曲面の写像類群の線型性問題の視覚化について

On visualization of the linearity problem for mapping class groups of surfaces

笠原 泰

高知工科大学

8 July 2016

予稿集の訂正

p.160, l.17 ，「$m \geq 1$ に対し」→「$m \geq 2$ に対し」

p.163, l.9 ，$H_1(\Sigma_g; \mathbb{Z}) \to H_1(\Sigma_g; K)$

p.164, l.8 ，“Nilsen” → “Nielsen”

p.166, l.7 ，「群が満たすべき」→「有限生成群が満たすべき」

p.166, l.12 ，「$X_{\mathrm{GL}(n,K)}$」 → 「$X_{\mathrm{GL}(n,\mathbb{C})}$」 (2 箇所とも)

Outline

1. Introduction
   - Mapping Class Group and its linearity problem
   - Known results
   - Some Difficulty in higher genera
2. Visualization
   - case of closed surface
   - case of once-punctured surface
     - some consequences

Introduction

Mapping Class Group (MCG)

the group of isotopy classes of the orientation preserving homeomorphisms of an oriented surface.

(w/ some variants)

A fundamental problem is its linearity.

- A group is linear $\Leftrightarrow$ it admits a faithful finite dimensional linear representation over $^{\exists}$ field.
- A linear representation is faithful $\Leftrightarrow$ it is injenctive as a homomophism into the corresponding linear transformation group.

In particular, a group is said to be K-linear if it admits a faithful finite dimensional linear representation over a field K.

The purpose of this talk is

- to make a (personally biased) review on known results on the linearity problem on MCG of surfaces,
- to derive two types of new linearity conditions for MCG of surface, one for closed surface and one for 1-punctured surface (NOT to claim the solution of the problem, unfortunately),

Notation

Σ_g : a closed oriented surrface of genus g

$\Sigma_{g,*}$: a pair of Σ_g and a fixed base point $* \in \Sigma_g$

$\Sigma_{g,n}$: a connected compact oriented surface of genus g with $n \geq 0$ boundary components

$\mathcal{M}_g$: MCG of Σ_g

$\mathcal{M}_{g,*}$: MCG of $\Sigma_{g,*}$
(homeo and isotopy are assumed to fix $*$)

$\mathcal{M}_{g,n}$: MCG of $\Sigma_{g,n}$
(homeo and isotopy are assumed to fix the boundary pointwise)

Known results

Classical. $\mathcal{M}_1 \cong \mathcal{M}_{1,*} \cong \mathrm{SL}(2,\mathbb{Z})$.

Therefore, $\mathcal{M}_1$ and $\mathcal{M}_{1,*}$ are $\mathbb{Q}$-linear.

For the genus 2 case:

Korkmaz ['00], Bigelow–Budney ['01]

$\mathcal{M}_2$ is linear.

Proof. the combination of:

- Artin's braid group B_n is linear (Bigelow ['01], Krammer ['02])
- the relation of B_n with MCG of $n+1$st punctured S^2,
- relation between $\mathcal{M}_2$ and MCG of 6th punctured S^2 (Birman–Hilden theory) □

All the other cases are unknown.

The linearity of $\mathcal{M}_g$ for $g \geq 3$ and also the linearity of $\mathcal{M}_{g,*}$ for $g \geq 2$ seem to remain open.

Lawrence representation of B_n

D^2 : a 2-disk
P_n : a set of fixed n points in $\mathrm{Int}(D^2)$
$D_n := (D^2, P_n)$
B_n : MCG of D_n
$\mathcal{C}_m(D_n)$: the configuration space of m points in $D^2 \setminus P_n$ (unordered)

B_n acts naturally on $H^1(\mathcal{C}_m(D_n); \mathbb{Z})$.

Lawrence['90].

$$H^1(\mathcal{C}_m(D_n); \mathbb{Z})^{B_n} \cong \begin{cases} \mathbb{Z} & (m = 1); \\ \mathbb{Z} \oplus \mathbb{Z} & (m \geq 2) \end{cases}$$

Take the mth homology of the covering of $\mathcal{C}_m(D_m)$ corresponding to the invariant part, to obtain

a linear representation $\rho^{(m)}$ of B_n over
$\mathbb{Z}[t^{\pm 1}]$ $(m = 1)$ or $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ $(m \geq 2)$.

Lawrence representation of B_n (2)

- $\rho^{(1)}$ is the Burau representation
  faithful for $n = 3$ (Magnus–Peluso ['69])
- $\rho^{(2)}$ is faithful for $n \geq 4$ (Krammer['00,'02], Bigelow['01])

What if one replaces D_n with $\Sigma_{g,1}$?

Non-existence of Lawrence rep for $\Sigma_{g,1}$

For $g \geq 1$, one can use a presentation of the braid group of $\Sigma_{g,1}$ (Bellingeri ['04]) to obtain as an $\mathcal{M}_{g,1}$-module,

$$H_1(\mathcal{C}_m(\Sigma_{g,1});\mathbb{Z}) \cong \begin{cases} H_1(\Sigma_{g,1};\mathbb{Z}) & (m=1); \\ H_1(\Sigma_{g,1};\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} & (m \geq 2) \end{cases}$$

where the action on $\mathbb{Z}/2\mathbb{Z}$ is trivial.

Therefore, one can not expect to derive information much more than the $\mathcal{M}_{g,1}$-action on the mth homology itself:

$$\rho_m : \mathcal{M}_{g,1} \to \mathrm{GL}(H_m(\mathcal{C}_m(\Sigma_{g,1});\mathbb{Z})).$$

On the other hand, the kernel of the latter representation is given by:

Moriyama['07]

$\{\mathrm{Ker}\,\rho_m\}_{m\geq 1}$ coincides with the Johnson filtration.

Some other related topics (c.f. 予稿集)

- Comparison with $\mathrm{Aut}(F_n)$
- Lattice in top. group VS Linearity
- Classification of low degree rep.'s over $\mathbb{C}$

Motivation for Visualization

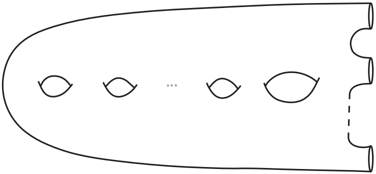
- Linearity problem seems quite subtle, as observed above.
- If an ad-hoc way is good enough, the problem might be solved even today by somebody.
- However, any systematic approach seems missing.

So, we tried to rephrase the linearity problem for MCG in terms of proper MCG geometry/topology,

in the hope to find new interesting problems, and further hopefully a clue to the linearity problem itself.

Compact surface case: Setting

$\Sigma_{g,n}$: the compact connected oriented surface of genus $g \geq 1$ and $n \geq 0$ boundary components.



$\mathcal{M}_{g,n}$: MCG of $\Sigma_{g,n}$ (id on ∂)

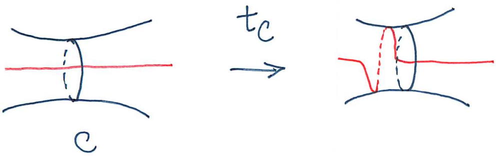
$\mathcal{S} = \mathcal{S}_{g,n}$: the set of isotopy classes of essential (unoriented) simple closed curves (SCC) on $\Sigma_{g,n}$

Here, essential means: not homotopic to a point nor parallel to any boudary component.

Note: $\mathcal{M}_{g,n}$ naturally acts on $\mathcal{S}$.

Dehn twist

For $C \in \mathcal{S}$, t_C denotes the (right-handed) Dehn twist $\in \mathcal{M}_{g,n}$:



Definition

We define a set mapping

$$\iota : \mathcal{S} \to \mathcal{M}_{g,n} \quad \text{by } \iota(C) := t_C \text{ for } C \in \mathcal{S}.$$

Fact

- ι is injective.
- For $f \in \mathcal{M}_{g,n}$, $f \cdot t_C \cdot f^{-1} = t_{f(C)}$, i.e.,

$$\iota(f(C)) = f \cdot \iota(C) \cdot f^{-1}.$$

Our starting point:

Lemma (K.)

For any group homomorphism $\varphi : \mathcal{M}_{g,n} \to G$*,*

$$\operatorname{Ker}\varphi \subset Z(\mathcal{M}_{g,n}) \quad \Leftrightarrow \quad \varphi \circ \iota \text{ is injective}$$

where Z *denotes the center of the group.*

Proof.

- $f \cdot t_C \cdot f^{-1} = t_{f(C)}$ $(f \in \mathcal{M}_{g,n})$
- $\operatorname{Ker}(\mathcal{M}_{g,n} \to \mathrm{Aut}(\mathcal{S})) = Z(\mathcal{M}_{g,n})$
- The action on $\mathcal{S}$ can detect $\mathcal{S}$. □

By making use of this lemma, we can "visualize" the linearity of $\mathcal{M}_{g,n}$, up to center.

To explain this, we introduce the following.

Module of curves

$K[\mathcal{S}]$: the vector space over K with basis $\mathcal{S}$

Definition (K.)

A module of curves (of type $\mathcal{S}$) is defnied as the pair of

- an $\mathcal{M}_{g,n}$-module M (over K),
- an $\mathcal{M}_{g,n}$-equivariant surjective homomorphism $p : K[\mathcal{S}] \to M$.

If p is clear, we will simply refer to M as a module of curves.

Module of curves (2)

We say a module of curves is of finite dimension if its dimension over K is finite.

- A module of curves is nothing but $K[\mathcal{S}]$ divided by skein type relations, i.e., some formal finite sums of finite numbers of SCCs.
- There is only one example of finite dimensional module of curves given, in terms of skein type relations (Luo['97]).

N.B. Not all $\mathcal{M}_{g,n}$-modules admit the structure of module of curves. E.g., any $\mathcal{M}_g$-equivariant homomorphism $K[\mathcal{S}] \to H_1(\Sigma_g; K)$ must be zero, if $\mathrm{char}(K) \neq 2$.

Any linear rep. of $\mathcal{M}_{g,n}$ induces a module of curves.

V: a finite dimensional vector space over K
$\rho : \mathcal{M}_{g,n} \to \mathrm{GL}(V)$: a given linear representation
$\mathrm{End}\,(V)$ is naturally an $\mathcal{M}_{g,n}$-module by

$$f_* X = \rho(f) X \rho(f)^{-1} \quad (f \in \mathcal{M}_{g,n},\ X \in \mathrm{End}\,(V)).$$

Definition

Let $M_\rho := \mathrm{Spann}_K(\rho \circ \iota(\mathcal{S})) \subset \mathrm{End}\,(V)$.

M_ρ is preserved under the $\mathcal{M}_{g,n}$-action

$$\left(\because \quad \begin{aligned} f_*(\rho \circ \iota(C)) &= \rho(f)\rho(t_C)\rho(f)^{-1} = \rho(f t_C f^{-1}) \\ &= \rho(t_{f(C)}) = \rho \circ \iota(f(C)) \end{aligned} \right)$$

M_ρ receives a structure of fin. dim. module of curves with

$$p_\rho : K[\mathcal{S}] \to M_\rho, \quad p_\rho(C) := \rho \circ \iota(C).$$

Visualization for closed surface

The construction of the module of curves associated to a linear representation, together with the Starting Lemma, implies:

Theorem (K.)

Let $g \geq 1$ and $n \geq 0$. Then $\mathcal{M}_{g,n}$ admits a finite dimensional linear representation over K with kernel $\subset Z(\mathcal{M}_{g,n})$ if and only if it has a finite dimensional module of curves $p : K[\mathcal{S}] \to M$ such that $p|_{\mathcal{S}}$ is an injection.

Since $Z(\mathcal{M}_{g,n}) = 1$ for $g \geq 3$ and $n = 0$, we have

Corollary (K.)

For $g \geq 3$, $\mathcal{M}_g$ is K-linear if and only if it admits a finite dimensional module of curves $p : K[\mathcal{S}] \to M$ such that $p|_{\mathcal{S}}$ is injective.

Some problems

For a module of curves

$$p : K[\mathcal{S}] \to M,$$

- When does M have finite dimensions over K?
- When is $\operatorname{Ker} p$ finitely generated as an $\mathcal{M}_{g,n}$-module?

once-punctured surface: Setting

Σ_g : the closed oriented surface of genus $g \geq 2$, with a fixed based point $* \in \Sigma_g$

$\Sigma_{g,*}$: the pair of Σ_g and the based point $*$

$\mathcal{M}_g$: the MCG of Σ_g

$\mathcal{M}_{g,*}$: the MCG of $\Sigma_{g,*}$
(All homeo.s' & isotopies are assumed to fix $*$.)

Aside from $\mathcal{S}$, there is another geometric object contained in $\mathcal{M}_{g,*}$ such that the natural $\mathcal{M}_{g,*}$-action is ~~almost~~ effective and coincides with the conjugation action in $\mathcal{M}_{g,*}$:

Namely, the fundamental group $\pi_1(\Sigma_g, *)$.
This is described by the so-called the Birman exact sequence.

Birman exact sequence

$\mathrm{Homeo}_+(\Sigma_g)$: the space of all orientation preserving homeo.s' of Σ_g
$\mathrm{ev} : \mathrm{Homeo}_+(\Sigma_g) \to \Sigma_g$: the evaluation at $* \in \Sigma_g$

- ev is a locally trivial fibration with fiber $\mathrm{Homeo}_+(\Sigma_{g,*})$
- each connected component of $\mathrm{Homeo}_+(\Sigma_g)$ is contractible (Hamstrom['69] & Luke–Mason ['72])

Take the lowest part of long exact sequence of homotopy groups to obtain:

Birman ['69]

$$1 \to \pi_1(\Sigma_g, *) \xrightarrow{i} \mathcal{M}_{g,*} \xrightarrow{j} \mathcal{M}_g \to 1$$

Birman exact sequence (2)

$$1 \to \pi_1(\Sigma_g, *) \xrightarrow{i} \mathcal{M}_{g,*} \xrightarrow{j} \mathcal{M}_g \to 1$$

More explicitly,

i sends a loop α to the ending homeomorphisms of the isotopy of Σ_g extending the isotopy of $*$ which corresponds to the loop α^{-1}.

j : the homomorphism of forgetting $*$.

The natural action of $\mathcal{M}_{g,*}$ on $\pi_1(\Sigma_g, *)$ is described, like the action on $\mathcal{S}$, by:

$$i(f_* \gamma) = f \cdot i(\gamma) \cdot f^{-1}.$$

We identify $i(\pi_1(\Sigma_g, *))$ with $\pi_1(\Sigma_g, *)$ via i.

Analogue of the Starting Lemma

Lemma (K.)

For any group homomorphism $\rho : \mathcal{M}_{g,*} \to G$,

$$\rho \text{ is injective} \Leftrightarrow \text{its restriction to } \pi_1(\Sigma_g, *) \text{ is injective.}$$

Proof of "$\Leftarrow$". Suppose ρ is injective on $\pi_1(\Sigma_g, *)$ and $f \in \operatorname{Ker} \rho$.

- For each $\gamma \in \pi_1(\Sigma_g, *)$,

$$\rho(f_* \gamma) = \rho(f \gamma f^{-1}) = \rho(\gamma).$$

- Then the assumption implies $f_*(\gamma) = \gamma$ for $\forall \gamma \in \pi_1(\Sigma_g, *)$.
- By the Dehn–Nielsen theorem, the action of $\mathcal{M}_{g,*}$ on $\pi_1(\Sigma_g, *)$ is effective, and therefore $f = 1$.

□

The next problem: when a rep. of $\pi_1(\Sigma_g, *)$ extends to $\mathcal{M}_{g,*}$?
—To describe this, terms of deformation space seem appropriate.

Deformation space

G : a group fixed.

$\mathrm{Hom}(\pi_1(\Sigma_g, *), G)$: the set of all homomorphisms $\pi_1(\Sigma_g, *) \to G$

$X_G := \mathrm{Hom}(\pi_1(\Sigma_g, *), G)/G$ the quotient by post conjugation

The MCG $\mathcal{M}_{g,*}$ acts on $\mathrm{Hom}(\pi_1(\Sigma_g, *), G)$ by

$$f \cdot \phi = \phi \circ f^{-1} \quad (f \in \mathcal{M}_{g,*},\ \phi \in \mathrm{Hom}(\pi_1(\Sigma_g, *), G)).$$

The action of $\gamma \in \pi_1(\Sigma_g, *) \subset \mathcal{M}_{g,*}$ coincides with the conjugation by $\phi(\gamma)$; i.e., the action of $\pi_1(\Sigma_g, *)$ on X_G is trivial.

$\Rightarrow$

This action descends to an action of $\mathcal{M}_g$ on X_G via the Birman exact sequence.

Deformation space (2)

For $\phi \in \mathrm{Hom}(\pi_1(\Sigma_g, *), G)$, its representing class in X_G is denoted by $[\phi]$.

Lemma (K.)

If $\phi \in \mathrm{Hom}(\pi_1(\Sigma_g, *), G)$ extends to a homomorphism $\mathcal{M}_{g,*} \to G$, then $[\phi] \in X_G$ is a global fixed point of the $\mathcal{M}_g$-action on X_G.

Proof. Clear from the definitions. □

The convererse of this Lemma is probably not true.
(Partly because the centralizer of $\mathrm{Im}\,\phi$ in G is not trivial.)

Nevertheless, in case of $G = \mathrm{GL}(n, K)$, any global fixed point of $\mathcal{M}_g$-action on $X_{\mathrm{GL}(n,K)}$ induces a linear representation of $\mathcal{M}_{g,*}$.

A global fixed point induces a representation

For $\phi \in \mathrm{Hom}\,(\pi_1(\Sigma_g, *), \mathrm{GL}(n, K))$, we define
Ad $\phi : \pi_1(\Sigma_g, *) \to \mathrm{GL}(\mathrm{End}\,(n, K))$ by

$$\mathrm{Ad}\,\phi(\gamma)(M) = \phi(\gamma) M \phi(\gamma)^{-1} \quad \text{for } \gamma \in \pi_1(\Sigma_g, *),\ M \in \mathrm{End}\,(n, K).$$

$V_\phi := K[\phi(\pi_1(\Sigma_g, *))]$
V_ϕ is clearly a $\pi_1(\Sigma_g, *)$-submodule of $\mathrm{End}\,(\mathrm{GL}(n, K))$, which implies a linear representation

$$\mathcal{A}\,\phi : \pi_1(\Sigma_g, *) \to \mathrm{GL}(V_\phi).$$

Lemma (K.)

If ϕ represents a global fixed point of the $\mathcal{M}_g$-action on $X_{\mathrm{GL}(n,K)}$, then the correspondence $\phi(\gamma) \mapsto \phi(f_* \gamma)$ ($\gamma \in \pi_1(\Sigma_g, *)$, $f \in \mathcal{M}_{g,*}$) defines a linear representation

$$\Psi : \mathcal{M}_{g,*} \to \mathrm{GL}(V_\phi),$$

which extends the representation $\mathcal{A}\,\phi : \pi_1(\Sigma_g, *) \to \mathrm{GL}(V_\phi)$.

Visualization for $\mathcal{M}_{g,*}$

Observe

- If ϕ is injective, so is Ψ (since
$$\mathrm{Ker}\,\mathcal{A}\,\phi = \{\gamma \in \pi_1(\Sigma_g, *)\,;\,[\gamma, \pi_1(\Sigma_g, *)] \subset \mathrm{Ker}\,\phi\}$$
and $\pi_1(\Sigma_g, *)$ is center-free, it follows from the injectivity Lemma.)
- Since $V_\phi \subset \mathrm{End}\,(n, K)$, $\dim_K V_\phi \leq n^2$.

Now we have:

Theorem (K.)

Let $g \geq 2$. Then, $\mathcal{M}_{g,*}$ is K-linear
$\Leftrightarrow$
For some n, the action of $\mathcal{M}_g$ on $X_{\mathrm{GL}(n,K)}$ has a global fixed point represented by a faithful representation of $\pi_1(\Sigma_g, *)$.

If this is the case, there exists a faithful K-linear rep. of $\mathcal{M}_{g,*}$ of dimensions at most n^2.

Some consequences

Fixed points in low dimensions:

- Korkmaz ['11] has shown: for $g \geq 3$, there exsit no faithful $\mathbb{C}$-linear representation of $\mathcal{M}_{g,*}$ of dimension $\leq 3g - 3$.
- this result implies via our theorem that there are no global fixed points in $X_{\mathrm{GL}(n,\mathbb{C})}$ represented by a faithful representation with

$$n \leq \sqrt{3g - 3}.$$

Some consequences (2)

Dynamical Properties of $\mathcal{M}_g$-action on $X_{\mathrm{GL}(n,K)}$:

- Recent studies have revealed that the action is a complicated mixture of properly discontinuous & ergodic actions (under suitable topology and measure).
- Our theorem states that the faithful finite-dim rep. can exits only in the extremely opposite to the properly discontinuous part.
- On the other hand, even the full ergodicity of the action is not enough to imply the non-lineariy. (can't prohibit a single global fixed point represented by a faithful rep. of $\pi_1(\Sigma_g, *)$.)

Some consequences (3)

A consequence of the fixed point Lemma:

- A well-known source of a faithful (projective) linear representation of $\pi_1(\Sigma_g, *)$ is a hyperbolic structure on Σ_g.
  (Take the holonomy representation to obtain
  $\phi : \pi_1(\Sigma_g, *) \to \mathrm{PSL}(2, \mathbb{R})$.)
- the stablizer of $[\phi] \in X_{\mathrm{PSL}(2,\mathbb{R})}$ is nothing but the orientation preserving isometry group of the hyperbolic structure, and therefore, a finite subgroup of $\mathcal{M}_g$.
- And therefore, by the fixed point Lemma, such a representation cannot extend to $\mathcal{M}_{g,*} \to \mathrm{PSL}(2, \mathbb{R})$.

A comparison with the preceding results

The visualization theorem for $\mathcal{M}_{g,*}$ seems new.
However, we can point out:

- A finitely generated linear group must be residually finite; (i.e., any non-identity element can be excluded by a finite index normal subgroup.)
- the residually finiteness of $\mathcal{M}_{g,*}$ was established by Baumslag ['63].
- the residually finiteness of $\mathcal{M}_g$ was established by Grossman ['74/'75].

Afterwards, Bass–Lubotky ['83]: the residually finiteness of $\mathcal{M}_g$ follows roughly from that of $\mathcal{M}_{g,*}$ together with the fact:

> *only mapping class of $\mathcal{M}_g$ which fixes all points in $X_{\mathrm{GL}(n,\mathbb{C})}$ represented by irreducible representations with n varying is the identity.*

This seems to form a contrast with our visualization.

A problem

There is a Birman exact sequence for $\mathrm{Aut}\,(F_n)$:

$$1 \to F_n \xrightarrow{i} \mathrm{Aut}\,(F_n) \xrightarrow{j} \mathrm{Out}\,(F_n) \to 1$$

The visualization theorem remains true with the same proof.

On the other hand, while $\mathrm{Aut}\,(F_2)$ is $\mathbb{C}$-linear (Krammer ['00]), $\mathrm{Aut}\,(F_n)$ is not linear for $n \geq 3$ (Formanek–Procesi ['92])
In particular, the visualization theorem implies:

> *If $n \geq 3$, there are no global fixed points in* Hom $F_n, \mathrm{GL}(m, K)/\mathrm{GL}(m, K)$ *represented by a faithful representation.*

Problem

Give a direct proof of this fact.

Summary

- $\mathcal{M}_g$ is K-linear if and only if there exists a finite dimensional module of curves
$$p : K[\mathcal{S}] \to M$$
such that $p|_{\mathcal{S}}$ is injective.

  (When is a module of curves finite dimensional, or has a finite MCG-generators, in general?)

- $\mathcal{M}_{g,*}$ is K-linear if and only if the $\mathcal{M}_g$-action on $X_{\mathrm{GL}(n,K)}$ has a global fixed point represented by a faithful representation of $\pi_1(\Sigma_g, *)$ for some n.

  (Find an alternate direct proof for the analogous results corresponding to $\mathrm{Aut}\,(F_n)$.)