the other hand, the $\mathbb{Z}_2$-poset structure of $B(G)$ determines the graph (up to isolated points) and the chromatic number.
- However, the poset structure of $B(G)$, or $N(G)$ does not determine $\chi(G)$.

Thank you very much for listening.