HOMOTOPY THEORY OF POLYHEDRAL PRODUCTS

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based on joint work with Kouyemon Iriye

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Setting

- Let K be an abstract simplicial complex over the vertex set

$$[m] = \{1, 2, \ldots, m\}.$$

- Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i \in [m]}$ be a collection of space pairs.
- For a subset $\sigma \subset [m]$, we put

$$(\underline{X}, \underline{A})^{\sigma} = Y_1 \times \cdots \times Y_m, \quad Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma. \end{cases}$$

Definition

The polyhedral product associated with K and $(\underline{X}, \underline{A})$ is defined by

$$Z_K(\underline{X}, \underline{A}) := \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^{\sigma} \quad (\subset X_1 \times \cdots \times X_m).$$

Examples

Example

The moment-angle complex is the special polyhedral product

$$Z_K := Z_K(D^2, S^1).$$

Then $T^m \curvearrowright Z_K$. When $K = \partial P$ for a simplicial polytope P,

each quasitoric manifold M over $P^* = Z_K/T$ for some $T \subset T^m$.

The equivariant topology of M is also detected by

$$M \times_{T^m/T} E(T^m/T) = ET^m \times_{T^m} Z_K \simeq Z_K(\mathbb{C}P^\infty, *).$$

Polyhedral products appear in other constructions of toric topology.

Example

By definition, the topology of polyhedral products is connected with combinatorics. This connection can be best seen through algebras.

For example, if K is the flag complex of a graph Γ, then

$$BW_\Gamma = Z_K(\mathbb{R}P^\infty, *)$$

where W_Γ is the right-angled Coxeter group of Γ. Moreover,

$$\mathbb{R}Z_K := Z_K(D^1, S^0) = B[W_\Gamma, W_\Gamma]$$

where $\mathbb{R}Z_K$ is called the real moment-angle complex.

Other combinatorially constructed algebras are also realized by homotopy invariants of polyhedral products.

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Other combinatorially constructed algebras are also realized by homotopy invariants of polyhedral products.

We study the homotopy types of polyhedral products in connection with combinatorics and toric topology.

Our object

Special polyhedral products

$$Z_K(C\underline{X}, \underline{X}) \quad \text{and} \quad Z_K(\underline{X}, *)$$

are particularly important. Since there is a homotopy fibration

$$Z_K(C\Omega\underline{X}, \Omega\underline{X}) \to Z_K(\underline{X}, *) \to X_1 \times \cdots \times X_m,$$

they are supplementary to each other, where

$$(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i\in[m]} \quad \text{and} \quad (\underline{X}, *) = \{(X_i, *)\}_{i\in[m]}.$$

Our object to study is the polyhrdral product

$$Z_K(C\underline{X}, \underline{X}).$$

BBCG decomposition

We start with a stable homotopy decomposition of $Z_K(C\underline{X}, \underline{X})$.

- Let $K_I = \{\sigma \in K \mid \sigma \subset I\}$ for $I \subset [m]$.
- Let $|L|$ denote the geometric realization of a simplicial complex L.
- Let $\widehat{X}^I = \bigwedge_{i \in I} X_i$ for $\emptyset \neq I \subset [m]$.

Theorem (Bahri-Bendersky-Cohen-Gitler '10)

There is a homotopy equivalence

$$\Sigma Z_K(C\underline{X}, \underline{X}) \simeq \Sigma \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I.$$

Remark

There is a very similar previous results due to Kamiyama and Tsukuda, where they did not use the word "polyhedral product".

Problems

Our final goal is to answer:

Problem

Describe the homotopy type of $Z_K(C\underline{X}, \underline{X})$ *from BBCG decomposition.*

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The most naive way to attack this problem is to desuspend BBCG decomposition, and there are previous results.

Theorem (Grbić-Theriault '13, Iriye-Kishimoto '13)

BBCG decomposition desuspends for dual shifted complexes.

Theorem (Grujić-Welker '15)

BBCG decomposition of $Z_K(D^n, S^{n-1})$ desuspends whenever K is dual vertex-decomposable.

Can we generalize?

The methods of the previous results are too ad-hoc to generalize. Then we go back to BBCG decomposition.

What is BBCG decomposition?

The proof says nothing on unsuspended $Z_K(C\underline{X}, \underline{X})$, so in fact, we have not understood BBCG decomposition as a property of $Z_K(C\underline{X}, \underline{X})$.

Then we must start from scratch, and pose:

Problem

1. *Find a structure of unsuspended $Z_K(C\underline{X}, \underline{X})$ which induces BBCG decomposition.*
2. *Investigate this structure and apply it to desuspension.*

Combinatorial aspect of the problems

I explain the combinatorial aspect of our problems.

Definition

The Stanley-Reisner ring of K over a ring R is defined by

$$R[K] = R[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_k} \mid \{i_1, \ldots, i_k\} \notin K)$$

where $|v_i| = 2$.

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where $|v_i| = 2$.

Proposition (Davis-Januszkiewicz '91)

There is a ring isomorphism

$$H^*(Z_K(\mathbb{C}P^\infty, *); R) \cong R[K].$$

Then combinatorial properties of simplicial complexes can be reduced to polyhedral products via Stanley-Reisner rings.

This reduction is made stronger by:

Proposition (Baskakov-Buchstaber-Panov '04)

There is a ring isomorphism

$$H^*(Z_K; R) \cong \mathrm{Tor}^*_{R[v_1,\ldots,v_m]}(R[K], R)$$

where the product of the RHS is induced from the Koszul resolution of R over $R[v_1, \ldots, v_m]$.

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Hochster proved that there is an R-module isomorphism

$$\mathrm{Tor}^*_{R[v_1,\ldots,v_m]}(R[K], R) \cong \bigoplus_{I\subset[m]} \widetilde{H}^*(K_I; R).$$

One can easily see the implication:

BBCG decomposition $\Rightarrow$ Hochster decomposition

What about ring structure?

Hochster described the product of $\mathrm{Tor}^*_{R[v_1,\ldots,v_m]}(R[K], R)$ through the above decomposition.

But this is not very useful. For example, even the following easiest case is not well understood still.

Definition

K is called Golod over R if all products and (higher) Massey products in $\mathrm{Tor}^*_{R[v_1,\ldots,v_m]}(R[K], R)$ are trivial.

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Golodness has been studied since 60's, and several important simplicial complexes are known to be Golod.

We return to our problem. If BBCG decomposition desuspends, then $Z_K(C\underline{X},\underline{X})$ is a suspension, implying:

desuspension of BBCG decomposition $\Rightarrow$ Golodness

Fat wedge filtration

- The *k-th fat wedge* of $\underline{X} = \{X_i\}_{i \in [m]}$ is defined by

$$T^k(\underline{X}) = \{(x_1, \dots, x_m) \in X_1 \times \cdots \times X_m \mid \text{at least } m-k \text{ of } x_i \text{ is a base point}\}.$$

Then there is a filtration

$$* = T^0(\underline{X}) \subset T^1(\underline{X}) \subset \cdots \subset T^m(\underline{X}) = X_1 \times \cdots \times X_m.$$

Definition

Put

$$Z_K^i(C\underline{X}, \underline{X}) = Z_K(C\underline{X}, \underline{X}) \cap T^i(C\underline{X}).$$

Then there is a filtration

$$* = Z_K^0(C\underline{X}, \underline{X}) \subset Z_K^1(C\underline{X}, \underline{X}) \subset \cdots \subset Z_K^m(C\underline{X}, \underline{X}) = Z_K(C\underline{X}, \underline{X})$$

which we call the *fat wedge filtration* (FWF) of $Z_K(C\underline{X}, \underline{X})$.

FWF of $\mathbb{R}Z_K$

$Z_K(C\underline{X}, \underline{X})$ is defined by on-off of the cone parameters and $\mathbb{R}Z_K$ can be regarded as the space of the cone parameters.

So we first investigate the FWF of $\mathbb{R}Z_K$ and then apply its properties to the FWF of $Z_K(C\underline{X}, \underline{X})$.

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- Put $\mathbb{R}Z_K^i = Z_K^i(D^1, S^0)$.

There is a piecewise linear embedding

$$i_c \colon |\mathrm{Sd}\,\Delta^{[m]}| \to (D^1)^m.$$

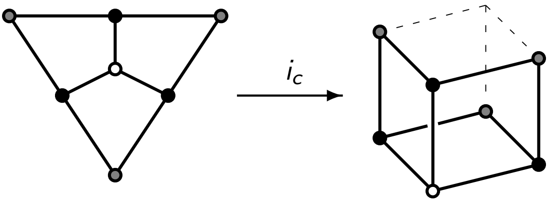

Figure: the case $m = 3$

Since K is a subcomplex of $\Delta^{[m]}$, we get an embedding

$$|\mathrm{Sd}\, K| \to (D^1)^m.$$

Lemma

This embedding yields a relative homeomorphism

$$\varphi_K \colon (|\mathrm{Cone}(\mathrm{Sd}\, K)|, |\mathrm{Sd}\, K|) \to (\mathbb{R}Z_K, \mathbb{R}Z_K^{m-1}).$$

By definition, the map $\varphi_K \colon |\mathrm{Sd}\, K| \to \mathbb{R}Z_K^{m-1}$ is explicitly given in a completely combinatorial way.

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Since

$$\mathbb{R}Z_K^i = \bigcup_{I\subset[m],\, |I|=i} \mathbb{R}Z_{K_I} \quad \text{and} \quad \mathbb{R}Z_K^{i-1} = \bigcup_{I\subset,\, |I|=i} \mathbb{R}Z_{K_I}^{i-1},$$

we deduce the following.

Theorem

$\mathbb{R}Z_K^i$ *is* $\mathbb{R}Z_K^{i-1}$ *with cones attached by the maps*

$$\varphi_{K_I} \colon |\mathrm{Sd}\, K_I| \to \mathbb{R}Z_{K_I}^{i-1} \subset \mathbb{R}Z_K^{i-1} \quad (I \subset [m], |I| = i)$$

FWF of $Z_K(C\underline{X}, \underline{X})$

By assigning $\mathbb{R}Z_K$ to cone parameters, we get a map

$$\mathbb{R}Z_K^{i-1} \times T^{i-1}(\underline{X}) \to Z_K^{i-1}(C\underline{X}, \underline{X}).$$

Then by composing with the map

$$\varphi_{K_I} \times \colon |\mathrm{Sd}\, K_I| \times T^{i-1}(\underline{X}) \to \mathbb{R}Z_K^{i-1} \times T^{i-1}(\underline{X})$$

for $I \subset [m]$ with $|I| = i$, we obtain:

Theorem

There is a relative homeomorphism

$$\coprod_{I\subset[m],\,|I|=i} \Phi_{K_I} \colon \coprod_{I\subset[m],\,|I|=i} (|\mathrm{Cone}(\mathrm{Sd}\, K_I)|, |\mathrm{Sd}\, K_I|) \times (\prod_{i\in I} X_i, T^{i-1}(\underline{X}_I)) \to (Z_K^i(C\underline{X}, \underline{X}), Z_K^{i-1}(C\underline{X}, \underline{X}))$$

where $\underline{X}_I = \{X_i\}_{i\in I}$.

FWF of Z_K

- Put $Z_K^i = Z_K^i(D^2, S^1)$.

The FWF of $Z_K(C\underline{X}, \underline{X})$ is not a cone decomposition in general. But we can prove the FWF of Z_K is a cone decomposition as follows.

We can construct a map

$$\bar{\varphi}_{K_I} \colon |\mathrm{Sd}\, K_I| * S^{|I|-1} \to Z_{K_I}^{|I|-1}$$

for $I \subset [m]$ by combining φ_{K_I} and higher Whitehead products.

Theorem
Z_K^i is Z_K^{i-1} with cones attached by

$$\bar{\varphi}_{K_I} \colon |\mathrm{Sd}\, K_I| * S^{i-1} \to Z_{K_I}^{i-1} \subset Z_K^{i-1}$$

for all $I \subset [m]$ with $|I| = i$.

Recovering BBCG decomposition

By James' retractile argument, the FWF of $Z_K(C\underline{X},\underline{X})$ splits after a suspension such that

$$\begin{aligned}\Sigma Z_K(C\underline{X},\underline{X}) &\simeq \Sigma \bigvee_{i=1}^{m} Z_K^i(C\underline{X},\underline{X})/Z_K^{i-1}(C\underline{X},\underline{X}) \\ &= \Sigma \bigvee_{i=1}^{m} \bigvee_{I\subset[m],\,|I|=i} |\Sigma K_I| \wedge \widehat{X}^I\end{aligned}$$

which is BBCG decomposition. So the FWF is the desired structure.

Desuspension of BBCG decomposition

We say that the FWF of $\mathbb{R}Z_K$ (resp. Z_K) is trivial if φ_{K_I} (resp. $\bar{\varphi}_{K_I}$) are null homotopic for all $\emptyset \neq I \subset [m]$.

Theorem

If the FWF of $\mathbb{R}Z_K$ is trivial, then BBCG decomposition of $Z_K(C\underline{X}, \underline{X})$ desuspends.

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Theorem

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The converse of this theorem is still open, but as for Z_K, we have:

Theorem

The following conditions are equivalent:

1. *the FWF of Z_K is trivial;*
2. *Z_K is a co-H-space;*
3. *there is a homotopy equivalence*

$$Z_K \simeq \bigvee_{\emptyset \neq I \subset [m]} \Sigma^{|I|+1}|K_I|.$$

Sequentially Cohen-Macaulayness

We now apply our results for specific simplicial complexes.

We must choose K from Golod complexes for desuspending BBCG decomposition.

Which Golod complex do we choose?

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Which Golod complex do we choose?

Golodness has been studied in connection with Cohen-Macaulayness. Recall that an algebra A over a field is called Cohen-Macaulay (CM) if

$$\dim A = \operatorname{depth} A.$$

K is called CM over $\Bbbk$ if so is $\Bbbk[K]$.

Recall that the Alexander dual of K is defined by

$$K^\vee = \{\sigma \subset [m] \mid [m] - \sigma \notin K\}.$$

Theorem (Stanley '96)

If K is dual CM over $\Bbbk$, that is, the Alexander dual of K is CM over $\Bbbk$, then K is Golod over $\Bbbk$.

There is a generalization of CMness.

Definition

K is sequentially Cohen-Macaulay (SCM) over a field $\Bbbk$ if a subcomplex of K generated by i-dimensional faces is CM over $\Bbbk$ for each i.

Theorem (Stanley '96)

If K is dual SCM over $\Bbbk$, then K is Golod over $\Bbbk$.

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Theorem (Stanley '96)

If K is dual SCM over $\Bbbk$, then K is Golod over $\Bbbk$.

There are subclasses of SCM complexes with the implications:

shifted $\Rightarrow$ vertex-decomposable $\Rightarrow$ shellable $\Rightarrow$ SCM

Since the previous results on desuspension of BBCG decomposition are on dual shifted and vertex-decomposable complexes,

we choose dual SCM complexes.

Property of φ_K

How do we show $\varphi_{K_I} \simeq *$?

Since φ_K is defined combinatorially, there are properties of φ_K related directly with combinatorial properties of K. For example, we have:

- A subset $\sigma \subset [m]$ is called a minimal non-face of K if $\sigma \notin K$ and $\sigma - v \in K$ for any $v \in \sigma$.
- Let $\widehat{K}$ be the simplicial complex obtained from K by adding all minimal non-faces.

Lemma

The map φ_K factors as

$$|\mathrm{Sd}\, K| \xrightarrow{\text{incl}} |\mathrm{Sd}\, \widehat{K}| \to \mathbb{R}Z_K^{m-1}.$$

So one way to apply the above criterion is to find a simplicial complex L such that

$$K \subset L \subset \widehat{K} \quad \text{and} \quad |L| \simeq *.$$

Shellable complex

- A facet of K is a maximal face of K.

Definition

A simplicial complex K is shellable if its facets can be arranged in order $F_1, \ldots, F_t$, a shelling, such that the subcomplex

$$\langle F_k \rangle \cap \langle F_1, \ldots, F_{k-1} \rangle$$

is pure and $(\dim F_k - 1)$-dimensional for $k = 2, \ldots, t$.

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Lemma

1. *If K is dual shellable, there are facets $F_1, \ldots, F_r$ such that*

$$|K \cup F_1 \cup \cdots \cup F_r| \simeq *.$$

2. *If K is dual shellabe, so is K_I for any $\emptyset \neq I \subset [m]$.*

Theorem

If K is dual shellable, the FWF of $\mathbb{R}Z_K$ is trivial, implying that BBCG decomposition desuspends.

Homology fillable complex

Definition

A simplicial complex K is homology fillability if

1. for each prime p, there are facets $F_1, \ldots, F_r$ such that
$$\widetilde{H}_*(K \cup \cdots \cup F_r; \mathbb{Z}/p) = 0,$$
2. $\widehat{K}$ is simply connected.

Theorem

If every component of K is homology fillable, then $\varphi_K \simeq *$.

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Theorem

If every component of K is homology fillable, then $\varphi_K \simeq *$.

Lemma

If K is dual SCM over $\mathbb{Z}$, then components of K_I are homology fillable for all $\emptyset \neq I \subset [m]$.

Theorem

If K is dual SCM over $\mathbb{Z}$, then the FWF of $\mathbb{R}Z_K$ is trivial, implying that BBCG decomposition desuspends.

Further problems

Problem

Is it true that if $\mathbb{R}Z_K$ is a suspension, then its FWF is trivial?

This is the converse problem of our criterion, and its difficulty comes from the non-triviality of $\pi_1(\mathbb{R}Z_K)$.

A deduction of this problem has been made, which only shows that the difficulty resembles Ganea conjecture. So it helps nothing now.

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Definition

K is *k*-neighborly whenever $\sigma \in K$ for any $\sigma \subset [m]$ with $|\sigma| = k+1$.

By analyzing the homotopy fiber of φ_K, we can prove:

Theorem

If K is $\lceil \frac{\dim K}{2} \rceil$-neighborly, then the FWF of $\mathbb{R}Z_K$ is trivial.

This theorem is applied to give a characterization of Golodness of triangulated surfaces. Recently, this characterization is algebraically generalized by Kätthan.

Problem

Find a class of simplicial complexes for which neighborliness and Golodness are equivalent.

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Find a class of simplicial complexes for which neighborliness and Golodness are equivalent.

The pinch map

$$Z_K(C\underline{X},\underline{X}) \to Z_K(\Sigma\underline{X},*)$$

is the (higher) Whitehead product if K is the boundary of a simplex.

When K is dual shifted, this map is shown to be iterated (higher) Whitehead product.

Problem

Describe this map when BBCG decomposition desuspends.