LINKING INVARIANT FOR ALGEBRAIC PLANE CURVES

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Introduction

The subject of our interest is the topology of algebraic plane curves (see [8, 12] for basic results). They are geometric objects defined from polynomial equations. More precisely, an *algebraic plane curve* is the zeros locus of a homogenous polynomial $\mathbb{C}[X_0, X_1, X_2]$. Abstractly, such a curve is defined by its combinatorics. Let $\mathrm{Irr}(\mathcal{C}) = \{(C_1, d_1), \dots, (C_n, d_n)\}$ be the set of algebraic plane curves C_i defined by the irreducible factors of the homogenous polynomial defining $\mathcal{C}$ with their degrees d_i; let also $\mathrm{Sing}(\mathcal{C}) = \{(P_1, \Sigma_1), \dots, (P_k, \Sigma_k)\}$ be the set singular points of $\mathcal{C}$ with their local topological types Σ_i. Finally, the combinatorics of a curve $\mathcal{C}$ is the data of $\mathrm{Irr}(\mathcal{C})$, $\mathrm{Sing}(\mathcal{C})$ and the relations between the local branches of Σ_i and the irreducible components of $\mathcal{C}$.

The question of the embedding determination of an algebraic plane curve in $\mathbb{CP}^2$ is not as easy. The embedding type, also called the topological type (or shortly the topology) of an algebraic plane curve $\mathcal{C}$, is defined as the homeomorphism type of the pair $(\mathbb{CP}^2, \mathcal{C})$. That is two curves $\mathcal{C}$ and $\mathcal{D}$ have the same topology if and only if there exists a homeomorphism $\phi : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $\phi(\mathcal{C}) = \mathcal{D}$. The topology is oriented if ϕ preserves the global orientation of $\mathbb{CP}^2$ and the local orientation around the meridians of $\mathcal{C}$ and $\mathcal{D}$. In the same way, the topology is ordered if the homeomorphism ϕ preserves fixed orders on the irreducible components of $\mathcal{C}$ and $\mathcal{D}$. In the case of an embedded curve, the combinatorics admits an equivalent definition with a more topological aspect. It can be defined as the homeomorphism type of $(\mathrm{Tub}(\mathcal{C}), \mathcal{C})$, where $\mathrm{Tub}(\mathcal{C})$ is a tubular neighbourhood of the curve (see [2]).

From these definitions, it is clear that the topological type determines the combinatorics of a curve. It is then natural to consider the inverse question: Is the topology determined by the combinatorics? The answer is known since the 30's, and it is negative. Indeed, in [21, 22, 23] Zariski constructs an explicit example of two curves with the same combinatorics and different topologies. We will describe explicitly this example in the next section. Such examples of two curves with the same combinatorics and different topologies are called Zariski pair (see [2]). The question is then: how to detect Zariski pair?

To differentiate topologies, several topological invariants were introduced, each one able to distinguish different kinds of "pathology". The most classical case of Zariski pair is the one detected by the Alexander polynomial, see Degtyarev [10], Libgober [16] or Akyol [1] for a classification of sextics such examples. Other examples with different "pathology" are known. Degtyarev [9, 11] and Shimada [19] construct Zariski pairs of sextics using the theory of $K3$-surfaces obtained from double branched covers of the curves. Studying some properties of double Galois cover Artal-Tokunaga [5] or Bannai [6] construct another kind of Zariski pair. In [11] Degtyarev constructs an example of Zariski pair with isomorphic fundamental group of the complement. Recently in [20], Shirane proves that Shimada's curves (see [18]) form k-plets of Zariski using the splitting numbers. We can conclude this non-exhaustive list with the Zariski pair constructed by Artal-Carmona-Cogolludo in [3] with homeomorphic complement but different braid monodromies.

We introduce here a new topological invariant: the *linking set*, it is first defined in [14] by Meilhan and the author. It is inspired by the linking numbers of knots theory, and generalized the $\mathcal{I}$-invariant introduced by Artal-Florens-Guerville in [4]. The idea is to consider γ an embedded oriented S^1 in the curve $\mathcal{C}$ (called a *cycle*) and to look its homology class in the complement of the sub-curve $\mathcal{C}_\gamma^c$ of $\mathcal{C}$ composed of all the irreducible components which not intersect the S^1 embedded. Unfortunately, this class is not invariant. We should then consider a quotient of $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}_\gamma^c)$ by a sub-group $\mathrm{Ind}_\gamma(\mathcal{C})$ deleting all the remaining indeterminacies. The linking set $\mathrm{lks}_\mathcal{C}(\gamma)$ is then define as the set of the classes in this quotient of all the cycles combinatorially equivalent to γ. It is an invariant of the oriented and ordered topology. In the particular case where the cycle γ is contained in a single irreducible component, we can remove the oriented hypothesis.

We apply this new invariant to the case of the curve introduced by Artal in [2] formed by a smooth cubic and three tangent lines in its inflexion points, and also to its generalization the k-Artal curves introduced in [7] composed of a smooth cubic and k tangent lines in its inflexion points. We then prove that there exists Zariski pair of k-Artal curve for $k = 3, 4, 5, 6$. These pairs are geometrically distinguished by the number of alignments in the set of the k inflexion points considered (this number is called the type of the arrangement). In order to obtain this result, we compute the linking set for a cycle contained in the cubic. Then we prove that for any order on the irreducible components of the k-Artal curves there is no homeomorphism of $\mathbb{CP}^2$ between two k-Artal curves of different types, for $k = 3, 4, 5, 6$. We can also apply this invariant to the Shimada's curves [18], and prove that they form a k-plets of Zariski with isomorphic fundamental groups. This is done in [15] proving (in some particular case) the equivalence between the splitting numbers and the linking set. This allows us to prove that the linking set is not determined by the fundamental group of the complement. Since the linking set is a generalization of the $\mathcal{I}$-invariant introduced in [4], it also distinguish the Zariski pair of line arrangements produced by the author in [13].

The following is organized as follows. Section 1 contains some details about the historical example of Zariski. The construction and invariance theorem of the linking set are done in Section 2. We conclude in Section 3 with the application of the linking set to the k-Artal curves.

1. Zariski example

In his historical papers [21, 22, 23], Zariski proves the following result:

Theorem 1.1 ([21]). *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two sextics with 6 cusps. Assume that these cusps lie on a conic for $\mathcal{C}_1$ and are in generic position for $\mathcal{C}_2$, then:*

$$\pi_1(\mathbb{CP}^2 \setminus \mathcal{C}_1) \simeq \mathbb{Z}_3 * \mathbb{Z}_2, \text{ and } \pi_1(\mathbb{CP}^2 \setminus \mathcal{C}_2) \simeq \mathbb{Z}_6.$$

Figure 1. Zariski sextics with 6 cusps

He proves in [22] the existence of such curves, and then provides the first example of Zariski pair. It is only in [17] that explicit equations of such curves will be given. Let us explain how to construct them.

Consider a smooth cubic C of $\mathbb{CP}^2$, and let P_1, P_2, P_3 be three of its nine inflexion points. We assume that the three tangent lines L_i of C passing through P_i are in generic position. We consider the coordinates of $\mathbb{CP}^2$ such that $L_1 : x = 0$, $L_2 : y = 0$ and $L_3 : z = 0$ (see Figure 2). Let p be the application defined by:

$$p : \left\{ \begin{array}{ccc} \mathbb{CP}^2 & \longrightarrow & \mathbb{CP}^2 \\ [X : Y : Z] & \longmapsto & [X^2 : Y^2 : Z^2] \end{array} \right.$$

It is a $4:1$ application outside of the axes. We denote by $\widetilde{C}$ the pre-image of the cubic C by p. It is a sextic with singular points contained in $p^{-1}(P_i)$. In order to differentiate the coordinates in the origin and the target of p, we will denote with capitals the coordinate in the target. Locally around P_1, the cubic C is of the form $y^3 = x$. Since p is ramified along the line $L_1 : x = 0$ then the pre-image of C near P_1 is of the form $Y^3 = X^2$. Furthermore p is $2:1$ over P_1, then $p^{-1}(P_1)$ is composed of 2 cusps. Using same arguments for P_2 and P_3, we obtain that $\widetilde{C}$ has exactly 6 cusps.

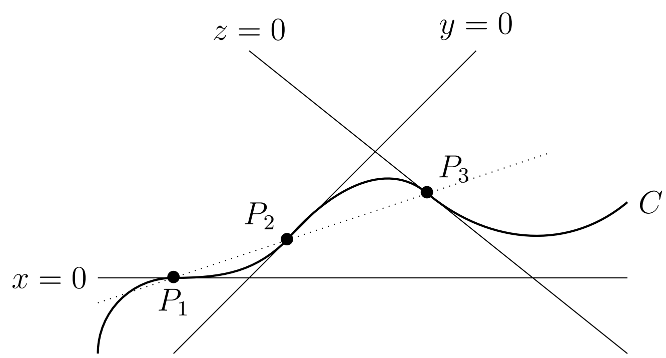

FIGURE 2. Cubic C with P_1, P_2 and P_3 collinear

To construct the two examples of Zarsiki (see Figure 1), we have to consider the case where the inflexion points are collinear, and the case where they are not. In the first case, consider the line L passing through P_1, P_2 and P_3, then $p^{-1}(L)$ is a smooth conic containing the 6 cusps of $\widetilde{C}$. In the second case, such a cubic does not exist.

Corollary 1.2. *The combinatorics does not determine the fundamental group of the complement of a curve.*

Remark 1.3. The corollary is also true for the complement and the topology of a curve.

2. Linking invariant

2.1. **Recall about linking numbers.** The *linking numbers* are classical invariants of the topology of oriented links. It can be defined as follows. Let $L = L_1 \cup \cdots \cup L_s$ an oriented link of $\mathcal{S}^3$ with s components. Let us recall that $\mathrm{H}_1(S^3 \setminus L_i)$ is isomorphic to $\mathbb{Z}$ and is generated by the meridian of L_i. The linking number of L_i with L_j, for $i \neq j$, is the numerical value of the class $[L_i]$ in $\mathrm{H}_1(S^3 \setminus L_j)$.

2.2. **Definition of the invariant.** The linking set is an adaptation of the linking number of knots theory to the case of algebraic curves. A *cycle* γ of a curve $\mathcal{C}$ is an oriented S^1 embedded in the curve, non homologically trivial in $\mathrm{H}_1(\mathcal{C};\mathbb{Z})$.

In order to compare two cycles, we need to define what could be comparable cycles. This is the notion of combinatorially equivalent cycles. Let $\gamma(t)$, for $t \in [0,1]$ be a parametrization of γ, with $\gamma(0)$ in the smooth part of $\mathcal{C}$. The combinatorial type of γ is the sequence

$$\Gamma(\gamma) = (C_{i_1}, P_{j_1}, \ldots, C_{i_k}, P_{j_k}),$$

with $C_{i_l} \in \mathrm{Irr}(\mathcal{C})$ and $P_{j_l} \in \mathrm{Sing}(\mathcal{C})$ for all $l \in \{1, \cdots, k\}$, such that there exists a set $\{t_1, \ldots, t_k\}$, with $0 < t_1 < \cdots < t_k < 1$ satisfying:

- for all $l \in \{1, \ldots, k\}$, $\gamma(t_l) = P_{j_l}$,
- for all $t \in (t_l, t_{l+1})$, $\gamma(t)$ is contained in $C_{i_{l+1}} \setminus (\mathrm{Sing}(\mathcal{C}) \cap C_{i_{l+1}})$ with $l \in \{1, \ldots, k-1\}$,
- for all $t \in [0, t_1) \cup (t_k, 1]$, $\gamma(t) \in C_{i_1}$.

Of course the sequence $\Gamma(\gamma)$ is defined up to cyclic permutation.

Definition 2.1. Let γ and μ be two cycles of $\mathcal{C}$. They are *combinatorially equivalent* if their combinatorial types are equal, that is $\Gamma(\gamma) = \Gamma(\mu)$.

We will now construct the sub-curve $\mathcal{C}^c_\gamma$ of $\mathcal{C}$ not intersecting γ. Let us first define the support and the internal support of a cycle.

Definition 2.2. The support of γ is:

$$\mathrm{Supp}(\gamma) = \bigcap_{g \sim \gamma} \left\{ C \in \mathrm{Irr}(\mathcal{C}) \mid C \cap g \neq \emptyset \right\},$$

its internal support is:

$$\overset{\circ}{\mathrm{Supp}}(\gamma) = \bigcap_{g \sim \gamma} \left\{ C \in \mathrm{Irr}(\mathcal{C}) \mid \left(C \overset{\circ}{\cap} g \right) \neq \emptyset \right\}.$$

A cycle is *minimal* if it is contained in $\bigcup\limits_{C \in \overset{\circ}{\mathrm{Supp}}(\gamma)} C$. Since for any cycle γ of $\mathcal{C}$ there exists a minimal cycle γ' such that γ and γ' are homotopically equivalent, we consider in all the following only minimal cycles. Remark that, if γ is a minimal cycle, then $\gamma \subset \mathbb{CP}^2 \setminus \bigcup\limits_{C \notin \mathrm{Supp}(\gamma)} C$.

The idea of the invariant is to consider the homology class of γ in the complement of $\mathcal{C}_\gamma^c = \bigcup_{C \notin \mathrm{Supp}(\gamma)} C$. Unfortunately, this class is not an invariant, indeed there exists homotopically equivalent cycles with non-equal homotopy classes in $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}_\gamma^c)$. In order to delete this problem, let us introduce the indeterminacy sub-group associated with γ.

Let P be a singular point of $\mathcal{C}$, we denote by $m_P^{C,b}$ the meridian around P in the local branch b of Σ_P contained in the component C. Remark that if C is not smooth at P then Σ_P admits several local branch b contained in C. See Figure 3 for an example when C is smooth in P.

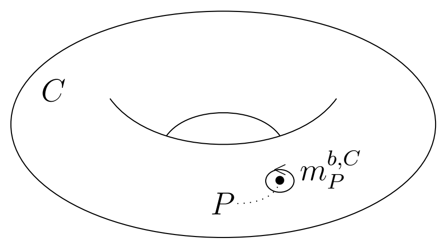

FIGURE 3. The meridian $m_P^{b,C}$ when $\mathrm{Br}(\Sigma_P, C) = \{b\}$

Definition 2.3. The *indeterminacy sub-group* of $\mathcal{C}$ associated with γ is the sub-group of $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}_\gamma^c)$ defined by:

$$\mathrm{Ind}_\gamma(\mathcal{C}) = \langle m_P^{b,C} \mid \forall C \in \mathring{\mathrm{Supp}}(\gamma), \forall P \in \mathrm{Sing}(\mathcal{C}) \cap C, \forall b \in \mathrm{Br}(\Sigma_P, C) \rangle,$$

where $\mathrm{Br}(\Sigma_P, C)$ is the local branch of Σ_P contained in C.

Proposition 2.4. *The value of $m_P^{b,C}$ in $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}_\gamma^c)$ is determined by the combinatorics, and we have:*

$$m_P^{b,C} = \sum_{D \notin \mathrm{Supp}(\gamma)} I_P(b, D) m_D,$$

where $I_P(b, D)$ is the multiplicity of the intersection of the local branch b and D at the point P, and m_D is the meridian of D in $\mathbb{CP}^2 \setminus \mathcal{C}_\gamma^c$.

We can define the linking invariant:

Definition 2.5. The *linking set* of a minimal cycle γ is defined as the set of classes in $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}_\gamma^c) / \mathrm{Ind}_\gamma(\mathcal{C})$ of the minimal cycles combinatorially equivalent to γ. It is denoted by $\mathrm{lks}_\mathcal{C}(\gamma)$.

If there is no ambiguity on the considered curve $\mathrm{lks}_\mathcal{C}(\gamma)$(resp. $\mathrm{Ind}_\gamma(\mathcal{C})$) is denoted by $\mathrm{lks}(\gamma)$ (resp. Ind_γ).

2.3. **Invariance theorem & corollaries.**

Theorem 2.6 ([14]). *Let $\mathcal{C}$ and $\mathcal{D}$ be two curves with the same oriented and ordered topology. If γ and μ are two combinatorially equivalent cycles of $\mathcal{C}$ and $\mathcal{D}$ respectively, then*

$$\phi_*(\mathrm{lks}(\gamma)) = \mathrm{lks}(\mu), \tag{1}$$

where $\phi_* : \mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}^c_\gamma)/\operatorname{Ind}_\gamma(\mathcal{C}) \to \mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D}^c_\mu)/\operatorname{Ind}_\mu(\mathcal{D})$ is the isomorphism induced by the equivalence of topology.

Remark 2.7. The isomorphism ϕ_* is the application sending meridian on meridian respecting the orders on $\mathrm{Irr}(\mathcal{C}) = \{C_1, \dots, C_n\}$ and $\mathrm{Irr}(\mathcal{D}) = \{D_1, \dots, D_n\}$; and respecting the orientation of the meridian (i.e. $\phi_*(m_{C_i}) = m_{D_i}$).

Corollary 2.8.

(1) *If $\mathcal{C}$ is a rational curve, then $\mathrm{lks}(\gamma) = \{\gamma\}$ and then Equation (1) becomes:*

$$\phi_*(\gamma) = \mu.$$

(2) *If $\overset{\circ}{\mathrm{Supp}}(\gamma) = \{C\}$ then we can remove the oriented hypothesis in Theorem 2.6.*

Corollary 2.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be two curves with the same combinatorics, and γ and μ be two combinatorially equivalent cycles of $\mathcal{C}$ and $\mathcal{D}$ respectively. If for any isomorphism*

$$\phi_* : \mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}^c_\gamma)/\operatorname{Ind}_\gamma(\mathcal{C}) \to \mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D}^c_\mu)/\operatorname{Ind}_\mu(\mathcal{D})$$

sending meridian on meridian and respecting their orientation, we have $\phi_*(\mathrm{lks}(\gamma)) \neq \mathrm{lks}(\mu)$, then $\mathcal{C}$ and $\mathcal{D}$ have distinct oriented topology.

Remark 2.10. If we also assume that the support (or the internal support) of γ and μ is a singleton, then we can remove the condition oriented in the conclusion of the previous corollary.

3. k-Artal curves

In [2], Artal introduces a Zariski pair where the curves are composed of a smooth cubic and three tangent lines in inflexion points. He proves that a difference of geometry (of the considered inflexion points) implies a difference of topology. Here, we will generalized this result to the case of the curves k-Artal.

Definition 3.1. A *curve k-Artal* is an algebraic curve composed of a smooth cubic C and k lines $L_1, \dots, L_k$ tangent in the inflexion points of C.

It is of Type l if there is exactly l subset composed of three collinear points in $\{P_i = L_i \cap C \mid i = 1, \cdots, k\}$.

Remark 3.2.

(1) For the 3-Artal case, the Zarsiki pair studied by Artal is formed by a curve of Type 0 and one of Type 1.
(2) This Zariski pair is related with the historical example of Zariski (see Section 1) by Figure 2. Indeed, the 3-Artal curve is the cubic C with the 3 axes, and the geometric difference between Type 0 and Type 1 induces the geometric difference between the sextics with the six cusps along (or not) a conic.

Theorem 3.3 ([7]). *Let $k \in \{3, \ldots, 6\}$, there exist k-Artal curves $\mathcal{C}$ and $\mathcal{D}$ of different Type such that they form a Zariski pair.*

To prove this theorem, we will compute their linking set (in a particular case) and then apply Corollary 2.9. To compute the linking set in each case, we will compute it in the case of the cubic with nine tangent lines and then consider restrictions to obtain the case with less than nine tangent lines. We denote by $\mathcal{C}_9$ the 9-Artal curve. It is known that the inflexion points of a smooth cubic have a the structure of $(\mathbb{F}_3)^2$. We assume that the points $P_i = L_i \cap C$ are as in Figure 4.

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ P_6 & P_3 & P_9 \\ \bullet & \bullet & \bullet \\ P_5 & P_1 & P_4 \\ \bullet & \bullet & \bullet \\ P_7 & P_2 & P_8 \end{array}$$

FIGURE 4. Representation of the inflexion points of C as $(\mathbb{F}^3)^2$

Let γ be a cycle contained in the smooth cubic C. It is clear that γ is minimal and that:

$$\mathrm{Supp}(\gamma) = \mathring{\mathrm{Supp}}(\gamma) = \{C\}\,.$$

The set of cycles combinatorially equivalent to γ is the set of cycles contained in C. Homologically this set is equal to $\mathrm{H}_1(C;\mathbb{Z}) \setminus \{0\}$. Let g_1 and g_2 be two cycles generating $\mathrm{H}_1(C;\mathbb{Z})$. In the case where C is defined by $x^3 - xz^2 - y^2z = 0$ it has been proved in [14], that there is

g_1 and g_2 such that:
(2)

$$\begin{array}{rcll} [g_1] & = & m_{L_2} - m_{L_3} - m_{L_6} + m_{L_7} + m_{L_8} - m_{L_9} & \text{in } \mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}_9), \\ [g_2] & = & -m_{L_4} + m_{L_5} + m_{L_6} + m_{L_7} - m_{L_8} - m_{L_9} & \text{in } \mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{C}_9). \end{array}$$

Consider the case $k = 3$, let $\mathcal{I}_1$ be the set $\{1, 2, 3\}$ and $\mathcal{I}_2$ be $\{1, 2, 4\}$. We denote by $\mathcal{C}_i$ the curve $C \cup \big(\bigcup_{j \in \mathcal{I}_i} L_j \big)$. By Figure 4, $\mathcal{C}_1$ is of Type 1 and $\mathcal{C}_2$ of Type 0. We have that $(\mathcal{C}_i)^c_\gamma = \bigcup_{j \in \mathcal{I}_i} L_j$. Using restrictions of Equation (2), we have that:

$$\begin{array}{llllll} [g_1]_1 & = m_{L_2} - m_{L_3}, & \text{and} & [g_2]_1 & = 0 & \text{in } \mathrm{H}_1(\mathbb{CP}^2 \setminus (\mathcal{C}_1)^c_\gamma), \\ [g_1]_2 & = m_{L_2} & \text{and} & [g_2]_2 & = -m_{L_4} & \text{in } \mathrm{H}_1(\mathbb{CP}^2 \setminus (\mathcal{C}_2)^c_\gamma). \end{array}$$

To compute the linking set, we need first to determine the indeterminacy sub-group of γ. Since it is combinatorial, then they are isomorphic for $\mathcal{C}_1$ and $\mathcal{C}_2$. In the case of the 3-Artal curves and a cycle γ contain in the cubic, the singular points contained in the cubic are the P_j, for $j \in \mathcal{I}_i$. These singular points are of type $\mathbb{A}_5$. This implies that $I_{P_j}(C, L_l) = 3\delta_{j,l}$, and then:

$$\mathrm{H}_1(\mathbb{CP}^2 \setminus (\mathcal{C}_i)^c_\gamma) / \operatorname{Ind}_\gamma(\mathcal{C}_i) \simeq \langle m_{L_j} \text{ for } j \in \mathcal{I}_i \mid 3m_{L_j}, \sum_{j \in \mathcal{I}_i} m_{L_j} \rangle \simeq (\mathbb{Z}_3)^2.$$

We denote by $\mathrm{lks}_i(\gamma)$ the linking set of γ in $\mathcal{C}_i$, for $i = 1, 2$. We have then:

$$\begin{array}{rl} \mathrm{lks}_1(\gamma) = & \{ m_{L_2} - m_{l_3}, -m_{L_2} + m_{l_3} \}, \\ \mathrm{lks}_2(\gamma) = & \{ m_{L_2}, m_{L_4}, -m_{L_2}, -m_{L_4}, m_{L_2} + m_{L_4}, \\ & m_{L_2} - m_{L_4}, -m_{L_2} + m_{L_4}, -m_{L_2} - m_{L_4} \}. \end{array}$$

It is clear (using an argument of cardinality) that for any isomorphism

$$\phi_* : \mathrm{H}_1(\mathbb{CP}^2 \setminus (\mathcal{C}_1)^c_\gamma) / \operatorname{Ind}_\gamma(\mathcal{C}_1) \to \mathrm{H}_1(\mathbb{CP}^2 \setminus (\mathcal{C}_2)^c_\gamma) / \operatorname{Ind}_\gamma(\mathcal{C}_2)$$

sending meridian on meridian and respecting the orientation, we have $\phi_*(\mathrm{lks}_1(\gamma)) \neq \mathrm{lks}_2(\gamma)$. Using Corollary 2.9 and Remark 2.10, the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ form a Zariski pair; then case $k = 3$ of Theorem 3.3 is proved.

Remark 3.4. With similar arguments, we can prove Theorem 3.3 for $k = 4, 5, 6$.

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